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Generic Submanifolds of Almost Contact Metric Manifolds

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Abstract

Ronsse introduced the notion of generic and skew CR-submanifolds of almost Hermitian manifolds in order to unify and generalize the notions of holomorphic, totally real, CR, slant, semi-slant and pseudo-slant submanifolds. Other authors, such as Tripathi, extended this notion to contact geometry, under the name of almost semi-invariant submanifolds. This class includes the one with the same name introduced by Bejancu (and studied also by Tripathi), but without being equal. The class of submanifolds that we introduce and study here in contact geometry is called by us generic submanifolds, in order to avoid the above confusion, and also since it is different from the class studied by Tripathi, because in our paper, the Reeb vector field is not necessarily tangent to the submanifold. We obtain necessary and sufficient conditions for the integrability and parallelism of some eigen-distributions of a canonical structure on generic submanifolds. Some properties of the Reeb vector field to be Killing and its curves to be geodesics are investigated. Totally geodesic and mixed geodesic results on generic submanifolds are established. We give necessary and sufficient conditions for a generic submanifold to be written locally as a product of the leaves of some eigen-distributions. Some examples on both generic submanifolds and skew CR-submanifolds of almost contact metric manifolds are constructed.

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1 Introduction

Our attempt here is to fill a gap in the literature by studying in almost contact geometry a corresponding notion of the generic submanifolds defined in the sense of Ronsse [18] in the Kähler context (which is different from the one defined in the sense of Chen in [11], Wells in [27], Yano in [28] and so on). In particular, we study in almost contact context a special class of generic submanifolds, namely skew CR-submanifolds. We are motivated to provide some insight into these special classes of submanifolds in almost contact manifolds (in particular Sasakian), in view of their geometric, topological and physical importance. String theory and many other applications in theoretical physics stimulated the development of the study of Sasakian spaces, especially after the duality conjecture between conformal field theory and supergravity on anti-de-Sitter space time, see [16]. Moreover, an important role of Sasakian manifolds is played in time-dependent Mechanics. Contact geometry is also used in optics, phase space of dynamical system, mechanics, thermodynamics and control theory, [1, 15, 17].

The generic submanifolds in Kähler manifolds studied by Chen in [11] were also investigated from topological point of view.

Definition 1 [11] A submanifold M of a Kähler manifold $(\tilde{M}, J, g)$ is called generic, if the vector space of holomorphic tangent vectors to M at $p \in M$,

$$\mathcal{H}_p(M) = T_p(M) \cap JT_p(M),$$

(i.e. the maximal complex subspace of $T_p\tilde{M}$ contained in T_pM), has constant dimension along M (i.e. $\mathcal{H}(M)$ defines a differentiable distribution on M).

Later on, Bejancu introduced in [5] the CR-submanifolds, as a generalization of the invariant and the anti-invariant submanifolds of a Kähler manifold. CR-submanifolds can be viewed as a special class of generic submanifolds defined in [11].

The notion of CR-submanifolds was extended from Kähler manifolds to the almost Hermitian manifolds by the first author in [4].

Another notion of generic submanifolds in Kähler manifolds was given by Ronsse in [18] as follows:

Definition 2 [18] Let M be a submanifold of a Kähler manifold $(\tilde{M}, J, g)$. For any $X \in \Gamma(TM)$, PX is defined as the tangent part of JX. Then M is called a generic submanifold if there exist an integer k and some functions α_i, $1 \leq i \leq k$, defined on M with values in $(0, 1)$ such that

- Let $0, 1, -\alpha_i^2(p)$, $1 \leq i \leq k$ be all the distinct eigenvalues of P^2 corresponding to the eigenspaces $\Delta_p^0, \Delta_p^1, \Delta_p^{\alpha_i}$, $1 \leq i \leq k$ such that the following orthogonal decomposition holds

$$T_pM = \Delta_p^0 \oplus \Delta_p^1 \oplus \Delta_p^{\alpha_1} \oplus \Delta_p^{\alpha_2} \oplus \cdots \oplus \Delta_p^{\alpha_k},\ for\ p \in M.$$

– The dimensions of $\Delta_p^0, \Delta_p^1, \Delta_p^{\alpha_1}, \Delta_p^{\alpha_2}, \ldots, \Delta_p^{\alpha_k}$, $1 \leq i \leq k$, are independent of $p \in M$.
  If in addition, each α_i is constant on M, then M is called a skew CR-submanifold.

Hence in Kähler geometry, there are different notions of generic submanifolds: one presented in Definition 1 by Chen [11] and the other one, given in Definition 2 by Ronsse [18], which implies the first one. More precisely, if a submanifold is generic in the sense of Ronsse, then it is generic in the sense of Chen.

Ronsse's notion was also studied in the context of complex space forms by Tripathi [23], and later on it was studied by the second author and et al. in the theory of submersions [19, 20]. Moreover, Ronsse introduced the notion of skew CR-submanifolds, which generalize CR-submanifolds defined by Bejancu in [5].

Also, Ronsse's notion was extended from Kähler geometry to framed metric manifolds (and in particular almost contact geometry) under the name of almost semi-invariant manifolds by Tripathi et al. in [22, 24, 25] and so on. This class of almost semi-invariant submanifolds includes the one with the same name introduced by [6], (and studied also in [21]), but without being equal. The class of submanifolds that we introduce and study here in contact geometry, is called by us generic submanifolds, in order to avoid the above confusion, and also since it is different from [2] and [22], because in our paper, the Reeb vector field is not necessarily tangent to the submanifold. We also cite here Lotta's paper [14], in the slant context.

We give now a brief overview of our new study on generic submanifolds which is done in contact geometry. After we introduce (slightly different from Tripathi's papers) generic submanifolds of almost contact metric manifolds in Definition 5 and in particular skew CR-submanifolds, we obtain necessary and sufficient conditions for the integrability of some eigen-distributions of a canonical operator, as well as some characterization of the parallelism of the above distributions. Some results here on the Reeb vector field are given, namely its property to be Killing and its integral curves to be geodesics. We establish when generic submanifolds are totally geodesic, or mixed geodesic. We give necessary and sufficient conditions for a generic submanifold to be written locally as a product of the leaves of some eigen-distributions. The last section is devoted to some examples on both generic submanifolds and skew CR-submanifolds of almost contact metric manifolds.

2 Preliminaries

This section is devoted to recall some basic notions mainly from the theory of submanifolds and almost contact geometry.

Definition 3 Let $\tilde{M}$ be a manifold.

(i) If $\tilde{\nabla}$ is a linear connection on $\tilde{M}$, then any (1,1)-tensor field T is called *parallel with respect to* $\tilde{\nabla}$, if $\tilde{\nabla}_X T = 0$, $\forall X \in \Gamma(T\tilde{M})$, where

$$(\tilde{\nabla}_X T)Y = \tilde{\nabla}_X(TY) - T\tilde{\nabla}_X Y,\ \forall X, Y \in \Gamma(T\tilde{M}).$$

(ii) Let $\tilde{\nabla}$ be as above. The distribution Δ on $\tilde{M}$ is called:

– *Parallel with respect to* $\tilde{\nabla}$ if

$$\tilde{\nabla}_X U \in \Delta, \ \forall U \in \Gamma(\Delta), \ X \in \Gamma(T\tilde{M});$$

– *Parallel with respect to a distribution* $\tilde{\Delta}$ on $\tilde{M}$, provided

$$\tilde{\nabla}_U V \in \Delta, \ \forall U \in \Gamma(\tilde{\Delta}), \ \forall V \in \Gamma(\Delta);$$

– *Parallel with respect to a vector field* $W \in \Gamma(T\tilde{M})$, if it is parallel with respect to the distribution span$\{W\}$, i.e.

$$\tilde{\nabla}_W Y \in \Delta, \ \forall Y \in \Gamma(\Delta).$$

(iii) On a Riemannian manifold $(\tilde{M}, g)$, when no linear connection $\tilde{\nabla}$ is specified, then ***(i)*** and ***(ii)*** apply to the the Levi-Civita connection of g.

