Bull. Malays. Math. Sci. Soc. (2022) 45:1093–1109
https://doi.org/10.1007/s40840-022-01262-8

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

New Modular Symmetric Function and its Applications: Modular s-Stirling Numbers

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Received: 25 October 2021 / Revised: 11 February 2022 / Accepted: 17 February 2022 /
Published online: 12 March 2022

Abstract
In this paper, we consider generalizations of the Stirling number of the first and the second kind by using a specialization of a new family of symmetric functions. We give combinatorial interpretations for these symmetric functions by means of weighted lattice path and tilings. We also present some new convolutions involving the complete and elementary symmetric functions. Additionally, we introduce different families of set partitions to give combinatorial interpretations for the modular s-Stirling numbers.

Keywords Symmetric functions · Generating functions · Stirling numbers

Mathematics Subject Classification 05A15 · 05A19

Communicated by Rosihan M. Ali.

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1 Introduction

Symmetric functions are ubiquitous in mathematics and mathematical physics. For example, they appear in elementary algebra, in the Viète's formulas that relate the coefficients of a polynomial to combinations of its roots. They are important objects to study in algebraic combinatorics. For example, symmetric functions are related to the representation theories of symmetric groups and general linear groups over the complex numbers or finite fields, and the enumeration of plane partitions [5].

Given a set of variables $x_1, x_2, \ldots, x_n$, the k-th *elementary and complete symmetric polynomials* are defined, respectively, by

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad 1 \le k \le n,$$
$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad k \ge 1,$$

with initial conditions $e_0(x_1, x_2, \ldots, x_n) = h_0(x_1, x_2, \ldots, x_n) = 1$. Note that $e_k(x_1, x_2, \ldots, x_n) = 0$ if $k > n$. The generating functions for the e_k and h_k are given by the expressions

$$\sum_{k=0}^{n} e_k(x_1, x_2, \ldots, x_n) z^k = \prod_{i=1}^{n} (1 + x_i z),$$
$$\sum_{k \ge 0} h_k(x_1, x_2, \ldots, x_n) z^k = \prod_{i=1}^{n} \frac{1}{1 - x_i z}.$$

A variety of combinatorial sequences can be obtained as evaluations of the symmetric polynomials at specific points (cf. [8, 16]). Particularly, the Stirling numbers of the first kind and the Stirling numbers of the second kind are given by

$$e_k(1, 2, \ldots, n) = \left[\begin{matrix} n+1 \\ n+1-k \end{matrix} \right] \quad \text{and} \quad h_k(1, 2, \ldots, n) = \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\}.$$

A *set partition* of a set $[n] := \{1, 2, \ldots, n\}$ is a collection of non-empty disjoint subsets, called *blocks*, whose union is $[n]$. The number of set partitions of $[n]$ into k non-empty blocks is counted by the *Stirling numbers of the second kind* $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. Similarly, the *Stirling numbers of the first kind* $\left[\begin{matrix} n \\ k \end{matrix} \right]$ count the number of permutations of $[n]$ into k cycles. The literature contains several generalizations of Stirling numbers of both kinds; see for example [4, 6, 15, 17–19].

In this work, we introduce an extension of the Stirling numbers of both kinds, called *s-modular Stirling numbers*, by introducing a new class of symmetric functions, and considering these new sequences as specializations of them. We give a combinatorial interpretation of these symmetric functions by using weighted lattice path and tilings. Similar symmetric functions were studied by Doty and Walker under the name of modular complete symmetric polynomials [7]. Most recently, Ahmia and Merca [1] introduced a variation of these symmetric functions. Independently, Grinberg [10] and Fu and Mei [9] introduced the same concept under the name of Petrie symmetric functions and truncated symmetric functions, respectively. Finally, we use set partitions to give a combinatorial interpretation to the s-modular Stirling numbers. Among other things, we give an interpretation (probably new) of the Stirling numbers of first kind in terms of set partitions. We also give a relationship with the Stirling numbers with higher level. This last sequence was recently studied in the context of special polynomials [12].

