Bull. Malays. Math. Sci. Soc. (2022) 45:2865–2883
https://doi.org/10.1007/s40840-022-01299-9

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

l-Clique Metric Dimension of Graphs

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Received: 15 April 2021 / Revised: 14 March 2022 / Accepted: 21 April 2022 /
Published online: 24 May 2022

Abstract

For an ordered non-empty subset $S = \{v_1, \ldots, v_k\}$ of vertices in a connected graph G and an l-clique V' of G, the l-clique metric S-representation of V' is the vector $r^l_G(V'|S) = (d_G(V', v_1), \ldots, d_G(V', v_k))$ where $d_G(V', v_i) = \min\{d_G(v, v_i) : v \in V'\}$. A non-empty subset S of $V(G)$ is an l-clique metric generator for G if all l-cliques of G have pairwise different l-clique metric S-representations. An l-clique metric generator of smallest order is an l-clique metric basis for G, its order being the l-clique metric dimension (l-CMD for short) $\mathrm{cdim}_l(G)$ of G. In this paper, we propose this concept as an extension of the 1-clique metric dimension which is known as the metric dimension, and also study some its properties. Moreover, l-CMD for $\Gamma(\mathbb{Z}_n)$ and the corona product of two graphs is investigated. Furthermore, we prove that computing the l-CMD of connected graphs is NP-hard and present an integer linear programming model for finding this parameter.

Keywords l-Clique metric dimension · Corona product graph

Mathematics Subject Classification 05C12 · 05C25 · 05C76

Communicated by Sanming Zhou.

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1 Introduction

All graphs considered in this paper are undirected and simple.

If $u, v \in V(G)$, then $d_G(u, v)$ denotes the number of edges on a shortest u, v-path in G. A clique V' is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent. Also, V' is called an l-clique if $|V'| = l$. For a vertex u and an l-clique V' of G, the distance between V' and u, denoted by $d_G(u, V')$, is defined as $\min\{d_G(u, v) : v \in V'\}$; in other words, $d_G(u, V') = \min\{d_G(u, v) : v \in V'\}$.

For an ordered non-empty subset $S = \{v_1, \ldots, v_k\}$ of vertices in a connected graph G and an l-clique V' of G, the *l-clique metric S-representation* of V' is the vector $r^l_G(V'|S) = (d_G(V', v_1), \ldots, d_G(V', v_k))$. A non-empty subset S of $V(G)$ is an *l-clique metric generator* for G if all l-cliques of G have pairwise different l-clique metric S-representations. l-Clique metric generators for special cases $l = 1$ and $l = 2$ are known as *metric generator* and *edge metric generator*, respectively. An l-clique metric generator of smallest order is an *l-clique metric basis* for G, its order being the *l-clique metric dimension* (l-CMD for short) $\mathrm{cdim}_l(G)$ of G.

Recall that the special case 1-clique metric dimension is called the metric dimension and denoted by $\dim(G)$ and also the special case 2-clique metric dimension is called the edge metric dimension and denoted by $\dim_e(G)$.

The concept of metric dimension was first introduced by Slater [21]. Since then lots of work has been done on this topic because of its wide range of applications in modeling of real world problems [13, 15]. For instance, Garey and Johnson [11], and Epstein et al. [10] studied NP-hardness of computing of metric dimension. Also, this invariant was investigated over the Cartesian product of graphs in [5], over the lexicographic product of graphs in [19], over the deleted lexicographic product of graphs in [9], and over the hierarchical product of graphs in [23]. Kelenc et al. [14] introduced the concept of edge metric dimension. In the present work, we expand the concept of metric dimension as l-clique metric dimension where l is a natural number. Note that in [12] resolving sets locate up to some fixed l, $l \geq 1$, vertices in a graph, while here resolving sets locate the l-cliques of a graph. The first section of this paper is dedicated to some properties of this parameter of graphs. In the second section, we compute l-CMD for $\Gamma(\mathbb{Z}_n)$. We also obtain the exact value of l-CMD of corona product of two graphs in the third section. [11, 14] showed the NP-completeness of l-CMD problems for $l = 1$ and $l = 2$, respectively. We prove the NP-completeness of l-CMD problems for $l \geq 3$ in the last section.

Throughout this paper, our notation is standard and taken mainly from [2].

2 Basic Results

In this section, we present some basic results on the l-clique metric dimension.

The following proposition gives the l-CMD of the complete graph K_n.

Proposition 2.1 *Let $n \geq 2$. We have*

$$\mathrm{cdim}_l(K_n) = \begin{cases} 1 & l = n \\ n - 1 & \text{otherwise.} \end{cases}$$

Proof If $l = n$, then clearly $\text{cdim}_l(K_n) = 1$. Let $l \in \{1, 2\}$. Then, by [14, Remark 1], we have $\text{edim}(K_n) = \dim(K_n) = n - 1$. Hence, in this situation, $\text{cdim}_l(K_n) = \text{edim}(K_n) = \dim(K_n) = n-1$. So we assume that $3 \leq l \leq n-1$ and $n \geq 4$. Let S be a minimal l-clique metric generator of K_n. If $|S| \leq n-2$, then there exist two distinct vertices $x, y \in K_n \backslash S$. Consider two l-cliques L_1 and L_2 such that $x \in L_1$, $y \in L_2$ and $L_1 \backslash \{x\} = L_2 \backslash \{y\}$. Then one can see that the l-clique metric S-representations of L_1 and L_2 are the same, which is impossible. Now, let $S \subseteq V(K_n)$ with $|S| = n-1$. Then, in this situation, for every two distinct cliques L_1 and L_2, there exists $s \in S$ such that $s \in L_1 \backslash L_2$. Therefore the component which is corresponding to s in the l-clique metric S-representations of L_1 and L_2 is 0 and 1, respectively, which implies that S is an l-clique metric generator for K_n. Hence $\text{cdim}_l(K_n) = n - 1$. □

Recall that the *wheel graph* $W_{1,n}$ is the graph obtained from a cycle C_n and the graph K_1 by adding all the edges between the vertex of K_1 and every vertex of C_n.

The least integer greater than or equal to a number m is denoted by $\lceil m \rceil$. Also, greatest integer less than or equal to a number m is denoted by $\lfloor m \rfloor$.

In the following proposition, we investigate the l-CMD $\text{cdim}_l(W_{1,n})$. Note that if $l = 1$, then $\text{cdim}_l(W_{1,n}) = \dim(W_{1,n})$, which is determined in [3], as follows.

$$\dim(W_{1,n}) = \begin{cases} 3 & n = 3, 6 \\ 2 & n = 4, 5 \\ \lfloor \frac{2n+2}{5} \rfloor & n \geq 6. \end{cases}$$

Also, if $l = 2$, then $\text{cdim}_l(W_{1,n}) = \text{edim}(W_{1,n})$, which is

$$\text{edim}(W_{1,n}) = \begin{cases} n & n = 3, 4 \\ n - 1 & n \geq 5, \end{cases}$$

see [14].

