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Positive Solutions for a Kirchhoff-Type Equation with Critical and Supercritical Nonlinear Terms

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Abstract

We consider a Kirchhoff-type equation with critical and supercritical nonlinear terms in a ball. By providing a method of decomposition of energy functional and subtle analysis, we show that every Palais–Smale sequence at a level below a certain energy threshold admits a subsequence that converges strongly to a nontrivial critical point of the variational functional.

Keywords Kirchhoff equations · Critical growth · Supercritical nonlinearity · Positive solution · Energy functional

Mathematics Subject Classification 35J35 · 35J60 · 35B33

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1 Introduction and Main Results

We consider the following Kirchhoff-type equation with critical growth:

$$\begin{cases} M\left(\int_{B_r} |\nabla u|^2 dx\right) \Delta u + u^5 + \frac{|x|^\beta}{1+|x|^\beta} u^{p-1} = 0, & \text{in } B_r, \\ u = 0, & \text{on } \partial B_r, \end{cases} \tag{1.1}$$

where $B_r \subset \mathbb{R}^3$ is an open ball centered at the origin, $0 < \beta < 1, 6 < p < 6 + 2\beta$. Denoting $\widehat{M}(s) = \int_0^s M(t)dt$, we make the following assumptions:

(M_1) $M \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $M(s) \geq a > 0$, a is a constant, $M(s)$ is increasing in s;
(M_2) $2M(s) \geq sM'(s)$ for each $s > 0$, and $\lim_{s\to+\infty} \frac{M(s)}{s^2} = 0$;
(M_3) for $s > 0$, $\widehat{M}(s) - \frac{1}{3}sM(s) \geq \frac{2}{3}as$ and $\frac{1}{s}(\frac{1}{2}\widehat{M}(s) - \frac{1}{6}sM(s))$ is nondecreasing in s.

A typical example of M is given by $M(t) = a+bt$ for $t \in \mathbb{R}^+$, where $a > 0, b \geq 0$. Problem (1.1) is often referred as being nonlocal because the presence of the term $M(\int_{B_r} |\nabla u|^2 dx)$ implies that problem (1.1) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which make the study of such a class of problem particularly interesting. Moreover, such a problem has physical motivation. Indeed, the Kirchhoff equation arises in nonlinear vibrations, namely

$$\begin{cases} u_{\text{tt}} - M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x, u), & \text{in } \Omega \times (0, T'), \\ u = 0, & \text{on } \partial\Omega \times (0, T'), \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \end{cases}$$

where $T' > 0$. Such a hyperbolic equation is related to the stationary analogue of the following equation:

$$u_{\text{tt}} - \left(a + b\int_\Omega |\nabla u|^2 dx\right) \Delta u = h(x, u),$$

where u denotes the displacement, h is the external force, and a is the initial tension, while b is related to the intrinsic properties of the string (such as Young's modulus). Such nonlocal elliptic problems like problem (1.1) have received a lot of attention, and some important and interesting results have been established by using the variational methods; see for example ([1–23]) and the references therein. In particular, Figueiredo in [1] studied the following semilinear equation with critical growth:

$$\begin{cases} M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u + |u|^4 u + \lambda f(x, u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where M satisfies the following conditions:

(A_1) The function M is increasing;
(A_2) There exists $M_0 > 0$ such that $M(t) \geq M_0 = M(0)$, for all $t \geq 0$. Under suitable conditions about f and the above assumptions, applying an appropriated truncated argument, the author proved that there exists a threshold value (here the threshold value is just $(\frac{1}{\theta} - \frac{1}{6})(M_0 S)^{\frac{3}{2}}$, where $4 < \theta < 6$ and S is the best constant for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$), only below this threshold value the functional associated with the problem (1.2) satisfies the Palais–Smale condition. Moreover, the author obtained that there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$, problem (1.2) admits a positive solution u_λ with $\lim_{\lambda\to\infty} \|u_\lambda\| = 0$.

After that, Wang et al. in [2] extended the above equation to the following p-Kirchhoff-type equation:

$$\begin{cases} M\left(\int_\Omega |\nabla u|^p dx\right) \Delta_p u + |u|^{p^*-2}u + \lambda f(x,u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The authors made the following assumptions $M(0) = 0$, and

(B_1) there exists $\theta \in (1, p^*/p)(p^* = \frac{Np}{N-p})$ such that $tM(t) \leq \theta \widetilde{M}(t) := \theta \int_0^t M(s)ds$ for all $t > 0$;
(B_2) for any $\tau > 0$ there exists $\kappa = \kappa(\tau) > 0$ such that $M(t) \geq \kappa$ for all $t \geq \tau$;
(B_3) there exists a constant $c > 0$ such that $M(t) \geq ct^{\theta-1}$ for all $t \in [0,1]$. The authors also obtained the same result in [1] by using the Mountain pass theorem.

Now, returning to problem (1.1), it is clear that problem (1.1) has a variational structure. We understand critical points of the associated energy functional acting on the Sobolev space H:

$$I(u) = \frac{1}{2}\widehat{M}\left(\int_{B_r} |\nabla u|^2 dx\right) - \frac{1}{6}\int_{B_r} |u|^6 dx - \frac{1}{p}\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta}|u|^p dx,$$

where $H := H^1_{0,\text{rad}}(B_r)$ is the first-order Sobolev space of radial functions equipped with the inner product and norm

$$\langle u, v\rangle = \int_{B_r} \nabla u \nabla v dx, \quad \|u\| = \langle u, v\rangle^{\frac{1}{2}}.$$

We say that the functional I satisfies the Palais–Smale ((PS) for short) condition if any (PS) sequence $\{u_n\} \subset H$, that is a sequence satisfying

$$\{I(u_n)\} \text{ bounded and } I'(u_n) \to 0 \text{ in } H^{-1} \text{ as } n \to +\infty,$$

admits a convergent subsequence. If $\{u_n\} \subset H$ is a bounded (PS) sequence of I, $\{u_n\}$ has a profile decomposition with containing finitely bubbles (see Lemma 2.4)

$$u_n = u + \sum_{k\in\Lambda} g_{n,k} V_N + \gamma_n,$$

where $V_N, \gamma_n \in D^{1,2}(\mathbb{R}^3)$ which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\varphi\|_D = \left(\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx\right)^{\frac{1}{2}}.$$

Λ is finite, say $\Lambda = 1, 2, ..., N$ (Λ may be empty and $N = 0$). In particular, if $N = 0$, then $u_n \to u$ strongly in H. In order to prove the result, we need to establish the exact threshold value, and only below this threshold value the functional I satisfies the (PS) condition.

Motivated by the above works, we study the existence of solutions for problem (1.1). Here, in order to overcome the lack of compactness induced by the presence of the Kirchhoff term and critical exponent, some delicate estimates are exploited which are totally different from those used in the papers mentioned above. Unlike [1, 2], due to the enough large constraint condition on the parameter λ, which plays a very important role, it causes us not to estimate the threshold value of the energy functional. Therefore, assume that M is an abstract function and the parameter λ is not restricted, in this case, it is very difficult to determine the threshold value of the energy functional. However, in this paper, by a concentration compactness analysis on the Palais–Smale sequence, we establish the threshold value to consequently prove the existence of positive solutions. Thus, the method used in those articles cannot be repeated here because we are working with M which is an abstract function and the parameter $\lambda \equiv 1$.

