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Relaxed DP-3-Coloring of Planar Graphs Without Some Cycles

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Abstract

Dvořák and Postle introduced the concept of DP-coloring to overcome some difficulties in list coloring. Sittitrai and Nakprasit combined DP-coloring and defective list coloring to define a new coloring—relaxed DP-coloring. For relaxed DP-coloring, Sribunhung et al. proved that planar graphs without 4- and 7-cycles are DP-(0, 2, 2)-colorable. Li et al. proved that planar graphs without 4, 8-cycles or 4, 9-cycles are DP-(1, 1, 1)-colorable. Lu and Zhu proved that planar graphs without 4, 5-cycles, or 4, 6-cycles, or 4, 7-cycles are DP-(1, 1, 1)-colorable. In this paper, we show that planar graphs without 4, 6-cycles or 4, 8-cycles are DP-(0, 2, 2)-colorable.

Keywords DP-coloring · Defective coloring · List coloring · Relaxed-DP-coloring

Mathematics Subject Classification 05C15

1 Introduction

All graphs in this paper are simple and undirected. Assume G is a plane graph, we use $V(G)$, $E(G)$, $F(G)$, and $\delta(G)$ to denote its vertex set, edge set, face set, and minimum degree in the graph G, respectively. We use $d(x)$ to denote the degree of x for each $x \in V(G) \cup F(G)$. We say that u is a d-vertex, d^+-vertex, or d^--vertex if $d(u) = d$, $d(u) \geq d$, or $d(u) \leq d$, respectively. Let $b(f)$ be the boundary of a face f and write $f = [v_1 v_2 \dots v_d]$, where $v_1, v_2, \dots, v_d$ are the boundary vertices of f in a cyclic order. If $d(f) = k$ ($d(f) \geq k$ or $d(f) \leq k$), then we call f a k-face (k^+-face or k^--face) of G. A face is called a *simple face* if its boundary is a cycle. A cycle of

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length k is called a *k-cycle*, and a 3-cycle is usually called as a triangle. Two cycles or faces are *adjacent* if they share at least one edge, or their boundaries share at least one edge, respectively. Two adjacent cycles (or faces) C_1 and C_2 are *normally adjacent* if $|V(C_1) \cap V(C_2)| = 2$.

We say that L is a k-list assignment for a graph G if it assigns a list $L(v)$ to each vertex v of G with $|L(v)| \geq k$. If G has a proper coloring ϕ such that $\phi(v) \in L(v)$ for each vertex v, then we say that G is *L-colorable*. A graph G is *k-choosable* if it is L-colorable for any k-list assignment L. The *list chromatic number* of G, denoted by $\chi_\ell(G)$, is the smallest integer k such that G is k-choosable.

Dvořák and Postle [2] introduced a generalization of list coloring. Let G be a graph and L be a list assignment on $V(G)$. A graph H_L, simply write H, is said to be a *cover* of G if it satisfies all the following two conditions.

(i) The vertex set of H is $\bigcup_{u \in V(G)}(\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$.
(ii) The edge set of H is $\mathscr{M} = \bigcup_{uv \in E(G)} \mathscr{M}_{uv}$, where $\mathscr{M}_{uv}$ is a matching between the sets $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

Let T be a subset of $V(H)$. If $|T \cap (\{u\} \times L(u))| = 1$ for each vertex u in G, then T is called a *transversal* of H. When a transversal is independent, it is a *DP-coloring*. If every cover H of G with a k-assignment L has a DP-coloring, then the least number k is the *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$. Note that DP-coloring is a generalization of list coloring. This implies that $\chi_\ell(G) \leq \chi_{DP}(G)$. Chen et al. [1] proved that every planar graph without 4-cycles adjacent to 6-cycles is DP-4-colorable. Recently, it is proved that every planar graph is DP-4-colorable if it does not contain i-cycles adjacent to j-cycles for distinct i and j from $\{3, 4, 5, 6\}$, see [1, 6, 10, 12]. More sufficient conditions for a planar graph to be DP-4-colorable, see [3, 9, 11, 15].

