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The Limiting Weak-type Behaviors of Intrinsic Square Functions

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Abstract

This paper is devoted to studying the limiting weak-type behaviors of intrinsic square function, which give a new way to find the lower bound of the best constant for weak type (1, 1) norm. Moreover, the corresponding results for intrinsic $g^*_{\lambda,\alpha}$ function are also established.

Keywords Best constant · Limiting weak-type behaviors · Intrinsic square function · Intrinsic $g_{\lambda,\alpha}{}^*$ function

Mathematics Subject Classification Primary 42B20; Secondary 42B25

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1 Introduction and Main Results

The study on the boundedness of singular integral and related operators is a central issue of harmonic analysis. The best constants in the strong-type and weak-type inequalities satisfied by these operators play an important role in determining the exact degrees of improved regularity and other geometric properties of solutions, their gradients and related nonlinear quantities for both linear and nonlinear PDEs in higher dimensions. In [1], Davis obtained the best constant for weak type $(1, 1)$ bound of the Hilbert transform. Janakiraman [9] extended Davis's results and gave the best constant for weak type (p, p) with $1 \leq p \leq 2$. In [10], the author presented that the weak-type $(1, 1)$ constant for the Riesz transform is at worst logarithmic with respect to dimension n. Janakiraman [11] further considered the limiting behaviors of weak type (1,1) bound of singular integral with homogeneous kernels, which gave a new way to find the lower bound of the best constant. Ding and Lai [2] extended the above results under more general L^1-Dini conditions. Guo, He and Wu [5] established optimal limiting weak-type behaviors of certain classical operators, which essentially improved and extended the previous results. Zhao and Guo [17] gave the corresponding results for factional maximal operators and fractional integrals without any smoothness assumption on the kernel. Readers can consult [3, 6–8, 13, 14] and related references therein for their recent developments.

Fefferman and Stein [4] proposed a conjecture: whether Lusin area function $S(f)$ is bounded from the weighted Lebesgue space $L^2_{\mathcal{M}(\nu)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_\nu(\mathbb{R}^n)$, where $0 \leq \nu \in L^1_{loc}(\mathbb{R}^n)$ and $\mathcal{M}(\nu)$ denotes the Hardy–Littlewood maximal function of ν. In order to settle the above conjecture, Wilcson [15] firstly introduced the following intrinsic square function.

Definition 1.1 Let $0 < \alpha \leq 1$. Suppose that $\varphi(x)$ supported in $B(0, 1) := \{x : |x| < 1\}$ satisfies

$$\int_{\mathbb{R}^n} \varphi(x)dx = 0,$$

and

$$|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha, \ \ for\ any\ x, x' \in \mathbb{R}^n. \tag{1.1}$$

We denote by $\mathcal{C}_\alpha$ if φ satisfies above conditions. Set $\varphi_t(x) = t^{-n}\varphi(x/t)$. For $(y, t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$, $f \in L^1_{loc}(\mathbb{R}^n)$, let

$$A_\alpha f(y, t) := \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.$$

For $\beta > 0$, we define the varying-aperture intrinsic square function by

$$S_{\alpha,\beta} f(x) := \Big(\iint_{\Gamma_\beta(x)} (A_\alpha f(y, t))^2 \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

where $\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$. Denote $S_{\alpha,1} f(x) =: S_\alpha f(x)$.

Let $\lambda > 1$. Define the following intrinsic $g^*_{\lambda,\alpha}$ function:

$$g^*_{\lambda,\alpha}(f)(x) := \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}.$$

The intrinsic square functions have an interesting feature. It follows from [15] that there is a pointwise relation between $S_{\alpha,\beta}(f)$ with different apertures:

$$S_{\alpha,\beta} f(x) \le \beta^{\frac{3n}{2}+\alpha} S_\alpha f(x), \quad \beta \ge 1.$$

In [15, 16], Wilson has proved that $S_\alpha f$ is bounded on $L^p(\omega)$ $(1 < p < \infty)$ and weighted weak type $(1,1)$. Lerner [12] established sharp $L^p(\omega)$ $(1 < p < \infty)$ norm inequalities for the intrinsic square function. It is nature to ask whether the lower bound of the best weak-type $(1,1)$ constant of intrinsic square function can be given. In this paper, we give a firm answer and establish the limiting weak-type behaviors of intrinsic square function.

