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Arithmetic Properties of 5-Tuple Partitions with 3-Cores

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Abstract

Let $A_{3,5}(n)$ denote the number of 5-tuple partitions of n with 3-cores. We establish some congruences modulo 2, 4, 5, 8 and 10 for $A_{3,5}(n)$ by employing q-series identities. For example, we prove for any prime $p \geq 5$, $\alpha \geq 1$, $\beta \geq 0$ and $n \geq 0$,

$$A_{3,5}\left(2^{2\alpha+2}p^{2\beta+2}n + \frac{(6j+p)\cdot 2^{2\alpha+1}p^{2\beta+1}-5}{3}\right) \equiv 0 \pmod{2},$$

where $1 \leq j \leq p-1$.

Keywords Partition · Congruence · k-tuple · t-core · Ramanujan's theta function

Mathematics Subject Classification Primary 11P83, Secondary 05A17

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers called parts whose sum is n. For convenience, we use the following notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad \text{and} \quad f_k = (q^k;q^k)_\infty.$$

A partition λ of a positive integer n is said to be t-core if it has no hook numbers that are multiples of t. Let $a_t(n)$ denote the number of t-core partitions of n. In [7,Eq.

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(2.1)], the generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1}. \tag{1.1}$$

A k-tuple partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a k-tuple of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that the sum of all the parts equals n. A k-tuple partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n with t-cores means that each λ_i is t-core for $i = 1, 2, \cdots, k$. Let $A_{t,k}(n)$ denote the number of k-tuple partitions of n with t-cores. The generating function of $A_{t,k}(n)$ is

$$\sum_{n=0}^{\infty} A_{t,k}(n)q^n = \frac{f_t^{kt}}{f_1^k}. \tag{1.2}$$

Many authors have studied arithmetic properties of $A_{3,2}(n)$ and obtained some Ramanujan-type congruences. Lin [11] established some infinite families of congruences modulo 4, 5, 7 and 8 for $A_{3,2}(n)$. Based on Lin's study, Xia [18] added proofs of several infinite families of congruences modulo 4, 8 and $\frac{4^k-1}{3}$ $(k \geq 2)$ for $A_{3,2}(n)$. For more results about $A_{3,2}(n)$, see [2, 5, 17, 18, 21].

Wang [16] proved some infinite families of identities and congruences for $A_{3,3}(n)$ by using some identities of q-series. In [17], Wang established some explicit formulas for $A_{3,2}(n)$ and $A_{3,3}(n)$. After that, Chern [5] extended the work of Wang [17] and studied some arithmetic identities of $A_{t,k}(n)$ for $(t, k) = (3, 4), (3, 6), (4, 2), (5, 1), (5, 2)$ and $(7, 1)$ by applying the theory of modular form.

Saikia and Boruah [13] proved some infinite families of congruences modulo 2, 3 for $A_{4,2}(n)$ and $A_{4,3}(n)$. Dasappa [6] discovered a nice congruence modulo 5^α $(\alpha \geq 1)$ for $A_{5,2}(n)$. Saikia and Boruah [14] also studied arithmetic properties of $A_{5,2}(n)$ and proved some congruences modulo 2 and 5. In sequel, they [15] established some Ramanujan-type congruences for $A_{t,k}(n)$ when $(t, k) = (3, 4), (3, 9), (4, 8), (5, 6), (8, 4), (9, 3)$ and $(9, 6)$ by employing q-series identities.

Zou [22] proved some congruences modulo 2 for $A_{t,2}(n)$; t is a prime such that $7 \leq t \leq 23$. In 2020, Naika and Nayaka [12] established some Ramanujan-type congruences modulo 5, 7 and 8 for $A_{t,4}(n)$, $t = 3, 5, 7, 25$.

In this paper, we mainly study arithmetic properties of $A_{3,5}(n)$. Its generating function is given by

$$\sum_{n=0}^{\infty} A_{3,5}(n)q^n = \frac{f_3^{15}}{f_1^5}. \tag{1.3}$$

We establish some results about congruences modulo 2, 4, 5, 8 and 10 for $A_{3,5}(n)$.

To be specific, by using some dissection formulae, we obtain some infinite families of congruences modulo 2 for $A_{3,5}(n)$ as follows.

Theorem 1.1 *For any integer $n \geq 0$, we have*

$$A_{3,5}(8n+5) \equiv 0 \pmod 2. \tag{1.4}$$

and for $\alpha \geq 1$,

$$A_{3,5}\left(2^{2\alpha+3}n + \frac{7 \cdot 2^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod{2}, \tag{1.5}$$

$$A_{3,5}\left(2^{2\alpha+4}n + \frac{13 \cdot 2^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod{2}. \tag{1.6}$$

Theorem 1.2 *For any prime* $p \geq 5$, $\alpha \geq 1$, $\beta \geq 0$, *and* $n \geq 0$,

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta}n + \frac{2^{2\alpha+1}p^{2\beta} - 5}{3}\right) q^n \equiv f(-q^4) \pmod{2}. \tag{1.7}$$

We deduce the following infinite families of congruences with two parameters α, β modulo 2 for $A_{3,5}(n)$.

Corollary 1.1 *For any prime* $p \geq 5$, $\alpha \geq 1$, $\beta \geq 0$, *if* n *cannot be represented as* $2k(3k+1)$ *for some integer* k, *then*

$$A_{3,5}\left(2^{2\alpha+2}p^{2\beta}n + \frac{2^{2\alpha+1}p^{2\beta} - 5}{3}\right) \equiv 0 \pmod{2}. \tag{1.8}$$

and for any integer $n \geq 0$,

$$A_{3,5}\left(2^{2\alpha+2}p^{2\beta+2}n + \frac{(6j+p) \cdot 2^{2\alpha+1}p^{2\beta+1} - 5}{3}\right) \equiv 0 \pmod{2}, \quad j \in \{1, 2 \cdots p-1\}. \tag{1.9}$$

$$A_{3,5}\left(7^2 \cdot 2^{2\alpha+2}p^{2\beta}n + \frac{(42k+7) \cdot 2^{2\alpha+1}p^{2\beta} - 5}{3}\right) \equiv 0 \pmod{2}, \quad k \in \{0, 2, 3, 4, 5, 6\}. \tag{1.10}$$

and

$$A_{3,5}\left(13^2 \cdot 2^{2\alpha+2}p^{2\beta}n + \frac{(78k+13) \cdot 2^{2\alpha+1}p^{2\beta} - 5}{3}\right) \equiv 0 \pmod{2},$$
$$k \in \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}. \tag{1.11}$$

We find the following congruences modulo 5 and 10 for $A_{3,5}(n)$ hold.

