Bull. Malays. Math. Sci. Soc. (2022) 45:2745–2763
https://doi.org/10.1007/s40840-022-01309-w

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

On Estimates of Some Coefficient Functionals for Certain Meromorphic Univalent Functions

Bappaditya Bhowmik[1] · **Firdoshi Parveen**[2]

Received: 10 August 2021 / Revised: 31 March 2022 / Accepted: 2 May 2022 /
Published online: 23 May 2022

Abstract

Let $\mathcal{V}_p(\lambda)$ be the class of all functions f defined on the open unit disc $\mathbb{D}$ of the complex plane having simple pole at $z = p$, $p \in (0, 1)$ and analytic in $\mathbb{D}\backslash\{p\}$ satisfying the normalizations $f(0) = 0 = f'(0) - 1$ such that $\left|(z/f(z))^2 f'(z) - 1\right| < \lambda$ for $z \in \mathbb{D}$, $\lambda \in (0, 1]$. In this article, we obtain sharp bounds of the Zalcman and the generalized Zalcman functionals for functions in $\mathcal{V}_p(\lambda)$ for some indices of these functionals. As consequences of the obtained results, we pose the Zalcman-like coefficient conjectures for this class of functions. In addition, we estimate bound for the generalised Fekete–Szegö functional along with bounds of the second- and the third-order Hankel determinants for this class of functions.

Keywords Taylor coefficients · Zalcman functional · Generalized Zalcman functional · Krushkal functional · Fekete–Szegö functional · Hankel determinant

Mathematics Subject Classification 30C45 · 30C50 · 30C55

Communicated by See Keong Lee.

The first author of this article would like to thank SERB, DST, India (Ref. No.- MTR/2018/001176) for its financial support through MATRICS grant.

✉ Bappaditya Bhowmik
bappaditya@maths.iitkgp.ac.in

Firdoshi Parveen
frd.par@gmail.com

1 Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

2 Department of Mathematics, SRM University-AP, Andhra Pradesh, Andhra Pradesh 522502, India

1 Introduction and Preliminaries

We shall use the following notations throughout the discussion of this article. Let $\mathbb{C}$ be the whole complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Let $\mathcal{A}$ be the class of all analytic functions f defined in $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$ and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent}\}$. Each $f \in \mathcal{S}$ has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

In the last century, the field of geometric function theory provided many interesting and fascinating facts. One of the main problems in this field was the Bieberbach conjecture, which was proposed in the year 1916. This conjecture states that each $f \in \mathcal{S}$ with the expansion (1.1) must satisfy the inequality $|a_n| \leq n$ for all $n \geq 2$. In the year 1985, de Branges (c.f. [9]) proved this conjecture. In order to settle the Bieberbach conjecture prior to the effort made by de Branges, many subclasses of $\mathcal{S}$ were introduced that are geometric in nature and the conjecture was being proved for these subclasses. Some of the special subclasses of $\mathcal{S}$ for which this conjecture was settled were the class of convex functions, starlike functions, and close-to-convex functions (c.f. [10]). Recently another subclass of $\mathcal{S}$, namely the class $\mathcal{U}(\lambda)$ got attention by many geometric function theorists. More precisely, the class $\mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, is defined as follows:

$$\mathcal{U}(\lambda) := \{f \in \mathcal{A} : |U_f(z)| < \lambda\},$$

where $U_f(z) := (z/f(z))^2 f'(z) - 1$, $z \in \mathbb{D}$. We refer to the articles [13, 20, 25] for more details about the class $\mathcal{U}(\lambda)$. There are several classical conjectures about the Taylor coefficients of functions belonging to certain classes of univalent functions; and till date, some of them are settled while others are not. One such conjecture is the famous Zalcman conjecture, (which we abbreviate as **ZC** throughout the discussion in this article), that was posed many years ago as an approach to prove the Bieberbach conjecture. More precisely, in the early 70's, L. Zalcman conjectured that the coefficients of $\mathcal{S}$ satisfy the sharp inequality $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for each $n \geq 2$, in which the equality holds only for the Koebe function $k(z) = z/(1-z)^2$, $z \in \mathbb{D}$ and its rotations. We mention here that the ZC implies the famous Bieberbach conjecture (see [5]). Also, the case $n = 2$ of the ZC, namely, $|a_2^2 - a_3| \leq 1$ for the class $\mathcal{S}$ is a simple consequence of the Gronwall area theorem (see for instance [10]). The ZC has been verified by a number of authors for certain subclasses of $\mathcal{S}$. For example, in 1986, Brown and Tsao (c.f. [5]) proved affirmatively for starlike functions and typically real functions. In [19], Ma proved for close-to-convex functions whenever $n \geq 4$, while Krushkal (c.f. [16]) proved for the case $n = 3$ and for $n = 4, 5, 6$ in [17]. We mention here that since the Koebe function $z/(1-z)^2$ belongs to the class of close-to-convex functions, the ZC has been settled by Krushkal (c.f. [16]) for $n = 3$ for this class.

In 1999, Ma proposed a generalized Zalcman conjecture (we abbreviate this as **gZC** from here on) (c.f. [18]) for $f \in \mathcal{S}$ that for $n \geq 2, m \geq 2$,

$$|a_m a_n - a_{m+n-1}| \leq (n-1)(m-1),$$

which is still an open problem. In the same article, Ma proved the gZC for starlike functions and univalent functions with real coefficients. In [11], Efraimidis and Vukotić proved that the gZC is asymptotically true for the class $\mathcal{S}$. Let us denote the Zalcman functionals and the generalized Zalcman functionals by μ_n and $\psi_{m,n}$ respectively, that is,

$$\mu_n := a_n^2 - a_{2n-1} \quad \text{and} \quad \psi_{m,n} := a_m a_n - a_{m+n-1}.$$

We note here that $\psi_{n,n} = \mu_n$ and $\psi_{m,n} = \psi_{n,m}$.

Motivated by the interesting work on functions in $\mathcal{U}(\lambda)$, a meromorphic analog of this class, namely $\mathcal{V}_p(\lambda)$ was introduced (see [2, 3]). We briefly demonstrate here about this class of functions. Let $\mathcal{A}(p)$ be the class that is defined as the collection of functions in $\mathbb{D}$ having a simple pole at $z = p$, where $p \in (0, 1)$ and analytic in $\mathbb{D}\backslash\{p\}$, satisfying the normalizations $f(0) = 0 = f'(0) - 1$. We define $\Sigma(p) := \{f \in \mathcal{A}(p) : f \text{ is univalent}\}$. In [3], $\mathcal{V}_p(\lambda)$ is defined as the class of all functions f in $\mathcal{A}(p)$ such that $\left|U_f(z)\right| < \lambda$, $\lambda \in (0, 1]$. In the same article, it is showed that $\mathcal{V}_p(\lambda) \subsetneq \Sigma(p)$ and many other results are obtained for functions in $\mathcal{V}_p(\lambda)$. As $f \in \mathcal{V}_p(\lambda)$ is analytic in $\mathbb{D}_p := \{z \in \mathbb{C} : |z| < p\}$, each $f \in \mathcal{V}_p(\lambda)$ has the following Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_p. \tag{1.2}$$