To be more precise, we have the following results:

Theorem 1.2 *Suppose $f \ge 0$ and $f \in L^1(\mathbb{R}^n)$. For $0 < \alpha \le 1$ and $\beta \ge 1$, we have*

$$\lim_{\lambda\to 0_+} \lambda m\big(x \in \mathbb{R}^n : |S_{\alpha,\beta} f(x)| > \lambda\big) \tag{1}$$
$$= m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big)\|f\|_{L^1};$$

$$\lim_{\lambda\to 0_+} \lambda m\Big(\Big\{x\in\mathbb{R}^n : \Big|S_{\alpha,\beta} f(x) - \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \|f\|_{L^1}\Big| > \lambda\Big\}\Big) = 0. \tag{2}$$

Theorem 1.3 *Suppose $f \ge 0$ and $f \in L^1(\mathbb{R}^n)$. For $0 < \alpha \le 1$ and $\lambda > 3 + (2\alpha)/n$, we have*

$$\lim_{\xi\to 0_+} \xi m\big(x \in \mathbb{R}^n : |g^*_{\lambda,\alpha}(f)(x)| > \xi\big) \tag{1}$$
$$= m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big)\|f\|_{L^1};$$

$$\lim_{\xi\to 0_+} \xi m\Big(\Big\{x \in \mathbb{R}^n : \Big|\Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} \|f\|_{L^1} \tag{2}$$
$$- g^*_{\lambda,\alpha}(f)(x)\Big| > \xi\Big\}\Big) = 0.$$

This paper is organized as follows. The proof of Theorem 1.2 will be presented in Sect. 2. In Sect. 3, we will give the proof of Theorem 1.3.

Throughout this paper, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables.

2 Proofs of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To do this, we need to establish the following key lemma.

Lemma 2.1 *Suppose $\beta \geq 1$ and $0 < \alpha \leq 1$. For a fixed $\eta > 0$, we have*

$$m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > \eta\Big\}\Big)$$
$$= \frac{1}{\eta} m\Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).$$

Proof By making the change of variable $t := rt_1$ and $y = ry_1$ with $r > 0$, we get

$$\int_0^\infty \int_{|x-y|<\beta t} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}} = \frac{1}{r^{3n}} \int_0^\infty \int_{|x-y|<r\beta t_1} \sup_{\varphi\in\mathcal{C}_\alpha} \Big|\frac{1}{t_1^n}\varphi(\frac{y}{rt_1})\Big|^2 \frac{dydt_1}{t_1^{n+1}}$$
$$= \frac{1}{r^{2n}} \int_0^\infty \int_{|x/r-y_1|<\beta t_1} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_{t_1}(y_1)|^2 \frac{dy_1dt_1}{t_1^{n+1}}.$$

Then

$$m\Big(\Big\{x \in \mathbb{R}^n : \Big(\int_0^\infty \int_{|x-y|<\beta t} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > \eta\Big\}\Big)$$
$$= m\Big(\Big\{x \in \mathbb{R}^n : \Big(\int_0^\infty \int_{|x/r-y_1|<\beta t_1} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_{t_1}(y_1)|^2 \frac{dy_1dt_1}{t_1^{n+1}}\Big)^{1/2} > r^n\eta\Big\}\Big)$$
$$= r^n m\Big(\Big\{x \in \mathbb{R}^n : \Big(\int_0^\infty \int_{|x-y|<\beta t} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > r^n\eta\Big\}\Big).$$

Taking $r^n = 1/\eta$ yields the conclusion. □

Proof of Theorem 1.2 Without loss of generality, we may assume $\|f\|_{L^1} = 1$. For $0 < \varepsilon \ll 1$, there exists $r_\varepsilon > 0$ such that

$$\int_{|x|<r_\varepsilon} f(x)dx > 1 - \varepsilon.$$

Let $f_1 := f(x)\chi_{B(0,r_\varepsilon)}$ and $f_2 := f(x)\chi_{B(0,r_\varepsilon)^c}$. For $\lambda > 0$, we denote

$$E_\lambda := \left\{x \in \mathbb{R}^n : S_{\alpha,\beta} f(x) > \lambda\right\},$$
$$E^1_\lambda := \left\{x \in \mathbb{R}^n : S_{\alpha,\beta} f_1(x) > \lambda\right\},$$

and

$$E^2_\lambda := \left\{x \in \mathbb{R}^n : S_{\alpha,\beta} f_2(x) > \lambda\right\}.$$

For $0 < \alpha \le 1$, take $\delta = \varepsilon^{\alpha/2}$ for the above ε. Using the sublinear of $S_{\alpha,\beta}$, we have

$$S_{\alpha,\beta} f_1(x) - S_{\alpha,\beta} f_2(x) \le S_{\alpha,\beta} f(x) \le S_{\alpha,\beta} f_1(x) + S_{\alpha,\beta} f_2(x).$$