Theorem 1.3 *For* $n \geq 0$, *we have*

$$A_{3,5}(4n+3) \equiv 0 \pmod{10}, \tag{1.12}$$

$$A_{3,5}(5n+k) \equiv 0 \pmod{5}, \quad k \in \{1, 2, 3, 4\}. \tag{1.13}$$

$$A_{3,5}(20n+15) \equiv 0 \pmod{5}, \tag{1.14}$$

and for $\alpha \geq 0$, *we get*

$$A_{3,5}\left(5 \cdot 2^{2\alpha} n + \frac{5 \cdot 2^{2\alpha} - 5}{3}\right) \equiv A_{3,5}(5n) \pmod{5}, \tag{1.15}$$

$$A_{3,5}\left(5 \cdot 2^{2\alpha+2} n + \frac{5^2 \cdot 2^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod{5}. \tag{1.16}$$

We also establish the following congruences modulo 4, 8 for $A_{3,5}(n)$.

Theorem 1.4 *For* $\alpha \geq 2$ *and* $n \geq 0$, *we have*

$$A_{3,5}\left(2^{2\alpha+1} n + \frac{5 \cdot 2^{2\alpha} - 5}{3}\right) \equiv 0 \pmod{4}. \tag{1.17}$$

Let $p \geq 3$ be a prime and a be an integer. The Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo} p \text{ and } a \not\equiv 0 \pmod{p}, \\ 0, & \text{if } a \equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo} p. \end{cases}$$

Theorem 1.5 *For any prime* $p \geq 5$ *such that* $\left(\frac{-6}{p}\right) = -1$, $\alpha \geq 0$, *and* $n \geq 0$,

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha} n + \frac{14 \cdot p^{2\alpha} - 5}{3}\right) q^n \equiv 2f(-q)\psi(q^2) \pmod{4}. \tag{1.18}$$

Corollary 1.2 *For any prime* $p \geq 5$ *such that* $\left(\frac{-6}{p}\right) = -1$, $\alpha \geq 0$, *if* n *cannot be represented as the sum of a pentagonal number and twice a triangular number, then*

$$A_{3,5}\left(16 \cdot p^{2\alpha} n + \frac{14 \cdot p^{2\alpha} - 5}{3}\right) \equiv 0 \pmod{4}. \tag{1.19}$$

and for any integer $n \geq 0$,

$$A_{3,5}\left(16 \cdot p^{2\alpha+2} n + \frac{(48j + 14p) \cdot p^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod{4}, \tag{1.20}$$

where $1 \leq j \leq p - 1$.

Theorem 1.6 *For any prime* $p \geq 5$ *such that* $\left(\frac{-18}{p}\right) = -1$, $\alpha \geq 0$, *and* $n \geq 0$,

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha} n + \frac{38 \cdot p^{2\alpha} - 5}{3}\right) q^n \equiv 2f(-q)\psi(q^6) \pmod{4}. \tag{1.21}$$

Corollary 1.3 *For any prime $p \geq 5$ such that $\left(\frac{-18}{p}\right) = -1$, $\alpha \geq 0$, if n cannot be represented as the sum of a pentagonal number and six times a triangular number, then*

$$A_{3,5}\left(16 \cdot p^{2\alpha} n + \frac{38 \cdot p^{2\alpha} - 5}{3}\right) \equiv 0 \pmod 4. \tag{1.22}$$

and for any integer $n \geq 0$,

$$A_{3,5}\left(16 \cdot p^{2\alpha+2} n + \frac{(48j + 38p) \cdot p^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod 4, \tag{1.23}$$

where $1 \leq j \leq p - 1$.

Theorem 1.7 *For $n \geq 0$, we have*

$$A_{3,5}(4n + 2) \equiv 0 \pmod 4, \tag{1.24}$$

$$A_{3,5}(16n + 13) \equiv 0 \pmod 8. \tag{1.25}$$

Theorem 1.8 *For any prime $p \geq 5$ such that $\left(\frac{-9}{p}\right) = -1$, $\alpha \geq 0$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} A_{3,5}\left(2 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha} - 5}{3}\right) q^n \equiv (-1)^{\alpha\left(\frac{\pm p - 1}{6}\right)} f(-q^2)\psi(q^6) \pmod 4. \tag{1.26}$$

Corollary 1.4 *For any prime $p \geq 5$ such that $\left(\frac{-9}{p}\right) = -1$, $\alpha \geq 0$, if n cannot be represented as the sum of a pentagonal number and three times a triangular number, then*

$$A_{3,5}\left(4 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha} - 5}{3}\right) \equiv 0 \pmod 4. \tag{1.27}$$

and for any integer $n \geq 0$,

$$A_{3,5}\left(2 \cdot p^{2\alpha+2} n + \frac{(6j + 5p) \cdot p^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod 4, \tag{1.28}$$

where $1 \leq j \leq p - 1$.

Theorem 1.9 *For any prime $p \geq 5$ such that $\left(\frac{-9}{p}\right) = -1$, $\alpha \geq 0$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} A_{3,5}\left(8 \cdot p^{2\alpha} n + \frac{20 \cdot p^{2\alpha} - 5}{3}\right) q^n \equiv 6(-1)^{\alpha\left(\frac{\pm p - 1}{6}\right)} f(-q^2)\psi(q^6) \pmod 8. \tag{1.29}$$

Corollary 1.5 *For any prime $p \geq 5$ such that $\left(\frac{-9}{p}\right) = -1$, $\alpha \geq 0$, if n cannot be represented as the sum of a pentagonal number and three times a triangular number, then*

$$A_{3,5}\left(16 \cdot p^{2\alpha} n + \frac{20 \cdot p^{2\alpha} - 5}{3}\right) \equiv 0 \pmod 8. \tag{1.30}$$

and for any integer $n \geq 0$,

$$A_{3,5}\left(8 \cdot p^{2\alpha+2} n + \frac{(24j + 20p) \cdot p^{2\alpha+1} - 5}{3}\right) \equiv 0 \pmod 8, \tag{1.31}$$

where $1 \leq j \leq p - 1$.

This paper is organized as follows. In Sect. 2, we shall prove some theorems about congruences for $A_{3,5}(n)$ modulo 2, 5 and 10. In Sect. 3, we give the proofs of remaining theorems about congruences modulo 4, 8 for $A_{3,5}(n)$.

2 Proofs of Theorems 1.1–1.3 and Corollary 1.1

Proof of Theorem 1.1. In order to prove Theorem 1.1, we first prove the following lemma.