In [3], the authors established the following representation for functions in $\mathcal{V}_p(\lambda)$, i.e., each $f \in \mathcal{V}_p(\lambda)$ can be expressed as

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z \int_0^z w(t)\mathrm{d}t,$$

where w is analytic in $\mathbb{D}$ with $|w(z)| \leq 1$, for $z \in \mathbb{D}$. In the above representation, if we take

$$w_1(z) = \int_0^z w(t)\mathrm{d}t,$$

then w_1 is analytic in $\mathbb{D}$, $w_1(0) = 0$ and $|w_1(z)| \leq |z|$. Also we have $|w_1'(z)| = |w(z)| \leq 1$ for every $z \in \mathbb{D}$. Therefore, the aforementioned representation takes the form

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z w_1(z), \quad z \in \mathbb{D}. \tag{1.3}$$

In [3, Corollary 1], it is proved that $|a_2| \leq (p^{-1} + \lambda p)$ and equality occurs in this inequality for the function $k_p^{\lambda}(z) = \frac{-pz}{(z-p)(1-\lambda pz)} = \sum_{n=1}^{\infty} A_n z^n$, where

$$A_n = \frac{1-(\lambda p^2)^n}{p^{n-1}(1-\lambda p^2)}, \quad n \geq 1. \tag{1.4}$$

Also, in [4, Theorem 1], it is established that if $f \in \mathcal{V}_p(\lambda)$ for some $0 < \lambda \leq 1$ and has the expansion of the form (1.2) in $\mathbb{D}_p$, then

$$|a_n| \leq A_n, \tag{1.5}$$

for $n = 3$, $p \in (0, 1/2]$ and $n \geq 4$, $p \in (0, 1/3]$. Equality holds in the above inequalities for the function k_p^λ. It is evident from this discussion that the coefficient problem for functions in the class $\mathcal{V}_p(\lambda)$ has not yet been settled for all $n \geq 3$ and for all $p \in (0, 1)$. This serves as a motivation to study the ZC and the gZC for functions in $\mathcal{V}_p(\lambda)$.

In [12], M. Fekete and G. Szegö proved that the inequality

$$|a_3 - \mu a_2^2| \leq 1 + 2e^{-2\mu/(1-\mu)},$$

holds for $f \in \mathcal{S}$ and for $0 \leq \mu \leq 1$, and that this inequality is sharp for each μ. The coefficient functional

$$\Lambda_\mu(f) := a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}(f''(0))^2\right),$$

on normalized analytic functions f in the unit disc $\mathbb{D}$ is important in the sense that it can represent various geometric quantities in function theory. For example, $\Lambda_0(f) = a_3$ and $\Lambda_1(f) = a_3 - a_2^2$ represents $\mathcal{S}_f(0)/6$, where $\mathcal{S}_f$ denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of f. In the literature, there exists a large number of results on bounds for $|\Lambda_\mu(f)|$ corresponding to various subclasses of $\mathcal{S}$. The problem of maximizing the absolute value of the functional $\Lambda_\mu(f)$ is known as the Fekete–Szegö problem. Many authors have considered this problem for typical classes of univalent functions (see, for instance [1, 7, 14] and references therein). In this article we consider the Fekete–Szegö problem with real parameter μ for functions in $\mathcal{V}_p(\lambda)$. We also consider the coefficient functional $a_n - a_2^{n-1}$, $n \geq 4$ for functions in $\mathcal{V}_p(\lambda)$ which can be seen as a generalisation of the well-known Fekete–Szegö functional $\Lambda_1(f) = a_3 - a_2^2$. In this article, we obtain sharp upper bound for the modulus of this generalised Fekete–Szegö functional when $n = 4, 5$.

We now move on to another interesting and well-known coefficient functional, namely the Hankel determinant. Let $f \in \mathcal{A}$ having the Taylor series expansion of the form (1.1) in $\mathbb{D}$. The qth Hankel determinant of f is defined (see [22, 23]) for $q \geq 1$, and $n \geq 1$ by

$$\mathcal{H}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinants $\mathcal{H}_q(n)$ are useful in showing that a function is of bounded characteristic in $\mathbb{D}$, i.e., a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational (see [6]). Also, the Hankel determinant plays an important role, for instance, in the study of the singularities and in the study of power series with integral coefficients (c.f. [6]). We observe that $\mathcal{H}_2(1) = \Lambda_1(f)$ is the classical Fekete–Szegö functional, which has been considered since the 1930's and is still of great interest, especially in a modified version: $\Lambda_\mu(f) = a_3 - \mu a_2^2$, where $\mu \in \mathbb{C}$. In recent years many mathematicians have investigated Hankel determinants for various classes of functions which are contained in $\mathcal{A}$. These studies focus primarily on the main subclasses of $\mathcal{S}$ namely, the class of convex, starlike and close-to-convex functions. In fact, the majority of papers discuss the determinants $\mathcal{H}_2(2)$ and $\mathcal{H}_3(1)$. An overview of results on the upper bounds of $|\mathcal{H}_2(2)|$ and $|\mathcal{H}_3(1)|$ can be found in [15, 22–24, 26] and references therein. In this paper, we provide an estimate of the upper bound of $|\mathcal{H}_2(2)|$ and a sharp estimate of upper bound of $|\mathcal{H}_3(1)|$ for functions in $\mathcal{V}_p(\lambda)$.

We organize the obtained results in this article as follows. In Sect. 2, we obtain the sharp upper bounds of $|\mu_n|$ whenever $n = 2,\ p \in (0, 1)$ and $n = 3,\ p \in (0, (\sqrt{15} - 3)/2]$ for functions in $\mathcal{V}_p(\lambda)$. We also determine the sharp estimates of $|\psi_{m,n}|$ for $m = 2,\ n = 3$ and $m = 2,\ n = 4$, which are the main content of Theorem 2. Next, in Theorem 3, we find the sharp upper bound of $|\psi_{m,n}|$, when $m = 2, n \geq 5$ for certain range of values of $p \in (0, 1)$. We also pose Zalcman-like conjectures for functions in $\mathcal{V}_p(\lambda)$. Next, in Sect. 3, we obtain the generalized Fekete–Szegö inequality for functions in $\mathcal{V}_p(\lambda)$ and estimate the generalized Fekete–Szegö functional for $n = 4, 5$. Finally in Sect. 4, we find some bounds for the Hankel determinants ($\mathcal{H}_2(2)$ and $\mathcal{H}_3(1)$) for functions in $\mathcal{V}_p(\lambda)$, respectively.

2 Zalcman-Like Conjectures for the Class $\mathcal{V}_p(\lambda)$

In the following theorem, we obtain the sharp upper bounds of $|\mu_n|$ whenever $n = 2,\ p \in (0, 1)$ and $n = 3,\ p \in (0, (\sqrt{15} - 3)/2]$ for functions in the class $\mathcal{V}_p(\lambda)$.