Remark 1 Any parallel distribution on a manifold is integrable.

2.1 Submanifolds of Riemannian Manifolds

In this section, we give a brief overview for submanifolds of Riemannian manifolds, [20].

Let M be a submanifold of a Riemannian manifold $(\tilde{M}, g)$ and let $\tilde{\nabla}$ be its the Levi-Civita connection. To fix notations, the Gauss and Weingarten formulas are written as

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{1}$$
$$\tilde{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \ \forall X, Y \in \Gamma(TM), \ \forall U \in \Gamma((TM)^{\perp}), \tag{2}$$

where the tangential component ∇ and A (resp. the normal component B and $\nabla^{\perp}$) are the induced connection on M and the Weingarten operator (resp. the second fundamental form of M and the normal connection). Hence

$$g(B(X, Y), U) = g(A_U X, Y), \ \forall X, Y \in \Gamma(TM), \ U \in \Gamma((TM)^{\perp}). \tag{3}$$

Definition 4 Let M^n be a submanifold of a Riemannian manifold $(\tilde{M}, g)$.

(i) The submanifold M is called *totally umbilical* if

$$B(X, Y) = g(X, Y)H, \ \forall X, Y \in \Gamma(TM),$$

where

$$H = \frac{1}{n}\text{trace}B,$$

H is the *mean curvature tensor field* of M in $\tilde{M}$.

(ii) Any distribution Δ on M is called *totally geodesic* if

$$B(U, V) = 0, \ \forall U, V \in \Gamma(\Delta).$$

In this case, M is called Δ *totally geodesic*. Proper umbilical means totally umbilical, but not totally geodesic.

(iii) We say that M is (Δ_1, Δ_2)-*mixed geodesic* if

$$B(U, V) = 0, \ \forall U \in \Gamma(\Delta_1), \ \forall V \in \Gamma(\Delta_2),$$

where Δ_1 and Δ_2 are two distributions on M.

2.2 Almost Contact Metric Manifolds

Let $\tilde{M}$ be a C^∞-differentiable manifold. An *almost contact structure* on $\tilde{M}$, denoted by (F, ξ, η), consists of a (1,1)-tensor field F (called the structure tensor field), a vector field ξ (called Reeb vector field) and a 1-form η (the dual of ξ) such that

$$F^2 = -I + \eta \otimes \xi \tag{4}$$

and

$$\eta(\xi) = 1, \tag{5}$$

where I denotes the identity endomorphism of the fibre bundle $T\tilde{M}$. In this case, $(\tilde{M}, F, \xi, \eta)$ is called an *almost contact manifold*. It follows that the manifold is of odd dimension and one has:

$$F\xi = 0, \quad \eta \circ F = 0. \tag{6}$$

If a Riemannian metric g on $\tilde{M}$ satisfies

$$g(F\tilde{X}, F\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y}), \ \forall \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}), \tag{7}$$

then g is said to be adapted to the almost contact structure (F, ξ, η). In this case, $(\tilde{F}, \xi, \eta, g)$ (resp. $(\tilde{M}, F, \xi, \eta, g)$) is called *almost contact metric structure* (resp. *almost contact metric manifold*). By using (4) - (7), one can obtain the following relation:

$$\eta(\tilde{X}) = g(\tilde{X}, \xi), \ \forall \tilde{X} \in \Gamma(T\tilde{M}). \tag{8}$$

Let $D = \mathrm{Im}F = \mathrm{Ker}\eta$ denote the contact distribution of the manifold $\tilde{M}$. Hence the tangent bundle decomposes into the direct orthogonal sum:

$$T\tilde{M} = D \oplus \mathrm{span}\{\xi\}. \tag{9}$$

From (4) and (7), it follows that F is skew-symmetric with respect to g, which allows one to define the 2-form Ω, called *the fundamental 2-form of the almost contact metric structure* on $\tilde{M}$, by

$$\Omega(\tilde{X}, \tilde{Y}) = g(\tilde{X}, F\tilde{Y}), \ \forall \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}), \tag{10}$$

see [8]. Hence, $(\tilde{M}, \Omega)$ is an almost symplectic manifold whose importance arises from classical and analytical mechanics. When $\Omega = d\eta$, then the manifold $(\tilde{M}, F, \xi, \eta, g)$ is called a *contact metric manifold*. An almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ is called *Sasakian* if

$$(\tilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X, \ \forall X, Y \in \Gamma(T\tilde{M}), \tag{11}$$

see [8]. Any Sasakian manifold is a contact metric manifold.

In the next section, we shall use the following:

Remark 2 On an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$, the Reeb vector field ξ is Killing, provided η is parallel, since

$$\begin{aligned}(\mathcal{L}_\xi g)(\tilde{X}, \tilde{Y}) &= g(\tilde{\nabla}_{\tilde{X}}\xi, \tilde{Y}) + g(\tilde{X}, \tilde{\nabla}_{\tilde{Y}}\xi)\\ &= (\tilde{\nabla}_{\tilde{X}}\eta)\tilde{Y} + (\tilde{\nabla}_{\tilde{Y}}\eta)\tilde{X}, \ \forall \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}).\end{aligned} \tag{12}$$

3 Generic Submanifolds

This section consists of the construction, investigation and existence process of the notion of generic submanifold.

Let $(\tilde{M}, F, \xi, \eta, g)$ be an almost contact metric manifold and let M be a Riemannian submanifold of $\tilde{M}$. For any $X \in \Gamma(TM)$, we may write

$$FX = PX + NX, \tag{13}$$

where $PX \in \Gamma(TM)$ and $NX \in \Gamma((TM)^\perp)$.

Proposition 1 *Let M be a submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ and let P be the operator defined by* (13). *Then*

- ***(i)*** *P is skew-symmetric with respect to g on M;*
- ***(ii)*** *P^2 is symmetric with respect to g on M;*
- ***(iii)*** *All eigenvalues of P^2 are contained in* $[-1, 0]$.

Proof ***(i)*** follows from the skew-symmetry of F.

(ii) P^2 is symmetric since P is skew-symmetric. Another way to show the symmetry of P^2 is provided in the sequel. The relation (13) yields

$$F^2X = P^2X + NPX + FNX, \ \forall X \in \Gamma(TM).$$

From the skew-symmetry of F with respect to g, it follows

$$g(P^2X, Y) = -g(X, Y) + \eta(X)\eta(Y) + g(NX, NY), \ \forall X, Y \in \Gamma(TM).$$

By interchanging X and Y, in the above equality we obtain

$$g(X, P^2Y) = g(P^2X, Y), \ \forall X, Y \in \Gamma(TM),$$

which shows that P^2 is symmetric.

(iii) From ***(ii)***, the eigenvalues of P^2 are real numbers at each point $p \in M$. If σ denotes an eigenvalue of P^2, then the following two cases arise:

- **Case 1:** $\sigma = 0$.
  Let Δ^0 be the eigen-distribution of P^2, corresponding to the eigenvalue $\sigma = 0$. From the skew-symmetry of F, it follows that $P\Delta^0 \perp F(TM)$.
- **Case 2:** $\sigma \neq 0$.
  In this case we may take $u \in \Gamma(TM)$ to be an arbitrary fixed eigenvector field of P^2, which is unitary, i.e. $\|u\| = 1$. From the skew symmetry of F and the relation (13), we have

$$g(Fu, Pu) = -g(u, FPu) = -g(u, P^2u) = -\sigma. \tag{14}$$

Since in this case $\sigma \neq 0$, then from the last equalities, it follows that Fu and Pu are nonzero. Hence, if θ denotes the angle between Fu and Pu, then we may write

$$\cos\theta = \frac{g(Fu, Pu)}{\|Fu\|\|Pu\|} = \frac{-\sigma}{\|Fu\|\|Pu\|}. \tag{15}$$

If α denotes the angle between ξ and u, then

$$\cos\alpha = \frac{g(\xi, u)}{\|\xi\|\|u\|} = \eta(u).$$

Since from (14) one can see that Fu is nonzero it follows that $u \neq \pm\xi$ and therefore $\sin\alpha \neq 0$, i.e. $\alpha \in (0, \pi)$. In (15) we replace $\|Fu\|$ and $\|Pu\|$, respectively, from

$$\begin{aligned}\|Fu\|^2 &= g(Fu, Fu) = -g(F^2u, u)\\ &= g(u - \eta(u)\xi, u)\\ &= \|u\|^2 - \eta^2(u) = 1 - \cos^2\alpha = \sin^2\alpha\end{aligned}$$

and

$$\|Pu\|^2 = g(Pu, Pu) = g(Fu, Pu) = \|Fu\|\|Pu\|\cos\theta = \|Pu\|\sin\alpha\cos\theta.$$

Hence we obtain

$$\cos\theta = \frac{-\sigma}{\sin^2\alpha\cos\theta},$$

which shows that $\sigma \in [-1, 0)$.