This implies $E^1_{(1+\delta)\lambda} \subseteq E_\lambda \bigcup E^2_{\delta\lambda}$ and $E_\lambda \subseteq E^1_{(1-\delta)\lambda} \bigcup E^2_{\delta\lambda}$. Therefore,

$$m(E^1_{(1+\delta)\lambda}) - m(E^2_{\delta\lambda}) \le m(E_\lambda) \le m(E^1_{(1-\delta)\lambda}) + m(E^2_{\delta\lambda}).$$

Since $S_{\alpha,\beta}$ is of weak type (1, 1), we obtain

$$m(E^2_{\delta\lambda}) = m(\{x \in \mathbb{R}^n : S_{\alpha,\beta} f_2(x) > \delta\lambda\}) \le \frac{C\|f_2\|_{L^1}}{\delta\lambda} \le \frac{C\sqrt{\varepsilon}}{\lambda}.$$

Then

$$m(E^1_{(1+\delta)\lambda}) - \frac{C\sqrt{\varepsilon}}{\lambda} \le m(E_\lambda) \le m(E^1_{(1-\delta)\lambda}) + \frac{C\sqrt{\varepsilon}}{\lambda}. \tag{2.1}$$

We need to estimate $m(E^1_{(1-\delta)\lambda})$ and $m(E^1_{(1+\delta)\lambda})$, respectively. Firstly, we deal with $m(E^1_{(1-\delta)\lambda})$. Set

$$I_1(x) := \left(\iint_{\Gamma_\beta(x)} \left(\sup_{\varphi\in\mathcal{C}_\alpha} \int_{\mathbb{R}^n} |\varphi_t(y-z) - \varphi_t(y)| f_1(z)dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{2.2}$$
$$I_2(x) := \left(\iint_{\Gamma_\beta(x)} \left(\sup_{\varphi\in\mathcal{C}_\alpha} \left|\int_{\mathbb{R}^n} \varphi_t(y) f_1(z)dz\right|\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}. \tag{2.3}$$

Using the triangle inequality, we have

$$I_2(x) - I_1(x) \le S_{\alpha,\beta} f_1(x) \le I_1(x) + I_2(x).$$

Let $R_\varepsilon = (1 + 1/\varepsilon)r_\varepsilon$. Denote

$$F^1_\lambda := \left\{|x| > R_\varepsilon : I_1(x) > \lambda\right\},$$

By Lemma 2.1, we get

$$m(E^1_{(1-\delta)\lambda}) \le m(\{x \in \mathbb{R}^n : I_2(x) > (1-2\delta)\lambda\}) + m(\{x \in \mathbb{R}^n : I_1(x) > \delta\lambda\})$$

$$\leq \frac{1}{(1-2\delta)\lambda} m\Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big)$$
$$+ m(F^1_{\delta\lambda}) + \gamma_n R^n_\varepsilon, \tag{2.4}$$

where γ_n is the volume of the unit ball in $\mathbb{R}^n$. Notice that $\varphi(x)$ supported in $B(0,1)$. For $|x| > R_\varepsilon$ and $t < |x|/(4\beta)$ and $|z| \leq r_\varepsilon$, we have

$$\varphi_t(y-z) - \varphi_t(y) = 0.$$

Then, using (1.1), we get

$$\begin{aligned} I_1(x) &\leq \Big(\int_{\frac{|x|}{4\beta}}^\infty \int_{|x-y|<\beta t} \Big(\frac{|z|^\alpha}{t^{n+\alpha}} \int_{\mathbb{R}^n} f_1(z)dz \Big)^2 \frac{dydt}{t^{n+1}} \Big)^{1/2} \\ &\leq \Big(\int_{\frac{|x|}{4\beta}}^\infty \int_{|x-y|<\beta t} \frac{|z|^{2\alpha}}{t^{3n+2\alpha+1}} dydt \Big)^{1/2} \\ &\leq C_{n,\alpha,\beta} \frac{r_\varepsilon^\alpha}{|x|^{n+\alpha}}. \end{aligned}$$

Recall that $\delta = \varepsilon^{\alpha/2}$. Then

$$\begin{aligned} m(F^1_{\delta\lambda}) &\leq \frac{1}{\delta\lambda} \int_{|x|>R_\varepsilon} I_1(x)dx \\ &\leq C_{n,\alpha,\beta} \frac{r_\varepsilon^\alpha}{\delta\lambda} \int_{|x|>R_\varepsilon} \frac{1}{|x|^{n+\alpha}} dx \\ &\leq C_{n,\alpha,\beta} \frac{r_\varepsilon^\alpha}{\delta\lambda R_\varepsilon^\alpha} \leq C_{n,\alpha,\beta} \frac{\varepsilon^{\alpha/2}}{\lambda}. \end{aligned}$$