Lemma 2.1 *For $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha} n + \frac{2^{2\alpha+1} - 5}{3}\right) q^n \equiv \frac{f_3^6}{f_1^2} \pmod 2. \tag{2.1}$$

Proof Hirschhorn, Garvan and Borwein [9] proved that

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \tag{2.2}$$

By the binomial theorem, for any positive integer k and any prime p,

$$f_1^{p^k} \equiv f_p^{p^{k-1}} \pmod{p^k}. \tag{2.3}$$

Substituting (2.2) into (1.3) and employing (2.3), we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(n) q^n &= \frac{f_3^{12}}{f_1^4} \frac{f_3^3}{f_1} \equiv \frac{f_6^6}{f_2^2} \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}\right) \\ &\equiv \frac{f_4^3 f_6^8}{f_2^4 f_{12}} + q \frac{f_6^6 f_{12}^3}{f_2^2 f_4} \pmod 2. \end{aligned} \tag{2.4}$$

Extracting the terms involving q^{2n+1} from both sides of (2.4) , dividing by q and replacing q^2 by q, then employing (2.3), we get

$$\sum_{n=0}^{\infty} A_{3,5}(2n+1)q^n \equiv \frac{f_3^6 f_6^3}{f_1^2 f_2} \equiv \frac{f_6^6}{f_2^2} \pmod{2}. \tag{2.5}$$

Hence,

$$\sum_{n=0}^{\infty} A_{3,5}(4n+1)q^n \equiv \frac{f_3^6}{f_1^2} \pmod{2}, \tag{2.6}$$

which is the case $\alpha = 1$ of (2.1).

Suppose that (2.1) holds for $\alpha \geq 1$. Utilizing (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha}n + \frac{2^{2\alpha+1}-5}{3}\right)q^n &\equiv \frac{f_3^6}{f_1^2} \equiv \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)^2 \\ &\equiv \frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + q^2\frac{f_{12}^6}{f_4^2} \pmod{2}. \end{aligned} \tag{2.7}$$

Hence,

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+1}n + \frac{2^{2\alpha+1}-5}{3}\right)q^n \equiv \frac{f_2^6 f_3^4}{f_1^4 f_6^2} + q\frac{f_6^6}{f_2^2} \equiv f_8 + q\frac{f_6^6}{f_2^2} \pmod{2}. \tag{2.8}$$

Extracting the terms involving q^{2n+1} from both sides of (2.8), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}n + \frac{2^{2\alpha+3}-5}{3}\right)q^n \equiv \frac{f_3^6}{f_1^2} \pmod{2}, \tag{2.9}$$

which is the case $\alpha + 1$ of (2.1). The proof of Lemma 2.1 is completed by induction on α.

Now, we turn to prove Theorem 1.1. It follows from (2.3) that

$$\sum_{n=0}^{\infty} A_{3,5}(4n+1)q^n \equiv \frac{f_3^6}{f_1^2} \equiv \frac{f_6^3}{f_2} \pmod{2}, \tag{2.10}$$

which implies (1.4).

Extracting the terms involving q^{2n} from both sides of (2.8) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}n + \frac{2^{2\alpha+1}-5}{3}\right)q^n \equiv f_4 \pmod{2}. \tag{2.11}$$

which implies that

$$A_{3,5}\left(2^{2\alpha+2}(2n+1)+\frac{2^{2\alpha+1}-5}{3}\right)\equiv 0 \pmod 2, \tag{2.12}$$

$$A_{3,5}\left(2^{2\alpha+2}(4n+2)+\frac{2^{2\alpha+1}-5}{3}\right)\equiv 0 \pmod 2. \tag{2.13}$$

We thus obtain (1.5) and (1.6). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 and Corollary 1.1.
We first recall that Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1. \tag{2.14}$$

Two important cases of $f(a,b)$ are the theta functions $\psi(q)$ and $f(-q)$ [1,p.36. Entry 22], which are given by

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}=\frac{f_2^2}{f_1}, \tag{2.15}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}=f_1. \tag{2.16}$$

Cui and Gu [4] proved the following p-dissection identities for $\psi(q)$ and $f(-q)$.

Lemma 2.2 [4,Theorem 2.1] *For any odd prime* p,

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}), \tag{2.17}$$

Furthermore, for $0\le k\le \frac{p-3}{2}$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod p.$$

Lemma 2.3 [4,Theorem 2.2] *For any prime* $p\ge 5$,

$$f(-q)=\sum_{k=-\frac{p-1}{2},\, k\neq\frac{\pm p-1}{6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \tag{2.18}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ *and* $k \neq \frac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

It follows from (2.11) and (2.16) that

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}n + \frac{2^{2\alpha+1}-5}{3}\right) q^n \equiv f(-q^4) \pmod 2, \tag{2.19}$$

which is the case $\beta = 0$ of (1.7).

Now, suppose that (1.7) holds for $\beta \geq 0$. Invoking Lemma 2.3, we get that for any prime $p \geq 5$,

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta}(pn + \frac{p^2-1}{6}) + \frac{2^{2\alpha+1}p^{2\beta}-5}{3}\right) q^n$$
$$= \sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta+1}n + \frac{2^{2\alpha+1}p^{2\beta+2}-5}{3}\right) q^n \equiv f(-q^{4p}) \pmod 2, \tag{2.20}$$

which implies that

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta+2}n + \frac{2^{2\alpha+1}p^{2\beta+2}-5}{3}\right) q^n \equiv f(-q^4) \pmod 2, \tag{2.21}$$

which is the case $\beta + 1$ of (1.7). The proof of Theorem 1.2 is completed by induction on β.

Combining (1.7) and (2.16), we have

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta}n + \frac{2^{2\alpha+1}p^{2\beta}-5}{3}\right) q^n \equiv \sum_{k=-\infty}^{\infty} (-1)^k q^{2k(3k+1)} \pmod 2, \tag{2.22}$$

which implies (1.8).

Applying (2.20) yields that for $j = 1, 2, \ldots, p-1$,

$$A_{3,5}\left(2^{2\alpha+2}p^{2\beta+1}(pn+j)+\frac{2^{2\alpha+1}p^{2\beta+2}-5}{3}\right)\equiv 0 \pmod 2,$$

which implies (1.9).