Proposition 2.2 *Let $W_{1,n}$ be a wheel graph. Then*

$$\text{cdim}_3(W_{1,n}) = \begin{cases} 3 & n = 3 \\ n - \lceil \frac{n}{3} \rceil & n \geq 4. \end{cases}$$

Proof By Proposition 2.1, we have $\text{cdim}_3(W_{1,3}) = \text{cdim}_3(K_4) = 3$ and $\text{cdim}_4(W_{1,3}) = \text{cdim}_4(K_4) = 1$. So assume that $n \geq 4$. Let $\{g_1, g_2, \ldots, g_n\}$ be the vertices of degree 3 in $W_{1,n}$. Clearly for each two distinct triangles L_1 and L_2 in $W_{1,n}$, either there exists $1 \leq i \leq n$ such that L_1 and L_2 have the common vertex g_i, or L_1 and L_2 have no common vertices from the set $\{g_1, g_2, \ldots, g_n\}$. In both of the situations, one can easily see that L_1 and L_2 have the same 3-clique metric S-representations if and only if their non-common vertices do not belong to S, where $S \subseteq V(W_{1,n})$. Now let S be a 3-clique metric basis of $W_{1,n}$. Clearly $S \subseteq \{g_1, g_2, \ldots, g_n\}$. We consider the following cases.

Case 1 $n = 3k$, where $k \geq 2$. Let S be a 3-clique metric basis of $W_{1,n}$. If there are two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$, say g_2 and g_3, such that $g_2, g_3 \notin S$, then we should have $g_4, g_5 \in S$ and $g_{3\lfloor \frac{n}{3} \rfloor}, g_1 \in S$. So, if $k = 2$, then $\text{cdim}_3(W_{1,6}) = 4$. Let $k > 2$. Since S is a 3-clique metric basis, without loss of generality, we may assume that $g_6 \notin S, g_7, g_8 \in S, \ldots, g_{3\lfloor\frac{n}{3}\rfloor-3} \notin S, g_{3\lfloor\frac{n}{3}\rfloor-2}, g_{3\lfloor\frac{n}{3}\rfloor-1} \in S$. Therefore, in this situation, $|S| = n - \lceil\frac{n}{3}\rceil$.

Now, assume that there exists a 3-clique metric basis of $W_{1,n}$, say S, such that for any two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ at least one of them belongs to S. Without loss of generality, assume that $g_3 \notin S$. Since S is a 3-clique metric basis, we may assume that

$$S = \{g_1, g_2, \ldots, g_n\}\backslash\{g_3, g_6, \ldots, g_{3i}, \ldots, g_{3\lfloor\frac{n}{3}\rfloor}\},$$

where $1 \le i \le \lfloor\frac{n}{3}\rfloor$. Clearly, in this situation we again have $|S| = n - \lceil\frac{n}{3}\rceil$.

Note that in either of the above situationes, by the structure that we obtain for a 3-clique metric basis of $W_{1,n}$, it is easy to see that any subset of $\{g_1, g_2, \ldots, g_n\}$ with less that $n - \lceil\frac{n}{3}\rceil$ elements is not a 3-clique metric generator of $W_{1,n}$. Therefore, in this case the 3-CMD of $W_{1,n}$ is equal to $n - \lceil\frac{n}{3}\rceil$.

Case 2 $n = 3k + 1$ or $n = 3k + 2$, where $k \ge 1$. First we show that for any 3-clique metric basis of $W_{1,n}$, say S, there exist two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ such that they do not belong to S. Assume on the contrary that for any two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$, at least one of them belongs to S. Without loss of generality, we may assume that $g_3 \notin S$. Since S is a 3-clique metric basis, we may assume that

$$S = \{g_1, g_2, \ldots, g_n\}\backslash\{g_3, g_6, \ldots, g_{3i}, \ldots, g_{3\lfloor\frac{n}{3}\rfloor}\},$$

where $1 \le i \le \lfloor\frac{n}{3}\rfloor$. Now consider the set $S' = S\backslash\{g_2\}$. One can easily see that S' is a 3-clique metric generator of $W_{1,n}$ with $|S'| < |S|$, which is a contradiction.

Now let S be a 3-clique metric basis of $W_{1,n}$. Then there are two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$, say g_2 and g_3, such that $g_2, g_3 \notin S$. By using a similar discussion as we used in Case 1, we obtain that

$$S = \{g_1, g_2, \ldots, g_n\}\backslash\{g_2, g_3, g_6, \ldots, g_{3i}, \ldots, g_{3\lfloor\frac{n}{3}\rfloor}\}$$

where $1 \le i \le \lfloor\frac{n}{3}\rfloor$ and $|S| = n - \lceil\frac{n}{3}\rceil$. Also, by the structure that we obtain for S, it is easy to see that any subset of $\{g_1, g_2, \ldots, g_n\}$ with less that $n - \lceil\frac{n}{3}\rceil$ elements, is not a 3-clique metric generator of $W_{1,n}$.

Therefore we have $\mathrm{cdim}_3(W_{1,n}) = n - \lceil\frac{n}{3}\rceil$, when $n \ge 4$. □

Similarly to the wheel graph, the *fan graph*, which is denoted by $F_{1,n}$, is the graph that is obtained from a path P_n and the graph K_1 by adding all the edges between the vertex of K_1 and every vertex of P_n. In [4, 14], $\dim(F_{1,n})$ and $\mathrm{edim}(F_{1,n})$ are determined as follows:

$$\dim(F_{1,n}) = \begin{cases} 1 & n = 1 \\ 2 & n = 2, 3 \\ 3 & n = 6 \\ \lfloor\frac{2n+2}{5}\rfloor & \text{otherwise} \end{cases}$$

and

$$\operatorname{edim}(F_{1,n}) = \begin{cases} n & n = 1, 2, 3 \\ n - 1 & n \geq 4. \end{cases}$$

In the following proposition, we investigate the l-CMD of $F_{1,n}$ in the case that $l = 3$.

Proposition 2.3 *For the fan graph $F_{1,n}$ we have*

$$\operatorname{cdim}_3(F_{1,n}) = \begin{cases} 1 & n = 1, 2 \\ n - \lceil \frac{n}{3} \rceil - 1 & n = 3k, 3k + 2 \text{ for } k \geq 1 \\ n - \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof Clearly if $n \in \{1, 2, 3\}$, we have $\operatorname{cdim}_3(F_{1,n}) = 1$. Let $\{g_1, g_2, \ldots, g_n\}$ be the vertices of the path P_n in the structure of $F_{1,n}$. Note that for each two distinct triangles L_1 and L_2 in $F_{1,n}$, they have the same 3-clique metric S-representations if and only if their non-common vertices do not belong to S, where $S \subseteq V(F_{1,n})$. Also clearly each 3-clique metric basis of $F_{1,n}$ is a subset of $\{g_1, g_2, \ldots, g_n\}$. Now we have the following cases:

Case 1 $n = 3k$, where $k \geq 2$. First we show that for any 3-clique metric basis of $F_{1,n}$, say S, there exist two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ such that they do not belong to S. Assume on the contrary that for any two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$, at least one of them belongs to S. If $g_1 \notin S$, then by using a similar method as we used in the proof of Proposition 2.2, we get that

$$S = \{g_1, g_2, \ldots, g_n\} \backslash \{g_1, g_4, \ldots, g_{3i+1}, \ldots, g_{3\lfloor \frac{n}{3} \rfloor - 2}\},$$

where $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$. But one can easily see that the set $S' = S \backslash \{g_{3\lfloor \frac{n}{3} \rfloor - 1}\}$ is a 3-clique metric generator of $F_{1,n}$ with $|S'| < |S|$, which is a contradiction. Now, let $g_1 \in S$. Then we may assume that

$$S = \{g_1, g_2, \ldots, g_n\} \backslash \{g_2, g_5, \ldots, g_{3i+2}, \ldots, g_{3\lfloor \frac{n}{3} \rfloor - 1}\},$$

where $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$. Again we see that the set $S' = S \backslash \{g_{3\lfloor \frac{n}{3} \rfloor}\}$ is a 3-clique metric generator of $F_{1,n}$ with $|S'| < |S|$, which is a contradiction. Therefore for any 3-clique metric basis of $F_{1,n}$, say S, there exist two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ such that they do not belong to S. Now it is easy to see that

$$S = \{g_1, g_2, \ldots, g_n\} \backslash \{g_1, g_2, g_5, \ldots, g_{3i+2}, \ldots, g_{3\lfloor \frac{n}{3} \rfloor - 1}\},$$

where $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$ is a 3-clique metric generator of $F_{1,n}$, and any subset of $\{g_1, g_2, \ldots, g_n\}$ with cardinality less than $|S| = n - \lceil \frac{n}{3} \rceil - 1$ is not a 3-clique metric generator for $F_{1,n}$. Hence in this case we have $\operatorname{cdim}_3(F_{1,n}) = n - \lceil \frac{n}{3} \rceil - 1$.