In [14], Sittitrai and Nakprasit combined DP-coloring and relaxed list coloring (defective list coloring) into a new coloring as follows. Let H be a cover of a graph G with a k-assignment L. A transversal T of H is a *$(d_1, d_2, \ldots d_k)$-coloring* if every $(v, i) \in T$ has degree at most d_i in $H[T]$. For any k-assignment L and any cover H_L, if H_L has a $(d_1, d_2, \ldots, d_k)$-coloring, then we say G is *DP-$(d_1, d_2, \ldots d_k)$-colorable*. For defective DP-coloring, we refer the readers to [4, 5, 7].

Li et al. [8] proved that every planar graph without 4, 8-cycles, or 4, 9-cycles is DP-(1, 1, 1)-colorable. Lu and Zhu [13] proved that every planar graph without 4, 5-cycles, or 4, 6-cycles, or 4, 7-cycles is DP-(1, 1, 1)-colorable. Sribunhung et al. [16] proved that every planar graph without 4, 7-cycles is DP-(0, 2, 2)-colorable. In this paper, we prove that every planar graph without 4, 6-cycles, or 4, 8-cycles is DP-(0, 2, 2)-colorable.

To prove the conclusion, we need some new definitions. Suppose B is a condition imposed on ordered vertices. A *DP-B-coloring* of H_L is a transversal T with ordered vertices from left to right such that each $(v, c) \in T$ satisfies condition B imposed on each element of H. Suppose T is a transversal of a cover H of G. We say that T is a *DP-B_A-coloring* if the vertices in T can be ordered from left to right such that:

(i) For each $(v, 1) \in T$, $(v, 1)$ has no neighbor on the left.
(ii) For each $(v, c) \in T$ where $c \neq 1$, (v, c) has at most one neighbor on the left and that neighbor (if it exists) is adjacent to at most one vertex on the left of (v, c).

We say that G is *DP-B_A-k-colorable* if every cover H_L of a graph G with a k-assignment L has a DP-B_A-coloring.

A graph is a *linear forest* if it is a forest with maximum degree at most two. It is easy to prove that a transversal T is a DP-B_A-coloring only if $H[T]$ is a linear forest and $\{(v, c) \in T : c = 1\}$ is independent in H. But the inverse is not true. For example, $T = \{(x, 1), (y, 2), (z, 1)\}$, where $(y, 2)$ is adjacent to $(x, 1)$ and $(z, 1)$ in H. Observe that T has no desired ordering as in the definition DP-B_A-coloring.

Theorem 1.1 *Every planar graph without* 4- *and* 8-*cycles is DP-B_A-*3-*colorable.*

Corollary 1.2 *If G is a planar graph without* 4- *and* 8-*cycles, then*

(i) *G is DP-*$(0, 2, 2)$*-colorable.*
(ii) *$V(G)$ can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set.*

Theorem 1.3 *Every planar graph without* 4- *and* 6-*cycles is DP-B_A-*3-*colorable.*

Corollary 1.4 *If G is a planar graph without* 4- *and* 6-*cycles, then*

(i) *G is DP-*$(0, 2, 2)$*-colorable.*
(ii) *$V(G)$ can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set.*

In order to prove results on DP-B_A-3-colorable graphs, Sribunhung et al. [16] gave some structural results.

Lemma 1.5 (Sribunhung et al. [16]) *If G is not DP-B_A-*3-*colorable, but all its proper induced subgraphs are DP-B_A-*3-*colorable, then $\delta(G) \geq 3$.*

Lemma 1.6 (Sribunhung et al. [16]) *Suppose G is not DP-B_A-*3-*colorable, but all its proper induced subgraphs are DP-B_A-*3-*colorable. If a* 3-*vertex u in G is adjacent to a* 3-*vertex, then u has two* 5^+-*neighbors. Moreover, if x is a* 5-*neighbor of u, then x has a* 4^+-*neighbor.*

We say that a 3-vertex is *bad* if it is adjacent to another 3-vertex; otherwise, it is a *good* 3-vertex.