Following Hirschhorn [10], for a positive integer k and a power series $\sum_{n=0}^{\infty} a(n)q^n$, we introduce an operator H_k, which acts on series of (positive and negative) powers of a single variable and picks out those terms in which the power is congruent to 0 modulo k. That is

$$H_k\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \sum_{n=0}^{\infty} a(kn)q^{kn}. \tag{2.23}$$

Garvan [8] proved the following results. Let

$$\xi = \frac{f(-q)}{q^2 f(-q^{49})} \quad \text{and} \quad T = \frac{f(-q^7)^4}{q^7 f(-q^{49})^4}. \tag{2.24}$$

We have

$$\begin{aligned} &H_7(\xi) = -1, \qquad H_7(\xi^2) = 1, \qquad H_7(\xi^3) = -7,\\ &H_7(\xi^4) = -4T-7, \quad H_7(\xi^5) = 10T+49, \quad H_7(\xi^6) = 49. \end{aligned} \tag{2.25}$$

Applying (2.3) and (2.24) in Theorem 1.2, for any prime $p \geq 5$, $\alpha \geq 1$ and $\beta \geq 0$, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta}n+\frac{2^{2\alpha+1}p^{2\beta}-5}{3}\right)q^n \equiv f(-q)^4 \equiv q^8 f(-q^{49})^4\xi^4 \pmod 2. \tag{2.26}$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}p^{2\beta}(7n+1)+\frac{2^{2\alpha+1}p^{2\beta}-5}{3}\right)q^{7n+1} &\equiv q^8 f(-q^{49})^4 H_7(\xi^4)\\ &\equiv q^8 f(-q^{49})^4 \pmod 2.\end{aligned} \tag{2.27}$$

Furthermore,

$$\sum_{n=0}^{\infty} A_{3,5}\left(7\cdot 2^{2\alpha+2}p^{2\beta}n+\frac{7\cdot 2^{2\alpha+1}p^{2\beta}-5}{3}\right)q^n \equiv qf(-q^7)^4 \pmod 2. \tag{2.28}$$

Therefore, for $k \in \{0, 2, 3, 4, 5, 6\}$, we get

$$A_{3,5}\left(7\cdot 2^{2\alpha+2}p^{2\beta}(7n+k)+\frac{7\cdot 2^{2\alpha+1}p^{2\beta}-5}{3}\right)\equiv 0 \pmod 2, \tag{2.29}$$

which implies (1.10).

Hirschhorn [11] stated the following results. Denote

$$\zeta = \frac{f(-q)}{q^7 f(-q^{169})} \quad and \quad S = \frac{f(-q^{13})^2}{q^{13} f(-q^{169})^2}. \tag{2.30}$$

We have

$$\begin{aligned} H_{13}(\zeta) &= 1, & H_{13}(\zeta^2) &= -2S - 1, \\ H_{13}(\zeta^3) &= 13, & H_{13}(\zeta^4) &= 2S^2 - 13. \end{aligned} \tag{2.31}$$

Applying (2.3) and (2.30) in Theorem 1.2, for any prime $p \geq 5$, $\alpha \geq 1$ and $\beta \geq 0$, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2} p^{2\beta} n + \frac{2^{2\alpha+1} p^{2\beta} - 5}{3}\right) q^n \equiv f(-q)^4 \equiv q^{28} f(-q^{169})^4 \zeta^4 \pmod 2. \tag{2.32}$$

Then,

$$\begin{aligned} \sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2} p^{2\beta}(13n+2) + \frac{2^{2\alpha+1} p^{2\beta} - 5}{3}\right) q^{13n+2} &\equiv q^{28} f(-q^{169})^4 H_{13}(\zeta^4) \\ &\equiv q^{28} f(-q^{169})^4 \pmod 2. \end{aligned} \tag{2.33}$$

Furthermore, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(13 \cdot 2^{2\alpha+2} p^{2\beta} n + \frac{13 \cdot 2^{2\alpha+1} p^{2\beta} - 5}{3}\right) q^n \equiv q^2 f(-q^{13})^4 \pmod 2. \tag{2.34}$$

Therefore, for $k \in \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

$$A_{3,5}\left(13 \cdot 2^{2\alpha+2} p^{2\beta}(13n+k) + \frac{13 \cdot 2^{2\alpha+1} p^{2\beta} - 5}{3}\right) \equiv 0 \pmod 2, \tag{2.35}$$

which implies (1.11). This completes the proof of Theorem 1.2 and Corollary 1.1.

Proof of Theorem 1.3. Using (2.2) in (1.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} A_{3,5}(n) q^n &= \frac{f_3^{15}}{f_1^5} = \left(\frac{f_3^3}{f_1}\right)^5 = \left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)^5 \\ &\equiv \frac{f_4^{15} f_6^{10}}{f_2^{10} f_{12}^5} + 5q\frac{f_4^{11} f_6^8}{f_2^8 f_{12}} + 5q^4 \frac{f_6^2 f_{12}^{11}}{f_2^2 f_4} + q^5 \frac{f_{12}^{15}}{f_4^5} \pmod{10}. \end{aligned} \tag{2.36}$$

Extracting the terms involving q^{2n+1} from both sides of (2.36), dividing by q and replacing q^2 by q, then employing (2.3), we get

$$\sum_{n=0}^{\infty} A_{3,5}(2n+1)q^n \equiv 5\frac{f_2^{11}f_3^8}{f_1^8 f_6} + q^2\frac{f_6^{15}}{f_2^5} \equiv 5f_2^7 f_6^3 + q^2\frac{f_6^{15}}{f_2^5} \pmod{10}, \quad (2.37)$$

which implies (1.12).

Utilizing (2.3) in (1.3), we have

$$\sum_{n=0}^{\infty} A_{3,5}(n)q^n = \frac{f_3^{15}}{f_1^5} \equiv \frac{f_{15}^3}{f_5} \pmod{5}. \quad (2.38)$$

This yields that for $k \in \{1, 2, 3, 4\}$, $A_{3,5}(5n+k) \equiv 0 \pmod{5}$, which is (1.13).

Combining (2.2) and (2.38), we have that

$$\sum_{n=0}^{\infty} A_{3,5}(5n)q^n \equiv \frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4} \pmod{5}. \quad (2.39)$$

Extracting the terms involving q^{2n+1} from both sides of (2.39), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} A_{3,5}(10n+5)q^n \equiv \frac{f_6^3}{f_2} \pmod{5}. \quad (2.40)$$

Then, we have

$$A_{3,5}(20n+15) \equiv 0 \pmod{5}, \quad (2.41)$$

which is (1.14). And we obtain

$$\sum_{n=0}^{\infty} A_{3,5}(20n+5)q^n \equiv \frac{f_3^3}{f_1} \pmod{5}. \quad (2.42)$$

In view of (2.39) and (2.42), we have

$$A_{3,5}(20n+5) \equiv A_{3,5}(5n) \pmod{5}. \quad (2.43)$$

Utilizing (2.43) and by mathematical induction on α, we get (1.15). Combining (2.41) and (1.15), we thus arrive at (1.16). This completes the proof of Theorem 1.3. □

3 Proofs of Theorems 1.4–1.9 and Corollaries 1.2–1.5

In this section, we prove the remaining theorems and corollaries about congruences modulo 4, 8 for $A_{3,5}(n)$.