Case 2 $n = 3k + 1$, where $k \geq 1$. Let S be a 3-clique metric basis. First assume that for any two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$, at least one of them belongs to S. If $g_1 \in S$, then if $g_2 \notin S$, then $S \backslash \{g_1\}$ is a 3-clique metric generator with less than $|S|$ elements which is impossible. Also if $g_2 \in S$, then $S \backslash \{g_2\}$ is a 3-clique metric generator with less than $|S|$ elements which is again impossible. So we have $g_1 \notin S$. In this situation, one can easily see that

$$S = \{g_1, g_2, \ldots, g_n\} \backslash \{g_1, g_4, \ldots, g_{3i+1}, \ldots, g_{3\lfloor \frac{n}{3} \rfloor + 1}\},$$

where $0 \leq i \leq \lfloor \frac{n}{3} \rfloor$ is a 3-clique metric basis for $F_{1,n}$, with $|S| = n - \lceil \frac{n}{3} \rceil$. Now, suppose that there exist two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ such that they do not belong to S. In this situation, we again have $|S| = n - \lceil \frac{n}{3} \rceil$. Therefore in this case we have $\text{cdim}_3(F_{1,n}) = n - \lceil \frac{n}{3} \rceil$.

Case 3 $n = 3k + 2$, where $k \geq 1$. Similar to Case 1, we can see that for any 3-clique metric basis of $F_{1,n}$, say S, there exist two adjacent vertices of the set $\{g_1, g_2, \ldots, g_n\}$ such that they do not belong to S. Now one can easily see that

$$S = \{g_1, g_2, \ldots, g_n\} \backslash \{g_1, g_2, g_5, \ldots, g_{3i+2}, \ldots, g_{3\lfloor \frac{n}{3} \rfloor + 2}\},$$

where $0 \leq i \leq \lfloor \frac{n}{3} \rfloor$, is a 3-clique metric generator of $F_{1,n}$, and any subset of $\{g_1, g_2, \ldots, g_n\}$ with cardinality less than $|S| = n - \lceil \frac{n}{3} \rceil - 1$ is not a 3-clique metric generator for $F_{1,n}$. Hence in this case we have $\text{cdim}_3(F_{1,n}) = n - \lceil \frac{n}{3} \rceil - 1$. □

Proposition 2.4 *Let G be a graph with n vertices such that the number of its l-cliques are t. Then if $t \geq 2$, we have* $\text{cdim}_l(G) \leq \min\{n, \binom{t}{2}\}$. *Otherwise* $\text{cdim}_l(G) = 1$.

Proof If $l = 1$ or $t \leq 1$, then clearly we are done. So assume that $l \geq 2$. Let $L_1, L_2, \ldots, L_t$ be the l-cliques of G. For each $1 \leq i < j \leq n$, consider a vertex $x_{i,j}$ which belongs to $L_i \backslash L_j$. Let $S = \{x_{i,j} \mid 1 \leq i < j \leq n\}$. Now one can see that S is an l-clique metric generator for G and $|S| \leq \binom{t}{2}$. Hence the result holds. □

The next corollary follows from Proposition 2.4.

Corollary 2.5 *Let G be a graph with at most two l-cliques. Then* $\text{cdim}_l(G) = 1$.

Proposition 2.6 *Let G be a graph with n vertices and $L_1, L_2, \ldots, L_t$ be the l-cliques of G such that $L_i \nsubseteq \bigcup_{i \neq j, j=1}^{t} L_j$, for $1 \leq i \leq t - 1$. Then* $\text{cdim}_l(G) \leq t - 1$.

Proof Let $x_i \in L_i \backslash \bigcup_{i \neq j, j=1}^{t} L_j$, for $1 \leq i \leq t - 1$. Set $S = \{x_i \mid 1 \leq i \leq t - 1\}$. Then the ith component of the l-clique metric S-representation of L_j is zero if and only if $i = j$, for $1 \leq i \leq t - 1$. Moreover, none of the components of the l-clique metric S-representation of L_t is zero. Hence S is an l-clique metric generator of G, and so $\text{cdim}_l(G) \leq t - 1$. □

If we consider disconnected graphs, then l-CMD could be easily defined by considering the distance between two vertices in two different components as infinite. In fact we have the following result.

Remark 2.7 Let G be a disconnected graph with components $G_1, \ldots, G_r$. If $I = \{i \mid G_i \text{ has one } l\text{−clique}\}$ and $J = \{i \mid G_i \text{ has at least two } l\text{−cliques}\}$, then

$$\mathrm{cdim}_l(G) = \sum_{i \in J} \mathrm{cdim}_l(G_i) + \begin{cases} 0 & |I| \leq 1 \\ |I| - 1 & |I| > 1 \end{cases}.$$

Recall that for two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is the graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2. The concept of join graph is generalized (in [17], it is called as a generalized composition graph). Assume that G is a graph on k vertices with $V(G) = \{v_1, v_2, \ldots, v_k\}$, and let $H_1, H_2, \ldots, H_k$ be k pairwise disjoint graphs. The *G-generalized join graph* $G[H_1, H_2, \ldots, H_k]$ of $H_1, H_2, \ldots, H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim v_j$ in the graph G. Now, if the graph G consists of two adjacent vertices, then the G-generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2.

Note that in the rest of this section, we assume that there exists at least a nontrivial H_i, with $1 \leq i \leq k$, in $G[H_1, H_2, \ldots, H_k]$.

In the following proposition, we study the l-CMD of the G-generalized join graph $G[H_1, H_2, \ldots, H_k]$, in the case that H_i's are empty graphs.