2 Plane Graphs without 4- and 8-Cycles

Firstly, we give some structural results on plane graphs without 4- and 8-cycles.

Lemma 2.1 *Let G be a plane graph without* 4- *and* 8-*cycles. Then the following statements hold.*

(i) *There are no adjacent* 3-*faces.*
(ii) *If a* 3-*face is adjacent to a* 5-*face, then they are normally adjacent.*
(iii) *If $\delta(G) \geq 3$ and a* 3-*face is adjacent to a* 6-*face, then they are normally adjacent.*
(iv) *If $\delta(G) \geq 3$, then each* 7-*face is not adjacent to any* 3-*face.*

(v) *If* $\delta(G) \geq 3$*, then there are no adjacent* 5*-faces.*
(vi) *If* $\delta(G) \geq 3$*, then each* 5*-face is adjacent to at most two* 3*-faces.*
(vii) *If* $\delta(G) \geq 3$*, then each* 6*-face is adjacent to at most one* 3*-face.*

Proof (i) If two 3-faces are adjacent, then G has a 4-cycle, a contradiction.

(ii) Suppose to the contrary that a 5-face $[v_1v_2v_3v_4v_5]$ is adjacent to a 3-face $[v_1v_2u]$. Since they are not normally adjacent, $u \in \{v_3, v_4, v_5\}$. Then the 5-cycle has a chord, a contradiction.

(iii) Suppose that a 6-face f is not a simple face. Then its boundary consists of two triangles. Let $f = [u'vuwvw']$ be a 6-face, where $[uvw]$ and $[u'vw']$ are two triangles. Observe that G has no adjacent triangles. Suppose that f is adjacent to a 3-face. Then either $[uvw]$ or $[u'vw']$ bounds a 3-face, and then there are at least two 2-vertices, a contradiction.

So we may assume that the 6-face f is a simple face. Suppose to the contrary that $f = [v_1v_2v_3v_4v_5v_6]$ is not normally adjacent to a 3-face $[v_1v_2u]$. Then $u \in \{v_3, v_4, v_5, v_6\}$. By symmetry, we need to consider two cases: $u = v_3$ or $u = v_4$. If $u = v_4$, then $[v_1v_2v_3v_4]$ is a 4-cycle, a contradiction. It follows that $u = v_3$. Since $[v_1v_2v_3]$ is a 3-face, we have that v_2 is a 2-vertex, a contradiction.

(iv) Assume that a 7-face f_1 is adjacent to a 3-face f_2. Observe that f_1 must be a simple face; otherwise, there is a 4-cycle in the boundary of f_1, a contradiction. Since $\delta(G) \geq 3$ and G does not have 4-cycle, f_1 and f_2 are normally adjacent. Now, $b(f_1) \cup b(f_2)$ contains an 8-cycle, a contradiction.

(v) Suppose to the contrary that a 5-face $[v_1v_2v_3v_4v_5]$ is adjacent to a 5-face $[v_1v_2uvw]$. Since there is no 8-cycle, $\{u, v, w\} \cap \{v_3, v_4, v_5\} \neq \emptyset$. Since $\delta(G) \geq 3$ and $[v_1v_2v_3v_4v_5]$ has no chord, we have $\{u, w\} \cap \{v_3, v_4, v_5\} = \emptyset$. By symmetry, we can obtain that $\{v_3, v_5\} \cap \{u, v, w\} = \emptyset$. If $v = v_4$, then $[vuv_2v_3]$ is a 4-cycle, a contradiction.

(vi) Suppose to the contrary that a 5-face f is adjacent to three 3-faces. If those three 3-faces share vertices outside f, then G has a 4-cycle, a contradiction. Then the boundaries of these four faces form an 8-cycle, a contradiction. Thus, each 5-face is adjacent to at most two 3-faces.