Proof of Theorem 1.4. In order to prove Theorem 1.4, we first prove the following lemma.

Lemma 3.1 *For $\alpha \geq 2$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha}n + \frac{2^{2\alpha+1}-5}{3}\right)q^n \equiv -\frac{f_6^3}{f_2} \pmod 4. \tag{3.1}$$

Proof Baruah and Ojah [3] established that

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{3.2}$$

Using (3.2) in (1.3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(n)q^n &= \frac{f_3^{15}}{f_1^5} = \frac{f_3^{16}}{f_1^4}\frac{1}{f_1 f_3} \\ &\equiv \frac{f_6^8}{f_2^2}\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right) \\ &\equiv \frac{f_6^4 f_8^2 f_{12}^5}{f_2^4 f_4 f_{24}^2} + q\frac{f_4^5 f_6^6 f_{24}^2}{f_2^6 f_8^2 f_{12}} \pmod 4. \end{aligned}\tag{3.3}$$

Xia and Yao [19] proved the following identity

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{3.4}$$

Extracting the terms involving q^{2n+1} from both sides of (3.3), dividing by q and replacing q^2 by q, then employing (3.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(2n+1)q^n &\equiv \frac{f_2^5 f_3^6 f_{12}^2}{f_1^6 f_4^2 f_6} \equiv \frac{f_2^5 f_{12}^2}{f_4^2 f_6}\left(\frac{f_3^2}{f_1^2}\right)^3 \\ &\equiv \frac{f_2^5 f_{12}^2}{f_4^2 f_6}\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right)^3 \\ &\equiv \frac{f_4^{10} f_6^2 f_{12}^8}{f_2^{10} f_8^3 f_{24}^3} + 2q\frac{f_4^7 f_6^3 f_{12}^5}{f_2^9 f_8 f_{24}} \pmod 4. \end{aligned}\tag{3.5}$$

Extracting the terms involving q^{2n} from both sides of (3.5), replacing q^2 by q and using (3.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(4n+1)q^n &\equiv \frac{f_2^{10} f_3^2 f_6^8}{f_1^{10} f_4^3 f_{12}^3} \equiv \frac{f_2^6 f_6^8}{f_4^3 f_{12}^3} \frac{f_3^2}{f_1^2} \\ &\equiv \frac{f_2^6 f_6^8}{f_4^3 f_{12}^3} \left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \right) \\ &\equiv \frac{f_2 f_4 f_6^9}{f_8 f_{12} f_{24}} + 2q \frac{f_2^2 f_6^{10} f_8 f_{24}}{f_4^2 f_{12}^4} \pmod 4. \end{aligned} \tag{3.6}$$

Xia and Yao [20] proved the following 2-dissection formula.

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{3.7}$$

Extracting the terms involving q^{2n} from both sides of (3.6), replacing q^2 by q and using (3.7), we get

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(8n+1)q^n &\equiv \frac{f_1 f_2 f_3^9}{f_4 f_6 f_{12}} \equiv \frac{f_2 f_6^3}{f_4 f_{12}} \cdot f_1 f_3 \\ &\equiv \frac{f_2 f_6^3}{f_4 f_{12}} \cdot \left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2} \right) \\ &\equiv \frac{f_2^2 f_6^2 f_8^2 f_{12}^3}{f_4^3 f_{24}^2} - q \frac{f_4^3 f_6^4 f_{24}^2}{f_8^2 f_{12}^3} \pmod 4. \end{aligned} \tag{3.8}$$

Extracting the terms involving q^{2n+1} from both sides of (3.8), dividing by q and replacing q^2 by q, then using (2.3), we get

$$\sum_{n=0}^{\infty} A_{3,5}(16n+9)q^n \equiv -\frac{f_2^3 f_3^4 f_{12}^2}{f_4^2 f_6^3} \equiv -\frac{f_6^3}{f_2} \pmod 4, \tag{3.9}$$

which is the case $\alpha = 2$ of (3.1).

Suppose that (3.1) is true for $\alpha \geq 2$. According to (3.1) and employing (2.2), we have

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+1}n + \frac{2^{2\alpha+1}-5}{3}\right) q^n \equiv -\frac{f_3^3}{f_1} \equiv -\frac{f_4^3 f_6^2}{f_2^2 f_{12}} - q \frac{f_{12}^3}{f_4} \pmod 4. \tag{3.10}$$

Extracting the terms involving q^{2n+1} from both sides of (3.10), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} A_{3,5}\left(2^{2\alpha+2}n + \frac{2^{2\alpha+3}-5}{3}\right)q^n \equiv -\frac{f_6^3}{f_2} \pmod 4, \tag{3.11}$$

which is the case $\alpha+1$ of (3.1). The proof of Lemma 3.1 is completed by induction on α.

Now, we turn to prove Theorem 1.4. According to (3.1), we have

$$A_{3,5}\left(2^{2\alpha}(2n+1) + \frac{2^{2\alpha+1}-5}{3}\right) \equiv 0 \pmod 4, \tag{3.12}$$

which is (1.17). This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5 and Corollary 1.2. Hirschhorn, Garvan and Borwein [9] proved that

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{3.13}$$

Extracting the terms involving q^{2n+1} from both sides of (3.5), dividing by q and replacing q^2 by q, then using (3.13), we get

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(4n+3)q^n &\equiv 2\frac{f_2^7 f_6^5}{f_4 f_{12}}\frac{f_3^3}{f_1^9} \equiv 2\frac{f_2^7 f_6^5}{f_4 f_{12}}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)^3 \\ &\equiv 2\frac{f_4^{17} f_6^{14}}{f_2^{20} f_{12}^7} + 2q\frac{f_4^{13} f_6^{12}}{f_2^{18} f_{12}^3} + 2q^2\frac{f_4^9 f_6^{10} f_{12}}{f_2^{16}} + 2q^3\frac{f_4^5 f_6^8 f_{12}^5}{f_2^{14}} \pmod 4.\end{aligned} \tag{3.14}$$

Extracting the terms involving q^{2n} from both sides of (3.14), replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} A_{3,5}(8n+3)q^n \equiv 2\frac{f_2^{17} f_3^{14}}{f_1^{20} f_6^7} + 2q\frac{f_2^9 f_3^{10} f_6}{f_1^{16}} \pmod 4. \tag{3.15}$$