2 Definitions and Properties

Let $s \geq 1$ be a positive integer. We define a *modular symmetric function* by

$$M_k^{(s)}(n) := M_k^{(s)}(x_1, \dots, x_n) = \sum_{\substack{a_1+\cdots+a_n=k \\ a_1,\dots,a_n \equiv (0,1) \mod (s+1)}} x_1^{a_1} \cdots x_n^{a_n}, \tag{1}$$

with $M_k^{(1)}(n) = h_k(n)$ and $M_k^{(s)}(0) = \delta_{k,0}$, where $\delta_{k,0}$ is the Kronecker delta.

For example, for $s = 2$ and $n = 1$ we have

$$M_0^{(2)}(1) = 1, \quad M_1^{(2)}(1) = x_1, \quad M_2^{(2)}(1) = 0, \quad M_3^{(2)}(1) = x_1^3 \quad M_4^{(2)}(1) = x_1^4, \quad M_5^{(2)}(1) = 0.$$

For $s = 2 = n$, we have

$$M_0^{(2)}(2) = 1, \quad M_1^{(2)}(2) = x_1 + x_2, \quad M_2^{(2)}(2) = x_1 x_2, \quad M_3^{(2)}(2) = x_1^3 + x_2^3.$$

From the definition of $M_k^{(s)}(n)$, we have the following theorem.

Theorem 2.1 *Let s and n be positive integers. Then,*

$$\sum_{k \geq 0} M_k^{(s)}(n) t^k = \prod_{i=1}^{n} \frac{1 + x_i t}{1 - (x_i t)^{s+1}}. \tag{2}$$

The modular symmetric function satisfies the following recurrence relations.

Theorem 2.2 *Let s and n be positive integers. Then,*

$$M_k^{(s)}(n) = \sum_{\substack{0 \leq j \leq k \\ j \equiv (0,1) \mod (s+1)}} x_n^j M_{k-j}^{(s)}(n-1) \quad \textit{and} \tag{3}$$

$$M_k^{(s)}(n) = x_n^{s+1} M_{k-s-1}^{(s)}(n) + x_n M_{k-1}^{(s)}(n-1) + M_k^{(s)}(n-1), \tag{4}$$

for $k \geq s + 1$.

Proof From (1) and Theorem 2.1, we have

$$\begin{aligned}
\sum_{k\geq 0} M_k^{(s)}(n)t^k &= \prod_{i=1}^{n} \frac{1+x_i t}{1-(x_i t)^{s+1}} = \prod_{i=1}^{n}\left((1+x_i t)\sum_{j\geq 0}(x_i t)^{(s+1)j}\right)\\
&= \prod_{i=1}^{n}\left(\sum_{j\geq 0}(x_i t)^{(s+1)j} + \sum_{j\geq 0}(x_i t)^{(s+1)j+1}\right)\\
&= \prod_{i=1}^{n}\left(\sum_{\ell\equiv(0,1)\mod (s+1)}(x_i t)^{\ell}\right)\\
&= \sum_{j\equiv(0,1)\mod (s+1)}(x_n t)^{j}\prod_{i=1}^{n-1}\left(\sum_{\ell\equiv(0,1)\mod (s+1)}(x_i t)^{\ell}\right)\\
&= \sum_{j\equiv(0,1)\mod (s+1)}(x_n t)^{j}\sum_{\ell=0}^{\infty} M_\ell^{(s)}(n-1)t^{\ell}\\
&= \sum_{k\geq 0} t^k \sum_{\substack{j+\ell=k\\ j\equiv(0,1)\mod (s+1)}} x_n^j M_\ell^{(s)}(n-1)\\
&= \sum_{k\geq 0} t^k \sum_{\substack{0\leq j\leq k\\ j\equiv(0,1)\mod (s+1)}} x_n^j M_{k-j}^{(s)}(n-1).
\end{aligned}$$

By comparing the k-th coefficient, we obtain (3). The relation (4) follows in a similar manner. □

Notice that from (3), we have the equality $h_k(n) = \sum_{j=0}^{k} x_n^j h_{k-j}(n-1)$.