We point out that both angles α and θ depend on u, which was not denoted explicitly for the sake of simplicity. From Cases 1 and 2 we complete the proof.

Remark 3 – From the above Proposition, P^2 has at each point the associated matrix diagonalizable;

- Based on the above proof, from now on any eigenvalue of P^2 will be denoted by $-\nu^2$, $\nu \in [0, 1]$;
- We may write $\nu = \sin\alpha(u)\cos\theta(u)$, $\alpha(u) \in [0, \pi]$, $\theta(u) \in [0, \frac{\pi}{2}]$ and $P^2X = -\nu^2X = \sin^2\alpha(u)\cos^2\theta(u)X$, for any $X \in \Delta^\nu$, where $\alpha(u)$ denotes the angle between ξ and u and $\theta(u)$ the angle between Fu and Pu, for any unitary eigenvector u of P^2;
- The existence of the Reeb vector field in the above proof shows that our work (in almost contact geometry) is different from [18], [20] in Kähler geometry and [3], [12], [19] in almost product geometry'
- Different from the almost product Riemannian case, in the almost contact framework, the (1,1)-tensor P is skew-symmetric.
- In the particular case, when ξ is tangent to M, the statement of the above proposition can be retrieved from Lemma 3.1. [25], which is given without proof.

From (13), (14), (6) and (7), we obtain

Corollary 1 *If M is a submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$, then the following conditions are equivalent:*

(i) *M is a leaf of the contact distribution of $\tilde{M}$;*
(ii) *The operator P coincides with the restriction of F to M;*
(iii) *The operator N is identically zero;*
(iv) *The only eigenvalue of P^2 is -1;*
(v) *The above angle $\alpha(u) = 0$, for any unitary eigenvector u of P^2;*
(vi) *ξ is orthogonal to M at any point $p \in M$.*

Despite the last statement of the Remark 3, the above corollary cannot be deduced from the study made in [25].

Let $-\nu^2$ be an eigenvalue of P^2 whose corresponding eigen-distribution will be denoted by Δ^ν. Since P^2 is diagonalizable we may take $-\nu_1^2(p), \ldots, -\nu_k^2(p)$ to be all distinct eigenvalues of P^2 at any $p \in M$, which yields the decomposition of T_pM into the direct orthogonal sum, i.e.

$$T_pM = \Delta_p^{\nu_1} \oplus \cdots \oplus \Delta_p^{\nu_k}. \tag{16}$$

Corresponding to the notions of generic and skew CR-submanifolds introduced by Ronsse in almost Hermitian context (see [18]), Uddin et al. gave in [26] the definition of generic and skew CR-submanifolds in almost contact framework under the condition when the Reeb vector field ξ of the almost contact manifold is tangent to the submanifold.

We give here a slightly more general definition (including the case when ξ is not necessarily tangent to the submanifold), as follows:

Definition 5 A submanifold M of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ is called *generic* if there exist some functions $\lambda_1, \ldots, \lambda_k : M \to (0, 1)$, for a positive integer k, such that at each $p \in M$:

(a) $-\lambda_1^2(p), \ldots, -\lambda_k^2(p)$ are distinct eigenvalues of P^2;

(b) the dimension of each $\Delta_p^0, \Delta_p^1, \Delta_p^{\lambda_1}, \ldots, \Delta_p^{\lambda_k}$ is independent of $p \in M$, where Δ_p^{λ} denotes the eigenspace corresponding to the eigenvalue $-\lambda(p)^2$ of P^2, for $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$;

(c) the tangent space decomposes into the direct orthogonal sum

$$T_pM = \Delta_p^0 \oplus \Delta_p^1 \oplus \Delta_p^{\lambda_1} \oplus \Delta_p^{\lambda_2} \oplus \cdots \oplus \Delta_p^{\lambda_k}.$$

When $\lambda_1, \ldots, \lambda_k$ are constant, we call M a *skew CR-submanifold*.

Remark 4 – In the above definition, Δ^0 (resp. Δ^1) is the maximal anti-invariant (resp. the maximal invariant) distribution with respect to F.

- We note that Definition 5 is a generalization of some classes of submanifolds, described as follows:
  Let $d_0, d_1, d_{\lambda_1}, \ldots, d_{\lambda_k}$ denote, respectively, the dimensions of the distributions $\Delta^0, \Delta^1, \Delta^{\lambda_1}, \Delta^{\lambda_2}, \ldots, \Delta^{\lambda_k}$.
  - If $d_1 = d_{\lambda_1} = \cdots = d_{\lambda_k} = 0$, then M is an anti-invariant submanifold, (see [13]);
  - if $d_0 = d_1 = 0$ and $k = 1$, then M is a proper slant submanifold which was first studied by A. Lotta, (see [14]);
  - If $d_{\lambda_1} = \cdots = d_{\lambda_k} = 0$, then M is a semi-invariant submanifold, (see [7]);
  - If $d_0 = 0$, $k = 1$ and λ_1 is constant, then M is a semi-slant submanifold, (see [9]);
  - If $d_1 = 0$, $k = 1$ and λ_1 is constant, then M is a pseudo-slant submanifold, (see [10]);
  - If $d_0 = d_1 = 0$, $k = 2$ and λ_1, λ_2 are constants , then M is a bi-slant submanifold, (see [9]);
- We emphasize two concepts: on one side the notion of CR-submanifolds in the Kähler context (see [5]), and on the other side its extension to the notion of the semi-invariant submanifolds in almost contact geometry (see [7]). In both these instances, the tangent space of the submanifold splits into two orthogonal distributions, one of which is invariant and the other one is anti-invariant with respect to the structure (1,1)-tensor field (namely the almost complex structure J in the first case and the almost contact structure F in the second case). Hence both these concepts are extended by the notion of skew CR-submanifold introduced in Definition 5.
- Different from [26], where Uddin et al. studied generic and skew CR-submanifolds in the warped product framework, in the present paper we follow a different direction of study and all results obtained here have no similarities with the ones provided in [26].
- The notion of “generic submanifold", introduced by the above definition corresponds to “almost semi-invariant submanifold” introduced in [25] in a different context. We prefer to call it “generic" since here we work under different conditions and also to avoid the confusion with the almost semi-invariant submanifolds studied in [7].

– The notion of skew CR-submanifold, introduced by the above definition corresponds to "almost semi-invariant* submanifold", introduced in [25] in a different context.

Proposition 2 *Let M be a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$. Then*

(a) *Any distribution $\Delta_p^{\nu_i}$ is P-invariant, for $i \in \{1, \dots, n\}$;*

(b) *For any nonzero eigenvalue, the corresponding eigen-distribution is even dimensional.*

Proof Let fix an arbitrary $i \in \{1, \dots, n\}$ and let $-\nu_i^2$ be an eigenvalue of P^2 whose associated eigen-distribution is $\Delta_p^{\nu_i}$. For any $j \in \{1, \dots, n\}$, $j \neq i$, the skew-symmetry of P yields:

$$\begin{aligned}\nu_i^2 g(PU, V) &= -\nu_i^2 g(U, PV) = g(P^2U, PV) = -g(PU, P^2V)\\ &= \nu_j^2 g(PU, V),\ \forall U \in \Gamma(\Delta_p^{\nu_i}),\ \forall V \in \Gamma(\Delta_p^{\nu_j}).\end{aligned}$$

Since the two eigenvalues are distinct, it follows

$$g(PU, V) = 0,\ \forall U \in \Gamma(\Delta_p^{\nu_i}),\ \forall V \in \Gamma(\Delta_p^{\nu_j}),$$

which shows ***(a)***.