(vii) Suppose to the contrary that a 6-face f is adjacent to two 3-faces. If those two 3-faces share vertices outside f, then G has a 4-cycle, a contradiction. Then the boundaries of these three faces form an 8-cycle, a contradiction. Thus, each 6-face is adjacent to at most one 3-face. □

Next, we prove the main result—Theorem 1.1.

Suppose to the contrary that G is a minimum counterexample to the statement. By Lemma 1.5, the minimum degree of G is at least three.

A 3-vertex is *special* if it is incident with a 3-face, a 5-face, and a 6-face.

Lemma 2.2 *Let* v *be a* 3*-vertex. If* v *is incident with a* 3*-face* $f_1 = [vv_1v_2]$*, a* 5*-face* $f_2 = [vv_2v_3v_4v_5]$*, and a* 6*-face* $f_3 = [vv_5v_6v_7v_8v_1]$*, then each of the following holds.*

(i) *There is only one possibility for the special* 3*-vertex* v*, as shown in Fig. 1, where* v_4 *and* v_7 *are identical. Furthermore,* f_2 *is adjacent to exactly one* 3*-face, say* f_1*.*
(ii) *There is no other special* 3*-vertex on the boundary of* f_2*.*

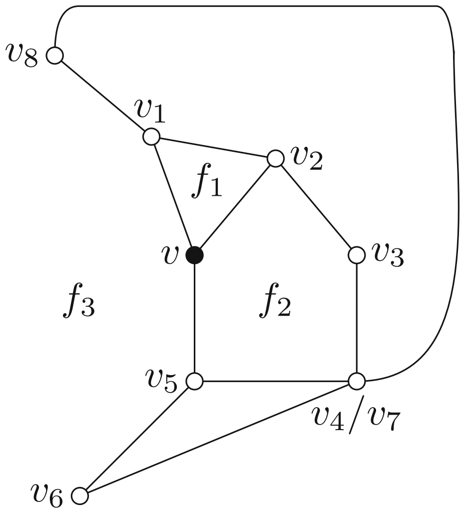

Fig. 1 Some cases in Lemma 2.2. Note that $[v_4v_5v_6]$ does not bound a 3-face

Proof (i) By Lemma 2.1(ii) and (iii), f_1 and f_2 are normally adjacent, f_1 and f_3 are normally adjacent. Note that the 6-face f_3 is adjacent to the 3-face f_1, the boundary of f_3 is a 6-cycle. Then $\{v, v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ are disjoint, $\{v, v_1, v_2\}$ and $\{v_5, v_6, v_7, v_8\}$ are disjoint. If $\{v_3, v_4\}$ and $\{v_6, v_7, v_8\}$ are disjoint, $[v_1v_2 \dots v_8]$ is an 8-cycle, a contradiction. So we may assume that $\{v_3, v_4\} \cap \{v_6, v_7, v_8\} \neq \emptyset$. If $v_3 \in \{v_6, v_7, v_8\}$, then there is a 4-cycle, a contradiction. It follows that $v_3 \notin \{v_6, v_7, v_8\}$. If $v_4 = v_8$, then $[vv_1v_8v_5]$ is a 4-cycle, a contradiction. It is observed that $v_4 \neq v_6$, for otherwise v_5 is a 2-vertex. Therefore, v_4 and v_7 are identical. Note that the 3-cycle $[v_4v_5v_6]$ does not bound a 3-face, for otherwise v_6 is a 2-vertex. Moreover, v_4v_5 cannot be incident with a 3-face; otherwise, there exists a 4-cycle with a chord v_4v_5. It is observed that vv_2 and v_4v_5 are is triangles; thus, no other edge on f_2 is contained in a triangle, for otherwise there exists an 8-cycle. It follows that f_2 is adjacent to exactly one 3-face.