Now our main result is as follows:

Theorem 1.1 *Assume that* $0 < \beta < 1, 6 < p < 6 + 2\beta$ *and* $(M_1), (M_2), (M_3)$ *hold, then problem* (1.1) *has a positive solution u. Moreover*

$$\lim_{r\to\infty} u(r) = 0.$$

Remark 1.2 Under the assumptions of Theorem 1.2, the following problem

$$\begin{cases} M\left(\int_{B_r} |\nabla u|^2 dx\right) \Delta u + u^5 + |x|^\beta u^{p-1} = 0, & \text{in } B_r, \\ u = 0, & \text{on } \partial B_r, \end{cases}$$

admits a positive solution u with $\lim_{r\to\infty} u(r) = 0$.

This paper is organized as follows. In Sect. 2, we give the compactness analysis and establish the Palais–Smale condition. In Sect. 3, we demonstrate the threshold value and conclude Theorem 1.2. In the proof, we use a same character C to denote several positive constants.

2 Concentration Compactness Analysis

In this section, we make concentration compactness analysis on the (PS) sequence of the functional I. The results will be used to deduce the system of coupled equations satisfied by the weak limit function of a (PS) sequence and the bubbles.

Lemma 2.1 *Assume that* $0 < \beta < 1, 6 < p < 6 + 2\beta$. *If* $\{u_n\}$ *is bounded and* $u_n \rightharpoonup u$ *in* H. *Then,*

$$\lim_{n\to\infty} \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta}|u_n|^p dx = \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta}|u|^p dx. \tag{2.1}$$

Proof From Lemma 2.2 in [24], we have

$$|u_n(x)| \le C\|u_n\| \frac{1}{|x|^{\frac{1}{2}}}, \quad \text{a.e. } x \in B_r.$$

By the boundedness of $\{u_n\}$, for $6 < p < 6 + 2\beta$, it follows that

$$\begin{aligned} \int_{|x|\le 1} \frac{|x|^\beta}{1+|x|^\beta}|u_n|^p dx &\le C \int_{|x|\le 1} \frac{|x|^\beta}{1+|x|^\beta} \frac{dx}{|x|^{\frac{p}{2}}} \\ &\le C \int_{|x|\le 1} \frac{dx}{|x|^{\frac{p}{2}-\beta}} \\ &= C \int_0^1 \frac{dt}{t^{\frac{p}{2}-\beta-2}} \\ &< +\infty \quad (\text{as } p < 6+2\beta) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \int_{|x|\ge 1} \frac{|x|^\beta}{1+|x|^\beta}|u_n|^p dx &\le C \int_{|x|\ge 1} \frac{|x|^\beta}{1+|x|^\beta} \frac{dx}{|x|^{\frac{p}{2}}} \\ &\le C \int_{|x|\ge 1} \frac{dx}{|x|^{\frac{p}{2}}} \\ &= C \int_0^{+\infty} \frac{dt}{t^{\frac{p}{2}-2}} \\ &< +\infty. \quad (\text{as } p > 6) \end{aligned} \tag{2.3}$$

Consequently, it leads to (2.1) by means of the Lebesgue dominated convergence theorem. The proof is complete. □

Lemma 2.2 *Any Palais–Smale sequence of* I *is bounded in* H.

Proof Let $\{u_n\} \subset H$ be such that

$$I'(u_n) \to 0, \quad I(u_n) \to c \text{ as } n \to \infty. \tag{2.4}$$

Combining with (M_3), one has

$$\begin{aligned} I(u_n) - \frac{1}{6}\langle I'(u_n), u_n\rangle &\geq \frac{1}{2}\widehat{M}(\|u_n\|^2) - \frac{1}{6}M(\|u_n\|^2)\|u_n\|^2 \\ &\quad + \left(\frac{1}{6} - \frac{1}{p}\right)\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta}|u_n|^p dx \\ &\geq \frac{a}{3}\|u_n\|^2. \end{aligned}$$

Therefore, $\{u_n\}$ is bounded in H. The proof is complete. □

To make the concentration compactness analysis, we introduce the dilation group $\mathfrak{D}$ in $\mathbb{R}^3$

$$\mathfrak{D} = \left\{g_{\sigma,y} | g_{\sigma,y}u(\cdot) = \sigma^{\frac{1}{2}}u(\sigma(\cdot - y)), \quad y \in \mathbb{R}^3, \ \sigma \in \mathbb{R}^+.\right\}$$

The dilation g in $\mathfrak{D}$ is an isometry in both $L^6(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$.

Let $\{u_n\} \subset H$ be a Palais–Smale sequence of the functional I. By Lemma 2.2, $\{u_n\}$ is bounded in H. According to Theorem 3.1 and Corollary 3.2 in [26](see also [25]), $\{u_n\}$ has a profile decomposition

$$u_n = u + \sum_{k\in\Lambda} g_{n,k}U_k + \gamma_n, \tag{2.5}$$

where $u \in H$, $U_k \in D^{1,2}(\mathbb{R}^3)$, $g_{n,k} = g_{\sigma_{n,k},y_{n,k}} \in \mathfrak{D}$, $\sigma_{n,k} > 0$, $y_{n,k} \in \overline{B}_r$, $\gamma_n \in D^{1,2}(\mathbb{R}^3)$, Λ is an index set, satisfy:

(1) $u_n \rightharpoonup u$ in H, $g_{n,k}^{-1}u_n \rightharpoonup U_k$ in $D^{1,2}(\mathbb{R}^3)$, as $n \to \infty$, $k \in \Lambda$;
(2) $g_{n,k} \rightharpoonup 0$ in $[D^{1,2}(\mathbb{R}^3)]^*$, $g_{n,k}^{-1}g_{n,l} \rightharpoonup 0$ in $[D^{1,2}(\mathbb{R}^3)]^*$ as $n \to \infty$, $k, l \in \Lambda$, $k \neq l$;
(3) $\|u_n\|_D^2 = \|u\|_D^2 + \sum_{k\in\Lambda}\|U_k\|_D^2 + \|\gamma_n\|_D^2 + o(1)$, as $n \to \infty$. From property (3) of (2.5), by the method of the Brézis–Lieb's lemma and [25], one has
(4) $\gamma_n \to 0$ in $L^6(\mathbb{R}^3)$ and

$$\int_{B_r} u_n^6 dx = \int_{B_r} u^6 dx + \sum_{k\in\Lambda}\int_{\mathbb{R}^3}|U_k|^6 dx + o(1), \text{ as } n \to \infty.$$

Here for a sequence $\{g_n\}$ of $\mathfrak{D}$, we say $g_n \rightharpoonup 0$ in $[D^{1,2}(\mathbb{R}^3)]^*$, if for all $v \in D^{1,2}(\mathbb{R}^3)$, $g_n v \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^3)$. Moreover, since $\{u_n\}$ is bounded in H, we have $\sigma_{n,k} \to \infty$ as $n \to \infty$, $k \in \Lambda$.

We deduce the system of coupled equations satisfied by the weak limit function u and the bubbles U_k, $k \in \Lambda$. Inspired by Lemma 2.1 in [30], we obtain the following conclusion.