Then ***(b)*** follows from skew-symmetry of P and the P-invariance of distributions.

By using (13), (7) and the skew-symmetry of P, it follows:

Lemma 1 *If M is a submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$, then under the above notations we have*

$$\|NX\|^2 = \|X\|^2 - (\eta(X))^2 + g(X, P^2X),\ \forall X \in \Gamma(TM). \tag{17}$$

We prefer to prove in detail the following statement, which is given in [25] slightly different.

Proposition 3 *If M is a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$, then $\Delta^0 = Ker P$ and $\Delta^1 = Ker N \cap D$.*

Proof The first equality follows from the skew-symmetry of P.

If $X \in T_pM$ is a tangent vector to M at a $p \in M$, then we have to prove following equivalence :

$$\textbf{\textit{(i)}}\ X \in \Delta^1 \quad \Leftrightarrow \quad \textbf{\textit{(ii)}}\ X \in Ker N \quad \text{and} \quad \textbf{\textit{(iii)}}\ X \in D.$$

If we assume ***(i)***, which means

$$P^2X = -X, \tag{18}$$

then (17) becomes

$$\|NX\|^2 = -(\eta(X))^2,$$

which shows that both NX and $\eta(X)$ vanish, i.e. ***(ii)*** and ***(iii)***.

Conversely, if we assume ***(ii)*** and ***(iii)***, then (17) becomes

$$\|X\|^2 + g(X, P^2X) = 0. \tag{19}$$

Since P^2 is symmetric, then there exists an orthonormal basis $\{e_i\}_i$ in T_pM, of eigenvectors of P^2, corresponding to distinct eigenvalues $\{-\nu_i^2(p)\}_i$. If we write

$$X = \sum_i X_i e_i,$$

then (19) becomes

$$\sum_i X_i^2(1 - \nu_i^2(p)) = 0.$$

From Definition 5 (b), one has $\nu_i \in [0, 1]$, $\forall i$, which yields that P^2 has only one eigenvalue $-\nu^2 = -1$. Hence we obtain ***(i)*** which complete the proof.

Remark 5 The existence of contact distribution D makes the above result different from the Kähler case studied by Ronsse [18].

From Proposition 3, we obtain:

Lemma 2 *Let M be a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$. If the Reeb vector field ξ is tangent to M, then $\Delta^0 \cap Ker N = \text{span}\{\xi\}$.*

Remark 6 When ξ is tangent to M, then the above Lemma yields $\text{span}\{\xi\} \subseteq \Delta^0$, which gives the following orthogonal decomposition

$$\Delta^0 = \widetilde{\Delta^0} \oplus \text{span}\{\xi\}, \tag{20}$$

where $\widetilde{\Delta^0}$ denotes the orthogonal complement of $\text{span}\{\xi\}$ in Δ^0. Hence, $\widetilde{\Delta^0}$ is contained in the contact distribution, which means $\widetilde{\Delta^0} \subseteq D$, or equivalently

$$\eta(\widetilde{\Delta^0}) = 0. \tag{21}$$

Proposition 4 *Let M be a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ with ξ tangent to M and Δ^0 parallel with respect to ξ. Then any integral curve of ξ is a geodesic on M if and only if $\widetilde{\Delta^0}$ is parallel with respect to ξ, (where the parallelism is considered with respect to the Levi-Civita connection ∇ on M).*

Proof When ξ is tangent to M, we assume that Δ^0 is parallel with respect to ξ, i.e. $\nabla_\xi X \in \Delta^0$, $\forall X \in \Gamma(\Delta^0)$. In particular:

$$\nabla_\xi \xi, \ \nabla_\xi W \in \Gamma(\Delta^0), \ \forall W \in \Gamma(\widetilde{\Delta^0}). \tag{22}$$

The Remark 6, Definition 3 and (22) yield, for any $W \in \Gamma(\widetilde{\Delta^0})$, the following equivalence:

$$\widetilde{\Delta^0} \textit{ is parallel with respect to } \xi \Leftrightarrow \nabla_\xi W \in \widetilde{\Delta^0} \Leftrightarrow g(\nabla_\xi W, \xi) = 0$$
$$\Leftrightarrow g(W, \nabla_\xi \xi) = 0 \Leftrightarrow \nabla_\xi \xi \in \operatorname{span}\{\xi\}.$$

As ξ is unitary, one has obviously

$$g(\nabla_\xi \xi, \xi) = 0. \tag{23}$$

From (22) and (23), it follows that $\widetilde{\Delta^0}$ is parallel with respect to ξ if and only if $\nabla_\xi \xi = 0$, which complete the proof.

Theorem 1 *Let M be a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ whose fundamental 2-form Ω is closed.*

(i) *Then the distribution $Ker P$ is integrable.*

(ii) *When ξ is tangent to M, then the distribution $\widetilde{\Delta^0}$ is integrable if and only if η restricted to $\widetilde{\Delta^0}$ is closed.*

(iii) *When ξ is tangent to M, then it is Killing on M (resp. on any leaf of $Ker P$), provided η is parallel on M (resp. on any leaf of $Ker P$). Moreover, $\mathcal{L}_\xi g = 0$ on any leaf of $\widetilde{\Delta^0}$ provided η restricted to $\widetilde{\Delta^0}$ is parallel.*

Proof ***(i)*** Let $Y, Z \in \Gamma(\Delta^0)$ and $U \in \Gamma(\Delta^1 \oplus \Delta^{\lambda_1} \oplus \Delta^{\lambda_2} \oplus \cdots \oplus \Delta^{\lambda_k})$. Since U is orthogonal to Δ^0, it follows that there exists $X \in \Gamma(TM)$, such that $PX = U$. By using $PY = PZ = 0$, we have

$$\begin{aligned} g([Y, Z], U) &= g([Y, Z], PX) = Xg(Z, PY) - Zg(X, PY) - Yg(X, PZ) \\ &\quad - g([X, Z], PY) - g([Y, X], PZ) + g([Y, Z], PX) \\ &= X\Omega(Y, Z) - Z\Omega(X, Y) - Y\Omega(X, Z) \\ &\quad - \Omega([X, Z], Y) - \Omega([Y, X], Z) - \Omega([Z, Y], X) \\ &= d\Omega(X, Y, Z) \\ &= 0, \end{aligned}$$

which shows that $[Y, Z] \in \Gamma(\Delta^0)$.

(ii) Let $U, V \in \Gamma(\widetilde{\Delta^0})$. From ***(i)***, it follows $[U, V] \in \Gamma(\Delta^0)$. By using (21), we have

$$g([U, V], \xi) = \eta([U, V]) = d\eta(U, V).$$

Hence, $[U, V] \in \Gamma(\widetilde{\Delta^0})$ if and only if $d\eta(U, V) = 0$.

(iii) When ξ is tangent to M, then from (1), (12) and ***(i)***, it follows that ξ is Killing on M (resp. on any leaf of $Ker P$), provided η is parallel on M (resp. on any leaf of $Ker P$). If moreover, η restricted to $\widetilde{\Delta^0}$ is parallel, then η restricted to $\widetilde{\Delta^0}$ is closed, and from ***(ii)***, it follows that $\widetilde{\Delta^0}$ is integrable. Hence from (12), we obtain $\mathcal{L}_\xi g = 0$ on any leaf of $\widetilde{\Delta^0}$, which complete the proof.