By (2.3), we see that

$$\frac{f_2^{17} f_3^{14}}{f_1^{20} f_6^7} \equiv f_2 f_4^3 \pmod 2, \tag{3.16}$$

$$\frac{f_2^9 f_3^{10} f_6}{f_1^{16}} \equiv f_2 f_{12}^3 \pmod 2. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we deduce that

$$\sum_{n=0}^{\infty} A_{3,5}(8n+3)q^n \equiv 2f_2f_4^3 + 2qf_2f_{12}^3 \pmod 4, \tag{3.18}$$

which yields

$$\sum_{n=0}^{\infty} A_{3,5}(16n+3)q^n \equiv 2f_1f_2^3 \pmod 4, \tag{3.19}$$

$$\sum_{n=0}^{\infty} A_{3,5}(16n+11)q^n \equiv 2f_1f_6^3 \pmod 4. \tag{3.20}$$

Using (2.3) in (2.15), we obtain

$$\psi(q) \equiv f_1^3 \pmod 4. \tag{3.21}$$

Furthermore, in view of (2.16) and (3.21), we get

$$\sum_{n=0}^{\infty} A_{3,5}(16n+3)q^n \equiv 2f(-q)\psi(q^2) \pmod 4, \tag{3.22}$$

$$\sum_{n=0}^{\infty} A_{3,5}(16n+11)q^n \equiv 2f(-q)\psi(q^6) \pmod 4. \tag{3.23}$$

By (3.22), we find (1.18) holds for $\alpha = 0$. Assume that (1.18) holds for $\alpha \geq 0$. Employing Lemmas 2.2 and 2.3 in (1.18), we consider the congruence

$$\frac{3k^2+k}{2} + 2\cdot\frac{m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p, \tag{3.24}$$

where $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $0 \leq m \leq \frac{p-3}{2}$.

The congruence (3.24) is equivalent to

$$(6k+1)^2 + 6(2m+1)^2 \equiv 0 \pmod p. \tag{3.25}$$

For $\left(\frac{-6}{p}\right) = -1$, the congruence (3.25) holds if and only if $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$.

Extracting the terms containing $q^{pn+\frac{7p^2-7}{24}}$ from both sides of (1.18), dividing by $q^{\frac{7p^2-7}{24}}$ and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} A_{3,5}\left(16\cdot p^{2\alpha+1}n + \frac{14\cdot p^{2\alpha+2}-5}{3}\right)q^n \equiv 2f(-q^p)\psi(q^{2p}) \pmod 4. \tag{3.26}$$

This implies that

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha+2}n + \frac{14 \cdot p^{2\alpha+2} - 5}{3}\right) q^n \equiv 2f(-q)\psi(q^2) \pmod 4, \quad (3.27)$$

which is the case $\alpha + 1$ of (1.18). The proof of Theorem 1.5 is completed by induction on α.

Combining (1.18), (2.15) and (2.16), we obtain

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha}n + \frac{14 \cdot p^{2\alpha} - 5}{3}\right) q^n \equiv 2 \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} q^{\frac{n(3n+1)}{2}+k(k+1)} \pmod 4, \quad (3.28)$$

which implies (1.19).

Applying (3.26) yields that for $j = 1, 2, \ldots, p-1$,

$$A_{3,5}\left(16 \cdot p^{2\alpha+1}(pn + j) + \frac{14 \cdot p^{2\alpha+2} - 5}{3}\right) q^n \equiv 0 \pmod 4, \quad (3.29)$$

which implies (1.20). This completes the proof of Theorem 1.5 and Corollary 1.2. □

Proof of Theorem 1.6 and Corollary 1.3. It follows from (3.23) that (1.21) holds for $\alpha = 0$. Suppose that (1.21) holds for $\alpha \geq 0$. Employing Lemma 2.2 and Lemma 2.3 in (1.21), we consider the congruence

$$\frac{3k^2 + k}{2} + 6 \cdot \frac{m^2 + m}{2} \equiv \frac{19p^2 - 19}{24} \pmod p, \quad (3.30)$$

where $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $0 \leq m \leq \frac{p-3}{2}$.

The congruence (3.30) is equivalent to

$$(6k + 1)^2 + 18(2m + 1)^2 \equiv 0 \pmod p. \quad (3.31)$$

For $\left(\frac{-18}{p}\right) = -1$, the congruence (3.31) holds if and only if $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$.

Extracting the terms containing $q^{pn+\frac{19p^2-19}{24}}$ from both sides of (1.21), dividing by $q^{\frac{19p^2-19}{24}}$ and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha+1}n + \frac{38 \cdot p^{2\alpha+2} - 5}{3}\right) q^n \equiv 2f(-q^p)\psi(q^{6p}) \pmod 4. \quad (3.32)$$

This implies that

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha+2}n + \frac{38 \cdot p^{2\alpha+2} - 5}{3}\right) q^n \equiv 2f(-q)\psi(q^6) \pmod 4, \quad (3.33)$$

which is the case $\alpha+1$ of (1.21). The proof of Theorem 1.6 is completed by induction on α.

Combining (1.21), (2.15) and (2.16), we have

$$\sum_{n=0}^{\infty} A_{3,5}\left(16\cdot p^{2\alpha}n+\frac{38\cdot p^{2\alpha}-5}{3}\right)q^n \equiv 2\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} q^{\frac{n(3n+1)}{2}+3k(k+1)} \pmod 4, \tag{3.34}$$

which implies (1.22).

Applying (3.32) yields that for $1\le j\le p-1$,

$$A_{3,5}\left(16\cdot p^{2\alpha+1}(pn+j)+\frac{38\cdot p^{2\alpha+2}-5}{3}\right)\equiv 0 \pmod 4, \tag{3.35}$$

which implies (1.23). This completes the proof of Theorem 1.6 and Corollary 1.3. □

Proof of Theorem 1.7 Xia and Yao [19] proved the following 2-dissection formulae.

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{3.36}$$

$$\frac{1}{f_1^2 f_3^2} = \frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}. \tag{3.37}$$

In view of (3.36), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(n)q^n &= \frac{f_3^{16}}{f_1^8}\frac{f_1^3}{f_3} \equiv \frac{f_6^8}{f_2^4}\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right)\\ &\equiv \frac{f_6^8 f_4^3}{f_2^4 f_{12}} - 3q\frac{f_6^6 f_{12}^3}{f_2^2 f_4} \pmod 8.\end{aligned} \tag{3.38}$$

Utilizing (2.3), we get

$$\sum_{n=0}^{\infty} A_{3,5}(2n)q^n \equiv \frac{f_3^8 f_2^3}{f_1^4 f_6} \equiv f_2 f_6^3 \pmod 4, \tag{3.39}$$

which implies (1.24).