(ii) Since every special 3-vertex is incident with a 3-face, the possible other special 3-vertex on f_2 is v_2. By Lemma 2.2(i), if v_2 is a special 3-vertex, then v_4 should be identified with a vertex on the 6-face incident with v_2, and v_3v_4 should be contained in a triangle, but this is impossible. □

Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face x, and let $\mu^*(x)$ denote the final charge of x after the discharging process. By the Euler's formula, $|V(G)| - |E(G)| + |F(G)| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$, we can derive the following identity: $\sum_{x \in V(G) \bigcup F(G)} \mu(x) = -8$. By the following discharging rules, we shall finally get $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Thus a contradiction is obtained and the proof is completed.

The discharging rules are as follows:

R1 Each 5^+-vertex gives $\frac{1}{4}$ to each adjacent bad 3-vertex.

R2 Each 5^+-face gives $\frac{1}{3}$ to each adjacent 3-face.

R3 Each 5-face gives $\frac{1}{6}$ to each incident good 3-vertex and $\frac{1}{12}$ to each incident bad 3-vertex.

R4 Let f be a 6-face or 7-face. Then f gives $\frac{1}{2}$ to each incident good 3-vertex and $\frac{1}{4}$ to each incident bad 3-vertex.
R5 Each 8^+-face gives $\frac{5}{6}$ to each incident good 3-vertex and $\frac{5}{12}$ to each incident bad 3-vertex.

Let $\beta(f)$ be the final charge of a 5-face f after applying the rules R1–R5.
R6 If v is a special 3-vertex, then the incident 5-face f additionally sends $\beta(f)$ to v.

Now, we give a lower bound of $\beta(f)$ in R6.

Lemma 2.3 *If f is a 5-face which is incident with a special 3-vertex, then $\beta(f) \geq \frac{1}{3}$.*

Proof By Lemma 2.2(i), the 5-face is adjacent to exactly one 3-face. If the 5-face is incident with at most two 3-vertices, then $\beta(f) \geq 1 - \frac{1}{3} - \frac{1}{6} \times 2 = \frac{1}{3}$ by R2 and R3. If the 5-face is incident with at least three 3-vertices, then the 5-face is incident with exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 1.6. It follows that $\beta(f) \geq 1 - \frac{1}{3} - \frac{1}{6} - \frac{1}{12} \times 2 = \frac{1}{3}$ by R2 and R3. □

Recall that every vertex v is a 3^+-vertex.

Consider a good 3-vertex v. If v is incident with at least two 6^+-faces, then $\mu^*(v) \geq \mu(v) + \frac{1}{2} \times 2 = 0$ by R4 and R5. So we may assume that v is incident with at least two 5^--faces. By Lemma 2.1(v) and (i), v is incident with a 3-face and a 5-face. If v is incident with an 8^+-face, then $\mu^*(v) = \mu(v) + \frac{1}{6} + \frac{5}{6} = 0$ by R3 and R5. Otherwise, v is incident with a 3-face, a 5-face f, and a 6-face by Lemma 2.1(iv), i.e., v is a special 3-vertex. Then $\mu^*(v) = \mu(v) + \frac{1}{6} + \frac{1}{2} + \beta(f) \geq 0$ by R3, R4, R6, and Lemma 2.3.

Consider a bad 3-vertex v. By Lemma 1.6, v is adjacent to two 5^+-vertices. If v is incident with at least two 6^+-faces, then $\mu^*(v) \geq \mu(v) + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 0$ by R1, R4, and R5. Then v is incident with at least two 5^--faces. By Lemma 2.1(v) and (i), v is incident with a 3-face and a 5-face. If v is incident with an 8^+-face, then $\mu^*(v) = \mu(v) + \frac{1}{4} \times 2 + \frac{1}{12} + \frac{5}{12} = 0$ by R1, R3, and R5. Otherwise, v is incident with a 3-face, a 5-face f, and a 6-face by Lemma 2.1(iv), i.e., v is a special 3-vertex. Then $\mu^*(v) = \mu(v) + \frac{1}{4} \times 2 + \frac{1}{12} + \frac{1}{4} + \beta(f) > 0$ by R1, R3, R4, R6, and Lemma 2.3.