Remark 7 The clue of the proof ***(i)*** above consists of the fact that the following two cases arise:

- Either ξ is tangent to M, in which case $\xi \in \Gamma(\Delta^0)$ from Lemma 2
- Or ξ is not tangent to M, in which case $\xi \notin \Gamma(\Delta^1 \oplus \Delta^{\lambda_1} \oplus \Delta^{\lambda_2} \oplus \cdots \oplus \Delta^{\lambda_k})$.

Corollary 2 *If M is a generic submanifold of a contact metric manifold $\tilde{M}$, then Δ^0 is integrable.*

Now we recall the following

Theorem 2 [14] *If $\tilde{M}$ is a contact metric manifold, whose Reeb vector field ξ is normal to a submanifold M, then M is anti-invariant.*

Based on Theorem 2, a particular case of Corollary 2 is commented on by the following:

Remark 8 If in Corollary 2, ξ is normal to M, then $\Delta^0 = TM$.

Theorem 3 *Let $\tilde{M}$ be an almost contact metric manifold. Let M be a generic submanifold, whose Levi-Civita connection is denoted by ∇, such that P^2 is parallel.*

(i) *Then, M is a skew CR-submanifold.*

(ii) *Corresponding to each eigenvalue $-\nu^2$ of P^2, the eigen-distribution Δ^ν, is parallel.*

(iii) *The distribution $\widetilde{\Delta^0}$ is parallel if and only if η is covariantly constant on $\widetilde{\Delta^0}$, i.e. $\nabla_X \eta = 0$ on $\widetilde{\Delta^0}$, $\forall X \in \Gamma(TM)$.*

Remark 9 The parallelism of the objects mentioned in Theorem 3 refers to the Levi-Civita connection ∇ on M and not on $\tilde{M}$.

Proof ***(i)*** Fix $-\nu^2$ to be an eigenvalue of P^2. For any $p \in M$, $v \in \Delta_p^\nu$ and $X \in \Gamma(TM)$, there exists a nonzero vector field $V \in \Gamma(\Delta^\nu)$ which is the parallel translate of v along the integral curves of X, i.e. $\nabla_X V = 0$. Therefore, from the parallelism of P^2, we obtain

$$\begin{aligned} X(-\nu^2)V &= \nabla_X(-\nu^2 V) + \nu^2 \nabla_X V \\ &= \nabla_X(P^2 V) + \nu^2 \nabla_X V \\ &= P^2 \nabla_X V + \nu^2 \nabla_X V = 0, \end{aligned}$$

which shows that ν^2 is constant.

(ii) From ***(i)***, Definitions 3 and 5, we obtain

$$P^2 \nabla_X U = \nabla_X P^2 U = \nabla_X(-\nu^2 U) = -\nu^2 \nabla_X U, \ \forall U \in \Gamma(\Delta^\nu), \ X \in \Gamma(TM),$$

which shows that

$$\nabla_X U \in \Gamma(\Delta^\nu), \ \forall U \in \Gamma(\Delta^\nu), \ X \in \Gamma(TM),$$

i.e. the distribution Δ^ν is parallel.

(iii) Let $X \in \Gamma(TM)$ and $W \in \Gamma(\widetilde{\Delta^0})$. Hence $\eta(W) = 0$ and from ***(i)***, we have $\nabla_X W \in \Gamma(\Delta^0)$. Since

$$g(\xi, \nabla_X W) = \eta(\nabla_X W) = -(\nabla_X \eta)W,$$

we obtain $\nabla_X W \in \Gamma(\widetilde{\Delta^0})$ if and only if $(\nabla_X \eta)W = 0$, which complete the proof.

Remark 10 The fact that P is parallel implies P^2 is parallel but the converse is not valid.

Corollary 3 *If M be a generic submanifold of an almost contact metric manifold $\tilde{M}$ such that P^2 is parallel, then M can be written locally as*

$$M = M_0 \times M_1 \times M_{\lambda_1} \times \cdots \times M_{\lambda_k},$$

where $M_0, M_1, M_{\lambda_1}, \ldots, M_{\lambda_k}$ are, respectively, the leaves of the eigen-distributions $\Delta^0, \Delta^1, \Delta^{\lambda_1}, \ldots, \Delta^{\lambda_k}$ of P^2.

Let $(\tilde{M}, F, \xi, \eta, g)$ be an almost contact manifold and let M be a Riemannian submanifold of $\tilde{M}$. For any $U \in \Gamma((TM)^{\perp})$, we may write

$$FU = tU + fU, \tag{24}$$

where $tU \in \Gamma(TM)$ and $fU \in \Gamma((TM)^{\perp})$.

Remark 11 When ξ is tangent to M, then the above corollary and the statements **(i)** and **(ii)** from Theorem 3 can be retrieved from Theorem 6.1 [25].

Notation: If M is a generic submanifold of an almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$, then we may write the Reeb vector field ξ decomposed as:

$$\xi = \xi_T + \xi_N, \tag{25}$$

where ξ_T and ξ_N are the tangent and the normal part of ξ to M, respectively.

Lemma 3 *Let M be a generic submanifold of a Sasakian manifold $(\tilde{M}, F, \xi, \eta, g)$. Then*

(i) *we have:*

$$\begin{aligned} P[X,Y] &= \nabla_X PY - \nabla_Y PX - A_{NY}X + A_{NX}Y \\ &\quad -\eta(Y)X + \eta(X)Y,\ \forall X, Y \in \Gamma(TM), \end{aligned} \tag{26}$$

$$\begin{aligned} N[X,Y] &= B(X, PY) - B(Y, PX) + \nabla^{\perp}_X NY \\ &\quad -\nabla^{\perp}_Y NX,\ \forall X, Y \in \Gamma(TM). \end{aligned} \tag{27}$$

(ii) *P is parallel with respect to ∇ if and only if*

$$A_{NY}Z - A_{NZ}Y + \eta(Z)Y + \eta(Y)Z = 0,\ \forall Y, Z \in \Gamma(TM); \tag{28}$$

(iii) *in particular, when* ξ *is normal to* M*, we have that* P *is parallel with respect to* ∇ *if and only if*

$$A_{NY}Z = A_{NZ}Y, \ \forall Y, Z \in \Gamma(TM). \tag{29}$$

Proof From (11), we have

$$(\tilde{\nabla}_X F)Y = g(X, Y)\xi - \eta(Y)X, \ \forall X, Y \in \Gamma(TM).$$

By using (1), (2), (13) and (24), it follows that

$$\begin{aligned} 0 &= \tilde{\nabla}_X FY - F\tilde{\nabla}_X Y - g(X, Y)\xi - \eta(Y)X \\ &= \nabla_X PY + B(X, PY) - A_{NY}X + \nabla^{\perp}_X NY - P\nabla_X Y \\ &\quad -N\nabla_X Y - tB(X, Y) - fB(X, Y) - g(X, Y)\xi - \eta(Y)X. \end{aligned} \tag{30}$$

If we decompose (30) into the tangent and the normal part to M, then by using (25), we obtain

$$P\nabla_X Y = \nabla_X PY - A_{NY}X - tB(X, Y) - g(X, Y)\xi_T - \eta(Y)X, \tag{31}$$

$$N\nabla_X Y = B(X, PY) + \nabla^{\perp}_X NY - fB(X, Y) - g(X, Y)\xi_N, \tag{32}$$

which yield (26) and (27), i.e. ***(i)*** is verified.

From Definition 3, (25) and (31), we obtain

$$(\nabla_X P)Y = A_{NY}X + tB(X, Y) + g(X, Y)\xi_T + \eta(Y)X,$$

where $X, Y \in \Gamma(TM)$. For any $Z \in \Gamma(TM)$, by using (8), (13), (3), the symmetry of A and the skew symmetry of F, we obtain

$$\begin{aligned} g((\nabla_X P)Y, Z) &= g(A_{NY}X + tB(X, Y) + g(X, Y)\xi_T + \eta(Y)X, Z) \\ &= g(A_{NY}X, Z) + g(tB(X, Y), Z) + g(X, Y)\eta(Z) + g(X, Z)\eta(Y) \\ &= g(A_{NY}Z - A_{NZ}Y + \eta(Z)Y + \eta(Y)Z, X), \end{aligned} \tag{33}$$

which gives ***(ii)***.