Extracting the terms involving q^{2n+1} from both sides of (3.38), dividing by q and replacing q^2 by q, then using (3.37), we get

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(2n+1)q^n &\equiv -3\frac{f_3^6 f_6^3}{f_1^2 f_2} \equiv 5\frac{f_3^6 f_6^3}{f_1^2 f_2} \equiv 5\frac{f_3^8 f_6^3}{f_2 f_1^2 f_3^2} \equiv 5\frac{f_6^7}{f_2}\frac{1}{f_1^2 f_3^2} \\ &\equiv 5\frac{f_6^7}{f_2}\left(\frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}\right) \\ &\equiv 5\frac{f_6^2 f_8^5 f_{24}^5}{f_2^6 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_6 f_{12}^4}{f_2^7} + 4q^4\frac{f_4^2 f_6^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^6 f_8 f_{24}} \pmod 8. \end{aligned} \tag{3.40}$$

In view of (2.3), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A_{3,5}(4n+1)q^n &\equiv 5\frac{f_3^2 f_4^5 f_{12}^5}{f_1^6 f_8^2 f_{24}^2} + 4q^2\frac{f_2^2 f_3^2 f_6^2 f_8^4 f_{24}^2}{f_1^6 f_4 f_{12}} \\ &\equiv 5\frac{f_3^2 f_4^5 f_{12}^5}{f_1^6 f_8^2 f_{24}^2} + 4q^2\frac{f_6 f_8^4 f_{24}^2}{f_2 f_4} \pmod 8. \end{aligned} \tag{3.41}$$

Using (3.13), we see that

$$\begin{aligned} 5\frac{f_3^2 f_4^5 f_{12}^5}{f_1^6 f_8^2 f_{24}^2} &\equiv 5\frac{f_4^5 f_{12}^5}{f_8^2 f_{24}^2}\left(\frac{f_3}{f_1^3}\right)^2 \equiv 5\frac{f_4^5 f_{12}^5}{f_8^2 f_{24}^2}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)^2 \\ &\equiv 5\frac{f_4^5 f_{12}^5}{f_8^2 f_{24}^2}\left(\frac{f_4^{12} f_6^6}{f_2^{18} f_{12}^4} + 6q\frac{f_4^8 f_6^4}{f_2^{16}} + 9q^2\frac{f_4^4 f_6^2 f_{12}^4}{f_2^{14}}\right) \\ &\equiv 5\frac{f_4^{17} f_6^6 f_{12}}{f_2^{18} f_8^2 f_{24}^2} + 6q\frac{f_4^{13} f_6^4 f_{12}^5}{f_2^{16} f_8^2 f_{24}^2} + 5q^2\frac{f_4^9 f_6^2 f_{12}^9}{f_2^{14} f_8^2 f_{24}^2} \pmod 8. \end{aligned} \tag{3.42}$$

Combining (3.41) and (3.42), extracting the terms involving q^{2n+1} from both sides of (3.41), dividing by q and replacing q^2 by q, we find

$$\sum_{n=0}^{\infty} A_{3,5}(8n+5)q^n \equiv 6\frac{f_2^{13} f_3^4 f_6^5}{f_1^{16} f_4^2 f_{12}^2} \equiv 6 f_2 f_6^3 \pmod 8, \tag{3.43}$$

which implies (1.25). This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8 and Corollary 1.4. By (3.39), using (2.16) and (3.21), we get

$$\sum_{n=0}^{\infty} A_{3,5}(2n)q^n \equiv f(-q^2)\psi(q^6) \pmod 4, \tag{3.44}$$

which is the case $\alpha = 0$ of (1.26).

Suppose that (1.26) holds for $\alpha \geq 0$. Employing Lemma 2.2 and Lemma 2.3 in (1.26), we consider the congruence

$$2 \cdot \frac{3k^2+k}{2} + 6 \cdot \frac{m^2+m}{2} \equiv \frac{20p^2-20}{24} \pmod{p}, \tag{3.45}$$

where $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $0 \leq m \leq \frac{p-3}{2}$.

The congruence (3.45) is equivalent to

$$(6k+1)^2 + 9(2m+1)^2 \equiv 0 \pmod{p}. \tag{3.46}$$

For $\left(\frac{-9}{p}\right) = -1$, the congruence (3.46) holds if and only if $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$.

Extracting the terms containing $q^{pn+\frac{5p^2-5}{6}}$ from both sides of (1.26), dividing by $q^{\frac{5p^2-5}{6}}$ and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} A_{3,5}\left(2 \cdot p^{2\alpha+1}n + \frac{5 \cdot p^{2\alpha+2}-5}{3}\right) q^n \\ \equiv (-1)^{(\alpha+1)(\frac{\pm p-1}{6})} f(-q^{2p})\psi(q^{6p}) \pmod{4}. \tag{3.47}$$

This implies that

$$\sum_{n=0}^{\infty} A_{3,5}\left(2 \cdot p^{2\alpha+2}n + \frac{5 \cdot p^{2\alpha+2}-5}{3}\right) q^n \\ \equiv (-1)^{(\alpha+1)(\frac{\pm p-1}{6})} f(-q^{2})\psi(q^{6}) \pmod{4}, \tag{3.48}$$

which is the case $\alpha+1$ of (1.26). The proof of Theorem 1.8 is completed by induction on α.

Combining (1.26), (2.15) and (2.16), we obtain

$$\sum_{n=0}^{\infty} A_{3,5}\left(4 \cdot p^{2\alpha}n + \frac{5 \cdot p^{2\alpha}-5}{3}\right) q^n \equiv \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} (-1)^{\alpha(\frac{\pm p-1}{6})+n} q^{\frac{n(3n+1)}{2}+3\frac{k(k+1)}{2}} \pmod{4},$$

which implies (1.27).

Applying (3.47) yields that for $j = 1, 2, \cdots, p-1$,

$$A_{3,5}\left(2 \cdot p^{2\alpha+1}(pn+j) + \frac{5 \cdot p^{2\alpha+2}-5}{3}\right) \equiv 0 \pmod{4}.$$

which implies (1.28). This finishes the proof of Theorem 1.8 and Corollary 1.4. □

Proof of Theorem 1.9 and Corollary 1.5. By (3.43), according to (2.16) and (3.21), we have

$$\sum_{n=0}^{\infty} A_{3,5}(8n+5)q^n \equiv 6f(-q^2)\psi(q^6) \pmod 8, \tag{3.49}$$

which is the case $\alpha = 0$ of (1.29).

Suppose that (1.29) holds for $\alpha \geq 0$. Employing Lemma 2.2 and Lemma 2.3 in (1.29), we consider the congruence

$$2 \cdot \frac{3k^2+k}{2} + 6 \cdot \frac{m^2+m}{2} \equiv \frac{20p^2-20}{24} \pmod p, \tag{3.50}$$

where $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $0 \leq m \leq \frac{p-3}{2}$.

The congruence (3.50) is equivalent to

$$(6k+1)^2 + 9(2m+1)^2 \equiv 0 \pmod p. \tag{3.51}$$

For $\left(\frac{-9}{p}\right) = -1$, the congruence (3.51) holds if and only if $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$.