Applying this along with (2.1), (2.4), we obtain

$$m(E_\lambda) \leq \frac{1}{(1-2\delta)\lambda} m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big)$$
$$+ C_{n,\alpha,\beta} \frac{\varepsilon^{\alpha/2}}{\lambda} + \gamma_n R^n_\varepsilon.$$

By letting $\lambda \to 0_+$ and $\varepsilon \to 0_+$, we have

$$\overline{\lim_{\lambda\to 0_+}} \lambda m(E_\lambda) \leq m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big). \tag{2.5}$$

On the other hand, by Lemma 2.1 and the estimate of $m(F^1_{\delta\lambda})$, we get

$$\begin{aligned} m(E^1_{(1+\delta)\lambda}) &\geq m\big(\big\{|x| > R_\varepsilon : S_{\alpha,\beta} f_1(x) > \lambda\big\}\big) \\ &\geq m(\{|x| > R_\varepsilon : I_2(x) > (1+2\delta)\lambda\}) - m(F^1_{\delta\lambda}) \end{aligned}$$

$$
\begin{aligned}
&\geq m(x \in \mathbb{R}^n : I_2(x) > (1+2\delta)\lambda\}) - \gamma_n R_\varepsilon^n - m(F_{\delta\lambda}^1) \\
&\geq \frac{1-\varepsilon}{(1+2\delta)\lambda} m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\
&\quad - \gamma_n R_\varepsilon^n - C_{n,\alpha,\beta} \frac{\varepsilon^{\alpha/2}}{\lambda}.
\end{aligned}
$$

Together with (2.1) implies

$$
\begin{aligned}
m(E_\lambda) \geq\ & \frac{1-\varepsilon}{(1+2\delta)\lambda} m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\
&- \gamma_n R_\varepsilon^n - C_{n,\alpha,\beta} \frac{\varepsilon^{\alpha/2}}{\lambda} - \frac{C\sqrt{\varepsilon}}{\lambda}.
\end{aligned}
$$

By taking $\lambda \to 0_+$ and $\varepsilon \to 0_+$, we obtain

$$
\varliminf_{\lambda\to 0_+} \lambda m(E_\lambda) \geq m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).
$$

Combining with (2.5), we get

$$
\lim_{\lambda\to 0_+} \lambda m(E_\lambda) = m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).
$$

Now, we turn to (2). Assume $\|f\|_{L^1} = 1$. For $\lambda > 0$, suppose

$$
G_\lambda := \Big\{x \in \mathbb{R}^n : \Big|S_{\alpha,\beta} f(x) - \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\Big| > \lambda\Big\}.
$$

It remains to prove

$$
\lim_{\lambda\to 0_+} \lambda m(G_\lambda) = 0.
$$

Employing the notations $I_1(x)$, $I_2(x)$ in (2.2) and (2.3), we have

$$
\begin{aligned}
&\Big|S_{\alpha,\beta} f(x) - \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\Big| \\
&\quad\leq S_{\alpha,\beta} f_2(x) + \Big|S_{\alpha,\beta} f_1(x) - \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\Big| \\
&\quad\leq S_{\alpha,\beta} f_2(x) + I_1(x) + \Big|I_2(x) - \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\Big| \\
&\quad\leq S_{\alpha,\beta} f_2(x) + I_1(x) + \varepsilon\Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},
\end{aligned}
$$

where the last inequality follows from $\|f_1\|_{L^1} > 1-\varepsilon$. Recall that $R_\varepsilon = (1+1/\varepsilon)r_\varepsilon$. By Lemma 2.1, the estimate of $m(F^1_{\delta\lambda})$ and $m(E^2_{\delta\lambda})$, we have

$$\begin{aligned} m(G_\lambda) &\le m\Big(\Big\{x\in\mathbb{R}^n : \varepsilon\Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > (1-2\delta)\lambda\Big\}\Big) \\ &\quad + m(F^1_{\delta\lambda}) + m(E^2_{\delta\lambda}) + \gamma R^n_\varepsilon \\ &\le \frac{\varepsilon}{(1-2\delta)\lambda} m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\ &\quad + C_{n,\alpha,\beta}\frac{\varepsilon^{\alpha/2}}{\lambda} + \frac{C\sqrt{\varepsilon}}{\lambda} + \gamma_n R^n_\varepsilon. \end{aligned}$$

By taking $\lambda\to 0_+$ and $\varepsilon\to 0_+$, we obtain

$$\lim_{\lambda\to 0_+} \lambda m(G_\lambda) = 0.$$

This completes the proof. □

Remark 2.2 We remark that the conclusion (2) in Theorem 1.2 is stronger than conclusion (1).