When ξ is normal to M, then $\eta(Z) = 0$, $\forall Z \in \Gamma(TM)$ and $P \equiv 0$. Hence (33) yields ***(iii)***.

Theorem 4 *Let* M *be a generic submanifold of a Sasakian manifold* $(\tilde{M}, F, \xi, \eta, g)$*, and let* ∇ *be the Levi-Civita connection on* M*.*

(i) Δ^1 *is integrable if and only if the following conditions hold:*

$$g(\nabla_X \xi_T, Y) = g(\nabla_Y \xi_T, X), \ \forall X, Y \in \Gamma(\Delta^1), \tag{34}$$

$$B(X, PY) = B(Y, PX), \ \forall X, Y \in \Gamma(\Delta^1). \tag{35}$$

(ii) $\Delta^0 \oplus \Delta^1$ *is integrable if and only if the following conditions hold:*

$$\nabla_X PY - \nabla_Y PX \in \Delta^1, \; \forall X, Y \in \Gamma(\Delta^1), \tag{36}$$

$$\nabla_X PY + A_{NX}Y \in \Delta^1, \; \forall X \in \Gamma(\Delta^0), \; \forall Y \in \Gamma(\Delta^1). \tag{37}$$

(iii) *When Δ^1 is integrable and M is $(\Delta^1, \Delta^0 \oplus \Delta^\lambda)$-mixed geodesic for $\lambda \in \{\lambda_1, \ldots, \lambda_k\}$, then the distribution $\Delta^0 \oplus \Delta^1$ is integrable if and only if Δ^1 is parallel with respect to Δ^0.*

Proof ***(i)*** Since $X, Y \in \Gamma(\Delta^1)$, based on Proposition 3, we have $\eta(X) = \eta(Y) = 0$, and hence:

$$\begin{aligned} g(\xi, [X, Y]) &= g(\xi, \tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = g(\xi, \tilde{\nabla}_X Y) - g(\xi, \tilde{\nabla}_Y X) \\ &= X\eta(Y) - g(\tilde{\nabla}_X \xi, Y) - Y\eta(X) + g(\tilde{\nabla}_Y \xi, X) \\ &= g(\tilde{\nabla}_Y \xi, X) - g(\tilde{\nabla}_X \xi, Y) \\ &= g(\nabla_Y \xi_T - A_{\xi_N} Y, X) - g(\nabla_X \xi_T - A_{\xi_N} X, Y) \\ &= g(\nabla_Y \xi_T, X) - g(B(Y, X), \xi_N) - g(\nabla_X \xi_T, Y) + g(B(X, Y), \xi_N) \\ &= -g(Y, \nabla_X \xi_T) + g(X, \nabla_Y \xi_T), \; \forall X, Y \in \Gamma(\Delta^1), \end{aligned}$$

from the symmetry of B. From (27), we obtain

$$N[X, Y] = B(X, PY) - B(Y, PX), \; \forall X, Y \in \Gamma(\Delta^1).$$

The statement follows from Proposition 3 and Frobenius theorem.

(ii) Since Δ^0 is integrable, then $[X, Y] \in \Gamma(\Delta^0)$ for any $X, Y \in \Gamma(\Delta^0)$. Based on Proposition 3, for any $X, Y \in \Gamma(\Delta^1)$, we have $NX = NY = 0$ and $\eta(X) = \eta(Y) = 0$. Hence in (26), it follows that

$$P[X, Y] = \nabla_X PY - \nabla_Y PX, \; \forall X, Y \in \Gamma(\Delta^0).$$

Thus $[X, Y] \in \Delta^0 \oplus \Delta^1$ if and only if

$$\nabla_X PY - \nabla_Y PX \in \Delta^1, \; \forall X, Y \in \Gamma(\Delta^1).$$

If $X \in \Gamma(\Delta^0)$ and $Y \in \Gamma(\Delta^1)$, from (26), we have

$$P[X, Y] = \nabla_X PY + A_{NX}Y + \eta(X)Y,$$

which shows that $[X, Y] \in \Delta^0 \oplus \Delta^1$ if and only if

$$\nabla_X PY + A_{NX}Y \in \Delta^1, \; \forall X \in \Gamma(\Delta^0), \forall Y \in \Gamma(\Delta^1).$$

(iii) Under the condition Δ^1 is integrable, then from Proposition 3, it follows that (36) is satisfied. From ***(ii)***, we obtain $\Delta^0 \oplus \Delta^1$ is integrable if and only if (37) holds, which is equal to

$$g(\nabla_X PY + A_{NX}Y, Z) = 0, \forall X \in \Gamma(\Delta^0),\ \forall Y \in \Gamma(\Delta^1),\ \forall Z \in \Gamma(\Delta^0 \oplus \Delta^\lambda),$$

where $\lambda \in \{\lambda_1, \ldots, \lambda_k\}$. Since M is $(\Delta^1, \Delta^0 \oplus \Delta^\lambda)$-mixed geodesic for $\lambda \in \{\lambda_1, \ldots, \lambda_k\}$, we have

$$\begin{aligned} g(\nabla_X PY, Z) &= -g(A_{NX}Y, Z) \\ &= g(B(Y, Z), NX) \\ &= 0, \end{aligned}$$

which shows that $\nabla_X PY \in \Delta^1$, $\forall X \in \Gamma(\Delta^0)$, $\forall Y \in \Gamma(\Delta^1)$, which complete the proof.

Remark 12 The case when ξ is tangent to M can be retrieved from Proposition 8 and 10 in [24], but we treated the arbitrary case when ξ is not necessarily tangent to M (i.e. ξ is tranversal to M) in a unitary way in the above Theorem.

Corollary 4 *There are no proper umbilical generic submanifolds in almost contact metric manifold $(\tilde{M}, F, \xi, \eta, g)$ with the distribution Δ^1 integrable, when ξ is tangent or normal to M.*

Remark 13 When ξ is tangent to M, the condition Δ^1 integrable is not necessary as one can see Theorem 15, [24].

Proposition 5 *Let M be a generic submanifold of a Sasakian manifold $(\tilde{M}, F, \xi, \eta, g)$ and let ∇ denote the Levi-Civita connection on M. Then P restricted to Δ^1 is parallel with respect to ∇, i.e. $(\nabla_X P)Y = 0,\ X, Y \in \Gamma(\Delta^1)$.*

Proof The statement is obtained from Proposition 3 and the relation (28), which are written for $Y, Z \in \Gamma(\Delta^1)$.

We recall the covariant derivative of the canonical structure N as in the following:

$$(D_X N)Y = \nabla^\perp_X NY - N\nabla_X Y, \tag{38}$$

where $X, Y \in \Gamma(TM)$. In this case, N is called *parallel* if

$$(D_X N)Y = 0,\ X, Y \in \Gamma(TM). \tag{39}$$

Lemma 4 *Let M be a generic submanifold of a Sasakian manifold $(\tilde{M}, F, \xi, \eta, g)$. Then: **(i)** N is parallel if and only if*

$$A_{fU}Y + A_U PY = \eta(U)Y,\ \forall Y \in \Gamma(TM),\ \forall U \in \Gamma(TM^\perp). \tag{40}$$

(ii) *in particular, when ξ is tangent to M, we have N is parallel if and only if*

$$A_{fU}Y + A_U PY = 0,\ \forall Y \in \Gamma(TM),\ \forall U \in \Gamma(TM^\perp). \tag{41}$$

Proof ***(i)*** Let ξ be arbitrary. By using (32) and (38), we have

$$(D_X N)Y = fB(X,Y) - B(X,PY) + g(X,Y)\xi_N. \tag{42}$$

Multiplication of the last equation with any normal vector $U \in \Gamma(TM^{\perp})$, by using (3), the symmetry of A and the skew symmetry of F, yields

$$\begin{aligned} g((D_X N)Y, U) &= g(fB(X,Y) - B(X,PY) + g(X,Y)\xi, U) \\ &= -g(B(X,Y), fU) - g(A_U X, PY) + g(X,Y)\eta(U) \\ &= g(-A_{fU}Y - A_U PY + \eta(U)Y, X), \end{aligned}$$

which gives (40).