3 Combinatorial Interpretation

The goal of this section is to present a combinatorial interpretation for the modular symmetric functions by means of weighted lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$. A *lattice path* Γ in the lattice plane $\mathbb{Z}\times\mathbb{Z}$, with steps in a given set $S\subset\mathbb{Z}^2$, is a concatenation of directed steps of S, that is $\Gamma = s_1 s_2\cdots s_\ell$, where $s_i\in S$, for each $1\leq i\leq \ell$. Let $\mathcal{P}_{n,k}$ denote the set of lattice paths from the point $(0,0)$ to the point $(k,n-1)$, with step set $S=\{H=(1,0), V=(0,1)\}$, such that the horizontal steps are labeled with the weight x_i, where $i-1$ is the level of the step. Let $\mathcal{P}_{n,k}^{(s)}$ denote the weighted lattice path in $\mathcal{P}_{n,k}$ such that the number of horizontal steps in each level is congruent to 0 or 1 modulo $s+1$. Given a weighted path Γ in $\mathcal{P}_{n,k}^{(s)}$, we denote by $\omega(\Gamma)$ the weight associated to the path Γ. For example, in Fig. 1 we show a lattice path in $\mathcal{P}_{6,12}^{(2)}$ of weight $x_2^6 x_4 x_5 x_6^4$.

From (1), we obtain the following combinatorial interpretation.

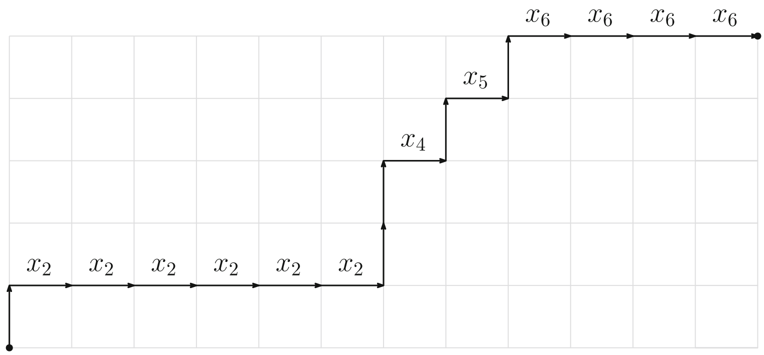

Fig. 1 Weighted lattice path in $\mathcal{P}^{(2)}_{6,12}$

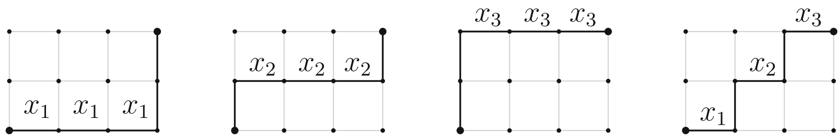

Fig. 2 The four paths associated to $M^{(2)}_3(x_1, x_2, x_3)$

Theorem 3.1 *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then,*

$$M^{(s)}_k(x_1, x_2, \ldots, x_n) = \sum_{\Gamma \in \mathcal{P}^{(s)}_{n,k}} \omega(\Gamma).$$

Figure 2 shows the weighted lattice path interpretation for $M^{(2)}_3(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + x_1x_2x_3$.

3.1 Tiling Interpretation

In this section, we use weighted tilings to give an additional combinatorial interpretation of the modular symmetric function. We define a *weighted tiling* as a tiling of a board of length n (n-board) by gray and black squares, such that each black square received the weight x_{m+1}, where m is equal to the number of gray squares to the left of that black square in the tiling. Let $\mathcal{T}^{(s)}_{n,k}$ denote the set of weighted tilings of an $(n+k-1)$-board using exactly k black squares and $n-1$ gray squares, such that the number of successive black squares is congruent to 0 or 1 modulo $s+1$. For a tiling T, we denote by $\omega(T)$ the weight of T.