Proposition 2.8 *Assume that G is a connected graph on k vertices with $V(G) = \{v_1, v_2, \ldots, v_k\}$, and let $H_1, H_2, \ldots, H_k$ be k pairwise disjoint empty graphs. If $\{v_1, v_2, \ldots, v_t\}$, where $0 \leq t \leq k$ are the vertices in G such that each of them belongs to an l-clique, then*

$$\sum_{i=1}^{t} |V(H_i)| - t \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_k]) \leq \mathrm{cdim}_l(G) + \sum_{i=1}^{t} |V(H_i)| - t.$$

Proof Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$, where $0 \leq t \leq k$ be the vertices in G such that each of them belongs to at least one l-clique. If $t = 0$, then $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_k]) = \mathrm{cdim}_l(G) = 1$. So assume that $t > 0$. Let $h_1, \ldots, h_t$ be arbitrary vertices in $H_1, \ldots, H_t$, respectively. Assume that S is an l-clique metric generator of the graph $G[H_1, H_2, \ldots, H_k]$. For each $1 \leq i \leq t$, we show that $V(H_i) \backslash \{h_i\} \subseteq S$. Suppose on the contrary that there exists $h'_i \in V(H_i)$ with $h'_i \neq h_i$ such that $h'_i \notin S$. Now consider two l-cliques L_1 and L_2 such that h_i is a vertex of L_1, h'_i is a vertex of L_2 and $L_1 \backslash \{h_i\} = L_2 \backslash \{h'_i\}$. Now, one can see that the l-clique metric S-representations of L_1 and L_2 are the same, which is a contradiction. Hence $V(H_i) \backslash \{h_i\} \subseteq S$, for each $1 \leq i \leq t$. Therefore we have

$$\sum_{i=1}^{t} |V(H_i)| - t \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_k]).$$

Let G' be the induced subgraph on vertex set $\{h_1, \ldots, h_t, v_{t+1}, \ldots, v_k\}$. Clearly G' is isomorphic to G. Now, let S' be an l-clique metric basis for G'. Since, for each

$h_j, h'_j \in V(H_j)$, where $t+1 \leq j \leq k$, we have $d(L, h_j) = d(L, h'_j)$, where L is an l-clique, $S' \cup \bigcup_{i=1}^{t}(V(H_i)\backslash\{h_i\})$ is an l-clique metric generator for $G[H_1, H_2, \ldots, H_k]$. So

$$\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_k]) \leq \mathrm{cdim}_l(G) + \sum_{i=1}^{t} |V(H_i)| - t.$$

□

In the following theorem, we determine the l-CMD of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are empty graphs and G is a path P_n. In fact the following theorem shows examples where the bounds in Proposition 2.8 are reached.

Theorem 2.9 *Assume that G is a path on $n \geq 2$ vertices with $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint empty graphs. Then $\sum_{i=1}^{n} |V(H_i)| - n \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) \leq \sum_{i=1}^{n} |V(H_i)| - n + 1$, when $l \in \{1, 2\}$. Also if $|V(H_i)| > 1$, for each $1 \leq i \leq n$, then we have*

$$\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \begin{cases} \sum_{i=1}^{n} |V(H_i)| - n + 1 & n = 3,\ l = 1, 2 \\ \sum_{i=1}^{n} |V(H_i)| - n & n \neq 3,\ l = 1, 2 \\ 1 & l \geq 3. \end{cases}$$

Proof If $l \geq 3$, then clearly $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = 1$. So let $l \in \{1, 2\}$. Let $h_1, \ldots, h_n$ be arbitrary vertices in $H_1, \ldots, H_n$, respectively. Set $S = \bigcup_{i=1}^{n}(V(H_i)\backslash\{h_i\})$, where h_i is an arbitrary vertex in H_i. By Proposition 2.8, every l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$ contains S. Also $S \cup \{h_1\}$ is an l-clique metric generator for $G[H_1, H_2, \ldots, H_n]$. Hence we have

$$\sum_{i=1}^{n} |V(H_i)| - n \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) \leq \sum_{i=1}^{n} |V(H_i)| - n + 1.$$

If $n = 3$, then we have $r^1_{G[H_1,H_2,H_3]}(h_1|S) = r^1_{G[H_1,H_2,H_3]}(h_3|S)$ and also we have $r^2_{G[H_1,H_2,H_3]}(h_1h_2|S) = r^2_{G[H_1,H_2,H_3]}(h_2h_3|S)$, which means that S is not an l-clique metric generator of $G[H_1, H_2, H_3]$, and as a consequence, $\mathrm{cdim}_l(G[H_1, H_2, H_3]) > |S| = \sum_{i=1}^{3} |V(H_i)| - 3$. Set $S' = S \cup \{h_1\}$. Now, one can see that S' is an l-clique metric basis of $G[H_1, H_2, H_3]$, and so $\mathrm{cdim}_l(G[H_1, H_2, H_3]) = \sum_{i=1}^{3} |V(H_i)| - 2$. Now, let $|V(H_i)| > 1$, for each $1 \leq i \leq n$ and, assume that $n \neq 3$. Then it is easy to see that S is an l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$, which implies that $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n$. □

In the following theorem, we determine the l-CMD of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are empty graphs and G is the complete graph K_n.

Theorem 2.10 *Assume that $G \cong K_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n > 2$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint empty graphs such that the number of trivial H_i's is $r < n$. Then we have*

$$\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \begin{cases} \sum_{i=1}^{n} |V(H_i)| - 1 & 2 \le l \le n-1 \\ \sum_{i=1}^{n} |V(H_i)| - n + r - 1 & l = 1,\ r > 0 \\ \sum_{i=1}^{n} |V(H_i)| - n & l = 1,\ r = 0 \\ \sum_{i=1}^{n} |V(H_i)| - n & l = n. \end{cases}$$

Proof Assume that $h_1, \ldots, h_n$ are arbitrary vertices in $H_1, \ldots, H_n$, respectively. Let $S = \bigcup_{i=1}^{n}(V(H_i)\backslash\{h_i\})$. By Proposition 2.8, every l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$ contains S, which implies that $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) \ge \sum_{i=1}^{n} |V(H_i)| - n$. First assume that $l = 1$. Since the places in which there is a 2, if exists, appears in the l-clique metric S-representation of each two distinct h_i and h_j, with $1 \le i \ne j \le n$, are different from each other, their l-clique metric S-representations are not equal. Without loss of generality, assume that $|V(H_1)| = \cdots = |V(H_r)| = 1$. Hence the l-clique metric S-representation of all h_i's, for $1 \le i \le r$ is equal. So, in this situation, any l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$ is of the form $S \cup \bigcup_{i=1, i\ne j}^{r}\{h_i\}$, for some $1 \le j \le r$. Hence we have $\mathrm{cdim}_1(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n + r - 1$, for $0 < r < n$. Clearly if $r = 0$, then S is a 1-clique metric basis of $G[H_1, H_2, \ldots, H_n]$, and so $\mathrm{cdim}_1(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n$.

Now, assume that $l \ge 2$. Let S' be an l-clique metric generator and L be an arbitrary l-clique of $G[H_1, H_2, \ldots, H_n]$. For each $x \in S'$, we have

$$d_{G[H_1, H_2, \ldots, H_n]}(L, x) = \begin{cases} 1 & x \notin L \\ 0 & x \in L. \end{cases}$$

So, for each two distinct l-cliques L_1 and L_2, $L_1 \cap S' = L_2 \cap S'$ if and only if L_1 and L_2 have the same l-clique metric S'-representations. If $l = n$, then, for each two distinct l-cliques L_1 and L_2, $L_1 \cap S = L_2 \cap S$ implies that $L_1 = L_2$. This implies that S is an l-clique metric basis, and so $\mathrm{cdim}_n(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n$. Now, assume that $2 \le l \le n-1$. If there are h_i and h_j with $1 \le i \ne j \le n$ such that they do not belong to an l-clique metric generator S', then consider two l-cliques L_1 and L_2 with $h_i \in L_1$, $h_j \in L_2$ and $L_1\backslash\{h_i\} = L_2\backslash\{h_j\}$. Since $L_1 \cap S' = L_2 \cap S'$, they have the same l-clique metric S'-representations, which is impossible. So in this situation, any l-clique metric generator is of the form $S \cup \bigcup_{i=1, i\ne j}^{n} V(H_i)$, for some $1 \le j \le n$. Thus we have $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - 1$. □

In the following theorem, we determine the l-CMD of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are empty graphs and G is isomorphic to the cycle C_n, where $n > 3$. Note that the case $n = 3$ is obtained by Theorem 3.3.