(ii) When ξ is tangent to M, then $\eta(U) = 0$, $\forall U \in \Gamma(TM^{\perp})$ and hence (40) yields (41), which complete the proof.

Proposition 6 *Let M be a generic submanifold of a Sasakian manifold $(\tilde{M}, F, \xi, \eta, g)$ with parallel canonical structure N and let ∇ denote the Levi-Civita connection on M.*

1) *When ξ is tangent to M, then:*

(i)

$$A_{fU}Y = 0,\ \forall Y \in \Gamma(\Delta^0),\ \forall U \in \Gamma(TM^{\perp}). \tag{43}$$

(ii) *M is $(\Delta^{\lambda}, \Delta^{\beta})$-mixed geodesic $\forall \lambda \neq \beta,\ \lambda,\ \beta \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$.*

Moreover, for any $Z \in \Gamma(\Delta^{\lambda})$ one of the followings holds:

- *$B(Z, Z) = 0$,*
- *$B(Z, Z)$ is an eigenvalue of f^2 with eigenvalue $-\lambda^2$.*

2) *When ξ is normal to M, then:*

(iii)

$$A_{f\xi}Y = Y,\ \forall Y \in \Gamma(TM). \tag{44}$$

(iv)

$$A_{fU}Y = \eta(U)Y,\ \forall Y \in \Gamma(TM),\ \forall U \in \Gamma(TM^{\perp}). \tag{45}$$

Proof **1)** If we write (41), for any $Y \in \Gamma(\Delta^0)$ and for any $U \in \Gamma(TM^{\perp})$, then we obtain (43).

Now, by using (42) under the condition ξ is tangent to M, we obtain

$$f^2 B(X,Y) = -\lambda^2 B(X,Y),\ \forall X, Y \in \Gamma(TM), \tag{46}$$

which proves ***(ii)***, for any $X \in \Gamma(\Delta^{\lambda})$ and for any $Y \in \Gamma(\Delta^{\beta})$, $\lambda \neq \beta,\ \lambda,\ \beta \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$.

Moreover, from (46), we obtain for any $Z \in \Gamma(\Delta^{\lambda})$, either $B(Z, Z) = 0$ or $B(Z, Z)$ is an eigenvector of f^2 with eigenvalue $-\lambda^2$.

2) When ξ is normal to M, then from Theorem 2, we have $\Delta^0 = TM$. If (40) is considered for $U = \xi$ and $Y \in \Gamma(\Delta^0)$, then (44) is obtained.

Writing (40) for $Y \in \Gamma(\Delta^0)$, it gives ***(iv)***, which complete the proof.

Remark 14 The first statement of **(ii)** can be proved by using Proposition 8 and Theorem 18 from [24].

4 Examples

Example 1 Let $\tilde{M}$ be the 7-dimensional unit sphere,

$$S^7 = \{x = (x_0, \dots, x_7) \in \mathbb{R}^8 |\, x_0^2 + \cdots + x_7^2 = 1\},$$

whose normal vector field is denoted by

$$\tilde{\mathbf{N}} = x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + x_6\partial_6 + x_7\partial_7,$$

and whose parallelization is given by the following tangent vector fields

$$\begin{aligned}
\tilde{X}_0 &= -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3 - x_5\partial_4 + x_4\partial_5 + x_7\partial_6 - x_6\partial_7,\\
\tilde{X}_1 &= -x_2\partial_0 + x_3\partial_1 + x_0\partial_2 - x_1\partial_3 - x_6\partial_4 - x_7\partial_5 + x_4\partial_6 + x_5\partial_7,\\
\tilde{X}_2 &= -x_3\partial_0 - x_2\partial_1 + x_1\partial_2 + x_0\partial_3 - x_7\partial_4 + x_6\partial_5 - x_5\partial_6 + x_4\partial_7,\\
\tilde{X}_3 &= -x_4\partial_0 + x_5\partial_1 + x_6\partial_2 + x_7\partial_3 + x_0\partial_4 - x_1\partial_5 - x_2\partial_6 - x_3\partial_7,\\
\tilde{X}_4 &= -x_5\partial_0 - x_4\partial_1 + x_7\partial_2 - x_6\partial_3 + x_1\partial_4 + x_0\partial_5 + x_3\partial_6 - x_2\partial_7,\\
\tilde{X}_5 &= -x_6\partial_0 - x_7\partial_1 - x_4\partial_2 + x_5\partial_3 + x_2\partial_4 - x_3\partial_5 + x_0\partial_6 + x_1\partial_7,\\
\tilde{X}_6 &= -x_7\partial_0 + x_6\partial_1 - x_5\partial_2 - x_4\partial_3 + x_3\partial_4 + x_2\partial_5 - x_1\partial_6 + x_0\partial_7,
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ for any $i \in \{0, \dots, 7\}$. An almost contact metric structure (F, ξ, η, g) can be defined on $\tilde{M}$ by

$$\begin{aligned}
\xi &= \tilde{X}_0,\\
F\tilde{X}_i &= -\tilde{X}_{i+1},\ \forall i \in \{1, 3, 5\},\\
F\tilde{X}_i &= \tilde{X}_{i-1},\ \forall i \in \{2, 4, 6\},
\end{aligned}$$

with the standard Euclidean metric g on S^7, and the 1-form η of ξ with respect to g, i.e.

$$\eta(\tilde{X}_0) = 1 \text{ and } \eta(\tilde{X}_i) = 0,\ i \in \{1, \dots, 6\}.$$

If we take

$$\begin{aligned}
M = S^3 &= \{x = (x_0, \dots, x_3) \in \mathbb{R}^4 |\, x_0^2 + \cdots + x_3^2 = 1\}\\
&\equiv \{x = (x_0, \dots, x_7) \in \mathbb{R}^8 \,|\, x_0^2 + \cdots + x_3^2 = 1 \text{ and } x_4 = \cdots = x_7 = 0\} \subset S^7,
\end{aligned}$$

then $\mathbf{N}$ (resp. Z_0, Z_1, Z_2) denotes a unit normal vector field (resp. a global frame) on S^3, where

$$\begin{aligned}\mathbf{N} &= x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\\ Z_0 &= x_1\partial_0 - x_0\partial_1 + x_3\partial_2 - x_2\partial_3,\\ Z_1 &= x_2\partial_0 - x_3\partial_1 - x_0\partial_2 + x_1\partial_3,\\ Z_2 &= x_3\partial_0 + x_2\partial_1 - x_1\partial_2 - x_0\partial_3.\end{aligned}$$

Hence,

$$TM = \Delta^0 \oplus \Delta^1,$$

where

$$\Delta^0 = \mathrm{span}\{Z_0\}, \;\; \Delta^1 = \mathrm{span}\{Z_1, Z_2\}.$$

The only eigenvalues of P^2 are 0 and -1.