Theorem 1 *If $f \in \mathcal{V}_p(\lambda)$ and has the expansion of the form (1.2) in $\mathbb{D}_p$, then*

(i) $|a_2^2 - a_3| \leq \lambda,\ p \in (0, 1)$,

(ii) $|a_3^2 - a_5| \leq \lambda \left(\frac{1}{p} + \lambda p\right)^2$ *for* $0 < p \leq (\sqrt{15} - 3)/2$.

Equalities hold in both the above inequalities for the function k_p^λ.

Proof Let the Taylor series expansion of w_1 in the representation formula (1.3) is $w_1(z) = \sum_{n=1}^{\infty} c_n z^n$. We have (see [21]) that if w_1 satisfies

$$w_1(0) = 0, |w_1(z)| \leq |z| \text{ and } |w_1'(z)| = |w(z)| \leq 1,$$

then

$$|c_1| \leq 1,\ |c_2| \leq \frac{1}{2}(1 - |c_1|^2) \quad \text{and} \quad |c_3| \leq \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|}\right). \tag{2.1}$$

Now Eq. (1.3) together with Eq. (1.2) give us

$$\begin{aligned} z &= \left(1 - a_2 z + \lambda z\{c_1 z + c_2 z^2 + c_3 z^3 + \cdots\}\right)\left(z + a_2 z^2 + a_3 z^3 + \cdots\right) \\ &= z + (a_3 - a_2^2 + \lambda c_1) z^3 + (a_4 - a_2 a_3 + \lambda c_1 a_2 + \lambda c_2) z^4 \\ &\quad + (a_5 - a_2 a_4 + \lambda c_1 a_3 + \lambda c_2 a_2 + \lambda c_3) z^5 + \cdots . \end{aligned} \tag{2.2}$$

Next, by comparing coefficient of z^3 from both sides of the above equation, we get $a_3 - a_2^2 + \lambda c_1 = 0$, which gives $|\mu_2| = |a_2^2 - a_3| = |\lambda c_1| \leq \lambda$. This completes the proof of the first part of the theorem . Next for the function k_p^λ, we compute $a_2^2 - a_3 = \lambda$. This shows that equality occurs in the inequality (i) for the function k_p^λ.

We now proceed to prove the second part of this theorem. By equating coefficients of z^4 and z^5 from both sides of Eq. (2.2), and by using $a_3 = a_2^2 - \lambda c_1$, we have

$$a_4 = a_2 a_3 - \lambda c_1 a_2 - \lambda c_2 = a_2^3 - 2\lambda c_1 a_2 - \lambda c_2$$

and

$$a_5 = a_2 a_4 - \lambda c_1 a_3 - \lambda c_2 a_2 - \lambda c_3 = a_2^4 - 3\lambda c_1 a_2^2 - 2\lambda c_2 a_2 + \lambda^2 c_1^2 - \lambda c_3.$$

Therefore,

$$\mu_3 = a_3^2 - a_5 = \lambda \left(c_3 + 2c_2 a_2 + c_1 a_3 + \lambda c_1^2\right). \tag{2.3}$$

Now by using the triangle inequality, the bounds for $|a_2|$, $|a_3|$ and Eq. (2.1), we have for $p \leq 1/2$,

$$\begin{aligned} &\left|c_3 + 2c_2 a_2 + c_1 a_3 + \lambda c_1^2\right| \\ &\quad \leq |c_3| + 2|c_2||a_2| + |c_1||a_3| + \lambda |c_1|^2 \\ &\quad \leq \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|}\right) + 2|c_2|\left(\frac{1}{p} + \lambda p\right) \\ &\qquad + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)|c_1| + \lambda |c_1|^2. \end{aligned}$$

Let us denote $x := |c_1|$ and $y := |c_2|$. Then $0 \leq x \leq 1$ and $0 \leq y \leq (1 - x^2)/2$. Let $\Omega := \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq (1 - x^2)/2\}$. Now let us define

$$\begin{aligned} f(x, y) = \frac{1}{3}\left(1 - x^2 - \frac{4y^2}{1 + x}\right) + 2\left(\frac{1}{p} + \lambda p\right) y \\ + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) x + \lambda x^2, \end{aligned}$$

for $(x, y) \in \Omega$. Then

$$\frac{\partial f}{\partial x} = -\frac{2x}{3} + \frac{4y^2}{3(1+x)^2} + 2\lambda x + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right),$$

which is always positive in Ω for $p \leq 1/2$. This shows that the extremum of $f(x, y)$ cannot be attained in the interior of the domain Ω. Since f is continuous and Ω is compact, the maximum of f will be attained at some boundary point of Ω. Now on the boundary $x = 0,\ 0 \leq y \leq 1/2$ of Ω, we have $f(0, y) = \left((1-4y^2)/3\right) + 2y\left(p^{-1} + \lambda p\right)$. Therefore, for $p \leq 1/2$ and $0 \leq y \leq 1/2$, we have

$$\frac{\partial f}{\partial y}(0, y) = -\frac{8y}{3} + 2\left(\frac{1}{p} + \lambda p\right) > 0.$$

This implies $f(0, y)$ is an increasing function of y and hence the maximum will be attained at $y = 1/2$, with the maximum value to be $\left(p^{-1} + \lambda p\right)$. Next, on the boundary $y = 0,\ 0 \leq x \leq 1$, we have

$$f(x, 0) = \frac{1}{3}(1 - x^2) + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) x + \lambda x^2.$$

Therefore, for $0 < p \leq 1/2$ and $0 \leq x \leq 1$, we have

$$\frac{\partial f}{\partial x}(x, 0) = -\frac{2x}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\lambda x > 0,$$

which implies $f(x, 0)$ is an increasing function of x and hence the maximum will be attained at $x = 1$ and the maximum value is $\left(p^{-1} + \lambda p\right)^2$. On the boundary $y = (1 - x^2)/2,\ 0 \leq x \leq 1$, we have

$$f\left(x, \frac{1}{2}(1 - x^2)\right) = \left(\frac{1}{p} + \lambda p\right) + \left\{\frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)\right\} x + \left\{\lambda - \left(\frac{1}{p} + \lambda p\right)\right\} x^2 - \frac{x^3}{3}.$$

Now for $0 \leq x \leq 1$,

$$\begin{aligned}\frac{\partial f}{\partial x}\left(x, \frac{1}{2}(1 - x^2)\right) &= \frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\left\{\lambda - \left(\frac{1}{p} + \lambda p\right)\right\} x - x^2 \\ &\geq \frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) - 2\left\{\left(\frac{1}{p} + \lambda p\right) - \lambda\right\} - 1 \\ &= \frac{(3 - 6p - 2p^2) + 6\lambda p^2(1 - p) + 3\lambda p^2 + 3\lambda^2 p^4}{3p^2},\end{aligned}$$

which is positive for $0 < p \leq (\sqrt{15}-3)/2$. Since $(\sqrt{15}-3)/2 < 1/2$, it follows that $f(x, (1-x^2)/2)$ is an increasing function of x whenever $0 < p \leq (\sqrt{15}-3)/2$ and hence the maximum will be attained at $x = 1$ and the maximum value is $\left(p^{-1}+\lambda p\right)^2$. We thus get the maximum value of $f(x, y)$ on Ω is $(p^{-1}+\lambda p)^2$ for $0 < p \leq (\sqrt{15}-3)/2$, and consequently, by using (2.3), we have

$$|a_3^2 - a_5| \leq \lambda \left(\frac{1}{p}+\lambda p\right)^2,$$

for $p \leq (\sqrt{15}-3)/2$. Now for the function k_p^λ, we have $a_3^2 - a_5 = \lambda(p^{-1}+\lambda p)^2$ which proves the sharpness of the inequality (ii) stated in the theorem. □