Lemma 2.3 *Let* $\{u_n\}$ *be a Palais–Smale sequence of* I*,* $A_n \triangleq \int_{B_r} |\nabla u_n|^2 dx \to A$ *as* $n \to \infty$.

(1) Assume $u_n \rightharpoonup u$ in H, then u satisfies the equation:

$$M(A)\int_{B_r} \nabla u \nabla \varphi dx = \int_{B_r} u^5 \varphi dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^{p-1}\varphi dx, \ \ for\ \varphi \in H. \quad (2.6)$$

(2) Let $g_n = g_{\sigma_n, y_n} \in \mathfrak{D}$, $\sigma_n \to \infty$ as $n \to \infty$ and $y_n \in \overline{B}_r$. Assume $\widetilde{u}_n = g_n^{-1}u_n \rightharpoonup U \neq 0$ in $D^{1,2}(\mathbb{R}^3)$. Then, U satisfies the equation:

$$M(A)\int_{\mathbb{R}^3} \nabla U \nabla \phi dx = \int_{\mathbb{R}^3} U^5 \phi dx, \ \ for\ \phi \in D^{1,2}(\mathbb{R}^3). \quad (2.7)$$

Proof (1) Since $\{u_n\}$ is (PS) sequence of I, by (2.1), we have

$$M\left(\int_{B_r} |\nabla u_n|^2 dx\right)\int_{B_r} \nabla u_n \nabla \varphi dx \\ -\int_{B_r} u_n^5 \varphi dx - \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_n^{p-1}\varphi dx = o(1) \quad (2.8)$$

for $\varphi \in H$. Letting $n \to \infty$, which deduce that (2.6) holds.

(2) Denote

$$d_n = \sigma_n \text{dist}(y_n, \partial B_r).$$

We first assume $d_n \to \infty$. Let φ be a smooth function in $\varphi \in C_0^\infty(\mathbb{R}^3)$ and $\psi = g_n\varphi = \sigma_n^{\frac{1}{2}}\varphi(\sigma_n(\cdot - x_n))$. For n large enough $\psi \in C_0^\infty(B_r)$. Taking ψ as a test function in (2.8), we have

$$M\left(\int_{B_r} |\nabla u_n|^2 dx\right)\int_{B_r} \nabla u_n \nabla \psi dx \\ = \int_{B_r} u_n^5 \psi dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_n^{p-1}\psi dx + o(1). \quad (2.9)$$

Making a variable change, we get

$$y = \sigma_n(x - y_n).$$

Set $\widetilde{u}_n = g_n^{-1}u_n$. In view of

$$g_n^{-1}u_n = \sigma_n^{-\frac{1}{2}} u_n(\sigma_n^{-1}(\cdot + x_n)),$$

we see that

$$\nabla g_n^{-1} u_n = \sigma_n^{-\frac{1}{2}} \frac{1}{\sigma_n} \nabla u_n.$$

Consequently, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla u_n \nabla (g_n \varphi) dx &= \int_{\Omega_n} \sigma_n \sigma^{\frac{1}{2}} \nabla g_n^{-1} u_n \sigma_n^{\frac{1}{2}} \sigma_n \nabla \varphi \frac{1}{\sigma_n^3} dy \ \ (\text{let } \sigma_n(x - x_n) = y) \\
&= \int_{\Omega_n} \nabla g_n^{-1} u_n \nabla \varphi dx \\
&= \int_{\Omega_n} \nabla \widetilde{u}_n \nabla \varphi dx, \\
\int_{\mathbb{R}^3} u_n^5 g_n \varphi dx &= \int_{\Omega_n} \sigma_n^{\frac{5}{2}} (\sigma_n^{-\frac{1}{2}} u_n (\frac{y}{\sigma_n} + x_n))^5 g_n \varphi(x) \frac{1}{\sigma_n^3} dy \\
&= \int_{\Omega_n} \sigma_n^{\frac{5}{2}} (\sigma_n^{-\frac{1}{2}} u_n (\frac{y}{\sigma_n} + x_n))^5 \sigma_n^{\frac{1}{2}} \varphi(y) \frac{1}{\sigma_n^3} dy \\
&= \int_{\Omega_n} \widetilde{u}_n^5 \varphi dx
\end{aligned}$$

where $\Omega_n = \{y | y \in \mathbb{R}^3, x = \sigma_n^{-1} y + y_n \in B_r\}$,

$$\int_{\mathbb{R}^3} \frac{|x|^\beta}{1 + |x|^\beta} u_n^{p-1} g_n \varphi dx = \sigma_n^{-\frac{6-p}{2}} \int_{\Omega_n} \frac{|x|^\beta}{1 + |x|^\beta} \widetilde{u}_n^{p-1} \varphi dx.$$

Consequently, (2.9) becomes

$$\begin{aligned}
&M\left(\int_{B_r} |\nabla u_n|^2 dx\right) \int_{\Omega_n} \nabla \widetilde{u}_n \nabla \varphi dx \\
&\quad = \int_{\Omega_n} \widetilde{u}_n^5 \varphi dx + \sigma_n^{-\frac{6-p}{2}} \int_{\Omega_n} \frac{|x|^\beta}{1 + |x|^\beta} \widetilde{u}_n^{p-1} \varphi dx + o(1), \qquad (2.10)
\end{aligned}$$

Since $\widetilde{u}_n = g_n^{-1} u_n \rightharpoonup U$ and $\sigma_n \to \infty$ as $n \to \infty$, taking the limit $n \to \infty$ in (2.10), we obtain

$$M(A) \int_{\mathbb{R}^3} \nabla U \nabla \varphi dx - \int_{\mathbb{R}^3} U^5 \varphi dx = 0, \ \text{for } \varphi \in C_0^\infty(\mathbb{R}^3).$$

By a density argument, we obtain

$$M(A) \int_{\mathbb{R}^3} \nabla U \nabla V dx - \int_{\mathbb{R}^3} U^5 V dx = 0$$

for $V \in D^{1,2}(\mathbb{R}^3)$.

Finally, in the case $d_n = \sigma_n \text{dist}(y_n, \partial B_r) \to d < +\infty$. Without loss of generality, we assume $d = 0$. In this case, we can prove that U satisfies $U = 0$ in $\mathbb{R}^3 \backslash \mathbb{R}^3_+$ and

$$M(A) \int_{\mathbb{R}^3_+} \nabla U \nabla V dx = \int_{\mathbb{R}^3_+} U^5 V dx, \text{ for } V \in D^{1,2}(\mathbb{R}^3), V = 0 \text{ in } \mathbb{R}^3 \backslash \mathbb{R}^3_+.$$

By the uniqueness theory in [27] for positive solutions of the equation

$$\begin{cases} \Delta u + u^{2^*-1} = 0, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{2.11}$$

$U \equiv 0$ in $\mathbb{R}^3$, which is a contradiction. The proof is complete. □

We continue the concentration compactness analysis on Palais–Smale sequences.