Theorem 2.11 *Assume that G is a cycle C_n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n > 3$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint empty graphs. Then*

$$\sum_{i=1}^{n} |V(H_i)| - n \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) \leq \sum_{i=1}^{n} |V(H_i)| - n + 2,$$

when $l \in \{1, 2\}$, and $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = 1$, for $l \geq 3$. Also, for $n = 4$ and $l \in \{1, 2\}$, we have $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n + 2$, and if $|V(H_i)| > 1$, for each $1 \leq i \leq n$, then we have $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n$, when $n > 4$ and $l \in \{1, 2\}$.

Proof Clearly if $l \geq 3$, then $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = 1$. So assume that $l \in \{1, 2\}$. Let $h_1, \ldots, h_n$ be arbitrary vertices in $H_1, \ldots, H_n$, respectively, and $S = \bigcup_{i=1}^{n} (V(H_i) \backslash \{h_i\})$. By Proposition 2.8, every l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$, contains S. Also $S \cup \{h_1, h_2\}$ is an l-clique metric generator of $G[H_1, H_2, \ldots, H_n]$. Hence $\sum_{i=1}^{n} |V(H_i)| - n \leq \mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) \leq \sum_{i=1}^{n} |V(H_i)| - n + 2$. If $n = 4$, then one can see that $S \cup \{h_1, h_2\}$ is an l-clique metric basis of $G[H_1, H_2, \ldots, H_n]$. So $\mathrm{cdim}_l(G[H_1, H_2, H_3, H_4]) = \sum_{i=1}^{4} |V(H_i)| - 2$.

Now, assume that $n \geq 5$. Let $|V(H_i)| > 1$, for each $1 \leq i \leq n$. Since $n \geq 5$ and $|V(H_i)| \geq 2$, for any two vertices $h_i, h_j \notin S$, the distance between h_i and any vertex belonging to $S \cap (V(H_{i-1}) \cup V(H_{i+1}))$ is one, while the distance between h_j and any vertex belonging to at least one of these two sets $S \cap V(H_{i-1})$ or $S \cap V(H_{i+1})$ is different than one. Thus, S is an 1-clique metric generator for $G[H_1, H_2, \ldots, H_n]$. Now, let L_1 and L_2 be two distinct 2-cliques. If $L_1 \cap S = \phi = L_2 \cap S$, then the places that 1 appears in their 2-clique metric S-representations are different. So, without loss of generality, assume that $s \in L_1 \cap S$. If $s \notin L_2$, then the corresponding components to s in the 2-clique metric S-representations of L_1 and L_2 are zero and nonzero, respectively. Thus, let $s \in L_2$. If $L_1 \subseteq S$ or $L_2 \subseteq S$, then clearly their 2-clique metric S-representations are different. Now, assume that $L_1 \nsubseteq S$ and $L_2 \nsubseteq S$. Then one can see that the places of 1 in their 2-clique metric S-representations are different. So S is an 2-clique metric generator for $G[H_1, H_2, \ldots, H_n]$. Hence we have $\mathrm{cdim}_l(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |V(H_i)| - n$. □

3 *l*-Clique Metric Dimension of $\Gamma(\mathbb{Z}_n)$

Let R be a commutative ring with nonzero identity. We denote the set of all unit elements and zero divisors of R by $U(R)$ and $Z(R)$, respectively. Also by $Z^*(R)$ we denote the set $Z(R)\backslash\{0\}$. Sharma and Bhatwadekar [20] defined the comaximal graph of a commutative ring R. The *comaximal graph* of R is a simple graph whose vertices consists of all elements of R, and two distinct vertices a and b are adjacent if and only if $aR + bR = R$, where cR is the ideal generated by c, for $c \in R$. Let $\Gamma(R)$ be an induced subgraph of the comaximal graph with nonunit elements of R as vertices. The properties of the graph $\Gamma(R)$ were studied in [16, 22, 25].

For two integers r and s, the notation (r, s) stands for the greatest common divisor of r and s. Also we denote the elements of the ring $\mathbb{Z}_n$, where $n > 1$, by $0, 1, 2, \ldots, n-1$.

For every nonzero element a in $\mathbb{Z}_n$, if $(a, n) = 1$, then a is a unit element; otherwise, $(a, n) \neq 1$, and so a is a zerodivisor. Therefore, $|U(\mathbb{Z}_n)| = \phi(n)$ and $|Z(\mathbb{Z}_n)| = n - \phi(n)$, where ϕ is the Euler's totient function.

An integer d is said to be a *proper divisor* of n if $1 < d < n$ and $d \mid n$. Now let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of n. For $1 \leq i \leq k$, set

$$A_{d_i} := \{x \in \mathbb{Z}_n \mid (x, n) = d_i\}.$$

Clearly, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_k}$ are pairwise disjoint and we have

$$Z^*(\mathbb{Z}_n) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}$$

and

$$V(\Gamma(\mathbb{Z}_n)) = \{0\} \cup A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}.$$

The following lemma is stated from [27].

Lemma 3.1 [27, Proposition 2.1] *Let* $1 \leq i \leq k$. *Then* $|A_{d_i}| = \phi(\frac{n}{d_i})$.

In this section, the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set A_{d_i} is denoted by $\Gamma(A_{d_i})$, where $1 \leq i \leq k$.

The following lemma states some adjacencies in $\Gamma(\mathbb{Z}_n)$.

Lemma 3.2 *The following statements hold:*

(i) *Two distinct vertices x and y are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if $(x, y) \in U(\mathbb{Z}_n)$.*
(ii) *For $1 \leq i \leq k$, $\Gamma(A_{d_i})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{d_i})}$.*
(iii) *For $1 \leq i \neq j \leq k$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} if and only if $(d_i, d_j) = 1$.*

Proof (i) First suppose that x and y are adjacent vertices in $\Gamma(\mathbb{Z}_n)$. Assume on the contrary that $d = (x, y) \notin U(\mathbb{Z}_n)$. So we have $x\mathbb{Z}_n \subseteq d\mathbb{Z}_n$ and $y\mathbb{Z}_n \subseteq d\mathbb{Z}_n$. Thus $x\mathbb{Z}_n + y\mathbb{Z}_n \subseteq d\mathbb{Z}_n \neq \mathbb{Z}_n$, and this means that x and y are not adjacent, which is a contradiction. Now, let $u = (x, y) \in U(\mathbb{Z}_n)$. So there exist $r, s \in \mathbb{Z}$ such that $u = rx + sy \in x\mathbb{Z}_n + y\mathbb{Z}_n$. Therefore we have $x\mathbb{Z}_n + y\mathbb{Z}_n = \mathbb{Z}_n$, which implies that x and y are adjacent.

(ii) For each two distinct elements $x, y \in A_{d_i}$, we have $(x, n) = d_i = (y, n)$. So $d_i \mid (x, y)$, which implies that $(x, y) \notin U(\mathbb{Z}_n)$. Hence by (i), we have that x and y are not adjacent. Therefore by Lemma 3.1, we have $\Gamma(A_{d_i}) \cong \overline{K}_{\phi(\frac{n}{d_i})}$.

