Bull. Malays. Math. Sci. Soc. (2022) 45:1903–1912
https://doi.org/10.1007/s40840-022-01296-y

BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Quasimöbius invariance of Loewner spaces

Tiantian Guan[1] **· Yaxiang Li**[2]

Received: 9 December 2021 / Revised: 26 March 2022 / Accepted: 15 April 2022 /
Published online: 3 May 2022

Abstract
In this paper, we show that Loewner spaces introduced by Heinonen and Koskela (Acta Math., 1998) are preserved under quasimöbius mappings between Ahlfors regular spaces.

Keywords Sphericalization · Flattening · Loewner condition · Quasimöbius.

Mathematics Subject Classification Primary: 30C65 · 30F45; Secondary: 30C20

1 Introduction and Main Results

In this note, we study the behavior of Loewner condition under quasimöbius mappings between metric measure spaces. The class of quasimöbius mappings, under which, in a certain sense, the cross ratio is quasi-invariant, was explicitly defined and investigated

Communicated by Saminathan Ponnusamy.

The first author was supported by Department of Education of Guangdong Province, China (No. 2021KTSCX116), by Guangdong Basic and Applied Basic Research Foundation (No. 2021A1515012289), and Research Fund of Guangdong-Hong Kong-Macao Joint Laboratory for Intelligent Micro-Nano Optoelectronic Technology (No. 2020B1212030010). The second author was supported by National Natural Science Foundation of Hunan Province (No. 2021JJ3016), by Scientific Research Fund of Hunan Provincial Education Department (Nos. 20B118, 18C0253), by NNSF of China (No. 11971124) and by NSF of Guangdong Province (No. 2021A1515010326).

✉ Yaxiang Li
yaxiangli@163.com

Tiantian Guan
ttguan93@163.com

1 School of Mathematics and Big Data, Foshan University, Foshan 528000, Guangdong, People's Republic of China

2 Department of Mathematics, Hunan First Normal University, Changsha 410205, Hunan, People's Republic of China

in [11]. The main motivation for introducing this class of mappings by Väisälä is that they give a handy tool when studying the relationship between quasisymmetric and quasiconformal mappings. However, this class has appeared before such as in the celebrated Mostow's work on rigidity. Recently, the motivation in the study of quasimöbius mappings comes from many sources, see e.g. [1, 3, 5, 9, 12–16] and the references therein.

There are two important classes of quasimöbius mappings, known as sphericalization and flattening ([1]), in geometric function theory. The original idea of these deformations comes from the work of Bonk and Kleiner [3] in defining a metric on the one point compactification of an unbounded locally compact metric space. The first class of deformation, sphericalization, is a generalization of the deformation from the Euclidean distance on $\mathbb{R}^n$ to the chordal distance on the unit sphere $\mathbb{S}^n$. The second class of flattening deformation is a generalization of inversion on the punctured sphere. It was shown in [5] that these two conformal transformations are dual in the sense that if one starts from a bounded metric space, and then performs a flattening transformation followed by a sphericalization, thus the object space is bilipschitzly equivalent to the original space. This duality comes from the idea that the stereographic projection between the Euclidean space and the Riemann sphere can be realized as a special case of inversion. Sphericalization and flattening have a lot of applications in the area of analysis on metric spaces, such as [1–3, 5].

Our motivation comes from Heinonen and Koskela's celebrated work on the equivalence of quasiconformality and quasisymmetry between metric measure spaces in [8]. They introduced the concept of Loewner spaces, which has many applications in studying Sobolev spaces and quasiconformal theory in metric spaces, see [2, 4, 7]. It should be noted that Tyson [10] answered positively to a conjecture proposed by Heinonen and Koskela [8,Sect. 8.7] in proving that the Q-Loewner condition is preserved under quasisymmetric maps between two Ahlfor Q-regular spaces.