We now observe that, for the function k_p^λ, we have

$$A_n^2 - A_{2n-1} = \lambda A_{n-1}^2,$$

where A_n is given in (1.4). Also, Theorem 1 gives

$$|a_2^2 - a_3| \leq \lambda = \lambda A_1^2, \quad p \in (0, 1),$$

and

$$|a_3^2 - a_5| \leq \lambda \left(\frac{1}{p}+\lambda p\right)^2 = \lambda A_2^2, \quad p \leq (\sqrt{15}-3)/2.$$

These patterns of inequalities lead us to propose the general form of the Zalcman-like coefficient conjecture for functions in the class $\mathcal{V}_p(\lambda)$.

Conjecture 1 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2) in $\mathbb{D}_p$. Then for $n \geq 3$,*

$$|a_n^2 - a_{2n-1}| \leq \lambda A_{n-1}^2,$$

for all $\lambda \in (0, 1]$ and $p \in (0, 1)$. Equality holds in the above inequalities for the function k_p^λ.

Remark We note that the conjectured bounds of $|\mu_n|$, $n \geq 3$ for the class $\mathcal{V}_p(\lambda)$ as $p \to 1^-$ and $\lambda = 1$ coincide with the corresponding bounds of $|\mu_n|$ for the class $\mathcal{S}$.

The next theorem deals with the sharp estimates of $|\psi_{m,n}|$ for the cases $m = 2$, $n = 3$ and $m = 2$, $n = 4$, whenever $f \in \mathcal{V}_p(\lambda)$.

Theorem 2 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2) in $\mathbb{D}_p$. Then for all $\lambda \in (0, 1)$, we have*

(i) $|a_2a_3 - a_4| \leq \lambda\left(\frac{1}{p}+\lambda p\right)$ *for* $0 < p < 1$,

(ii) $|a_2a_4 - a_5| \leq \lambda\left(\frac{1}{p^2}+\lambda+\lambda^2p^2\right)$ *for* $0 < p \leq 1/2$.

Equalities hold in both the above inequalities for the function k_p^λ.

Proof (i) Comparing the coefficient of z^4 from both sides of (2.2), we have $a_2a_3-a_4=\lambda(c_1a_2+c_2)$. Therefore, the bound for $|a_2|$ and (2.1) together imply

$$|a_2a_3-a_4|\leq\lambda\left\{\left(\frac{1}{p}+\lambda p\right)|c_1|+\frac{1}{2}(1-|c_1|^2)\right\}. \tag{2.4}$$

Let $|c_1|=x$ and $f(x)=(p^{-1}+\lambda p)x+(1-x^2)/2$. Then $0\leq x\leq 1$, and $f'(x)=(p^{-1}+\lambda p)-x$ which is greater than 0 for $0<p<1$. This shows that the maximum of f is attained at $x=1$ and the maximum value is $(p^{-1}+\lambda p)$. Therefore, from Eq. (2.4), we have $|a_2a_3-a_4|\leq\lambda(p^{-1}+\lambda p)$ for $0<p<1$.

(ii) From Eq. (2.2), we obtain $a_2a_4-a_5=\lambda(c_1a_3+c_2a_2+c_3)$. Now by using the triangle inequality, the bounds for $|a_2|,\ |a_3|$ and Eq. (2.1), we have for $p\leq 1/2$,

$$\begin{aligned}&|a_2a_4-a_5|\\&\quad\leq\lambda\left[\frac{1}{3}\left(1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right)+\left(\frac{1}{p}+\lambda p\right)|c_2|+\left(\frac{1}{p^2}+\lambda+\lambda^2p^2\right)|c_1|\right].\end{aligned} \tag{2.5}$$

Let us denote $|c_1|=:x,\ |c_2|=:y$. Then (2.1) implies $0\leq x\leq 1, 0\leq y\leq(1-x^2)/2$. We consider $\Omega:=\{(x,y):0\leq x\leq 1\ \text{ and }\ 0\leq y\leq(1-x^2)/2\}$ and we define

$$f(x,y)=\frac{1}{3}\left(1-x^2-\frac{4y^2}{1+x}\right)+\left(\frac{1}{p}+\lambda p\right)y+\left(\frac{1}{p^2}+\lambda+\lambda^2p^2\right)x,$$

for $(x,y)\in\Omega$. We then get

$$\frac{\partial f}{\partial x}=-\frac{2x}{3}+\frac{4y^2}{3(1+x)^2}+\left(\frac{1}{p^2}+\lambda+\lambda^2p^2\right),$$

which is always positive in Ω for $p\leq 1/2$. This shows that the extremum of $f(x,y)$ cannot be attained at some interior point of the domain Ω. As f is continuous and Ω is compact, the maximum of f will be attained at some boundary point of Ω. Now on the boundary $x=0,\ 0\leq y\leq 1/2$ of Ω, we have $f(0,y)=\big((1-4y^2)/3\big)+\big(p^{-1}+\lambda p\big)\,y$. Therefore for $p\leq 1/2$ and $0\leq y\leq 1/2$, we have

$$\frac{\partial f}{\partial y}(0,y)=-\frac{8y}{3}+\left(\frac{1}{p}+\lambda p\right)>0.$$

This implies $f(0,y)$ is an increasing function of y and hence the maximum is attained at $y=1/2$ and the maximum value is $\big(p^{-1}+\lambda p\big)/2$. Next on the boundary $y=0,\ 0\leq x\leq 1$, $f(x,0)=\big((1-x^2)/3\big)+\big(p^{-2}+\lambda+\lambda^2p^2\big)\,x$. For $0<p\leq 1/2$ and $0\leq x\leq 1$, we have

$$\frac{\partial f}{\partial x}(x,0)=-\frac{2x}{3}+\left(\frac{1}{p^2}+\lambda+\lambda^2p^2\right)>0.$$

Thus $f(x, 0)$ is an increasing function of x and hence the maximum value is $f(1, 0) = \left(p^{-2} + \lambda + \lambda^2 p^2\right)$. On the boundary $y = (1 - x^2)/2,\ 0 \leq x \leq 1$, we have

$$f\left(x, \frac{1}{2}(1 - x^2)\right) = \frac{1}{2}\left(\frac{1}{p} + \lambda p\right) + \left\{\frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)\right\} x \\ -\frac{1}{2}\left(\frac{1}{p} + \lambda p\right) x^2 - \frac{x^3}{3}.$$