For example, in Fig. 3 we show a weighted tiling in $\mathcal{T}^{(2)}_{6,12}$ of weight $x_2^6x_4x_5x_6^4$.

There is a bijection between the sets $\mathcal{P}^{(s)}_{n,k}$ and $\mathcal{T}^{(s)}_{n,k}$. Indeed, each vertical step V is replaced by a gray square and each horizontal step is replaced by a black square. Since the bijection between lattice paths and tiling is weight-preserving, we obtain the following result.

Fig. 3 Weighted tiling in $\mathcal{T}_{6,12}^{(2)}$

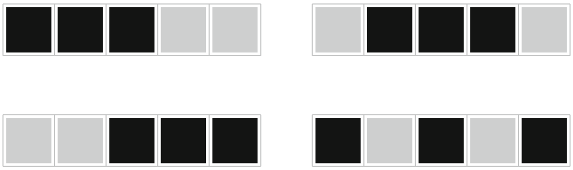

Fig. 4 The four tilings associated to $M_3^{(2)}(x_1, x_2, x_3)$

Theorem 3.2 *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then,*

$$M_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathcal{T}_{n,k}^{(s)}} \omega(T).$$

Figure 4 shows the tiling interpretation for $M_3^{(2)}(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + x_1 x_2 x_3$. In Theorem 3.3, we give a combinatorial expression for the sequence $M_k^{(s)}(\underbrace{1, 1, \ldots, 1}_{n \text{ times}})$.

Theorem 3.3 *For $n, k \geq 0$, $s \geq 1$, we have*

$$M_k^{(s)}(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}) = \sum_{j=0}^{\left\lfloor \frac{k}{s+1} \right\rfloor} \binom{n}{k - j(s+1)} \binom{j+n-1}{n-1}.$$

Proof. From the combinatorial interpretation $M_k^{(s)}(\overbrace{1, 1, \ldots, 1}^{n \text{ times}})$ counts the number of weighted tilings of a $(n+k-1)$-board using exactly k black squares and $n-1$ gray squares, such that the number of successive black squares is congruent to 0 or 1 modulo $s+1$. On the other hand, let j be the number of successive black squares multiples of $s+1$. Notice that $0 \leq j \leq \lfloor k/(s+1) \rfloor$. Then, there are $j+n-1$ gray and black blocks tiles. Such a tiling with $j+n-1$ tiles, exactly j of which are black blocks of size congruent to 0 module $s+1$ is $\binom{j+n-1}{j}$. The remaining $k - j(s+1)$ black squares can be inserted before to each gray square or to the end of the tiling. Since there are $n-1$ gray squares we have $\binom{n}{k-j(s+1)}$ ways to insert the black squares. Hence, there are $\binom{n}{k-j(s+1)}\binom{j+n-1}{j}$ tilings altogether. Summing over all j gives the total number of weighted tiling in $\mathcal{T}_{n,k}^{(s)}$, which implies the identity. □

Notice that we can also give an algebraic proof for the above result. Indeed, from the generating function given in Theorem 2.1 we have

$$\begin{aligned}\sum_{k\geq 0} M_k^{(s)}(1,1,\ldots,1)t^k &= \frac{(1+t)^n}{\left(1-t^{s+1}\right)^n} = \sum_{j=0}^{n}\binom{n}{j}t^j\sum_{\ell\geq 0}\binom{\ell+n-1}{n-1}t^{\ell(s+1)}\\ &= \sum_{j=0}^{n}\sum_{\ell\geq 0}\binom{n}{j}\binom{\ell+n-1}{n-1}t^{j+\ell(s+1)}\\ &= \sum_{k\geq 0}t^k\sum_{j+\ell(s+1)=k}\binom{n}{j}\binom{\ell+n-1}{n-1}\\ &= \sum_{k\geq 0}t^k\sum_{\ell=0}^{\left\lfloor\frac{k}{s+1}\right\rfloor}\binom{n}{k-\ell(s+1)}\binom{\ell+n-1}{n-1}.\end{aligned}$$

By comparing the k-th coefficient, we obtain the desired result.