In fact, let E_λ, G_λ be as before and $\|f\|_{L^1} = 1$. By Lemma 2.1, we obtain that for any $0<\eta<1$,

$$\begin{aligned} m(E_\lambda) &\le m(G_{\eta\lambda}) + m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > (1-\eta)\lambda\Big\}\Big) \\ &\le m(G_{\eta\lambda}) + \frac{1}{(1-\eta)\lambda} m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big). \end{aligned}$$

Using conclusion (2), taking $\lambda\to 0_+$, $\eta\to 0_+$, we get

$$\overline{\lim_{\lambda\to 0_+}} \lambda m(E_\lambda) \le m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big). \tag{2.6}$$

On the other hand, for any $0<\eta<1$, from Lemma 2.1, we obtain

$$\begin{aligned} m(E_\lambda) &\ge m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > (1+\eta)\lambda\Big\}\Big) - m(G_{\eta\lambda}) \\ &\ge \frac{1}{(1+\eta)\lambda} m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) - m(G_{\eta\lambda}). \end{aligned}$$

Letting $\lambda\to 0_+$, $\eta\to 0_+$, we get

$$\underline{\lim}_{\lambda\to 0_+} \lambda m(E_\lambda) \geq m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\Gamma_\beta(x)} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).$$

This together (2.6) implies that the conclusion (1) holds.

3 Proofs of Theorem 1.3

In this section, we give the proof of Theorem 1.3. At first, we need establish the following lemma.

Lemma 3.1 *Let* $0 < \alpha \leq 1$. *For a fixed* $\eta > 0$, *we have*

$$\begin{aligned} &m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > \eta\Big\}\Big) \\ &\quad = \frac{1}{\eta} m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big). \end{aligned}$$

Proof For $r > 0$, we obtain

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}} \\ &\quad = \frac{1}{r^{2n}} \int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{rt_1}{rt_1+|x-ry_1|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_{t_1}(y_1)|^2 \frac{dy_1dt_1}{{t_1}^{n+1}} \\ &\quad = \frac{1}{r^{2n}} \int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{t_1}{t_1+|x/r-y_1|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_{t_1}(y_1)|^2 \frac{dy_1dt_1}{{t_1}^{n+1}}. \end{aligned}$$

where in the first equality we make the variable change $t := rt_1$ and $y := ry_1$. Then

$$\begin{aligned} &m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > \eta\Big\}\Big) \\ &\quad = m\Big(\Big\{x \in \mathbb{R}^n : \Big(\int_0^\infty \int_{\mathbb{R}^n} \Big(\frac{t_1}{t_1+|x/r-y_1|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_{t_1}(y_1)|^2 \frac{dy_1dt_1}{{t_1}^{n+1}}\Big)^{1/2} > r^n\eta\Big\}\Big) \\ &\quad = r^n m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > r^n\eta\Big\}\Big). \end{aligned}$$

By taking $r^n = 1/\eta$, we finish the proof. □

Proof of Theorem 1.3 Without loss of generality, we may assume $\|f\|_{L^1} = 1$. For $0 < \varepsilon \ll 1$, there exists $r_\varepsilon > 0$ such that

$$\int_{|x|<r_\varepsilon} f(x)dx > 1 - \varepsilon.$$

Let $f_1 := f(x)\chi_{B(0,r_\varepsilon)}$ and $f_2 := f(x)\chi_{B(0,r_\varepsilon)^c}$. For $\xi > 0$, set

$$H_\xi := \left\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha}(f)(x)) > \xi\right\},$$
$$H^1_\xi := \left\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha}(f_1)(x) > \xi\right\},$$

and

$$H^2_\xi := \left\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha}(f_2)(x) > \xi\right\}.$$

For $0 < \alpha \leq 1$, take $\delta = \varepsilon^{\alpha/2}$ for the above ε. By the sublinear of $g^*_{\lambda,\alpha}$, we get

$$g^*_{\lambda,\alpha}(f_1)(x) - g^*_{\lambda,\alpha}(f_2)(x) \leq g^*_{\lambda,\alpha}(f)(x) \leq g^*_{\lambda,\alpha}(f_1)(x) + g^*_{\lambda,\alpha}(f_2)(x).$$