Lemma 2.4 *Let $\{u_n\}$ be a Palais–Smale sequence of I. Assume the profile decomposition (2.5) holds, namely*

$$u_n = u + \sum_{k \in \Lambda} g_{n,k} U_k + \gamma_n.$$

Then,

(1) the index set Λ is finite, say $\Lambda = \{1, 2, ..., N\}$ (Λ may be empty and $N = 0$).
(2) There exist $V_N \in D^{1,2}(\mathbb{R}^3)$ and $g_n \in \mathfrak{D}$, $k = 1, 2, ..., N$ such that

(2_a) $U_k = g_k V_N$, $k = 1, 2..., N$ and the profile decomposition (2.5) reduces to

$$u_n = u + \sum_{k=1}^{N} g_{n,k} V_N + \gamma_n. \tag{2.12}$$

(2_b) *u and V_N satisfy the system*

$$\begin{cases} M(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) \int_{B_r} \nabla u \nabla \varphi dx = \int_{B_r} (u^5 + \frac{|x|^\beta}{1+|x|^\beta} u^{p-1}) \varphi dx, \varphi \in H, \\ M(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) \int_{\mathbb{R}^3} \nabla V_N \nabla \phi dx = \int_{\mathbb{R}^3} V_N^5 \phi dx, \phi \in D^{1,2}(\mathbb{R}^3). \end{cases}$$

(2_c) *There hold that*

$$\begin{cases} \int_{B_r} |\nabla u_n|^2 dx = \int_{B_r} |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx + o(1), \\ \int_{B_r} u_n^6 dx = \int_{B_r} u^6 dx + N \int_{\mathbb{R}^3} V_N^6 dx + o(1), \ \textit{as } n \to \infty. \end{cases}$$

Proof (1) By Lemma 2.3, we have the system

$$\begin{cases} M(A)\int_{B_r}\nabla u\nabla\varphi dx=\int_{B_r}u^5\varphi dx+\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^{p-1}\varphi dx,\ \varphi\in H,\\ M(A)\int_{\mathbb{R}^3}\nabla U_k\nabla\phi dx=\int_{\mathbb{R}^3}U_k^5\phi dx,\ \phi\in D^{1,2}(\mathbb{R}^3),\ k\in\Lambda, \end{cases} \tag{2.13}$$

where $A=\lim_{n\to\infty}\int_{B_r}|\nabla u_n|^2dx$. Taking $\phi=U_k$ as test function in the second equation of (2.13), we have

$$\begin{aligned} a\int_{\mathbb{R}^3}|\nabla U_k|^2dx\le m(A)\int_{\mathbb{R}^3}|\nabla U_k|^2dx&=\int_{\mathbb{R}^3}U_k^6dx\\ &\le S^{-3}\left(\int_{\mathbb{R}^3}|\nabla U_k|^2dx\right)^3, \end{aligned}$$

where S is the Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$. We deduce that

$$\int_{\mathbb{R}^3}|\nabla U_k|^2dx\ge\sqrt{aS^3}. \tag{2.14}$$

By the property (3) of the profile decomposition (2.5), Λ is a finite set, say $\Lambda=\{1,2,\ldots,N\}$.

(2) By the second equation of (2.13) and the uniqueness theory [27] of the positive solutions of the equation (2.11), there exist $V_k\in D^{1,2}(\mathbb{R}^3)$, $g_k\in\mathfrak{D}$, $k=1,2,\ldots,N$ such that $U_k=g_kV_N$ and V_N satisfies

$$M(A)\int_{\mathbb{R}^3}\nabla V_N\nabla\phi dx=\int_{\mathbb{R}^3}V_N^5\phi dx,\ \phi\in D^{1,2}(\mathbb{R}^3).$$

Replacing $g_{n,k}$ by $g_{n,k}g_k$ in the profile decomposition in (2.5), we obtain

$$u_n=u+\sum_{k=1}^N g_{n,k}V_N+\gamma_n.$$

Noting that u_n satisfies the inequality

$$\begin{aligned} \int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u_n^{p-1}vdx=M\left(\int_{B_r}|\nabla u_n|^2dx\right)\int_{B_r}\nabla u_n\nabla vdx\\ -\int_{B_r}u_n^5vdx+o(1) \end{aligned}$$

for $v\in H$. Taking $v=u_n$ as test function in the above inequality, it yields

$$M\left(\int_{B_r}|\nabla u_n|^2dx\right)\int_{B_r}|\nabla u_n|^2dx$$

$$-\int_{B_r} u_n^6 dx - \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_n^{p-1} dx = o(1). \tag{2.15}$$

By (2.13)

$$\begin{cases} M(A) \int_{B_r} |\nabla u|^2 dx = \int_{B_r} u^6 dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx, \\ M(A) \int_{\mathbb{R}^3} |\nabla U_k|^2 dx = \int_{\mathbb{R}^3} U_k^6 dx, \ k \in \Lambda \end{cases} \tag{2.16}$$

and the property (4) of the profile decomposition (2.5)

$$\int_{B_r} u_n^6 dx = \int_{B_r} u^6 dx + \sum_{k=1}^{N} \int_{\mathbb{R}^3} U_k^6 dx + o(1). \tag{2.17}$$

Notice that

$$\int_{B_r} |\nabla u_n|^2 dx \to A, \quad \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_n^p dx \to \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx$$

as $n \to \infty$. It follows from (2.15), (2.16) and (2.17) that

$$\int_{B_r} |\nabla u_n|^2 dx = \int_{B_r} |\nabla u|^2 dx + \sum_{k=1}^{N} \int_{\mathbb{R}^3} |\nabla U_k|^2 dx + o(1). \tag{2.18}$$

Finally, since $g_k \in \mathfrak{D}$, $k = 1, 2, ..., N$ are isometry in both $L^6(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |\nabla U_k|^2 dx = \int_{\mathbb{R}^3} |\nabla V_N|^2 dx,$$

and

$$\int_{\mathbb{R}^3} V_k^6 dx = \int_{\mathbb{R}^3} V_N^6 dx,$$

where $k = 1, 2, ..., N$. Hence, (2.17) and (2.18) can be rewritten as, respectively.

$$\int_{B_r} |\nabla u_n|^2 dx = \int_{B_r} |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx + o(1),$$

and

$$\int_{B_r} u_n^6 dx = \int_{B_r} u^6 dx + N \int_{\mathbb{R}^3} V_N^6 dx + o(1)$$

as $n \to \infty$. In particular,

$$A = \lim_{n\to\infty} \int_{B_r} |\nabla u_n|^2 dx = \int_{B_r} |\nabla u|^2 dx + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx,$$

and u, V_N satisfy the system in (2_b). The proof is complete. □

3 Threshold Value and the Proof of Theorem 1.2

In this section, we determine the threshold value, below which the functional I satisfies the $(PS)_c$ condition, and then show that this level is less than the threshold value. Consequently, we prove the existence of the mountain pass-type solution.