4 Modular *s*-Stirling Numbers

The *Stirling numbers of the second kind* $\left\{{n \atop k}\right\}$ can be determined by the recurrence relation $\left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\}$, with the initial conditions $\left\{{0 \atop 0}\right\} = 1$ and $\left\{{n \atop 0}\right\} = \left\{{0 \atop n}\right\} = 0$ for $n \geq 1$. It is well known that the $\left\{{n \atop k}\right\}$ are determined by the identities $x^n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^{\underline{k}}$, $n \geq 0$, where $x^{\underline{n}} = x(x-1)\cdots(x-(n-1))$ for $n \geq 1$ and $x^{\underline{0}} = 1$ or equivalently by the generating function

$$\sum_{n\geq k}\left\{{n \atop k}\right\}x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, \quad k \geq 0. \tag{5}$$

Using (5), it is not difficult to show that the Stirling numbers of the second kind are the specialization of the complete symmetric function given by

$$\left\{{n+k \atop n}\right\} = h_k(1,2,\ldots,n) = \sum_{a_1+\cdots+a_n=k} 1^{a_1}\cdots n^{a_n}. \tag{6}$$

Equation (6) can be interpreted by considering the following algorithm:

Algorithm 1 Interpretation of (6).

(1) Start with the partition of [1] given by $\{1\}$.
(2) Take every integer from 2 to $1+a_1$ and put it in the block of 1, so you end up having $\{1, 2, \ldots, 1+a_1\}$.
(3) Then you place $2+a_1$ in a new block and for every integer in between $3+a_1$ and $3+a_1+a_2$ you have 2 options, either you place this number in the first block or in the second one. You place $3+a_1+a_2$ in a new block and so now you will have 3 options.
(4) You keep doing this until you have placed $n+k$ elements.

For example, for $n = 3$ and $k = 5$ the term $1^2 2^1 3^2$ corresponds to a partition that looks like $\{1, 2, 3, *\}$, $\{4, *\}$, $\{6, *\}$, where 5 can go in either of the first 2 blocks and 7, 8 can go in any block (there are 3 of them). Giving a total of $2^1 \cdot 3^2$ options.

Notice then that the a_i integers have a direct relationship with the minimal elements in each of the blocks of a partition. To see this, consider the following construction: let $\Pi(n, k)$ denote the set of partitions of $[n]$ having k blocks. Let $\pi \in \Pi(n, k)$ be represented as $\pi = B_1/B_2/\cdots/B_k$, where B_i denotes the i-th block, with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. Call $m_i = \min(B_i)$ and define the vector of consecutive differences by

$$d(\pi) := (d_1, \ldots, d_k) = (m_2 - m_1 - 1, m_3 - m_2 - 1, \ldots, m_k - m_{k-1} - 1, n - m_k).$$

In the example above, notice that d_i corresponds to a_i because these are exactly the number of elements that we have to place in i blocks and so there are a total of i^{d_i} ways to do this. Notice, further, that since $m_1 = 1$, we have that $d_1 + \cdots + d_k = n - k$. If we impose the modularity conditions on the d_i's, we get the modular symmetric function defined in Eq. (1).