(iii) Let $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. First assume that $x \in A_{d_i}$ and $y \in A_{d_j}$ are adjacent vertices. If $(d_i, d_j) = d \neq 1$, then $(n, d) = d$. Since $(x, n) = d_i$ and $(y, n) = d_j$, we have that $d \mid x, y$. Hence $Rx + Ry \subseteq Rd \neq R$, which is impossible. Now suppose that $(d_i, d_j) = 1$. Let $x \in A_{d_i}$ and $y \in A_{d_j}$ be arbitrary vertices. If $d = (x, y) \notin U(\mathbb{Z}_n)$, then $t = (d, n) \neq 1$. Since $t \mid x, y, n$, we have $t \mid (d_i, d_j)$ and this is impossible. Hence $(x, y) \in U(\mathbb{Z}_n)$ which means that x and y are adjacent. □

Now, we introduce a simple graph G_n, which plays an important role in the structure of $\Gamma(\mathbb{Z}_n)$. The graph G_n is the simple graph with vertex set $\{d_1, d_2, \ldots, d_k\}$, where d_i's, $1 \le i \le k$, are the proper divisors of n, and two distinct vertices d_i and d_j are adjacent if and only if $(d_i, d_j) = 1$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ be the factorization of n to its prime powers, where $t, \alpha_1, \ldots, \alpha_t$ are positive integers and $p_1, \ldots, p_t$ are distinct prime numbers. Every divisor of n is of the form $p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t}$, for some integers $\beta_1, \ldots, \beta_t$, where $0 \le \beta_i \le \alpha_i$ for each $i \in \{1, 2, \ldots, t\}$. Hence the number of proper divisors of n is equal to $\prod_{i=1}^{t}(n_i + 1) - 2$. Therefore we have $k = |V(G_n)| = \prod_{i=1}^{t}(n_i + 1) - 2$.

Let $\Gamma^*(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)\backslash\{0\}$. Consider the graph G_n and replace each vertex d_i of G_n by $\Gamma[A_{d_i}]$. In view of Lemma 3.1, we have

$$\Gamma^*(\mathbb{Z}_n) = G_n\left[\overline{K}_{\phi\left(\frac{n}{d_1}\right)}, \overline{K}_{\phi\left(\frac{n}{d_2}\right)}, \ldots, \overline{K}_{\phi\left(\frac{n}{d_k}\right)}\right].$$

Now, since the zero element is adjacent to none of the vertices of $\Gamma^*(\mathbb{Z}_n)$, we have

$$\Gamma(\mathbb{Z}_n) = (K_1 \cup \Gamma^*(\mathbb{Z}_n)).$$

In the following theorem, we study the l-CMD of $\Gamma(\mathbb{Z}_n)$.

Theorem 3.3 *Assume that $\{d_1, d_2, \ldots, d_t\}$, where $1 \le t \le k$, are those vertices of G_n that each of them belongs to an l-clique. Then for $l = 1$ we have*

$$\sum_{i=1}^{k} \phi\left(\frac{n}{d_i}\right) - k + r \le \mathrm{cdim}_l(\Gamma(\mathbb{Z}_n)) \le \mathrm{cdim}_l(G_n) + \sum_{i=1}^{k} \phi\left(\frac{n}{d_i}\right) - k + r$$

and for $l > 1$,

$$\sum_{i=1}^{t} \phi\left(\frac{n}{d_i}\right) - t \le \mathrm{cdim}_l(\Gamma(\mathbb{Z}_n)) \le \mathrm{cdim}_l(G_n) + \sum_{i=1}^{t} \phi\left(\frac{n}{d_i}\right) - t,$$

where r is the number of isolated vertices of G_n.

Proof Note that the graph G_n is not connected in general. Let r be the number of isolated vertices of G_n. Since 0 is the isolated vertex of $\Gamma(\mathbb{Z}_n)$, we assume that $0, a_1, \ldots, a_r$ are the isolated vertices of $\Gamma(\mathbb{Z}_n)$. By Remark 2.7, we have

$$\mathrm{cdim}_1(\Gamma(\mathbb{Z}_n)) = \mathrm{cdim}_1(\Gamma(\mathbb{Z}_n)\backslash\{0, a_1, \ldots, a_r\}) + r.$$

Now, the results follow from Proposition 2.8 and Remark 2.7. □

Example 3.4 Consider the ring $\mathbb{Z}_{12}$. We have $d_1 = 2, d_2 = 3, d_3 = 4$, and $d_4 = 6$. Then G_{12} is the graph $2 \sim 3 \sim 4 \cup \{6\}$, which is isomorphic to $P_3 \cup K_1$. Hence we have

$$\Gamma(\mathbb{Z}_{12}) = K_1 \cup G_{12}[\overline{K}_2, \overline{K}_2, \overline{K}_2, K_1]$$

and, by Theorems 2.9 and 3.3 , we have

$$\mathrm{cdim}_l(\Gamma(\mathbb{Z}_{12})) = \begin{cases} 5 & l = 1 \\ 4 & l = 2 \\ 1 & l \geq 3. \end{cases}$$

In the rest of this section, we discuss the CMD of $\Gamma(\mathbb{Z}_n)$, for (i) $n = p^t$, (ii) $n = pq$ and (iii) $n = p^2q$, where p and q are distinct prime numbers and t is a positive integer.

(i) Let $n = p^t$. Then $\Gamma(\mathbb{Z}_{p^t})$ is an empty graph with $p^t - \phi(p^t) = p^{t-1}$ vertices, and so $\Gamma(\mathbb{Z}_{p^t}) = \overline{K_{p^{t-1}}}$. Now, by Remark 2.7 we have

$$\mathrm{cdim}_l(\Gamma(\mathbb{Z}_{p^t})) = \begin{cases} p^{t-1} - 1 & l = 1 \\ 1 & l \geq 2. \end{cases}$$

(ii) Let $n = pq$, where p and q are distinct prime numbers. Since the only proper divisors of n are p and q, the graph G_{pq} is $p \sim q$. So we have

$$\Gamma(\mathbb{Z}_{pq}) = K_1 \cup G_{pq}[\overline{K}_{\phi(q)}, \overline{K}_{\phi(p)}].$$

Now, by Theorem 2.9, we have

$$\mathrm{cdim}_l(\Gamma(\mathbb{Z}_{pq})) = \begin{cases} p + q - 4 & l = 1, 2 \\ 1 & l \geq 3. \end{cases}$$

(iii) Let $n = p^2q$, where p and q are distinct prime numbers. Since p, q, and pq are the proper divisors of n, the graph G_{p^2q} is $p \sim q \sim p^2 \cup \{pq\}$. Hence we have

$$\Gamma(\mathbb{Z}_{p^2q}) = K_1 \cup G_{p^2q}[\overline{K}_{\phi(pq)}, \overline{K}_{\phi(p^2)}, \overline{K}_{\phi(q)}, \overline{K}_{\phi(p)}].$$

Since $\phi(pq) = pq - p - q + 1$ and $\phi(p^2) = p^2 - p$, by Theorem 2.9 and Remark 2.7,

$$\mathrm{cdim}_l(\Gamma(\mathbb{Z}_{p^2q})) = \begin{cases} p^2 + pq - p - 3 & l = 1 \\ p^2 + pq - 2p - 2 & l = 2 \\ 1 & l \geq 3. \end{cases}$$

4 *l*-Clique Metric Dimension Over Corona Product

Let G and H be two graphs with the vertex sets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$, respectively. The corona of G and H, denoted by $G \circ H$, is the graph whose vertex and edge sets are defined as below:

$$
\begin{aligned}
V(G \circ H) &= V(G) \cup (\cup_{i=1}^{n}\{h_{1_i}, \ldots, h_{m_i}\}),\\
E(G \circ H) &= E(G) \cup \{h_{j_i}h_{l_i} : h_jh_l \in E(H) \,\&\, 1 \leq i \leq n\}\\
&\quad \cup \{g_ih_{j_i} : 1 \leq j \leq m,\ 1 \leq i \leq n\}.
\end{aligned}
$$

The metric dimension (1-CMD) of corona product graphs was investigated in [26]. After that Peterin and Yero studied the edge metric dimension (2-CMD) over corona product in [18]. In this section, we give a formula for the l-CMD of corona product of two graphs G and H for $l \geq 3$. In what follows, we say the vertex v distinguishes two l-cliques U and W if $d(v, U) \neq d(v, W)$.