Assume that $\{u_n\}$ is a $(PS)_c$ sequence of I and the profile decomposition (2.12) holds, namely

$$u_n = u + \sum_{k=1}^{N} g_{n,k} V_N + \gamma_n.$$

By Lemma 2.4, we have

$$\begin{aligned}
\lim_{n\to\infty} I(u_n) &= \frac{1}{2}\widehat{M}(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) - \frac{1}{p} \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx \\
&\quad - \frac{1}{6}\left(\int_{B_r} u^6 dx + N \int_{\mathbb{R}^3} V_N^6 dx \right) \\
&= \frac{1}{2}\widehat{M}(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) - \left(\frac{1}{p} - \frac{1}{6}\right) \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx \\
&\quad - \frac{1}{6} M(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) \left(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx \right),
\end{aligned} \tag{3.1}$$

where we have used the fact that,

$$\begin{cases} M(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) \int_{B_r} \|u\|^2 = \int_{B_r} u^6 dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx, \\ M(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx) \int_{\mathbb{R}^3} |\nabla V_N|^2 dx = \int_{\mathbb{R}^3} V_N^6 dx. \end{cases}$$

Noting that

$$\begin{aligned}
S^{-3} \left(\int_{\mathbb{R}^3} |\nabla V_N|^2 dx \right)^3 &= \int_{\mathbb{R}^3} V_N^6 dx \\
&= M\left(\|u\|^2 + N \int_{\mathbb{R}^3} |\nabla V_N|^2 dx \right) \int_{\mathbb{R}^3} |\nabla V_N|^2 dx.
\end{aligned} \tag{3.2}$$

Using the following lemma, we can solve equation (3.2) for $\int_{\mathbb{R}^3} |\nabla V_N|^2 dx$.

Lemma 3.1 *Give $s \geq 0$, the equation $M(s+Nt) = S^{-3}t^2$ has a unique positive solution $t := \mathcal{F}_N(s)$. The function $\mathcal{F}_N$ is continuously differentiable. Moreover, $\mathcal{F}_N(s) \geq \mathcal{F}_1(0) := T$, where T is the unique positive solution of the equation $M(t) = S^{-3}t^2$.*

Proof By the assumption (M_2), the function

$$g(t,s) = \frac{M(s+Nt)}{t^2} = \frac{M(s+Nt)}{(s+Nt)^2}\frac{(s+Nt)^2}{t^2}$$

is strictly decreasing in t, and

$$\lim_{t\to+\infty} g(t,s) = 0, \quad \lim_{t\to 0^+} g(t,s) = +\infty.$$

Hence, there exists a unique $t > 0$, denoted by $\mathcal{F}_N(s)$, satisfies the equation $g(t,s) = S^{-3}$, that is,

$$M(s+Nt) = S^{-3}t^2.$$

Since M is a continuously differentiable function and

$$\frac{\partial}{\partial t} g(t,s) = \frac{1}{t^3}(NtM'(s+Nt) - 2M(s+Nt)) < 0,$$

so is the function $t = \mathcal{F}_N(s)$ by the implicit function theorem. Finally, by the assumption (M_1) for $t = \mathcal{F}_N(s)$

$$\frac{M(s+\mathcal{F}_N(s))}{\mathcal{F}_N^2(s)} = S^{-3} = g(t) = \frac{M(s+Nt)}{t^2} \geq \frac{M(t)}{t^2},$$

and by assumption (M_3), $\mathcal{F}_N(s) = t \geq T = \mathcal{F}_1(0)$. The proof is complete. □

As a result of (3.2) and Lemma 3.1, we obtain

$$\int_{\mathbb{R}^3} |\nabla V_N|^2 dx = \mathcal{F}_N(\|u\|^2) \tag{3.3}$$

and rewrite Formula (3.1) as

$$
\begin{aligned}
\lim_{n\to\infty} I(u_n) &= \frac{1}{2}\widehat{M}(\|u\|^2 + N\int_{\mathbb{R}^3} |\nabla V_N|^2 dx) - \frac{1}{p}\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx \\
&\quad -\frac{1}{6}\left(\int_{B_r} u^6 dx + N\int_{\mathbb{R}^3} V_N^6 dx\right) \\
&= \frac{1}{2}\widehat{M}(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)) - \frac{1}{p}\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx \quad \text{(by (3.3))} \\
&\quad -\frac{1}{6}\left(\int_{B_r} u^6 dx + NS^{-3}\mathcal{F}_N^3(\|u\|^2)\right) \\
&= I_N(u)
\end{aligned}
\tag{3.4}
$$

where

$$
\int_{\mathbb{R}^3} V_N^6 dx = S^{-3}\left(\int_{\mathbb{R}^3} |\nabla V_N|^2 dx\right)^3 = S^{-3}\mathcal{F}_N^3(\|u\|^2).
$$

Also, we rewrite equation (in Lemma 2.4 (2_b)) satisfied by u as

$$
M(\|u\|^2 + N\mathcal{F}_N(\|u\|^2))\int_{B_r} \nabla u \nabla \varphi dx = \int_{B_r} u^5 \varphi dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^{p-1}\varphi dx \tag{3.5}
$$

for $\varphi \in H$. Define

$$
\begin{aligned}
\Sigma_N &= \{u | u \in H,\ u \text{ satisfiesthattheequation(3.5)}\}, \\
\mu_N &= \inf\{I_N(u) | u \in \Sigma_N\}.
\end{aligned}
$$

The following lemma gives the lower bound for μ_N.

Lemma 3.2 *There exists a constant C_1 (independent of N) such that $\mu_N \geq ND$, where $D = \frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T$ and $T = \int_{\mathbb{R}^3} |\nabla V_N|^2 dx$.*

Proof Let $u \in \Sigma_N$, it follows that

$$
M(\|u\|^2 + N\mathcal{F}_N(\|u\|^2))\|u\|^2 = \int_{B_r} u^6 dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx.
$$

It follows from (3.2) and (3.3) that

$$
M(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)) = S^{-3}\mathcal{F}_N^2(\|u\|^2).
$$

At this moment, by (3.4), we have

$$
\begin{aligned}
I_N(u) &= \frac{1}{2}\widehat{M}(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)) - \frac{1}{p}\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^p dx \\
&\quad -\frac{1}{6}\left(\int_{B_r} u^6 dx + NS^{-3}\mathcal{F}_N^3(\|u\|^2)\right) \\
&= \frac{1}{2}\widehat{M}(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)) - \frac{1}{6}M(\|u\|^2 + N\mathcal{F}_N(\|u\|^2))\|u\|^2 \\
&\quad +\left(\frac{1}{6}-\frac{1}{p}\right)\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^p dx - \frac{1}{6}NS^{-3}\mathcal{F}_N^3(\|u\|^2)) \\
&= \frac{1}{2}\widehat{M}(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)) + \left(\frac{1}{6}-\frac{1}{p}\right)\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^p dx \\
&\quad -\frac{1}{6}M(\|u\|^2 + N\mathcal{F}_N(\|u\|^2))(\|u\|^2 + N\mathcal{F}_N(\|u\|^2)). \qquad (3.6)
\end{aligned}
$$

Let

$$h(s) = \frac{1}{2}\widehat{M}(s) - \frac{1}{6}M(s)s.$$

By the assumption (M_3), one has

$$h(a+b) \geq h(a) + h(b), \quad \text{for } a, b \in \mathbb{R}^+. \qquad (3.7)$$

In fact

$$
\begin{aligned}
h(a+b) &= a\cdot\frac{h(a+b)}{a+b} + b\cdot\frac{h(a+b)}{a+b} \\
&\geq a\cdot\frac{h(a)}{a} + b\cdot\frac{h(a)}{b} \\
&= h(a) + h(b).
\end{aligned}
$$