From this, we have $H^1_{(1+\delta)\xi} \subseteq H_\xi \bigcup H^2_{\delta\xi}$ and $H_\xi \subseteq H^1_{(1-\delta)\xi} \bigcup H^2_{\delta\xi}$. Therefore,

$$m(H^1_{(1+\delta)\xi}) - m(H^2_{\delta\xi}) \leq m(H_\xi) \leq m(H^1_{(1-\delta)\xi}) + m(H^2_{\delta\xi}).$$

For $\lambda > 3 + (2\alpha)/n$, it is easy to check that

$$\begin{aligned} g^*_{\lambda,\alpha}(f)(x) &\leq C\Big(S_\alpha f(x) + \sum_{k=1}^{\infty} 2^{-k\lambda n/2} S_{\alpha,2^k} f(x)\Big) \\ &\leq C\Big(S_\alpha f(x) + \sum_{k=1}^{\infty} 2^{k(3n+2\alpha-\lambda n)/2} S_\alpha f(x)\Big) \\ &\leq C_{n,\alpha} S_\alpha f(x). \end{aligned}$$

By noting that S_α is of weak type (1, 1), we have that $g^*_{\lambda,\alpha}(f)$ is of weak type (1, 1). Then

$$m(H^2_{\delta\xi}) = m(\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha}(f_2)(x) > \delta\xi\}) \leq \frac{C\|f_2\|_{L^1}}{\delta\xi} \leq \frac{C\sqrt{\varepsilon}}{\xi}.$$

Therefore,

$$m(H^1_{(1+\delta)\xi}) - \frac{C\sqrt{\varepsilon}}{\xi} \leq m(H_\xi) \leq m(H^1_{(1-\delta)\xi}) + \frac{C\sqrt{\varepsilon}}{\xi}. \tag{3.1}$$

We will estimate $m(H^1_{(1-\delta)\xi})$ and $m(H^1_{(1+\delta)\xi})$, respectively. For $m(H^1_{(1-\delta)\xi})$, let

$$\begin{aligned} J_1(x) := \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \times \\ \Big(\sup_{\varphi\in\mathcal{C}_\alpha} \int_{\mathbb{R}^n} |\varphi_t(y-z) - \varphi_t(y)| f_1(z) dz\Big)^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}, \end{aligned} \tag{3.2}$$

$$J_2(x) := \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \Big(\sup_{\varphi\in\mathcal{C}_\alpha} \Big| \int_{\mathbb{R}^n} \varphi_t(y) f_1(z)dz\Big|\Big)^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}. \quad (3.3)$$

It is easy to see

$$J_2(x) - J_1(x) \le g^*_{\lambda,\alpha}(f_1)(x) \le J_1(x) + J_2(x).$$

Set $R_\varepsilon = (1+1/\varepsilon)r_\varepsilon$. Denote

$$K^1_\xi := \big\{|x| > R_\varepsilon : J_1(x) > \xi\big\},$$

This together with Lemma 3.1 shows

$$\begin{aligned} m(H^1_{(1-\delta)\xi}) &\le m(\{x\in\mathbb{R}^n : J_2(x) > (1-2\delta)\xi\}) + m(\{x\in\mathbb{R}^n : J_1(x) > \delta\xi\}) \\ &\le \frac{1}{(1-2\delta)\xi}\times \\ &\quad m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\ &\quad + m(K^1_{\delta\xi}) + \gamma_n R^n_\varepsilon. \end{aligned} \quad (3.4)$$

For $|x| \ge R_\varepsilon$, we obtain that

$$\begin{aligned} J_1(x) \le{}& \Big(\int_0^{\frac{|x|}{4}}\int_{|x-y|<\frac{|x|}{4}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big(\sup_{\varphi\in\mathcal{C}_\alpha}\int_{\mathbb{R}^n}|\varphi_t(y-z)-\varphi_t(y)|f_1(z)dz\Big)^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &+\Big(\int_0^{\frac{|x|}{4}}\int_{|x-y|\ge\frac{|x|}{4}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big(\sup_{\varphi\in\mathcal{C}_\alpha}\int_{\mathbb{R}^n}|\varphi_t(y-z)-\varphi_t(y)|f_1(z)dz\Big)^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &+\Big(\int_{\frac{|x|}{4}}^\infty\int_{|x-y|<t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big(\sup_{\varphi\in\mathcal{C}_\alpha}\int_{\mathbb{R}^n}|\varphi_t(y-z)-\varphi_t(y)|f_1(z)dz\Big)^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\ &+\Big(\int_{\frac{|x|}{4}}^\infty\int_{|x-y|\ge t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big(\sup_{\varphi\in\mathcal{C}_\alpha}\int_{\mathbb{R}^n}|\varphi_t(y-z)-\varphi_t(y)|f_1(z)dz\Big)^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\ =:{}& J_{11}(x)+J_{12}(x)+J_{13}(x)+J_{14}(x). \end{aligned}$$