If v is a 4-vertex, then it is not involved in a discharging process and thus $\mu^*(v) = \mu(v) = 0$.

Consider a 5-vertex v. If v is adjacent to a bad 3-vertex, say u, then v has a 4^+-neighbor by Lemma 1.6. Consequently, v is adjacent to at most four bad 3-vertices. Then $\mu^*(v) \geq \mu(v) - \frac{1}{4} \times 4 = 0$ by R1.

Consider a d-vertex v where $d \geq 6$. Then $\mu^*(v) \geq \mu(v) - d \times \frac{1}{4} = (d-4) - d \times \frac{1}{4} > 0$ by R1.

Let f be a k-face.

- $k = 3$. It follows from Lemma 2.1(i) that f is adjacent to three 5^+-faces, and $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by R2.
- $k = 4$. Since G does not contain a 4-cycle, it does not contain a 4-face.
- $k = 5$. It follows from Lemma 2.1(vi) that f is adjacent to at most two 3-faces. Suppose that f is incident with a special 3-vertex. By Lemma 2.2(ii), f is incident with exactly one special 3-vertex. By R6 and Lemma 2.3, we get $\mu^*(f) = 0$. So we may assume that f is not incident with a special 3-vertex. If f is incident with

at most two 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 = 0$ by R2 and R3. If f is incident with at least three 3-vertices, then f is incident with exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 1.6. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 2 - \frac{1}{6} - \frac{1}{12} \times 2 = 0$ by R2 and R3.

- $k = 6$. It follows from Lemma 2.1(vii) that f is adjacent to at most one 3-face. If f is incident with at most three 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} - \frac{1}{2} \times 3 = \frac{1}{6} > 0$ by R2 and R4. If f is incident with at least four 3-vertices, then f is incident with exactly four 3-vertices in which all of them are bad 3-vertices by Lemma 1.6. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{3} - \frac{1}{4} \times 4 = \frac{2}{3} > 0$ by R2 and R4.
- $k = 7$. If f is not a simple face, then G contains a 4-cycle, a contradiction. So we may assume that f is a simple face. Then f is bounded by a 7-cycle. It follows from Lemma 2.1(iv) that f is not adjacent to any 3-face. By Lemma 1.6, f is incident with at most four 3-vertices. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{2} \times 4 > 0$ by R4.
- $k = 8$. If f is a simple face, then G contains an 8-cycle, a contradiction. So f is not a simple face, its boundary consists of a 3-cycle and a 5-cycle, or two 3-cycles and a cut edge. It follows from Lemma 2.1(i) and (vi) that f is adjacent to at most two 3-faces. By Lemma 1.6, f is incident with at most five 3-vertices. If f is incident with at most four 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 2 - \frac{5}{6} \times 4 = 0$ by R2 and R5. If f is incident with five 3-vertices, then at least four of the 3-vertices are bad by Lemma 1.6. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 2 - \frac{5}{6} - \frac{5}{12} \times 4 = \frac{5}{6} > 0$ by R2 and R5.
- $k \geq 9$. It follows from Lemma 2.1(i) that a 3-vertex is incident with at least two 4^+-faces. If f is a 9-face incident with exactly four good 3-vertices, then f is adjacent to at most five 3-faces and f is not incident with a bad 3-vertex, thus $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 5 - \frac{5}{6} \times 4 = 0$ by R2 and R5. So we may assume that f is not a 9-face incident with exactly four good 3-vertices. In what follows, if f is a 9-face, then it is incident with at most three good 3-vertices.