The submanifold constructed in [9] with a different purpose can be adapted here to obtain a new example, as follows:

Example 2 Let $\tilde{M} = \mathbb{R}^{11} = \left\{(x_1, \ldots, x_5, y_1, v, y_5, z) | \, x_i, y_i, z \in \mathbb{R}, \; i = 1, \ldots, 5\right\}$ be endowed with the Sasakian structure (F, ξ, η, g), where

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{5}((dx_i)^2 + (dy_i)^2), \quad \xi = 2\frac{\partial}{\partial z}, \; \eta = \frac{1}{2}dz \quad and$$
$$F(\partial x_i) = -\partial y_i, \quad F(\partial y_i) = \partial x_i, \quad F(\partial z) = 0,$$

with $\partial \bullet_i = \frac{\partial}{\partial \bullet_i}$ for any $i \in \{1, \ldots, 5\}$. Let M be the 7-dimensional submanifold defined by

$$x(u, v, w, r, s, t, z) = 2(u, 0, v, 0, w, 0, r, sins, -coss, t, z),$$

where $s \neq 0$. One can see that

$$\begin{aligned}&E_1 = 2\partial x_1, \quad E_2 = 2\partial y_2, \quad E_3 = 2coss\partial y_3 + 2sins\partial y_4,\\ &E_4 = 2\partial x_3, \quad E_5 = 2\partial x_5, \quad E_6 = 2\partial y_5, \quad E_7 = 2\partial z\end{aligned}$$

restricted to M form a global orthonormal frame of TM.

$$\mathbf{N} = a\partial x_2 + b\partial x_4 + c\partial y_1 + d(-sins\partial y_3 + coss\partial y_4), \; (a, b, c, d \in \mathbb{R})$$

denotes a normal vector field on M. Since

$$\begin{aligned}P(E_1) &= 0, \qquad P(E_2) = 0, \qquad P(E_3) = coss E_4,\\ P(E_4) &= -coss E_3, \quad P(E_5) = -E_6, \quad P(E_6) = E_5,\end{aligned}$$

we have

$$TM = \widetilde{\Delta^0} \oplus \mathrm{span}\{\xi\} \oplus \Delta^1 \oplus \Delta^\lambda,$$

where the eigenvalues of P^2 are 0, -1, $\lambda = coss$ and

$$\widetilde{\Delta^0} = \mathrm{span}\{E_1, E_2\},\ \Delta^1 = \mathrm{span}\{E_5, E_6\},\ \Delta^\lambda = \mathrm{span}\{E_3, E_4\}.$$

Therefore, M is a generic submanifold of the Sasakian manifold $\tilde{M}$.

The lowest dimensional skew CR-submanifold is 5, as one can see in the following example:

Example 3 Let $\tilde{M} = S^1 \times \cdots \times S^1$ be the 7-dimensional torus with the product Riemannian metric g and let $\mathbb{B} = (X_1, \ldots, X_7)$ be a global frame of orthonormal vector fields on T^7, each of them tangent, respectively, to each cycle. Let M be the 5-dimensional torus, embedded as $T^5 \times \{0\}$ in $\tilde{M}$, having $X_1, \ldots, X_5$ tangent to M. We take $\xi = X_1$ and η its dual 1-form. With respect to $\mathbb{B}$, we define

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & \pi & t \\ 0 & 0 & N & f \end{pmatrix},$$

where we denote by the same letters as the (1,1)-tensor fields in (13) and (24), respectively, the matrices:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & \pi \end{pmatrix},\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pi = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix},$$
$$N = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix},\ t = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix},$$

with the real numbers $\lambda \neq 0, 1$, $\beta \neq 0$, $u = (\lambda^2 - 1)/\beta$. Hence, (F, ξ, η, g) is an almost contact structure on $\tilde{M}$. Since with respect to $X_1, \ldots, X_5$,

$$P^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -\lambda^2 I \end{pmatrix},$$

where I is the unit 2-dimensional matrix, it follows

$$TM = \Delta^0 \oplus \Delta^1 \oplus \Delta^\lambda,$$

with $\Delta^0 = \mathrm{span}\{\xi\}$, $\Delta^1 = \mathrm{span}\{X_2, X_3\}$ and $\Delta^\lambda = \mathrm{span}\{X_4, X_5\}$, all these vector fields being a restricted to M. Therefore, M is a skew CR-submanifold of $\tilde{M}$.

We provide now an easy example to show that the Reeb vector field ξ of an almost contact Riemannian manifold could be neither tangent nor normal to a generic submanifold.

Example 4 Let U be a 1-dimensional Riemannian manifold. Let

$$\tilde{M} = U \times U \times U = \{(p_0, p_1, p_2) |\ p_0, p_1, p_2 \in U\}$$

be the Riemannian product endowed with the Riemannian metric g.

Let $\{X_0, X_1, X_2\}$ be an orthonormal basis with respect to g, such that X_i is tangent respectively to $\{(p_1, 0, 0)|p_1 \in U\}$, $\{(0, p_2, 0)|p_2 \in U\}$ and $\{(0, 0, p_3)|p_3 \in U\}$, $i \in \{0, 1, 2\}$.

We take $\xi = X_0$ and with respect to this basis, we define

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let M be a submanifold of $\tilde{M}$ defined by

$$M = \{(p_0, p_1, p_2) |\ p_0 = p_1 \text{ and } p_0, p_1, p_2 \in U\} = \{(p_0, p_0, p_2)|p_0, p_2 \in U\}.$$

Then $X_0 + X_1$ and X_2 (restricted to M) is a basis of tangent vector fields on M. Hence

$$\xi = X_0 = \frac{1}{2}\Big[(X_0 + X_1) + (X_0 - X_1)\Big] = \xi_T + \xi_N,$$

where

$$\xi_T = \frac{1}{2}(X_0 + X_1),$$
$$\xi_N = \frac{1}{2}(X_0 - X_1).$$

We have

$$P(X_0 + X_1) = F(X_0 + X_1) = -X_2; \tag{47}$$
$$F(X_2) = X_1 = \frac{1}{2}(X_0 + X_1) - \frac{1}{2}(X_0 - X_1)$$

and hence

$$P(X_2) = \frac{1}{2}(X_0 + X_1). \tag{48}$$

From (47) and (48), we have

$$P^2(X_2) = \frac{1}{2}P(X_0 + X_1) = -\frac{1}{2}X_2$$

and

$$P^2(X_0 + X_1) = P(-X_2) = -\frac{1}{2}(X_0 + X_1),$$

from which we obtain

$$\Delta^{\frac{\sqrt{2}}{2}} = \mathrm{span}\{X_0 + X_1, X_2\} = TM.$$

Based on the idea used in Example 4, we may construct a more sophisticated example, of a generic submanifold $\mathcal{M}$ in an almost contact Riemannian manifold $\tilde{M}$, such that the Reeb vector field ξ is neither tangent nor normal to $\mathcal{M}$.

Example 5 Let $\tilde{M}$ be the manifold endowed with the almost contact Riemannian structure (F, ξ, η, g) constructed in Example 3 and let $\mathcal{M}$ be a submanifold of $\tilde{M}$ defined by:

$$\mathcal{M} = \{(p_1, \ldots, p_5, 0, 0) \in \tilde{M} | p_1 = p_2 \text{ and } p_1, \ldots, p_5 \in S^1\}.$$

If $\{X_1, \ldots, X_7\}$ is the global frame of orthonormal vector fields on $\tilde{M}$ given in Example 3, then $\{X_1 + X_2, X_3, X_4, X_5\}$, restricted to $\mathcal{M}$, is a basis of tangent vector fields on $\mathcal{M}$ and we have

$$\xi = X_1 = \frac{1}{2}\Big[(X_1 + X_2) + (X_1 - X_2)\Big] = \xi_T + \xi_N,$$

where

$$\xi_T = \frac{1}{2}(X_1 + X_2),$$
$$\xi_N = \frac{1}{2}(X_1 - X_2).$$

We calculate

$$P(X_1 + X_2) = F(X_1 + X_2) = -X_3$$
$$F(X_3) = X_2 = \frac{1}{2}(X_1 + X_2) - \frac{1}{2}(X_1 - X_2),$$

from which:

$$P(X_3) = \frac{1}{2}(X_1 + X_2).$$

We also have

$$P(X_4) = -\lambda X_5 \text{ and } P(X_5) = \lambda X_4,$$
$$\Delta^{\frac{\sqrt{2}}{2}} = \mathrm{span}\{X_1 + X_2, X_3\} \text{ and } \Delta^{\lambda} = \mathrm{span}\{X_4, X_5\},$$

and the rest can easily be deduced, in the same way as in the last two examples.

Finally, we remark that the first three examples have Δ^0 odd-dimensional, while the last two examples have Δ^0 zero-dimensional (hence even-dimensional).

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