In their recent work [9], Li and Shanmugalingam studied the invariance of bounded geometry under sphericalization and flattening transformations; in particular, doubling measure, Ahlfors regularity and Poincaré inequality. It is natural to ask whether the Loewner condition is preserved under sphericalization and flattening, more general quasimöbius mappings. Indeed, this question has been investigated by Brania and Yang [4] by introducing the notion of controlled modulus condition. They proved that quasimöbius mapping preserves the n-Loewner condition in the extended Euclidean space $\mathbb{R}^n \cup \{\infty\}$, see [4,Proposition 3.1]. In the setting of metric spaces, they also showed that the Loewner condition is preserved under quasimöbius mappings, provided the spaces are simultaneously bounded or unbounded and the mapping sends infinity to infinity, see [4,Corollary 3.4].

We obtain the following as our main theorem.

Theorem 1.1 *Let (X, d, μ) and (Y, σ, ν) be locally compact Ahlfors Q-regular metric measure spaces with $Q > 1$. Suppose that $f : X \to Y$ is a quasimöbius homeomorphism. If X is Q-Loewner, then Y is also Q-Loewner.*

We remark that Theorem 1.1 is a generalization of [4,Proposition 3.1 and Corollary 3.4] and our proof is different. Also, we explain the connection between Theorem 1.1 and Li-Shanmugalingam's results in [9]:

First, Heinonen and Koskela demonstrated that Q-Loewner condition and $(1, Q)$-Poincaré inequality are equivalent in proper, Ahlfors Q-regular and φ-convex metric measure spaces, see [8,Corollary 5.13]. In [9,Theorem 1.1], it was shown by Li and Shanmugalingam that for a complete doubling metric measure space which admits a $(1, p)$-Poincaré inequality with $1 \leq p < \infty$, if in addition the sphericalized (or flattened) space is annular quasiconvex, then the deformed space also admits a $(1, p)$-Poincaré inequality. Notice that we do not need any extra assumptions concerning the connectivity or completeness of the spaces in Theorem 1.1. By using a deformed cross ratio introduced in [2], our proof is direct and simple.

We conclude with reviewing certain examples of Loewner spaces, see [7, 8] and the references therein. For instance, Euclidean spaces and compact Riemannian manifolds, Carnot groups equipped with its Carnot-Carathéodory metrics, Riemannian manifolds of non-negative Ricci curvature, and so on. Indeed, by using Theorem 1.1, we can obtain many new Loewner spaces via sphericalization and flattening transformations.

The rest of this paper is organized as follows. In Sect. 2, we introduce the necessary terminology. The proof of Theorem 1.1 is given in Sect. 3.

2 Preliminary and Notations

Following [4, 5, 7, 11], we introduce certain terminology and recall useful results in this section.

For real numbers s and t, we set

$$s \wedge t = \min\{s, t\} \quad \text{and} \quad s \vee t = \max\{s, t\}.$$

Let (X, d) be a metric space. A *curve* in X means a continuous map $\gamma : I = [a, b] \to X$ from an interval $I \subset \mathbb{R}$ to X. We denote the image set $\gamma(I)$ of γ by γ. The length of γ is denoted by

$$\ell_d(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1}))\right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < t_2 < \ldots < t_n = b$. A curve γ is called *rectifiable* if its length $\ell_d(\gamma) < \infty$.

A metric space X is called *rectifiably connected* if every pair of points in X can be joined with a rectifiable curve γ. The length function associated with a rectifiable curve γ: $[a, b] \to X$ is z_γ: $[a, b] \to [0, \ell(\gamma)]$, given by $z_\gamma(t) = \ell(\gamma|_{[a,t]})$. For any rectifiable curve $\gamma : [a, b] \to X$, there is a unique map $\gamma_s : [0, \ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ z_\gamma$. Obviously, $\ell(\gamma_s|_{[0,t]}) = t$ for $t \in [0, \ell(\gamma)]$. The curve γ_s is called the *arclength parametrization* of γ.

For a rectifiable curve γ in X, the line integral over γ of each Borel function $\varrho : X \to [0, \infty)$ is

$$\int_\gamma \varrho \, ds = \int_0^{\ell(\gamma)} \varrho \circ \gamma_s(t) \, dt.$$

2.1 Metric Measure Spaces

Following [8], (X, d, μ) denotes a metric measure space with X a locally compact, rectifiably connected metric space and μ a Borel regular measure with dense support.