Now, for $0 \leq x \leq 1,\ 0 < p \leq 1/2$,

$$\begin{aligned} \frac{\partial f}{\partial x}\left(x, \frac{1}{2}(1 - x^2)\right) &= \frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) - \left(\frac{1}{p} + \lambda p\right) x - x^2 \\ &\geq \frac{1}{3} + \left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) - \left(\frac{1}{p} + \lambda p\right) - 1 \\ &= \frac{(3 - 3p - 2p^2) + 3\lambda p^2(1 - p) + 3\lambda^2 p^4}{3p^2} \\ &> 0, \end{aligned}$$

which implies that $f(x, (1 - x^2)/2)$ is an increasing function of x and hence the maximum will be attained at $x = 1$ and the maximum value is $\left(p^{-2} + \lambda + \lambda^2 p^2\right)$. Therefore from the above, we get the maximum value of $f(x, y)$ on Ω is $\left(p^{-2} + \lambda + \lambda^2 p^2\right)$ and hence from Eq. (2.5), we get for $p \leq 1/2$, $|a_2 a_4 - a_5| \leq \lambda\left(p^{-2} + \lambda + \lambda^2 p^2\right)$. Now a little calculation shows that $a_2 a_3 - a_4 = \lambda(p^{-1} + \lambda p)$ and $a_2 a_4 - a_5 = \lambda\left(p^{-2} + \lambda + \lambda^2 p^2\right)$ for the function k_p^λ defined in (1.4). This proves the sharpness of both inequalities stated in the theorem. □

We now determine the upper bounds for $|\psi_{m,n}|$ when $m = 2$, $n \geq 5$ for some range of values of p.

Theorem 3 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2). Then for $n \geq 5$, we have*

$$|a_2 a_n - a_{n+1}| \leq \lambda A_{n-1},$$

whenever $p \leq 1/3$. Equalities hold in the above inequalities for the function k_p^λ.

Proof Inserting the Taylor expansion for functions f and w_1 in (1.3) and then equating coefficients of z^{n+1} from both sides, we have

$$a_{n+1} = a_2 a_n - \lambda(c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{n-2} a_2 + c_{n-1} a_1).$$

Therefore, $a_2 a_n - a_{n+1} = \lambda \sum_{k=1}^{n-1} c_{n-k} a_k$, which gives

$$|a_2 a_n - a_{n+1}| \leq \lambda \left(\sum_{k=1}^{n-1} |c_{n-k}||a_k|\right). \tag{2.6}$$

Being the Taylor coefficients of the unimodular analytic function w_1, it is known that $c_n,\ n \geq 1$ satisfy the following inequalities (see [8, Lemma 2.1]):

$$|c_1| \leq 1 \quad \text{and} \quad |c_n| \leq 1 - |c_1|^2 \quad \text{for} \quad n \geq 2.$$

Now for $p \leq 1/3$, the above inequality and (1.5) together yield

$$\begin{aligned}\sum_{k=1}^{n-1} |c_{n-k}||a_k| &= \sum_{k=1}^{n-2} |c_{n-k}||a_k| + |c_1||a_{n-1}| \\ &\leq \sum_{k=1}^{n-2}(1-|c_1|^2)A_k + |c_1|A_{n-1} \\ &\leq 2(1-|c_1|)\sum_{k=1}^{n-2} A_k + |c_1|A_{n-1},\end{aligned} \tag{2.7}$$

as we know that $1 - |c_1|^2 \leq 2(1-|c_1|)$ for $0 \leq |c_1| \leq 1$. Now

$$\begin{aligned}2\sum_{k=1}^{n-2} A_k &= \frac{2}{(1-\lambda p^2)}\left[\sum_{k=1}^{n-2}\frac{1}{p^{k-1}} - \sum_{k=1}^{n-2}\lambda^k p^{k+1}\right] \\ &= \frac{2}{(1-\lambda p^2)}\left[\frac{1-p^{n-2}}{p^{n-3}(1-p)} - \lambda p^2 \frac{1-(\lambda p)^{n-2}}{1-\lambda p}\right].\end{aligned}$$

So the inequality $2\sum_{k=1}^{n-2} A_k \leq A_{n-1}$ is equivalent to

$$\frac{2}{(1-\lambda p^2)}\left[\frac{1-p^{n-2}}{p^{n-3}(1-p)} - \lambda p^2 \frac{1-(\lambda p)^{n-2}}{1-\lambda p}\right] \leq \frac{1-(\lambda p^2)^{n-1}}{p^{n-2}(1-\lambda p^2)},$$

which yields

$$\frac{1-3p+2p^{n-1}}{p^{n-2}(1-p)} + \frac{2\lambda p^2 - 3\lambda^{n-1}p^n + \lambda^n p^{n+1}}{1-\lambda p} \geq 0.$$

We observe that $(1 - 3p + 2p^{n-1}) \geq 0$ for $p \leq 1/3$ and we see

$$\begin{aligned}&2\lambda p^2 - 3\lambda^{n-1}p^n + \lambda^n p^{n+1} \\ &\quad = (1-\lambda p)\{2\lambda p^2(1+\lambda p+\lambda^2 p^2 + \cdots + \lambda^{n-3}p^{n-3}) - \lambda^{n-1}p^n\} \\ &\quad \geq 0,\end{aligned}$$

for $n \geq 5$, $p \in (0, 1)$. Therefore from (2.6) and (2.7), it is clear that for $p \leq 1/3$,

$$\begin{aligned}|a_2a_n - a_{n+1}| &\leq \lambda\{(1-|c_1|)A_{n-1} + |c_1|A_{n-1}\} \\ &= \lambda A_{n-1}.\end{aligned}$$

Next, for the function k_p^λ, it is a simple exercise to check that

$$A_2 A_n - A_{n+1} = \lambda A_{n-1}.$$

Equality holds in the inequality stated in the theorem for the function k_p^λ. This completes the proof of the theorem. □

We notice here that for the function k_p^λ, we have

$$\psi_{m,n} = a_m a_n - a_{m+n-1} = \lambda A_{m-1} A_{n-1}.$$

In Theorem 2, we have obtained the sharp estimates of $|\psi_{m,n}|$ for the cases $m = 2, n = 3$ in $0 < p < 1$ and $m = 2, n = 4$ for $p \leq 1/2$. Also in Theorem 3, we have determined the upper bounds of $|\psi_{m,n}|$ when $m = 2, n \geq 5$ and $p \leq 1/3$. These results lead us to the following conjecture.

Conjecture 2 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2) in $\mathbb{D}_p$. Then for all $m \geq 2,\ n \geq 3$, we have*

$$|a_m a_n - a_{m+n-1}| \leq \lambda A_{m-1} A_{n-1}$$

for all $\lambda \in (0, 1]$ and $p \in (0, 1)$. The above inequality is sharp for the function k_p^λ.

Remark When $p \to 1^-$ and $\lambda = 1$, the above inequality becomes

$$|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1).$$

We see that this is the same as the gZC for the class $\mathcal{S}$ provided by Ma (c.f. [18]).