We start the estimate of $J_{11}(x)$. Observing that $\varphi_t(y-z)$ and $\varphi_t(y)$ both vanish when $t < |x|/4$, $|x-y| < |x|/4$ and $|z| \le r_\varepsilon$. We have $J_{11}(x) = 0$.

Next, we turn to estimate $J_{12}(x)$. By (1.1) and $\lambda > 3 + (2\alpha)/n$, we get

$$\begin{aligned} J_{12}(x) &\le |z|^\alpha\Big(\int_0^{\frac{|x|}{4}}\sum_{k=-2}^\infty\int_{2^k|x|\le|x-y|<2^{k+1}|x|}\frac{t^{\lambda n-3n-2\alpha-1}}{2^{k\lambda n}|x|^{\lambda n}}dydt\Big)^{1/2} \\ &\le C_n|z|^\alpha\Big(\int_0^{\frac{|x|}{4}}\frac{t^{\lambda n-3n-2\alpha-1}}{|x|^{\lambda n-n}}dt\sum_{k=-2}^\infty 2^{-k\lambda n+kn}\Big)^{1/2} \\ &\le C_{\lambda,n,\alpha}\frac{r^\alpha_\varepsilon}{|x|^{n+\alpha}}. \end{aligned}$$

For $J_{13}(x)$, using (1.1), we obtain

$$J_{13}(x) \le \Big(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\frac{|z|^{2\alpha}}{t^{3n+2\alpha+1}}dydt\Big)^{1/2} \le C_{n,\alpha}\frac{r_\varepsilon^\alpha}{|x|^{n+\alpha}}.$$

For $J_{14}(x)$, by (1.1) and $\lambda > 3 + (2\alpha)/n$ again, we have

$$\begin{aligned}J_{14}(x) &\le |z|^\alpha\Big(\int_{\frac{|x|}{4}}^{\infty}\sum_{k=0}^{\infty}\int_{2^k t\le|x-y|<2^{k+1}t}2^{-k\lambda n}\frac{|z|^{2\alpha}}{t^{3n+2\alpha+1}}dydt\Big)^{1/2}\\ &\le C_n|z|^\alpha\Big(\int_{\frac{|x|}{4}}^{\infty}\frac{1}{t^{2n+2\alpha+1}}dt\sum_{k=0}^{\infty}2^{-k\lambda n+kn}\Big)^{1/2}\\ &\le C_{\lambda,n,\alpha}\frac{r_\varepsilon^\alpha}{|x|^{n+\alpha}}.\end{aligned}$$

Combining the estimates of $J_{11}(x)$, $J_{12}(x)$, $J_{13}(x)$ and $J_{14}(x)$, we further obtain that for $|x| \ge R_\varepsilon$,

$$J_1(x) \le C_{\lambda,n,\alpha}\frac{r_\varepsilon^\alpha}{|x|^{n+\alpha}}.$$

Recall that $\delta = \varepsilon^{\alpha/2}$. For $|z| \le r_\varepsilon$, we have

$$\begin{aligned}m(K^1_{\delta\xi}) &\le \frac{1}{\delta\xi}\int_{|x|>R_\varepsilon}J_1(x)dx \le C_{\lambda,n,\alpha}\frac{r_\varepsilon^\alpha}{\delta\xi}\int_{|x|>R_\varepsilon}\frac{1}{|x|^{n+\alpha}}dx\\ &\le C_{\lambda,n,\alpha}\frac{r_\varepsilon^\alpha}{\delta\xi R_\varepsilon^\alpha} \le C_{\lambda,n,\alpha}\frac{\varepsilon^{\alpha/2}}{\xi}.\end{aligned}$$

Together with (3.4), (3.1), we get

$$\begin{aligned}m(H_\xi) \le{}& \frac{1}{(1-2\delta)\xi}\\ & m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big)\\ &+ C_{\lambda,n,\alpha}\frac{\varepsilon^{\alpha/2}}{\xi} + \gamma_n R_\varepsilon^n.\end{aligned}$$