Definition 2.1 ([6]) Let (X, d, μ) be a metric measure space. Given $Q > 1$, we say that X is *Ahlfors Q-regular* if there exists a constant $C > 0$ such that for each $x \in X$ and $0 < r \leq \mathrm{diam}(X)$,

$$r^Q/C \leq \mu(B(x,r)) \leq Cr^Q,$$

where $\mathrm{diam}(X)$ means the diameter of X and $B(x,r) = \{y \in X;\ d(x,y) < r\}$.

For instance, the Euclidean space $\mathbb{R}^n$ with Lebesgue measure satisfies the Ahlfors n-regularity.

Definition 2.2 Let $Q > 1$. We define the *Q-modulus* of a family Γ of curves in a metric measure space (X, d, μ) by

$$mod_Q \Gamma = \inf \int_X \rho^Q \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ satisfying

$$\int_\gamma \rho \, ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma$. Then the Q-modulus of a pair of disjoint nonempty compact sets $E, F \subset X$ is

$$mod_Q(E, F; X) = mod_Q \Delta(E, F; X),$$

where $\Delta(E, F; X)$ is the family of all curves in X joining the sets E and F.

In studying the equivalence of quasiconformality and quasisymmetry, Heinonen and Koskela [8] introduced the notion of Loewner spaces. Moreover, they showed some examples for Loewner, such as Euclidean n-space $\mathbb{R}^n$, a Carnot group admits a $(1, p)$-Poincaré inequality for all $p > 1$ and so on. Note that a metric measure space (X, d, μ) is called *Q-Loewner*, with $Q > 1$, provided the Loewner control function

$$\varphi(t) = \inf\{mod_Q(E, F; X) : \Delta_d(E, F) \leq t\}$$

is strictly positive for all $t > 0$; here E and F are non-degenerate disjoint continua in X and

$$\Delta_d(E, F) = \frac{\mathrm{dist}_d(E, F)}{\mathrm{diam}_d(E) \wedge \mathrm{diam}_d(F)},$$

where $\mathrm{dist}_d(E, F)$ means the distance between E and F in the metric d.

2.2 Mappings on Metric Spaces

Given a metric space (X, d), the *cross ratio* $r(x, y, z, w)$ of each four distinct points $x, y, z, w \in X$ is defined as

$$r(x, y, z, w) = \frac{d(x, z)d(y, w)}{d(x, y)d(z, w)}.$$

It is often convenient to consider cross ratios also in the extended space $\dot{X} = X \cup \{\infty\}$. If x, y, z, w are points in $\dot{X}$ and if one of the points x, y, z, w is ∞, the cross ratio is defined by deleting the distances from ∞. For example,

$$r(x, y, z, \infty) = \frac{d(x, z)}{d(x, y)}.$$

Let (X_1, d_1) and (X_2, d_2) be two metric spaces, and let $f : X_1 \to X_2$ be a homeomorphism. We say that f is *L-bilipschitz* if there exists $L \geq 1$ such that

$$d_1(x, y)/L \leq d_2(f(x), f(y)) \leq Ld_1(x, y).$$

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that f is *η-quasisymmetric* if for all distinct points $x, y, z \in X_1$, we have

$$\frac{d_2(f(x), f(z))}{d_2(f(x), f(y))} \leq \eta\left(\frac{d_1(x, z)}{d_1(x, y)}\right).$$

Moreover, we say that f is *η-quasimöbius* if for all distinct points $x, y, z, w \in X_1$, we have

$$r(f(x), f(y), f(z), f(w)) \leq \eta\big(r(x, y, z, w)\big).$$

For the properties of quasimöbius and quasisymmetric mappings see [7, 11, 12]. We also need the following result.

Lemma 2.3 ([11,Theorem 3.10]) *Suppose that X is unbounded and that $f : X \to Y$ is θ-quasimöbius between two metric spaces. Then f is θ-quasisymmetric if and only if $f(x) \to \infty$ as $x \to \infty$.*

Our main tool in this paper is the following useful notation introduced by Bonk and Kleiner in [2]:

$$\langle x, y, z, w\rangle = \frac{d(x, z) \wedge d(y, w)}{d(x, y) \wedge d(z, w)}.$$

They also established a relation between $r(x, y, z, w)$ and $\langle x, y, z, w\rangle$ as follows.