Theorem 4.1 *Let G and H be two connected graphs of order n and m, respectively, and $l \geq 3$ be an integer number. If $\{V_1(H), \ldots, V_k(H)\}$ is the $(l-1)$-clique set of H, then*

$$
\mathrm{cdim}_l(G \circ H) = \begin{cases} \mathrm{cdim}_l(G) & \text{if } \omega(H) < l-1 \\ \dim(G) & \text{if } k = 1 \text{ and } \omega(G) < l \end{cases},
$$

where $\omega(G)$ and $\omega(H)$ are the clique numbers of G and H, respectively.

Proof Let $V(G) = \{g_1, \ldots, g_n\}$ and H_i be the i-th copy of H in $G \circ H$, $1 \leq i \leq n$. Then $G \circ H$ is obtained by joining each vertex of the i-copy of H to the i-th vertex, g_i, of G.

Let S_G be an l-clique metric basis of G and $\{V_1(G), \ldots, V_t(G)\}$ be the l-clique set of G. Also, let $V_{j_i}(H)$ denote the i-the copy of $V_j(H)$ in $G \circ H$, for $1 \leq i \leq n$ and $1 \leq j \leq k$. Thus, it is clear that $V'_{j_i}(H) = V_{j_i}(H) \cup \{g_i\}$, $1 \leq i \leq n$, is an l-clique in $G \circ H$.

First, we prove that if $\omega(H) < l-1$ (or $k = 0$), then $\mathrm{cdim}_l(G \circ H) = \mathrm{cdim}_l(G)$. To do this, we prove that S_G is also an l-clique metric basis of $G \circ H$. Clearly S_G is an l-clique metric generator for $G \circ H$ and so $\mathrm{cdim}_l(G \circ H) \leq \mathrm{cdim}_l(G)$. Suppose that S is an l-clique metric basis of $G \circ H$. We claim that $|S \cap V(H_i)| \leq 1$ for $1 \leq i \leq n$. To prove this claim, suppose, on the contrary that there exist $u, z \in S \cap V(H_i)$. Then $S' = S \backslash \{u\}$ is not an l-clique metric generator for $G \circ H$. Thus there exist two l-cliques U and W in $G \circ H$ such that $d_{G\circ H}(v, U) = d_{G\circ H}(v, W)$ for each $v \in S'$. Hence $d_{G\circ H}(z, U) = d_{G\circ H}(z, W)$. On the other hand, since $\omega(H) < l-1$, then $d_{G\circ H}(z, U) = d_{G\circ H}(z, W) = d_G(g_i, U) + 1 = d_G(g_i, W) + 1$. Also, since $\omega(H) < l-1$, then $d_{G\circ H}(u, U) = d_{G\circ H}(u, W) = d_G(g_i, U) + 1 = d_G(g_i, W) + 1$. Therefore S is not an l-clique metric generator for $G \circ H$ which is a contradiction.

Now suppose that $u \in S \cap V(H_i)$. Then $S' = (S - \{u\}) \cup \{g_i\}$ is also an l-clique metric basis of $G \circ H$. Because $d_{G\circ H}(u, V_j(G)) = d_G(g_i, V_j(G)) + 1$ for each $1 \leq j \leq t$. By repeating this technique, we reach an l-clique metric basis S'' of $G \circ H$ with this property that all vertices of S'' are in G. Therefore, $\mathrm{cdim}_l(G \circ H) \geq \mathrm{cdim}_l(G)$.

Now, suppose that $\omega(G) < l$, $k = 1$ and $V_1(H)$ is the $(l-1)$-clique of H. Let S_G be a 1-clique metric basis of G. We claim that S_G is an l-clique metric generator for $G \circ H$. Then, since $d_{G\circ H}(V'_{1_i}(H), v) = d_G(g_i, v)$ for each $v \in S_G$, then every pair of l-cliques $V_{1_i}(H)$'s, $1 \leq i \leq n$, is distinguished by a vertex of S_G. Therefore, S_G is an l-clique metric generator for $G \circ H$ and so $\mathrm{cdim}_l(G \circ H) \leq |S_G| = \dim(G)$. Then, it is sufficient to show that $\mathrm{cdim}_l(G \circ H) \geq \dim(G)$. To do this, suppose that S' is an l-clique metric basis of $G \circ H$. By the above argument, if $|S' \cap V(G)| = |S'|$, then we have nothing to prove. Otherwise, there exists $v \in S'$ such that $v \in V_{1_i}$ for an $i \in \{1, \ldots, n\}$. Since $d_G(v, V'_{1_j}) = d_G(g_i, V'_{1_j}) + 1$ for $i \neq j \in \{1, \ldots, n\}$, then $S'' = (S - v) \cup \{g_i\}$ is also an l-clique metric basis of $G \circ H$. We use this technique to reach an l-clique metric basis S''' of $G \circ H$ with this property that $|S''' \cap V(G)| = |S'''|$. Therefore, $\mathrm{cdim}_l(G \circ H) \geq \dim(G)$. □

The concept of global forcing sets for maximal matchings was presented in [24]. Here we need to introduce an extension of the idea of global forcing sets for l-cliques of a graph.

A global forcing set for l-cliques of a graph G is a subset S of $V(G)$ with this property that $V_1 \cap S \neq V_2 \cap S$ for any two l-cliques V_1 and V_2 of G. A global forcing set for l-cliques of G with minimum cardinality is called a minimum global forcing set for l-cliques of G, and its cardinality, denoted by φ_l, is the global forcing number for l-cliques of G.

We can find a global forcing set for l-cliques of G by the following ILP.

Let G be a graph with $V(G) = \{v_1, \ldots v_n\}$ and let $\{V_1, \ldots, V_k\}$ be the set of all l-cliques of G. Let $D_G = [d_{ij}]$ be a $k \times n$ matrix, where $d_{ij} = 1$ if $v_j \in V_i$, and $d_{ij} = 0$ otherwise. Let $F : \{0, 1\}^n \to \mathbb{N}_0$ be defined by

$$F(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

Then our goal is to determine min F subject to the constraints

$$|d_{i1} - d_{j1}|x_1 + |d_{i2} - d_{j2}|x_2 + \cdots + |d_{in} - d_{jn}|x_n > 0, \quad 1 \leq i < j \leq k.$$

Note that if $x'_1, \ldots, x'_n$ is a set of values for which F attains its minimum, then $S = \{v_i : x'_i = 1\}$ is a minimum global forcing set for l-cliques of G.