By (3.7), (3.6) can be reduced to

$$
\begin{aligned}
I_N(u) &\geq \frac{1}{2}\widehat{M}(\|u\|^2) + \left(\frac{1}{6}-\frac{1}{p}\right)\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^p dx - \frac{1}{6}M(\|u\|^2)\|u\|^2 \\
&\quad +\frac{1}{2}\widehat{M}(N\mathcal{F}_N(\|u\|^2)) - \frac{1}{6}M\left(N\mathcal{F}_N(\|u\|^2)\right)N\mathcal{F}_N(\|u\|^2) \\
&\triangleq J(u) + G_N(u),
\end{aligned} \qquad (3.8)
$$

where

$$J(u) = \frac{1}{2}\widehat{M}(\|u\|^2) + \left(\frac{1}{6}-\frac{1}{p}\right)\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}u^p dx - \frac{1}{6}M(\|u\|^2)\|u\|^2,$$

and

$$G_N(u) = \frac{1}{2}\widehat{M}(N\mathcal{F}_N(\|u\|^2)) - \frac{1}{6}M(N\mathcal{F}_N(\|u\|^2))N\mathcal{F}_N(\|u\|^2).$$

Since $p > 6$, using the assumption (M_3) again, we get

$$\begin{aligned} J(u) &= \frac{1}{2}\widehat{M}(\|u\|^2) + \left(\frac{1}{6} - \frac{1}{p}\right)\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u^p dx - \frac{1}{6}M(\|u\|^2)\|u\|^2 \\ &\geq \frac{1}{2}\widehat{M}(\|u\|^2) - \frac{1}{6}M(\|u\|^2)\|u\|^2 \\ &\geq \frac{1}{3}a\|u\|^2 \\ &\geq 0. \end{aligned} \tag{3.9}$$

If $N \geq 1$, by Lemma 3.1, one obtains

$$N\mathcal{F}_N(\|u\|^2) \geq NT \geq T.$$

Hence, by the assumption (M_3) again,

$$\begin{aligned} &\frac{\frac{1}{2}\widehat{M}(N\mathcal{F}_N(\|u\|^2)) - \frac{1}{6}M\left(N\mathcal{F}_N(\|u\|^2)\right)N\mathcal{F}_N(\|u\|^2)}{N\mathcal{F}_N(\|u\|^2)} \\ &\geq \frac{\frac{1}{2}\widehat{M}(NT) - \frac{1}{6}M(NT)T}{NT} \\ &\geq \frac{\frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T}{T}. \end{aligned}$$

As a result,

$$\begin{aligned} G_N(u) &= \frac{1}{2}\widehat{M}(N\mathcal{F}_N(\|u\|^2)) - \frac{1}{6}M\left(N\mathcal{F}_N(\|u\|^2)\right)N\mathcal{F}_N(\|u\|^2) \\ &\geq \left[\frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T\right]\frac{N\mathcal{F}_N(\|u\|^2)}{T} \\ &\geq N\left[\frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T\right] \\ &\triangleq ND. \end{aligned} \tag{3.10}$$

Therefore, the estimate for μ_N follows from relations (3.8), (3.9) and (3.10), that is,

$$\begin{aligned} \mu_N &= \inf\{I_N(u)|u \in \Sigma_N\} \\ &\geq \inf_{u\in\Sigma_N} J(u) + \inf_{u\in\Sigma_N} G_N(u) \\ &\geq ND. \end{aligned}$$

The proof is complete. □

Lemma 3.3 *μ_1 is achieved at the point 0 and $\mu_1 = D$, where D is defined in Lemma 3.2.*

Proof Since $0 \in \Sigma_1$, and

$$\begin{aligned} I_1(0) &= \frac{1}{2}\widehat{M}(\mathcal{F}_1(0)) - \frac{1}{6}S^{-3}\mathcal{F}_1^3(0) \\ &= \frac{1}{2}\widehat{M}(\mathcal{F}_1(0)) - \frac{1}{6}M(\mathcal{F}_1(0))\mathcal{F}_1(0) \\ &= \frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T \\ &= D. \end{aligned} \tag{3.11}$$

By Lemma 3.2, for any $u \in \Sigma_1$, we know that

$$\begin{aligned} I_1(u) &\geq J(u) + G_1(u) \\ &\geq G_1(u) \\ &\geq D. \end{aligned}$$

Hence, by the definition of μ_1 and (3.11), which is achieved at 0 and $\mu_1 = D$, the proof is complete. □

Lemma 3.4 *Under the assumptions of Theorem 1.1, the functional I satisfies the $(PS)_c$ condition provided $c < D$, where D is defined in Lemma 3.2.*

Proof Let $\{u_n\} \subset H$ be such that

$$I(u_n) \to c, \quad I'(u_n) \to 0$$

as $n \to \infty$. By Lemma 2.4, $\{u_n\}$ has the profile decomposition (2.12)

$$u_n = u + \sum_{k=1}^{N} g_{n,k}V_N + \gamma_n.$$

By Lemma 3.2, we deduce that

$$D > c = \lim_{n\to\infty} I(u_n) = I_N(u) \geq J(u) + G_N \geq G_N(u) \geq \mu_N \geq ND.$$

which implies that $N = 0$. Consequently, $u_n = u + o(1)$ in H as $n \to \infty$. That is,

$$\int_{B_r} |\nabla u_n|^2 dx \to \int_{B_r} |\nabla u|^2 dx, \quad \int_{B_r} u_n^6 dx \to \int_{B_r} u^6 dx$$

as $n \to 0$. The proof is complete. □

In the following, we estimate the threshold value of I. Denote

$$U(x) = \frac{3^{\frac{1}{4}}}{(1+|x|^2)^{\frac{1}{2}}}, \quad U_\varepsilon(x) = \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+|x|^2)^{\frac{1}{2}}}, \; x \in \mathbb{R}^3, \; \varepsilon > 0. \tag{3.12}$$

U (and U_ε) satisfies the limit equation

$$\Delta U + U^5 = 0, \quad U > 0 \text{ in } \mathbb{R}^3.$$

Choose $\eta \in C_0^\infty(B_1(x_0), [0,1])$ where $B_1(x_0) \subset B_r$ such that $\eta(x) = 1$ near $x = x_0$. Denote $\varphi_\varepsilon = U_\varepsilon \eta$.

Lemma 3.5 *Assume that* (M_1) *and* (M_2) *hold. Then,* $\sup_{t\geq 0} I(t\varphi_\varepsilon) < D$ *for sufficient small* $\varepsilon > 0$*, where* D *is defined in Lemma 3.2.*

Proof From Lemma 1.1 in [28], we know

$$\begin{cases} \int_{B_r} |\nabla\varphi_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |\nabla U|^2 dx + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon), \\ \int_{B_r} \varphi_\varepsilon^6 dx = \int_{\mathbb{R}^3} U^6 dx + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3). \end{cases}$$

Therefore, we deduce that

$$\int_{B_r} |\nabla\varphi_\varepsilon|^2 dx \to \int_{\mathbb{R}^3} |\nabla U|^2 dx \text{ as } \varepsilon \to 0.$$

Consequently, according to the continuity of $\widehat{M}$, we obtain

$$\widehat{M}\left(\int_{B_r} |\nabla\varphi_\varepsilon|^2 dx\right) = \widehat{M}\left(\int_{\mathbb{R}^3} |\nabla U|^2 dx\right) + O(\varepsilon).$$