Lemma 2.4 ([2,Lemma 2.3]) *For any distinct points x, y, z, w in a metric space X, we have*

$$\langle x, y, z, w\rangle \leq \theta_0(r(x, y, z, w)),$$

where $\theta_0(t) = 3(t \vee \sqrt{t})$.

3 Proof of Theorem 1.1

We begin the proof of Theorem 1.1 by showing that the sphericalization of an unbounded metric measure space preserves the Q-Loewner condition. Following [5, 9], we first introduce certain terminology.

Bonk and Kleiner introduced sphericalization of an unbounded metric measure space, see [3,Lemma 2.2, p87]. We briefly recall their work. Given an unbounded locally compact metric space (X, d) and a base point $a \in X$, we consider the one-point compactification $\dot{X} = X \cup \{\infty\}$ and define the function $d_a : \dot{X} \times \dot{X} \to [0, \infty)$ as follows

$$d_a(x, y) = d_a(y, x) = \begin{cases} \frac{d(x,y)}{[1+d(x,a)][1+d(y,a)]}, & \text{if } x, y \in X, \\ \frac{1}{1+d(x,a)}, & \text{if } y = \infty \text{ and } x \in X, \\ 0, & \text{if } x = \infty = y. \end{cases}$$

Note that, this distance function is an analog of the chordal metric on the Riemann sphere. Unfortunately, $d_a(x, y)$ will not satisfy the triangle inequality in general. In fact, generally, d_a is not a metric on $\dot{X}$ and however a quasimetric. There is a standard procedure, known as *chain construction*, to construct a metric from a quasimetric as follows. Define

$$\widehat{d_a}(x, y) := \inf \sum_{j=0}^{k} d_a(x_j, x_{j+1}),$$

where the infimum is taken over all finite sequences $x = x_0, x_1, \ldots, x_k, x_{k+1} = y$ from $\dot{X}$. Then $(\dot{X}, \widehat{d_a})$ is a metric space and called the *sphericalization* of (X, d) associated to the point $a \in X$. Moreover, by [5,(3.3)] we have for all $x, y \in \dot{X}$

$$\frac{1}{4} d_a(x, y) \leq \widehat{d_a}(x, y) \leq d_a(x, y). \tag{3.1}$$

If (X, d) is a rectifiably connected unbounded metric space, then the Borel function $\rho_a : X \to [0, \infty)$ is defined as

$$\rho_a(x) = \frac{1}{[1 + d(a, x)]^2}.$$

A similar argument as [5,(4.1)], we obtain that for any rectifiable curve γ in X joining x and y,

$$\ell_{\widehat{d}_a}(\gamma) = \int_\gamma \rho_a(z)\, ds, \tag{3.2}$$

where ds means the element of the length. If (X, d, μ) is Ahlfors Q-regular, then the associated spherical measure μ_a on a Borel set $A \subset X$ is given by

$$\mu_a(A) = \int_A \rho_a(z)^Q\, d\mu(z). \tag{3.3}$$

Lemma 3.1 *Suppose that (X, d, μ) is an unbounded Ahlfors Q-regular Q-Loewner locally compact metric measure space with $Q > 1$ and $a \in X$. Then, the sphericalized space $(X, \widehat{d}_a, \mu_a)$ is a bounded Q-regular Q-Loewner metric measure space.*

Proof Since (X, d, μ) is Ahlfors Q-regular, we see from [9,Proposition 3.1] that $(X, \widehat{d}_a, \mu_a)$ is also Ahlfors Q-regular. We first prove that the sphericalized transformation preserves Q-modulus, that is,

$$mod_Q(\Gamma, d, \mu) = mod_Q(\Gamma, \widehat{d}_a, \mu_a) \tag{3.4}$$

for all rectifiable family of curves Γ in X. Indeed, we know from (3.2) and (3.3) that for all nonnegative Borel function $\rho : X \to [0, \infty]$

$$\int_X \rho^Q\, d\mu = \int_X \frac{\rho^Q}{\rho_a^Q}\, d\mu_a,$$

and for any rectifiable curve $\gamma \in \Gamma$

$$\int_\gamma \rho\, ds = \int_\gamma \frac{\rho}{\rho_a}\, ds_a,$$

where ds_a is the arc-length element with respect to $\widehat{d}_a$. From the definition of Q-modulus, we know that (3.4) holds true.