Let $v_1, v_2, \ldots, v_k$ be the vertices on the boundary of f, and let f_i be the face sharing an edge $v_i v_{i+1}$ with f, where all the subscripts are taken modulo k. In order to easily check the final charge of f, we treat some transfer from an element to another element via some agents. Firstly, f sends $\frac{1}{2}$ to each vertex v_i and sends an extra $\frac{1}{6}$ to each good 3-vertex v_i. Next, v_i may play the role of agent. If f_i is a 3-face, then the agent v_i sends $\frac{1}{6}$ to f_i, and the agent v_{i+1} sends $\frac{1}{6}$ to f_i, which corresponds to R2 that f sends $2 \times \frac{1}{6} = \frac{1}{3}$ to f_i.

Suppose that v_i is a 3-vertex incident with 4^+-vertices v_{i-1} and v_{i+1}. Then the agent v_{i-1} sends $\frac{1}{4}$ to v_i if f_{i-1} is a 4^+-face; otherwise, the agent v_{i-1} sends $\frac{1}{4} - \frac{1}{6}$ to v_i. Similarly, the agent v_{i+1} sends $\frac{1}{4}$ or $\frac{1}{4} - \frac{1}{6}$ to v_i. Note that the 3-vertex v_i is incident with at most one 3-face; thus, f sends at least $(\frac{1}{2} - \frac{1}{6}) + \frac{1}{6} + \frac{1}{4} + (\frac{1}{4} - \frac{1}{6}) = \frac{5}{6}$ to v_i, which corresponds to the first part of R5.

Suppose that v_i is a 3-vertex, and one of v_{i-1} and v_{i+1} is also a 3-vertex. By symmetry, let v_{i+1} be a 3-vertex. Then the agent v_{i-1} sends $\frac{1}{4}$ or $\frac{1}{4} - \frac{1}{6}$ to v_i, and then f sends at least $(\frac{1}{2} - \frac{1}{6}) + (\frac{1}{4} - \frac{1}{6}) = \frac{5}{12}$ in total to v_i, which corresponds to the second part in R5.

For each 4^+-vertex v_i, when it plays the role of agent, it receives $\frac{1}{2}$ from f and gives at most $2(\frac{1}{4} - \frac{1}{6}) + 2 \times \frac{1}{6} = \frac{1}{2}$.

So we can treat f sends $\frac{1}{2}$ to each vertex v_i, and v_i maybe plays the role of agent to redistribute at most $\frac{1}{2}$ to incident 3-faces and 3-vertices; additionally, f sends an extra $\frac{1}{6}$ to each good 3-vertices.

If f is a 9-face incident with at most three good 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{2} \times 9 - \frac{1}{6} \times 3 = 0$. If f is a 10^+-face, then f is incident with at most $\frac{k}{2}$ good 3-vertices, and then $\mu^*(f) \geq \mu(f) - \frac{1}{2} \times k - \frac{1}{6} \times \frac{k}{2} = \frac{1}{6} > 0$.

This completes the proof.

3 Plane Graphs without 4- and 6-Cycles

In this section, we prove the second main result—Theorem 1.3.

Assume that G is a counterexample to Theorem 1.3, but all of its proper induced subgraphs are DP-B_A-3-colorable. By Lemma 1.5, the minimum degree of G is at least three. Since G has no 4- or 6-cycles, the following statements hold.

Lemma 3.1 *A 3-face is not adjacent to a 6^--face.*

Proof If two 3-faces are adjacent, then G has a 4-cycle, a contradiction.

Suppose that a 5-face $[v_1 v_2 v_3 v_4 v_5]$ is adjacent to a 3-face $[v_1 v_2 u]$. Since there is no 6-cycle, $u \in \{v_3, v_4, v_5\}$. But the 5-cycle has a chord, then there is a 4-cycle, a contradiction.