By the definition of φ_ε, we infer that

$$\begin{aligned}
\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta}\varphi_\varepsilon^p dx &= C\int_{|x|\le 1} \frac{|x|^\beta}{1+|x|^\beta}\frac{\varepsilon^{\frac{p}{2}}}{(\varepsilon^2+|x|^2)^{\frac{p}{2}}}dx \\
&\ge C\int_{|x|\le 1} \frac{|x|^\beta}{2}\frac{\varepsilon^{\frac{p}{2}}}{(\varepsilon^2+|x|^2)^{\frac{p}{2}}}dx \\
&= C\varepsilon^{\beta+3-\frac{p}{2}}\int_0^{1/\varepsilon}\frac{t^{2+\beta}}{(1+t^2)^{\frac{p}{2}}}dt \\
&= C\varepsilon^{\beta+3-\frac{p}{2}}\int_0^{1}\frac{t^{2+\beta}}{(1+t^2)^{\frac{p}{2}}}dt + C\varepsilon^{\beta+3-\frac{p}{2}}\int_0^{1/\varepsilon}\frac{t^{2+\beta}}{(1+t^2)^{\frac{p}{2}}}dt \\
&= C\varepsilon^{\beta+3-\frac{p}{2}}\int_0^{1}\frac{t^{2+\beta}}{(1+t^2)^{\frac{p}{2}}}dt \\
&\ge C\varepsilon^{\beta+3-\frac{p}{2}}\int_0^{1} t^{2+\beta}dt \\
&= C\varepsilon^{\beta+3-\frac{p}{2}}
\end{aligned}$$

for some $C > 0$. Since $M(t) = o(t^2)$, $\widehat{M}(t) = o(t^3)$ as $t \to +\infty$, $I(t\varphi_\varepsilon) \to -\infty$ as $t \to \infty$. We can assume there exist $0 < t_1 < t_2$ such that

$$\sup_{t\ge 0} I(t\varphi_\varepsilon) = \sup_{t\in[t_1,t_2]} I(t\varphi_\varepsilon).$$

From the above information, there holds

$$\begin{aligned}
I(t\varphi_\varepsilon) &= \frac{1}{2}\widehat{M}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx - \frac{t^p}{p}\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}\varphi_\varepsilon^p dx + O(\varepsilon) \\
&\le \frac{1}{2}\widehat{M}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx - \frac{t_1^p}{p}\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}\varphi_\varepsilon^p dx + O(\varepsilon) \\
&\le \frac{1}{2}\widehat{M}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx - C\varepsilon^{\beta+3-\frac{p}{2}} + C\varepsilon.
\end{aligned} \tag{3.13}$$

Define

$$g(t) = \frac{1}{2}\widehat{M}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx.$$

Then,

$$\begin{aligned}
g'(t) &= tM\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx - t^5\int_{\mathbb{R}^3}U^6dx \\
&= tM\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx - t^5S^{-3}\left(\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^3 \\
&= t\int_{\mathbb{R}^3}|\nabla U|^2dx\left[M\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - S^{-3}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^2\right].
\end{aligned}$$

Let $t_0 > 0$ be the unique positive zero, according to $g'(t_0) = 0$, one has

$$M\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) = S^{-3}\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^2. \tag{3.14}$$

By Lemma 3.1 and (3.14), we have

$$t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx = \mathcal{F}_1(0).$$

Furthermore, we have

$$\begin{aligned} g(t_0) &= \frac{1}{2}\widehat{M}\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{t_0^6}{6}\int_{\mathbb{R}^3}U^6dx \\ &= \frac{1}{2}\widehat{M}\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right) - \frac{1}{6}S^{-3}\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^3 \\ &= \frac{1}{2}\widehat{M}\left(\mathcal{F}_1(0)\right) - \frac{1}{6}S^{-3}\mathcal{F}_1^3(0) \\ &= \frac{1}{2}\widehat{M}\left(\mathcal{F}_1(0)\right) - \frac{1}{6}M(\mathcal{F}_1(0))\mathcal{F}_1(0) \\ &= \frac{1}{2}\widehat{M}(T) - \frac{1}{6}M(T)T \\ &= \mu_1 = D. \end{aligned} \tag{3.15}$$

Moreover, we have

$$\begin{aligned} g''(t) = M\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx \\ +2t^2M'\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\left(\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^2 - 5t^4\int_{\mathbb{R}^3}U^6dx. \end{aligned}$$

Noting that

$$0 = g'(t_0) = t_0M\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx - t_0^5\int_{\mathbb{R}^3}U^6dx,$$

and hence by the assumption (M_1), (M_2), we have

$$\begin{aligned} g''(t_0) = &-4M\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx \\ &+ 2t_0^2M'\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\left(\int_{\mathbb{R}^3}|\nabla U|^2dx\right)^2 \\ \le &- 2t_0^2M'\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx \\ < &\ 0 \end{aligned}$$

provided $M'\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)>0$. In case

$$M'\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)=0,$$

then

$$g''(t_0)=-4M\left(t_0^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)\int_{\mathbb{R}^3}|\nabla U|^2dx.$$

Again, we obtain $g''(t_0)<0$. Since t_0 is the unique stationary point of g and $g''(t_0)<0$, which implies that g achieves its maximum at t_0, namely

$$g(t)\le g(t_0),\quad \text{for } t\in[t_1,t_2].$$

Therefore, by (3.13) and (3.15), we have for $t\in[t_1,t_2]$

$$\begin{aligned}I(t\varphi_\varepsilon)&\le\frac{1}{2}\widehat{M}\left(t^2\int_{\mathbb{R}^3}|\nabla U|^2dx\right)-\frac{t^6}{6}\int_{\mathbb{R}^3}U^6dx-C\varepsilon^{\beta+3-\frac{p}{2}}+C\varepsilon\\&\le g(t_0)+C\varepsilon-C\varepsilon^{\beta+3-\frac{p}{2}}\\&<D\end{aligned}$$

for some $C>0$ and $\varepsilon>0$ is small enough. This leads us to the proof. □

Proof of Theorem 1.1. It is easy to verify that I has a mountain pass geometry. Indeed, by condition (M_1), we have

$$\begin{aligned}I(u)&\ge\frac{a}{2}\|u\|^2-\frac{1}{6}\int_{B_r}|u|^6dx-\frac{1}{p}\int_{B_r}\frac{|x|^\beta}{1+|x|^\beta}|u|^pdx\\&\ge\frac{a}{2}\|u\|^2-C_1\|u\|^6-C_2\|u\|^p\ \ (\text{by (2.2), (2.3)}),\end{aligned}$$

where $C_1,C_2>0$. Thus, there exist $\alpha,\rho>0$ and $e\in H$ with $\|e\|>\rho$ such that $I(u)>\alpha$ for all $\|u\|=\rho$, and $I(e)<0$ (since $\lim_{t\to+\infty}I(tu)\to-\infty$). Applying the mountain pass lemma [29], there is a sequence $\{u_n\}\subset H$ such that

$$I(u_n)\to c>0\quad\text{and}\quad I'(u_n)\to 0,$$

where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),$$

and

$$\Gamma=\{\gamma\in C([0,1],H):\gamma(0)=0,\gamma(1)=e\}.$$

From Lemmas 3.4 and 3.5, $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) and there exists $u_* \in H$ such that $u_n \to u_*$ in H. Moreover, u_* is a solution of problem (1.1) and

$$0 < c = \lim_{n\to\infty} I(u_n) = I(u_*) < D.$$

Therefore, we infer that $u_* \not\equiv 0$. By the strong maximum principle, we obtain $u_* > 0$ in B_r.