In order to show that $(X, \widehat{d}_a, \mu_a)$ is Q-Loewner, we only need to find a lower bound of the Q-modulus associated to any pair of disjoint, non-degenerate continua E and F in X. Since (X, d, μ) is Q-Loewner, by (3.4) we know that it suffices to find an increasing function $\psi : (0, \infty) \to (0, \infty)$ satisfying

$$\Delta_{\widehat{d}_a}(E, F) \ge \psi\big(\Delta_d(E, F)\big). \tag{3.5}$$

To this end, take $x \in E$ and $y \in F$ such that

$$\mathrm{dist}_{\widehat{d}_a}(E, F) = \widehat{d}_a(x, y).$$

Without loss of generality, we may assume that $\operatorname{diam}_{\widehat{d}_a}(E) \le \operatorname{diam}_{\widehat{d}_a}(F)$. Moreover, choose $z \in E$ such that $\operatorname{diam}_{\widehat{d}_a}(E) \le 2\widehat{d}_a(x, z)$ and choose $w \in F$ such that $\operatorname{diam}_{\widehat{d}_a}(F) \le 2\widehat{d}_a(y, w)$.

By [3,Lemma 2.2], we see that the identity map $\mathrm{id} : (X, d) \to (X, \widehat{d_a})$ is θ-quasimöbius with $\theta(t) = 16t$. It thus follows from Lemma 2.4 and (3.1) that

$$\begin{aligned}\frac{d(x, y) \wedge d(z, w)}{d(x, z) \wedge d(y, w)} &\le \theta_0\Big(\frac{d(x, y)d(z, w)}{d(x, z)d(y, w)}\Big)\\ &\le \theta_0\Big(16\frac{\widehat{d}_a(x, y)\widehat{d}_a(z, w)}{\widehat{d}_a(x, z)\widehat{d}_a(y, w)}\Big)\\ &\le \eta\Big(\frac{\widehat{d}_a(x, y) \wedge \widehat{d}_a(z, w)}{\widehat{d}_a(x, z) \wedge \widehat{d}_a(y, w)}\Big),\end{aligned}$$

where $\theta_0(t) = 3(t \vee \sqrt{t})$ and $\eta(t) = \theta_0\big(\frac{16}{\theta_0^{-1}(1/t)}\big)$. Consequently, from the above facts we get

$$\begin{aligned}\Delta_d(E, F) &= \frac{\operatorname{dist}_d(E, F)}{\operatorname{diam}_d(E) \wedge \operatorname{diam}_d(F)}\\ &\le \frac{d(x, y) \wedge d(z, w)}{d(x, z) \wedge d(y, w)}\\ &\le \eta\Big(\frac{\widehat{d}_a(x, y) \wedge \widehat{d}_a(z, w)}{\widehat{d}_a(x, z) \wedge \widehat{d}_a(y, w)}\Big)\\ &\le \eta\Big(\frac{2\operatorname{dist}_{\widehat{d}_a}(E, F)}{\operatorname{diam}_{\widehat{d}_a}(E) \wedge \operatorname{diam}_{\widehat{d}_a}(F)}\Big).\end{aligned}$$

Hence, (3.5) is true by letting $\psi(t) = \frac{1}{2}\eta^{-1}(t)$. We complete the proof of Lemma 3.1. □