3 Generalized Fekete–Szegö Inequality for the Class $\mathcal{V}_p(\lambda)$

The following theorem deals with the upper bound of $|\Lambda_\mu(f)| := |a_3 - \mu a_2^2|$ whenever $f \in \mathcal{V}_p(\lambda)$ and μ is a real number.

Theorem 4 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2) in $\mathbb{D}_p$. Then*

$$|\Lambda_\mu(f)| \leq \begin{cases} \frac{1}{p^2}(1-\mu) + (1-2\mu)\lambda + (1-\mu)\lambda^2 p^2 \ \textit{for}\ \mu \leq 0,\ \textit{and}\ 0 < p \leq 1/2 \\ \frac{1}{p^2}(\mu-1) + (2\mu-1)\lambda + (\mu-1)\lambda^2 p^2 \ \textit{for}\ \mu \geq 1,\ \textit{and}\ 0 < p < 1. \end{cases}$$

Equalities hold in the above inequalities for the function k_p^λ.

Proof We follow some initial lines of proofs of the Theorem 1. Next by comparing the coefficients of z^3 and z^4 from both sides of Eq. (2.2), we get

$$a_3 = a_2^2 - \lambda c_1 \quad \text{and} \quad a_4 = a_2 a_3 - \lambda c_1 a_2 - \lambda c_2. \tag{3.1}$$

From the above equations, we have $a_3 - \mu a_2^2 = (1-\mu)a_3 - \lambda\mu c_1$. Now from (1.5) and (2.1), we have

$$|a_3 - \mu a_2^2| \le |1-\mu|\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + \lambda|\mu|, \quad \text{for } 0 < p \le 1/2. \tag{3.2}$$

Case 1: Let $\mu \le 0$, $p \in (0, 1/2]$. We first note that $|1-\mu| = 1-\mu$, $|\mu| = -\mu$ and from (3.2) we have

$$\begin{aligned}|a_3 - \mu a_2^2| \le (1-\mu)\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) - \lambda\mu = \frac{1}{p^2}(1-\mu)\\ + (1-2\mu)\lambda + (1-\mu)\lambda^2 p^2.\end{aligned}$$

Next, for the function k_p^λ, we compute

$$\begin{aligned}a_3 - \mu a_2^2 = (1-\mu)(p^{-2} + \lambda + \lambda^2 p^2) - \lambda\mu = (1-\mu)p^{-2}\\ + (1-2\mu)\lambda + (1-\mu)\lambda^2 p^2.\end{aligned}$$

This shows the equality in the first inequality stated in the theorem.
Case 2: Let $\mu \ge 1$, $p \in (0, 1)$. Therefore $|1-\mu| = \mu - 1$. We see that as $|a_2| \le p^{-1} + \lambda p$ for all $p \in (0, 1)$,

$$\begin{aligned}|a_3 - \mu a_2^2| &= |(1-\mu)a_2^2 - \lambda c_1|\\ &\le (\mu-1)|a_2^2| + \lambda|c_1|\\ &\le (\mu-1)\left(\frac{1}{p^2} + 2\lambda + \lambda^2 p^2\right) + \lambda\\ &= \frac{1}{p^2}(\mu-1) + (2\mu-1)\lambda + (\mu-1)\lambda^2 p^2.\end{aligned}$$

It is a simple exercise to check that for the function k_p^λ, we have

$$a_3 - \mu a_2^2 = -\left[\frac{1}{p^2}(\mu-1) + (2\mu-1)\lambda + (\mu-1)\lambda^2 p^2\right].$$

This shows the sharpness of the second inequality stated in the theorem, hence the proof of the theorem is complete. □

Remark (i) We mention here that for $0 < \mu < 1$ and $0 < p \le 1/2$, we get an upper bound of the Fekete–Szegö functional from (3.2) as below:

$$\begin{aligned}|a_3 - \mu a_2^2| &\le (1-\mu)\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + \lambda\mu\\ &= \frac{1}{p^2}(1-\mu) + \lambda + (1-\mu)\lambda^2 p^2.\end{aligned}$$

We are not able to establish whether the above estimate is sharp or not. Also for the case when $p \in (1/2, 1)$ and $\mu \leq 0$, the problem of estimating the Fekete–Szegö functional $\Lambda_\mu(f)$ for functions in $\mathcal{V}_p(\lambda)$ remains open.

(ii) The case $\mu = 1$ of Theorem 4 has been obtained for all $p \in (0, 1)$ in Theorem 1.

In the following theorem, we further obtain the sharp estimates of the generalized Fekete–Szegö functional $a_n - a_2^{n-1}$ for $n = 4$ and $n = 5$.

Theorem 5 *If $f \in \mathcal{V}_p(\lambda)$ and has the expansion of the form (1.2) in $\mathbb{D}_p$, then*

(i) $|a_4 - a_2^3| \leq 2\lambda\left(\frac{1}{p} + \lambda p\right)$ *for* $0 < p < 1$;

(ii) $|a_5 - a_2^4| \leq \lambda\left(2\lambda + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)\right)$ *for* $0 < p \leq 1/2$.

Equalities hold in both the above inequalities for the function k_p^λ.

Proof (i) From (3.1), after a little computation, we get $a_4 - a_2^3 = -2\lambda c_1 a_2 - \lambda c_2$. Then, by using (2.1) and the bound of $|a_2|$, we have

$$|a_4 - a_2^3| \leq \lambda(2|c_1||a_2| + |c_2|) \leq \lambda\left(2|c_1|\left(\frac{1}{p} + \lambda p\right) + \frac{1}{2}(1 - |c_1|^2)\right). \quad (3.3)$$

Let $x = |c_1|$ and $f(x) = 2(p^{-1} + \lambda p)x + (1 - x^2)/2$. Then $x \in [0, 1]$ and $f'(x) = 2(p^{-1} + \lambda p) - x > 0$, which implies that f is an increasing function of x. Therefore, $\max_{x\in[0,1]} f(x) = f(1) = 2(p^{-1} + \lambda p)$ and hence from (3.3), we get

$$|a_4 - a_2^3| \leq 2\lambda\left(\frac{1}{p} + \lambda p\right).$$

Next, for the function k_p^λ, we compute $a_4 - a_2^3 = -2\lambda(p^{-1} + \lambda p)$, which proves the sharpness of the first inequality stated in the theorem.