By the Pohozaev equality, there holds

$$\begin{aligned}\frac{1}{2}M(\|u_*\|^2)\|u_*\|^2 = \frac{1}{2}\int_{B_r} u_*^6 dx + \frac{3}{p}\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_*^p dx \\ -\frac{1}{2}M(\|u_*\|^2)\int_{\partial B_r} (\frac{\partial u_*}{\partial n})^2 \cdot x \cdot n d\delta.\end{aligned}$$

Noting that $x \cdot n \geq 0$, we have

$$\frac{1}{2}M(\|u_*\|^2)\|u_*\|^2 \leq \frac{1}{2}\int_{B_r} u_*^6 dx + \frac{3}{p}\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_*^p dx. \tag{3.16}$$

Since u_* is a positive solution of problem (1.1), it follows that

$$M(\|u_*\|^2)\|u_*\|^2 = \int_{B_r} u_*^6 dx + \int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_*^p dx. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\left(\frac{1}{2} - \frac{3}{p}\right) M(\|u_*\|^2)\|u_*\|^2 \leq \left(\frac{1}{2} - \frac{3}{p}\right)\int_{B_r} u_*^6 dx.$$

By condition (M_1), we deduce that

$$a\|u_*\|^2 \leq M(\|u_*\|^2)\|u_*\|^2 \leq \int_{B_r} u_*^6 dx \leq S^3\|u_*\|^6,$$

that is,

$$\|u_*\|^2 \geq \sqrt{aS^3}.$$

Besides, by (3.17) and condition (M_1)

$$
\begin{aligned}
D > c = I(u_*) - \frac{1}{6}\langle I'(u_*), u_*\rangle
&= \frac{1}{2}\widehat{M}(\|u_*\|^2) - \frac{1}{6}M(\|u_*\|^2)\|u_*\|^2 \\
&\quad + \left(\frac{1}{6} - \frac{1}{p}\right)\int_{B_r} \frac{|x|^\beta}{1+|x|^\beta} u_*^p dx \\
&\geq \frac{a}{4}\|u_*\|^2,
\end{aligned}
$$

which implies that

$$
\sqrt{aS^3} \leq \|u_*\|^2 \leq \frac{4D}{a} < +\infty.
$$

Therefore,

$$
0 \leq \lim_{r\to\infty} u_*(r) \leq \lim_{r\to\infty} \frac{C\|u_*\|}{\sqrt{r}} = 0,
$$

that is, $\lim_{r\to\infty} u_*(r) = 0$. The proof is complete. □

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References

1. Figueiredo, G.M.: Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument. J. Math. Anal. Appl. **401**, 706–713 (2013)
2. Wang, L., Xie, K., Zhang, B.L.: Existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems. J. Math. Anal. Appl. **458**, 361–378 (2018)
3. Figueiredo, G.M., Ikoma, N., Santos Júnior, J.R.: Existence and concentration result for the Kirchhoff type equations with general nonlinearities. Arch. Ration. Mech. Anal **213**, 931–979 (2014)
4. Autuori, G., Fiscella, A., Pucci, P.: Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity. Nonlinear Anal. **125**, 699–714 (2015)
5. Alves, C.O., Boudjeriou, T.: Existence of solution for a class of nonvariational Kirchhoff type problem via dynamical methods. Nonlinear Anal. **197**, 111851 (2020)
6. Fan, H.N.: Positive solutions for a Kirchhoff-type problem involving multiple competitive potentials and critical Sobolev exponent. Nonlinear Anal. **198**, 111869 (2020)
7. Fiscella, A., Pucci, P.: p-fractional Kirchhoff equations involving critical nonlinearities. Nonlinear Anal.: Real World Appl. **35**, 350–378 (2017)
8. Zhang, Y.P., Tang, X.H., Qin, D.D.: Infinitely many solutions for Kirchhoff problems with lack of compactness. Nonlinear Anal. **197**, 111856 (2020)
9. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)
10. He, Y., Li, G.B., Peng, S.J.: Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Adv. Nonlinear Stud. **14**, 441–468 (2014)
11. Fiscella, A., Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. **94**, 156–170 (2014)
12. He, X.M., Zou, W.M.: Existence and concentration of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differ. Equ. **252**, 1813–1834 (2012)

13. Huang, Y.S., Liu, Z., Wu, Y.: On Kirchhoff type equations with critical Sobolev exponent. J. Math. Anal. Appl. **462**, 483–504 (2018)
14. Júnior, J.R.S., Siciliano, G.: Positive solutions for a Kirchhoff problem with vanishing nonlocal term. J. Differ. Equ. **265**, 2034–2043 (2018)
15. Lei, C.Y., Liao, J.F., Tang, C.L.: Multiple positive solutions for Kirchhoff type of problem with singularity and critical exponents. J. Math. Anal. Appl. **421**, 521–538 (2015)
16. Faraci, F., Farkas, C.: On a critical Kirchhoff-type problem. Nonlinear Anal. **192**, 111679 (2020)
17. Xie, Q.L.: Bounded state solution of degenerate Kirchhoff type problem with a critical exponent. J. Math. Anal. Appl. **479**, 1–24 (2019)
18. Naimen, D.: The critical problem of Kirchhoff type elliptic equations in dimension four. J. Differ. Equ. **257**, 1168–1193 (2014)
19. Perera, K., Zhang, Z.T.: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246–255 (2006)
20. Khodabakhshi, M., Vaezpour, S.M., Tavani, M.H.R.: Existence results for a Kirchhoff-type problem with singularity, Miskolc. Math. Notes **22**, 35–362 (2021)
21. Ragusa, M.A.: On weak solutions of ultraparabolic equations. Nonlinear Anal. **47**, 503–511 (2001)
22. Papageorgiou, N., Scapellato, A.: Nonlinear Robin problems with general potential and crossing reaction, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30**, 1–29 (2019)
23. Zhao, F., Liu, Z.Y., Liang, S.H.: Multiple solutions for critical fourth-order elliptic equations of Kirchhoff type. Bull. Malays. Math. Sci. Soc. **44**, 1057–1064 (2021)
24. Marcos do Ó, J., Rul, B., Ubilla, P.: On supercritical Sobolev type inequalities and related elliptic equations. Calc. Var. Partial Differ. Equ. 55–83 (2016)
25. Tintarev, K.: Concentration analysis and Compactness, In: Adimurthi, Sandeep K., Schindler I., Tintarev C. (eds) Concentration Analysis and Applications to PDE.(pp 117-141) Trends in Mathematics. Birkhuäser, Basel
26. Tintarev, K., Fieseler, K.H.: Concentration Compactness: Functional-Analytic Grounds and Applications. Imperial College Press, United Kingdom (2007)
27. Li, Y.Y., Zhu, M.: Uniqueness theorem through the method of moving spheres. Duke Math. J. **80**, 383–417 (1995)
28. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent. Comm. Pure. Appl. Math **36**, 437–477 (1983)
29. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
30. Liu, J.Q., Liu, X.Q., Wang, Z.Q.: Sign-changing solutions for coupled nonlinear Schrödinger equations with critical growth. J. Differ. Equ. **261**, 7194–7236 (2016)