Theorem 4.2 *Let G and H be two connected graphs with $|V(G)| = n$, and $l \geq 3$ be an integer number. If $\{V_1(H), \ldots, V_k(H)\}$ is the $(l-1)$-clique set of H and $\omega(H) = l-1$, then for $k \geq 2$ we have*

$$\mathrm{cdim}_l(G \circ H) = n \cdot \varphi_{l-1}(H).$$

Proof Let S be an l-clique metric generator for $G \circ H$. Suppose, on the contrary that there exists H_i, a copy of H in $G \circ H$, that $|S \cap V(H_i)| < \varphi_{l-1}(H)$. Then there exist two $(l-1)$-cliques $V_{j_i}(H)$ and $V_{q_i}(H)$ in H_i such that $S \cap V_{j_i}(H) = S \cap V_{q_i}(H)$. Hence $d_{G\circ H}(u, V_{j_i}(H)) = d_{G\circ H}(u, V_{q_i}(H)) = 0$ for each $u \in S \cap V_{j_i}(H)$, and $d_{G\circ H}(u, V_{j_i}(H)) = d_{G\circ H}(u, V_{q_i}(H)) = 1$ for each $u \in S \cap (V(H_i) \setminus V_{j_i}(H))$. On the other hand, it is not difficult to check that $d_{G\circ H}(u, V_{j_i}(H)) = d_{G\circ H}(u, V_{q_i}(H))$ for each $u \in S \setminus V(H_i)$. Thus, $d_{G\circ H}(u, V_{j_i}(H)) = d_{G\circ H}(u, V_{q_i}(H))$ for each $u \in S$, which is contrary to our assumption. Therefore, $\mathrm{cdim}_l(G \circ H) \geq n \cdot \varphi_{l-1}(H)$.

It remains to prove that $\mathrm{cdim}_l(G \circ H) \leq n \cdot \varphi_{l-1}(H)$. Let S_H be a minimum global forcing set for $(l-1)$-cliques of H, and let S_{H_i} be the i-th copy of S_H in $G \circ H$. Then, it is easy to check that $S' = \bigcup_{i=1}^{n} S_{H_i}$ is an l-clique metric generator for $G \circ H$. Therefore, $\mathrm{cdim}_l(G \circ H) \leq n \cdot \varphi_{l-1}(H)$. □

5 Complexity Issues

The clique problem is the optimization problem of finding a clique of maximum size in a graph. As a decision problem, we ask simply whether a clique of a given size k exists in the graph.

Theorem 5.1 [8] *The clique problem is NP-complete.*

Therefore, the problem of finding all l-cliques in a graph is NP-hard. Hence, throughout this section we are assuming that all the l-cliques of the graph are given.

In this section, we prove the l-CMD problem is NP-complete. Recall that for $l = 1, 2$, l-CMD problems are the metric dimension and the edge metric dimension problems, respectively. On the other hand, Garey and Johnson [11] proved that the decision version of the metric dimension problem is NP-complete on connected graphs. Also, NP-completeness of computing the edge metric dimension of connected graphs was proved in [14]. Moreover, Epstein, Levin, and Woeginger showed that for split graphs, bipartite graphs, co-bipartite graphs, and line graphs of bipartite graphs, the problem of computing the metric dimension of the graph is NP-hard [10]. Then, we prove NP-completeness of computing the l-CMD of connected graphs for $L \geq 3$. Let us start with the below decision problem.

l-CMD problem: For a given positive integer l. Let G be a connected graph with n where $n \geq 3$, X be the set of all distinct l-cliques of G, and let r be a positive integer such that $1 \leq r \leq n-1$. Is $\mathrm{cdim}_l(G) \leq r$?

Note that the l-CMD problem is the decision version of the problem of computing $\mathrm{cdim}_l(G)$ for a given connected graph G.

Our proof for showing that the NP-completeness of l-CMD problem is based on a reduction from the metric dimension problem on connected bipartite graphs. We recommend [7] for more details on the reduction technique. Now, we are ready to prove that the l-CMD problem is NP-complete.

Theorem 5.2 *The l-CMD problem, for $l \geq 3$, is NP-complete.*

Proof Note that the l-CMD problem is clearly in NP because we can check its feasibility as a l-clique metric generator in polynomial time.

For showing NP-hardness of this problem, we present a reduction from the metric dimension for connected bipartite graphs.

Let G be a connected bipartite graph where $V(G) = \{g_1, \ldots, g_n\}$. Now, we construct graph G' from G by taking one copy of G and n copies of the complete graph K_{l-1} and by joining each vertex of the i-th copy of K_{l-1} to the i-th vertex of G, $i = 1, \ldots, n$. In other words, $G' = G \circ K_{l-1}$. For more illustration, see an example of G and G' in Fig. 1. Since G is bipartite, then $\omega(G) < 3$. Thus by Theorem 4.1, $\mathrm{cdim}_l(G') = \mathrm{cdim}_l(G \circ H) = \dim(G)$. Moreover, it is easy to see that constructing G'

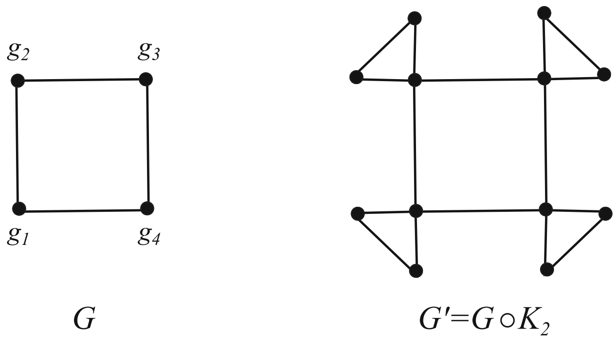

Fig. 1 The graph G' constructed from G for $l = 3$

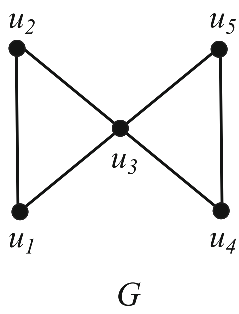

Fig. 2 Graph G

from G can be done in polynomial time. Therefore, if there exists a polynomial-time algorithm for computing $\mathrm{cdim}_l(G')$, then there exists a polynomial-time algorithm for computing $\dim(G)$. □

An integer linear programming (ILP) model for the classical metric dimension problem was presented in [6]. Motivated by this work and using its notations, we consider here an IPL model for computing $cdim_l(G)$ for a given connected graph G and its l-cliques. Let $G = (V, E)$ be a connected graphs with $V = \{u_1, \ldots, u_n\}$. Let $V_1, \ldots, V_k$ be the l-cliques of G. Also, suppose that $D_G = [d_{ij}]$ is a $k \times n$ matrix such that $d_{ij} = d_G(V_i, u_j)$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$. Consider the binary decision variables x_i for $i \in \{1, \ldots, n\}$ where $x_i \in \{0, 1\}$. By x_i, we mean the vertex u_i is a member of an l-clique metric generator of G and $x_i = 0$ for otherwise. we define the objective function F by

$$F(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

Minimize F subject to the following constraints

$$|d_{i1} - d_{j1}|x_1 + |d_{i2} - d_{j2}|x_2 + \cdots + |d_{in} - d_{jn}|x_n > 0, \quad 1 \le i < j \le k$$

is equivalent to finding a basis in the sense that if $x'_1, \ldots, x'_n$ is a set of values for which F attains its minimum, then $W = \{u_i \mid x'_i = 1\}$ is a basis for G.

For example, consider graph G shown in Fig. 2 with 3-cliques $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{u_3, u_4, u_5\}$. Then, $D_G = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$. Therefore, minimize $F(x_1, x_2, x_3) = x_1+x_2+x_3+x_4+x_5$ subject to the constraints $x_1+x_2+x_4+x_5 > 0, x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$. Thus F attains its minimum for $x_1 = 1$, $x_2 = x_3 = x_4 = x_5 = 0$, hence $W = \{u_1\}$ is a 3-clique metric basis for G.

Acknowledgements The authors are very grateful to the referees for their comments and suggestions.

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