Next, we consider the flatting transformations on bounded metric spaces. Given a bounded metric space (X, d) and a base point $c \in X$, we consider the space $X^c = X \setminus \{c\}$ and define the function $d^c : X^c \times X^c \to [0, \infty)$ as follows

$$d^c(x, y) = d^c(y, x) = \frac{d(x, y)}{d(x, c)d(y, c)}.$$

We also deform the quasimetric d^c by a chain construction and obtain from [5,(3.1)] that there is a metric $\widehat{d^c}$ on X^c satisfying

$$\frac{1}{4}d^c(x, y) \le \widehat{d^c}(x, y) \le d^c(x, y).$$

The metric space $(X^c, \widehat{d^c})$ is said to be the *flattening* of (X, d) associated to the point c. In the case that (X, d) is a rectifiably connected bounded metric space, we define the Borel function $\rho^c : X^c \to [0, \infty)$ to be

$$\rho^c(x) = \frac{1}{d(c, x)^2}.$$

Thus, by [5,(4.1)], we obtain that for any rectifiable curve γ in X^c joining x and y,

$$\ell_{\widehat{d^c}}(\gamma) = \int_\gamma \rho^c(z)\, ds.$$

If (X, d, μ) is Ahlfors Q-regular, then the corresponding flattening measure μ^c is defined by

$$\mu^c(A) = \int_A \rho^c(z)^Q \, d\mu(z),$$

where $A \subset X^c$ is a Borel set.

Moreover, we demonstrate that the Q-Loewner condition is preserved under the flattening of a bounded metric measure space. Since the argument for this result is completely similar to the proof of Lemma 3.1, we do not provide the proof.

Lemma 3.2 *Suppose that (X, d, μ) is a bounded Ahlfors Q-regular Q-Loewner metric measure space with $Q > 1$ and $c \in X$. Then the flattening space $(X^c, \widehat{d^c}, \mu^c)$ is an unbounded Q-regular Q-Loewner metric measure space.*

Now, we are ready to prove Theorem 1.1 by using Lemmas 3.1 and 3.2.

Proof of Theorem 1.1 Let (X, d, μ) and (Y, σ, ν) be locally compact Ahlfors Q-regular metric measure spaces with $Q > 1$. Suppose that $f : X \to Y$ is a quasimöbius homeomorphism and X is Q-Loewner. We need to show that Y is Q-Loewner.

For this, we only consider the case whenever X and Y are both bounded; for the other cases, it is easy to deal with and the proof is rather similar. No loss of generality we assume that $\mathrm{diam}_\sigma(Y) = 1$.

Fix $c \in X$ and $c' = f(c) \in Y$. It follows from Lemma 3.2 that the flattening spaces, $(X^c, \widehat{d^c}, \mu^c)$ and $(Y^{c'}, \widehat{\sigma^{c'}}, \nu^{c'})$, are both unbounded Ahlfors Q-regular metric spaces. By [5,Lemma 3.1], we find that the identity mappings

$$\varphi_X : (X^c, d) \to (X^c, \widehat{d^c}) \quad \text{and} \quad \varphi_Y : (Y^{c'}, \sigma) \to (Y^{c'}, \widehat{\sigma^{c'}})$$

are both θ-quasimöbius with $\theta(t) = 16t$, $\varphi_X(x) \to \infty$ as $x \to c$ and $\varphi_Y(y) \to \infty$ as $y \to c'$. Because the inverse and the composition of quasimöbius mappings are also quasimöbius, we obtain a quasimöbius mapping:

$$\widehat{f} = \varphi_Y \circ f \circ \varphi_X^{-1} : (X^c, \widehat{d^c}) \to (Y^{c'}, \widehat{\sigma^{c'}})$$

with $\widehat{f}(x) \to \infty$ as $x \to \infty$. Thus $\widehat{f}$ is quasisymmetric by using Lemma 2.3.

On the one hand, again by Lemma 3.2, we see that $(X^c, \widehat{d^c}, \mu^c)$ is Q-Loewner because (X, d, μ) is Q-Loewner. Therefore, appealing to [10,Corollary 1.6] we see that $(Y^{c'}, \widehat{\sigma^{c'}}, \nu^{c'})$ is Q-Loewner.