Since there is no 6-cycle in G, the boundary of a 6-face consists of two triangles. Let $f = [u'vuwvw']$ be a 6-face, where $[uvw]$ and $[u'vw']$ are two triangles. Observe that G has no adjacent triangles. Suppose that f is adjacent to a 3-face. Then either $[uvw]$ or $[u'vw']$ bounds a 3-face, and then there are at least two 2-vertices, a contradiction. □

We once again use the discharging method to complete the proof. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face x, and let $\mu^*(x)$ denote the final charge of x after the discharging procedure. According to the Euler's formula and handshaking theorem, the sum of the initial charge is -8. By the following discharging rules, we should finally get $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Thus a contradiction is obtained and the counterexample does not exist.

The discharging rules are as follows:

R1 Each 5^+-vertex gives $\frac{1}{4}$ to each adjacent bad 3-vertex.

R2 Each 7^+-face gives $\frac{1}{3}$ to each adjacent 3-face.

R3 Each 5-face gives $\frac{1}{3}$ to each incident good 3-vertex and $\frac{1}{6}$ to each incident bad 3-vertex.

R4 Each 6^+-face gives $\frac{1}{2}$ to each incident good 3-vertex and $\frac{1}{4}$ to each incident bad 3-vertex.

Recall that every vertex v is a 3^+-vertex.

Consider a good 3-vertex v. If v is not incident with a 3-face, then v is incident with three 5^+-faces; thus, $\mu^*(v) \geq \mu(v) + \frac{1}{3} \times 3 = 0$ by R3 and R4. If v is incident with a 3-face, then the other two faces are 7^+-faces by Lemma 3.1; thus, $\mu^*(v) = \mu(v) + \frac{1}{2} \times 2 = 0$ by R4.

Consider a bad 3-vertex v. If v is not incident with a 3-face, then v is incident with three 5^+-faces; thus, $\mu^*(v) \geq \mu(v) + \frac{1}{4} \times 2 + \frac{1}{6} \times 3 = 0$ by R1, R3, and R4. If v is incident with a 3-face, then the other two faces are 7^+-faces by Lemma 3.1, and then $\mu^*(v) = \mu(v) + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 0$ by R1 and R4.

If v is a 4-vertex, then it does not involve in a discharging process and then $\mu^*(v) = \mu(v) = 0$.

Consider a 5-vertex v. If v is adjacent to a bad 3-vertex, say u, then v has a 4^+-neighbor by Lemma 1.6. Consequently, v is adjacent to at most four bad 3-vertices. Then $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{4} = 0$ by R1.

Consider a d-vertex v where $d \geq 6$. Then $\mu^*(v) \geq \mu(v) - d \times \frac{1}{4} = (d-4) - d \times \frac{1}{4} > 0$ by R1.

Let f be a k-face.

- $\boldsymbol{k = 3}$. It follows from Lemma 3.1 that f is adjacent to three 7^+-faces. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by R2.
- $\boldsymbol{k = 4}$. Since G does not contain a 4-cycle, it does not contain a 4-face.
- $\boldsymbol{k = 5}$. It follows from Lemma 3.1 that f is not adjacent to any 3-face. If f is incident with at most two 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} \times 2 > 0$ by R3. If f is incident with at least three 3-vertices, then it is incident with exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 1.6. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{6} \times 2 - \frac{1}{3} > 0$ by R3.
- $\boldsymbol{k = 6}$. It follows from Lemma 3.1 that f is not adjacent to a 3-face. Since there are no 6-cycles in G, the boundary of f consists of two triangles, and f is incident with at most four 3-vertices. If f is incident with at most three 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{2} \times 3 = \frac{1}{2} > 0$ by R4. If f is incident with four 3-vertices, then each 3-vertex is bad; thus, $\mu^*(f) = \mu(f) - \frac{1}{4} \times 4 = 1 > 0$ by R4.
- $\boldsymbol{k \geq 7}$. Similar to the proof in Theorem 1.1, we can treat f sends $\frac{3}{7}$ to each incident vertex and redistribute at most $\frac{3}{7}$ to incident 3-faces and 3-vertices. Thus, $\mu^*(f) \geq \mu(f) - \frac{3}{7} \times k \geq 0$.

This completes the proof.

Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

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