We now proceed to prove the second part of this theorem. Equating coefficients of z^5 from both sides of (2.2), and by using (3.1), we have

$$\begin{aligned} a_5 - a_2^4 &= -3\lambda c_1 a_2^2 - 2\lambda c_2 a_2 + \lambda^2 c_1^2 - \lambda c_3 \\ &= -3\lambda c_1(a_2^2 - \lambda c_1) - 2\lambda^2 c_1^2 - 2\lambda c_2 a_2 - \lambda c_3 \\ &= -3\lambda c_1 a_3 - 2\lambda^2 c_1^2 - 2\lambda c_2 a_2 - \lambda c_3. \end{aligned}$$

Now, by using the triangle inequality, the bounds for $|a_2|$, $|a_3|$ and (2.1), we have for $p \leq 1/2$,

$$\begin{aligned} |a_5 - a_2^4| \leq \lambda\Bigg[&\frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|}\right) + 2\left(\frac{1}{p} + \lambda p\right)|c_2| \\ &+3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)|c_1| + 2\lambda|c_1|^2\Bigg]. \end{aligned} \quad (3.4)$$

Let us denote $x := |c_1|$ and $y := |c_2|$. Then $0 \le x \le 1$ and $0 \le y \le (1-x^2)/2$. Let $\Omega := \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le (1-x^2)/2\}$. Now let us define

$$f(x, y) = \frac{1}{3}\left(1 - x^2 - \frac{4y^2}{1+x}\right) + 2\left(\frac{1}{p} + \lambda p\right) y + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) x + 2\lambda x^2,$$

for $(x, y) \in \Omega$. Then

$$\frac{\partial f}{\partial x} = -\frac{2x}{3} + \frac{4y^2}{3(1+x)^2} + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 4\lambda x,$$

which is always positive in Ω for $p \le 1/2$. This shows that the extremum of $f(x, y)$ cannot be attained in the interior of the domain Ω. Since f is continuous and Ω is compact, the maximum of f will be attained at some boundary point of Ω. Now on the boundary $x = 0$, $0 \le y \le 1/2$ of Ω, we have $f(0, y) = \left((1-4y^2)/3\right) + 2y\left(p^{-1} + \lambda p\right)$. Therefore for $p \le 1/2$ and $0 \le y \le 1/2$, we have

$$\frac{\partial f}{\partial y}(0, y) = -\frac{8y}{3} + 2\left(\frac{1}{p} + \lambda p\right) > 0.$$

This implies $f(0, y)$ is an increasing function of y and hence

$$\max_{0 \le y \le 1/2} f(0, y) = f\left(0, \frac{1}{2}\right) = \left(\frac{1}{p} + \lambda p\right).$$

Next, on the boundary $y = 0$, $0 \le x \le 1$, we have

$$f(x, 0) = \frac{1}{3}(1 - x^2) + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) x + 2\lambda x^2.$$

Therefore, for $0 < p \le 1/2$ and $0 \le x \le 1$, we have

$$\frac{\partial f}{\partial x}(x, 0) = -\frac{2x}{3} + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 4\lambda x > 0,$$

which implies $f(x, 0)$ is an increasing function of x and hence

$$\max_{0 \le x \le 1} f(x, 0) = f(1, 0) = 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\lambda.$$

On the boundary $y = (1 - x^2)/2,\ 0 \leq x \leq 1$, we have

$$f(x, (1-x^2)/2) = \left(\frac{1}{p} + \lambda p\right) + \left\{\frac{1}{3} + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right)\right\} x + \left\{2\lambda - \left(\frac{1}{p} + \lambda p\right)\right\} x^2 - \frac{x^3}{3}.$$

Now for $0 \leq x \leq 1$, we compute

$$\begin{aligned} &\frac{\partial f}{\partial x}\left(x, \frac{1}{2}(1-x^2)\right) \\ &= \frac{1}{3} + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\left\{2\lambda - \left(\frac{1}{p} + \lambda p\right)\right\} x - x^2 \\ &\geq \frac{1}{3} + 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) - 2\left\{\left(\frac{1}{p} + \lambda p\right) - 2\lambda\right\} - 1 \\ &= \frac{(9 - 6p - 2p^2) + 6\lambda p^2(1-p) + 3\lambda p^2 + 9\lambda^2 p^4 + 12\lambda p^2}{3p^2}, \end{aligned}$$

which is positive for $0 < p \leq 1/2$. Therefore, $f(x, (1-x^2)/2)$ is an increasing function of x whenever $0 < p \leq 1/2$ and hence,

$$\max_{0 \leq x \leq 1} f\left(x, \frac{1}{2}(1-x^2)\right) = f(1, 0) = 3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\lambda.$$

Hence, the maximum value of $f(x, y)$ on Ω is $3(p^{-2} + \lambda + \lambda^2 p^2) + 2\lambda$ for $0 < p \leq 1/2$, and consequently, by using (3.4), we get

$$|a_5 - a_2^4| \leq \lambda \left[3\left(\frac{1}{p^2} + \lambda + \lambda^2 p^2\right) + 2\lambda\right],$$

for $p \leq 1/2$. Now for the function k_p^λ, it is a simple exercise to check that

$$a_5 - a_2^4 = -\lambda[3(p^{-2} + \lambda + \lambda^2 p^2) + 2\lambda],$$

which proves the sharpness of the inequality (ii) stated in the theorem. This completes the proof of the theorem. ◻

4 Hankel Determinant for the Class $\mathcal{V}_p(\lambda)$

In the following theorem, we establish the upper bounds for the Hankel determinants $|\mathcal{H}_2(2)|$ and $|\mathcal{H}_3(1)|$ for the functions f belonging to the class $\mathcal{V}_p(\lambda)$.

Theorem 6 *Let $f \in \mathcal{V}_p(\lambda)$ be of the form (1.2) in $\mathbb{D}_p$. Then for all $p \in (0, 1)$ and $\lambda \in (0, 1]$, we have*

(i) $|H_2(2)| \leq \frac{\lambda}{2}\left(\frac{1}{p} + \lambda p\right)$;

(ii) $|H_3(1)| \leq \frac{\lambda^2}{4}$.

Equality holds in (ii) *for the function*

$$l_p^\lambda(z) = \frac{z}{1 - \left(\frac{1}{p} + \frac{\lambda p^2}{2}\right) z + \frac{\lambda}{2} z^3}, \; z \in \mathbb{D}.$$

Proof From the definition of the Hankel determinant, we know that $H_2(2) = a_2a_4 - a_3^2$. Now by using (3.1), we get

$$H_2(2) = a_2a_4 - a_3^2 = -\lambda(a_2c_2 + \lambda c_1^2). \tag{4.1}$$

Note that

$$\begin{aligned} |a_2c_2 + \lambda c_1^2| &\leq \frac{1}{2}|a_2|(1 - |c_1|^2) + \lambda|c_1|^2 \\ &\leq \frac{1}{2}\left(\frac{1}{p} + \lambda p\right)(1 - |c_1|^2) + \lambda|c_1|^2. \end{aligned}$$

Let us denote $x := |c_1|$, so $0 \leq x \leq 1$. We now consider $r(x) := \frac{1}{2}\left(\frac{1}{p} + \lambda p\right)(1 - x^2) + \lambda x^2$, $x \in [0, 1]$. Then

$$r'(x) = -\left(\frac{1}{p} + \lambda p\right) x + 2\lambda x, \quad r''(x) = -\left(\frac{1}{p} + \lambda p\right) + 2\lambda.$$