On the other hand, take another point $p' \in Y$ with $\sigma(p', c') \geq \operatorname{diam}(Y)/2 = 1/2$. Consider the sphericalized space $(Y^{c'}, \widehat{(\sigma^{c'})}_{p'}, (\nu^{c'})_{p'})$ of $(Y^{c'}, \widehat{\sigma^{c'}}, \nu^{c'})$ with respect to p'. By Lemma 3.1, it follows that $(Y^{c'}, \widehat{(\sigma^{c'})}_{p'}, (\nu^{c'})_{p'})$ is also Ahlfors Q-regular and Q-Loewner. According to [5,Proposition 3.5], we see that the identity map id : $(Y, \sigma) \to (Y, \widehat{(\sigma^{c'})}_{p'})$ with c' corresponding to the point ∞ in the extended space $(\dot{Y}^{c'}, \widehat{\sigma^{c'}})$, is 256-bilipschitz. Since Q-Loewner is a bilipschitz invariant, we find that (Y, σ, ν) is also Q-Loewner. The proof of Theorem 1.1 is complete.

Remark 3.3 As we point out in Sect. 1, by Theorem 1.1, it is easy to get many Loewner spaces via sphericalization and flattening transformations. For instance, we obtain that the spherical surface $\mathbb{S}^n$ is a Loewner space, since every Euclidean space is a Loewner space. It follows from the space $(\mathbb{B}^2 \setminus \{0\}, |\cdot|)$ being a Loewner space that $(\mathbb{R}^2 \setminus \mathbb{B}^2, |\cdot|)$ is also a Loewner space by Theorem 1.1 and flattening transformation.

References

1. Balogh, Z., Buckley, S.: Sphericalization and flattening. Conform. Geom. Dyn. **9**, 76–101 (2005)
2. Bonk, M., Kleiner, B.: Quasisymmetric parametrizations of two-dimensional metric spheres. Invent. Math. **150**, 127–183 (2002)
3. Bonk, M., Kleiner, B.: Rigidity for quasi-Möbius group actions. J. Differential Geom. **61**, 81–106 (2002)
4. Brania, A., Yang, S.: Domains with controlled modulus and quasiconformal mappings. Nonlinear Stud. **9**, 57–74 (2002)
5. Buckley, S.M., Herron, D., Xie, X.: Metric space inversions, quasihyperbolic distance, and uniform spaces. Indiana Univ. Math. J. **57**, 837–890 (2008)
6. David, G., Semmes, S.: Fractured fractals and broken dreams: self-similar geometry through metric and measure. Oxford lecture series in mathematics and its applifications, 7. Clarendon Press, Oxford (1997)
7. Heinonen, J.: Lectures on analysis on metric spaces. Springer-Verlag, Berlin-Heidelberg-New York (2001)
8. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)
9. Li, X., Shanmugalingam, N.: Preservation of bounded geometry under sphericalization and flattening. Indiana. Math. J. **64**, 1303–1341 (2015)
10. Tyson, J.: Quasiconformality and quasisymmetry in metric measure spaces. Ann. Acad. Sci. Fenn. Math. **23**, 525–548 (1998)
11. Väisälä, J., Quasi-Möbius maps, *J. Anal. Math.* **44**, 218–234 (1984/85)
12. Wang, X., Zhou, Q.: Quasimöbius maps, weakly quasimöbius maps and uniform perfectness in quasi-metric spaces. Ann. Acad. Sci. Fenn. Ser. AI Math. **42**, 257–284 (2017)
13. Zhou, Q., Li, X., Li, Y.: Sphericalizations and applications in Gromov hyperbolic spaces. J. Math. Anal. Appl. **509**, 125948 (2022)
14. Zhou, Q., Li, X., Li, Y.: Deformations on symbolic Cantor sets and ultrametric spaces. Bull. Malays. Math. Sci. Soc. **43**(4), 3259–3270 (2020)
15. Zhou, Q. and Ponnusamy, S.: Gromov hyperbolic type metrics and quasimöbius invariance of uniform domains, *Submitted*
16. Zhou, Q., Rasila, A.: Quasimöbius invariance of uniform domains. Stud. Math. **261**(1), 1–24 (2021)