Extracting the terms containing $q^{pn+\frac{5p^2-5}{6}}$ from both sides of (1.29), dividing by $q^{\frac{5p^2-5}{6}}$ and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} A_{3,5}\left(8 \cdot p^{2\alpha+1}n + \frac{20 \cdot p^{2\alpha+2}-5}{3}\right)q^n \equiv 6(-1)^{(\alpha+1)(\frac{\pm p-1}{6})} f(-q^{2p})\psi(q^{6p}) \pmod 8. \tag{3.52}$$

This implies that

$$\sum_{n=0}^{\infty} A_{3,5}\left(8 \cdot p^{2\alpha+2}n + \frac{20 \cdot p^{2\alpha+2}-5}{3}\right)q^n \equiv 6(-1)^{(\alpha+1)(\frac{\pm p-1}{6})} f(-q^{2})\psi(q^{6}) \pmod 8, \tag{3.53}$$

which is the case $\alpha+1$ of (1.29). The proof of Theorem 1.9 is completed by induction on α.

Combining (1.29), (2.16) and (3.21), we have

$$\sum_{n=0}^{\infty} A_{3,5}\left(16 \cdot p^{2\alpha}n + \frac{20 \cdot p^{2\alpha}-5}{3}\right)q^n \equiv 6\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{\alpha(\frac{\pm p-1}{6})+n} q^{\frac{n(3n+1)}{2}+3\frac{k(k+1)}{2}} \pmod 8,$$

which implies (1.30).

Applying (3.52) yields that for $1 \leq j \leq p-1$,

$$A_{3,5}\left(8 \cdot p^{2\alpha+1}(pn+j) + \frac{20 \cdot p^{2\alpha+2} - 5}{3}\right) \equiv 0 \pmod 8, \tag{3.54}$$

which implies (1.31). This completes the proof of Theorem 1.9 and Corollary 1.5. □

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Declarations

Conflict of interest I have no relevant financial or non-financial interests to disclose.

References

1. Berndt, B.C.: Ramanujan's Notebooks. Part III. Springer-Verlag, New York (1991)
2. Baruah, N.D., Nath, K.: Infinite families of arithmetic identities and congruences for bipartitions with 3-cores. J. Number Theory **149**, 92–104 (2015). https://doi.org/10.1016/j.jnt.2014.10.010
3. Baruah, N.D., Ojah, K.K.: Analogues of Ramanujan's partition identities and congruences arising from his theta functions and modular equations. Ramanujan J. **28**, 385–407 (2012). https://doi.org/10.1007/s11139-011-9296-z
4. Cui, S.P., Gu, N.S.S.: Arithmetic properties of l-regular partitions. Adv. Appl. Math. **51**(4), 507–523 (2013). https://doi.org/10.1016/j.aam.2013.06.002
5. Chern, S.: Formulas for partition k-tuples with t-cores. J. Math. Anal. Appl. **437**(2), 841–852 (2016). https://doi.org/10.1016/j.jmaa.2016.01.040
6. Dasappa, R.: On a Ramanujan-type congruence for bipartitions with 5-cores. J. Integer Seq. **19**(16.8), 1 (2016)
7. Garvan, F., Kim, D., Stanton, D.: Cranks and t-cores. Invent. Math. **101**(1), 1–17 (1990). https://doi.org/10.1007/BF01231493
8. Garvan, F.G.: A simple proof of Watson's partition congruences for powers of 7. J. Aust. Math. Soc. (Series A) **36**(06), 316–334 (1984). https://doi.org/10.1017/s1446788700025386
9. Hirschhorn, M.D., Garvan, F., Borwein, J.: Cubic analogs of the Jacobian cubic theta function $\theta(z, q)$. Can. J. Math. **45**(4), 673–694 (1993). https://doi.org/10.4153/CJM-1993-038-2
10. Hirschhorn, M.D.: Ramanujan's tau function In : Analytic Number Theory, Modular Forms and q-Hypergeometric Series, pp. 311–328, Springer Proceedings in Mathematics and Statistics, 221, Springer, Cham, (2017)
11. Lin, B.L.S.: Some results on bipartitions with 3-core. J. Number Theory **139**, 44–52 (2014). https://doi.org/10.1016/j.jnt.2013.12.007
12. Naika, M.S.M., Nayaka, S.S.: Congruences for partition quadruples with t-cores. Acta Math. Vietnam. **45**(4), 795–806 (2020). https://doi.org/10.1007/s40306-019-00356-z
13. Saikia, N., Boruah, C.: Congruences for bipartition and partition triples with 4-core. Afr. Matematika **28**(1–2), 199–206 (2016). https://doi.org/10.1007/s13370-016-0437-9
14. Saikia, N., Boruah, C.: Some congruences modulo 2 and 5 for bipartition with 5-core. Arab. J. Math. Sci. **23**(2), 124–132 (2017). https://doi.org/10.1016/j.ajmsc.2016.05.003
15. Saikia, N., Boruah, C.: New congruences for k-tuples t-core partitions. J. Anal. **26**, 27–37 (2018). https://doi.org/10.1007/s41478-017-0065-2
16. Wang, L.: Arithmetic identities and congruences for partition triples with 3-cores. Int. J. Number Theory **12**(4), 995–1010 (2015). https://doi.org/10.1142/S1793042116500627
17. Wang, L.: Explicit formulas for partition pairs and triples with 3-cores. J. Math. Anal. Appl. **434**(2), 1053–1064 (2016). https://doi.org/10.1016/j.jmaa.2015.09.074

18. Xia, E.X.W.: Arithmetic properties of bipartitions with 3-cores. Ramanujan J. **38**(3), 529–548 (2014). https://doi.org/10.1007/s11139-014-9643-y
19. Xia, E.X.W., Yao, O.X.M.: Analogues of Ramanujan's partition identities. Ramanujan J. **31**(3), 373–396 (2013). https://doi.org/10.1007/s11139-012-9439-x
20. Xia, E.X.W., Yao, O.X.M.: Some modular relations for the Göllnitz -Gordon functions by an even -odd method. J. Math. Anal. Appl. **387**(1), 126–138 (2012). https://doi.org/10.1016/j.jmaa/2011.08.059
21. Yao, O.X.M.: Infinite families of congruences modulo 3 and 9 for bipartitions with 3-cores. Bull. Aust. Math. Soc. **91**(1), 47–52 (2015). https://doi.org/10.1017/S0004972714000586
22. Zou, Q.: Arithmetic properties of bipartitions with *l*-core. J. Interdiscip. Math. **21**, 1533–1546 (2018). https://doi.org/10.1080/09720502.2017.1400796