Now it is a simple exercise to check that $r'(x) = 0$ at $x = 0$ and $r''(0) < 0$ as

$$\left(\frac{1}{p} + \lambda p\right) - 2\lambda = \frac{1}{p}\left(1 - 2\lambda p + \lambda p^2\right) = \frac{1}{p}\left((1 - \lambda p)^2 + \lambda p^2(1 - \lambda)\right) > 0,$$

for all $p \in (0, 1)$ and $\lambda \in (0, 1]$. This implies that the function $r(x)$ has maximum at $x = 0$ and the maximum value is $(p^{-1} + \lambda p)/2$. Hence, from the above estimate and (4.1), we get

$$|H_2(2)| \leq \frac{\lambda}{2}\left(\frac{1}{p} + \lambda p\right),$$

which proves first part of the theorem. Now we proceed to prove the second part of the theorem. From the definition of the Hankel determinant, we get

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

Now from the above equality combined with (2.2) and (3.1),

$$H_3(1) = \lambda^2(c_1c_3 - c_2^2).$$

Next, by using the triangle inequality and (2.1), we get

$$\begin{aligned}|c_1c_3-c_2^2| &\le |c_1||c_3|+|c_2|^2\\ &\le \frac{1}{3}|c_1|\left[1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right]+|c_2|^2\\ &= \frac{1}{3}\left[|c_1|-|c_1|^3+\frac{3-|c_1|}{1+|c_1|}|c_2|^2\right]\\ &\le \frac{1}{3}\left[|c_1|-|c_1|^3+\frac{3-|c_1|}{4(1+|c_1|)}(1-|c_1|^2)^2\right]\\ &= \frac{1}{12}\left[3-2|c_1|^2-|c_1|^4\right]\le \frac{1}{4}.\end{aligned}$$

Consequently, $|H_3(1)|=\lambda^2|c_1c_3-c_2^2|\le\lambda^2/4$. Now the function $l_p^\lambda(z)$ is in $\mathcal{V}_p(\lambda)$, because $l_p^\lambda(p)=\infty$ and $U_{l_p^\lambda}(z)=-\lambda z^3$. For this function, we have

$$a_2=\left(\frac{1}{p}+\frac{\lambda p^2}{2}\right),\quad a_3=\left(\frac{1}{p}+\frac{\lambda p^2}{2}\right)^2,\quad a_4=-\frac{\lambda}{2}+\left(\frac{1}{p}+\frac{\lambda p^2}{2}\right)^3,$$

and

$$a_5=-\lambda\left(\frac{1}{p}+\frac{\lambda p^2}{2}\right)+\left(\frac{1}{p}+\frac{\lambda p^2}{2}\right)^4,$$

which after a little calculation yields

$$H_3(1)=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2)=-\lambda^2/4.$$

Hence, the equality in the second inequality stated in the theorem holds for the function l_p^λ. This completes the proof of the theorem. □

Remark We do not know at present whether inequality (i) of Theorem 6 is sharp or not.

Acknowledgements We would like to thank the reviewers for their careful reading of the manuscript and valuable suggestions.

References

1. Bhowmik, B., Ponnusamy, S., Wirths, K.-J.: On the Fekete-Szegö problem for concave univalent functions. J. Math. Anal. Appl. **373**, 432–438 (2011)
2. Bhowmik, B., Parveen, F.: On a subclass of meromorphic univalent functions. Complex Var. Elliptic Equ. **62**, 494–510 (2017)
3. Bhowmik, B., Parveen, F.: Sufficient conditions for univalence and study of a class of meromorphic univalent functions. Bull. Korean Math. Soc. **55**(3), 999–1006 (2018)
4. Bhowmik, B., Parveen, F.: On some results for a subclass of meromorphic univalent functions with nonzero pole. Results Math. **74**(4), 1–21 (2019)

5. Brown, J.E., Tsao, A.: On the Zalcman conjecture for starlike and typically real functions. Math. Z. **191**(3), 467–474 (1986)
6. Cantor, D.G.: On the power series with integral coefficients. Bull. Am. Math. Soc. **69**, 362–366 (1963)
7. Choi, J.H., Kim, Y.C., Sugawa, T.: A general approach to the Fekete-Szegö problem. J. Math. Soc. Japan **59**, 707–727 (2007)
8. Dai, S., Pan, Y.: Note on Schwarz-Pick estimates for bounded and positive real part analytic functions. Proc. Am. Math. Soc. **136**(2), 635–640 (2008)
9. De Branges, L.: A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152 (1985)
10. Duren, P.L.: Univalent Functions. Springer, Berlin (1983)
11. Efraimidis, I., Vukotić, D.: Applications of Livingston-type inequalities to the generalized Zalcman functional. Math. Nachr. **291**(10), 1502–1513 (2018)
12. Fekete, M., Szegö, G.: Eine Bemerkung über ungerade schlichte Funktionen. J. Lond. Math. Soc. **8**, 85–89 (1933)
13. Fournier, R., Ponnusamy, S.: A class of locally univalent functions defined by a differential inequality. Complex Var. Elliptic Equ. **52**(1), 1–8 (2007)
14. Koepf, W.: On the Fekete-Szegö problem for close-to-convex functions. Proc. Am. Math. Soc. **101**, 89–95 (1987)
15. Kowalczyk, B., Lecko, A., Sim, Y.J.: The sharp bound of the Hankel determinant of the third kind for convex functions. Bull. Aust. Math. Soc. **97**, 435–445 (2018)
16. Krushkal, S.L.: Univalent functions and holomorphic motions. J. Anal. Math. **66**, 253–275 (1995)
17. Krushkal, S.L.: Proof of the Zalcman conjecture for initial coefficients. Georgian Math. J. **17**(4), 663–681 (2010)
18. Ma, W.: Generalized Zalcman conjecture for starlike and typically real functions. J. Math. Anal. Appl. **234**, 328–339 (1999)
19. Ma, W.: The Zalcman conjecture for close-to-convex functions. Proc. Am. Math. Soc. **104**(3), 741–744 (1988)
20. Obradović, M., Ponnusamy, S., Wirths, K.-J.: Logarithmic coefficients and a coefficient conjecture for univalent functions. Monatsh Math. **185**(3), 489–501 (2018)
21. Obradovic, M., Tuneski, N.: Some properties of the class $\mathcal{U}$. Ann. Univ. Mariae Curie- Skodowska. Sectio A Math. **73**(1), 49–56 (2019)
22. Pommerenke, C.: On the coefficients and Hankel determinant of univalent functions. J. Lond. Math. Soc. **41**, 111–122 (1966)
23. Pommerenke, C.: On the Hankel determinants of univalent functions. Mathematika **14**, 108–112 (1967)
24. Pommerenke, C.: Hankel determinants and meromorphic functions. Mathematika **16**(2), 158–166 (1969)
25. Ponnusamy, S., Wirths, K.-J.: Coefficient problems on the class $\mathcal{U}(\lambda)$. Probl. Anal. Issues Anal. **7**(25), 87–103 (2018)
26. Wang, Z.-G., Raza, M., Arif, M., Ahmad, K.: On the third and fourth Hankel determinants of a subclass of analytic functions. Bull. Malays. Math. Sci. Soc. (2021). https://doi.org/10.1007/s40840-021-01195-8