By taking $\xi \to 0_+$ and $\varepsilon \to 0_+$, we have

$$\begin{aligned}&\overline{\lim_{\xi\to 0_+}}\,\xi m(H_\xi)\\ &\le m\Big(\Big\{x\in\mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big). \quad (3.5)\end{aligned}$$

On the other hand, by Lemma 3.1 and the estimate of $m(H^1_{\delta\xi})$, we obtain

$$\begin{aligned}
m(H^1_{(1+\delta)\xi}) &\geq m\Big(\Big\{|x| > R_\varepsilon : g^*_{\lambda,\alpha}(f_1)(x) > \xi\Big\}\Big) \\
&\geq m(\{|x| > R_\varepsilon : J_2(x) > (1+2\delta)\xi\}) - m(H^1_{\delta\xi}) \\
&\geq m(x \in \mathbb{R}^n : J_2(x) > (1+2\delta)\xi\}) - \gamma_n R^n_\varepsilon - m(H^1_{\delta\xi}) \\
&\geq \frac{1-\varepsilon}{(1+2\delta)\xi} \times \\
&\quad m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\
&\quad - \gamma_n R^n_\varepsilon - C_{\lambda,n,\alpha} \frac{\varepsilon^{\alpha/2}}{\xi}.
\end{aligned}$$

This together with (3.1) leads

$$\begin{aligned}
m(H_\xi) \geq &\frac{1-\varepsilon}{(1+2\delta)\xi} \times \\
&m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big) \\
&- \gamma_n R^n_\varepsilon - C_{\lambda,n,\alpha} \frac{\varepsilon^{\alpha/2}}{\xi} - \frac{C\sqrt{\varepsilon}}{\xi}.
\end{aligned}$$

By taking $\xi \to 0_+$ and $\varepsilon \to 0_+$, we obtain

$$\underline{\lim}_{\xi \to 0_+} \xi m(H_\xi) \geq m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).$$

This combining with (3.5) implies

$$\lim_{\xi \to 0_+} \xi m(H_\xi) = m\Big(\Big\{x \in \mathbb{R}^n : \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2} > 1\Big\}\Big).$$

Now, we turn to (2). Assume $\|f\|_{L^1} = 1$. For $\xi > 0$, set

$$L_\xi := \Big\{x \in \mathbb{R}^n : \Big|g^*_{\lambda,\alpha}(f)(x) - \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\alpha} |\varphi_t(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\Big| > \xi\Big\}.$$

We only need to check

$$\lim_{\xi \to 0_+} \xi m(L_\xi) = 0.$$

Employing the notations $J_1(x)$, $J_2(x)$ in (3.2) and (3.3), we get

$$
\begin{aligned}
&\left|g^*_{\lambda,\alpha}(f)(x) - \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\right| \\
&\quad\le g^*_{\lambda,\alpha}(f_2)(x) + \left|g^*_{\lambda,\alpha}(f_1)(x) - \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\right| \\
&\quad\le g^*_{\lambda,\alpha}(f_2)(x) + J_1(x) + \left|J_2(x) - \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\right| \\
&\quad\le g^*_{\lambda,\alpha}(f_2)(x) + J_1(x) + \varepsilon\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}.
\end{aligned}
$$

Recall that $R_\varepsilon = (1+1/\varepsilon)r_\varepsilon$. It follows from Lemma 3.1 and the estimate of $m(K^1_{\delta\xi})$, $m(H^2_{\delta\xi})$ that

$$
\begin{aligned}
m(L_\xi) &\le m\Big(\Big\{x\in\mathbb{R}^n : \varepsilon\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} > (1-2\delta)\xi\Big\}\Big) \\
&\quad + m(K^1_{\delta\xi}) + m(H^2_{\delta\xi}) + \gamma R^n_\varepsilon \\
&\le \frac{\varepsilon}{(1-2\delta)\xi}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\sup_{\varphi\in\mathcal{C}_\alpha}|\varphi_t(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2} \\
&\quad + C_{\lambda,n,\alpha}\frac{\varepsilon^{\alpha/2}}{\xi} + \frac{C\sqrt{\varepsilon}}{\xi} + \gamma_n R^n_\varepsilon.
\end{aligned}
$$

By taking $\xi \to 0_+$ and $\varepsilon \to 0_+$, we obtain

$$
\lim_{\xi\to 0_+}\xi m(L_\xi) = 0.
$$

This completes the proof. □

Remark 3.2 We remark that the conclusion (2) in Theorem 1.3 is stronger than conclusion (1).

The proof of Remark 3.2 follows from the same arguments in Remark 2.2. We omit the details.

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