Notice that (6) can be written as $\left\{ n \atop k \right\} = h_{n-k}(1, 2, \ldots, k)$. From this last equation and the combinatorial motivation of $d(\pi)$, we introduce a new kind of Stirling numbers. For all integers $n \geq 0$ and all k with $0 \leq k \leq n$, *the modular s-Stirling numbers of the second kind*, denoted as $\left\{ n \atop k \right\}^{(s)}$, are defined by the expression

$$\left\{ n \atop k \right\}^{(s)} = M_{n-k}^{(s)}(1, 2, \ldots, k). \tag{7}$$

It is clear that for $s = 1$ we recover the Stirling numbers of the second kind, that is, $\left\{ n \atop k \right\}^{(1)} = \left\{ n \atop k \right\}$. From Theorem 2.2, we have the following recurrence relation:

$$\left\{ n \atop k \right\}^{(s)} = \left\{ n-1 \atop k-1 \right\}^{(s)} + k \left\{ n-2 \atop k-1 \right\}^{(s)} + k^{s+1} \left\{ n-s-1 \atop k \right\}^{(s)}, \tag{8}$$

with the initial conditions $\left\{ {n \atop 0} \right\}^{(s)} = \delta_{0,n}$ and $\left\{ {0 \atop k} \right\}^{(s)} = \delta_{k,0}$. Moreover, we have the following generating function:

$$\sum_{n \geq k} \left\{ {n \atop k} \right\}^{(s)} x^{n-k} = \prod_{r=1}^{k} \frac{1 + rx^s}{1 - (rx)^{s+1}}.$$

In Theorem 4.1, we give a combinatorial interpretation for the modular s-Stirling numbers.

Theorem 4.1 *The number of set partitions π in $\Pi(n,k)$, such that the entries in the vector $d(\pi)$ satisfies $d_i \equiv 0, 1 \pmod{s+1}$ for each $1 \leq i \leq k$ is given by the modular s-Stirling numbers $\left\{ {n \atop k} \right\}^{(s)}$.*

Proof By imposing the modularity conditions on the vector of consecutive differences given above and applying the Algorithm described on Page 7, the theorem follows. □

For example, $\left\{ {5 \atop 2} \right\}^{(2)} = 9$ corresponding to the set partitions

$$\begin{array}{lllll} 1234/5, & 1345/2, & 134/25, & 135/24, & 13/245, \\ 145/23, & 14/235, & 15/234, & 1/2345. & \end{array}$$

If you restrict the difference vector to have elements of the form $d_i \equiv 0 \pmod{s+1}$, then the number of such partitions is given by $h_{\lfloor \frac{n-k}{s+1} \rfloor}(1^{s+1}, \dots, k^{s+1})$.

On the other hand, the (unsigned) *Stirling numbers of the first kind* satisfy the recurrence relation $\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]$, with the initial conditions $\left[{0 \atop 0} \right] = 1$ and $\left[{n \atop 0} \right] = \left[{0 \atop n} \right] = 0$ for $n \geq 1$. This sequence can also be defined as the connection constants in the polynomial identity

$$x(x+1)\cdots(x+(n-1)) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k. \tag{9}$$

Theorem 4.2 *Let n and k be nonnegative integers and $s > 0$. If r is the remainder of k when divided by $s+1$, then the following equation holds*

$$\left\{ {n+k \atop n} \right\}^{(s)} = \sum_{i=0}^{\min\{\lfloor \frac{n-r}{s+1} \rfloor, \lfloor \frac{k}{s+1} \rfloor\}} h_{\lfloor \frac{k}{s+1} \rfloor - i}(1^{s+1}, \dots, n^{s+1}) \left[{n+1 \atop n+1-r-i(s+1)} \right].$$

Proof From Eq. (1) consider the subset of the variables $\{a_i\}_{i \in [n]}$ such that $a_j \equiv 1 \pmod{s+1}$. Call $J \subseteq [n]$ the set of their subindices in such a way that $j \in J$ if and only if $a_j \equiv 1 \pmod{s+1}$. Notice that for each $j \in J$, one must have that $a_j = b_j(s+1)+1$ for some $b_j \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
\left\{ {n+k \atop n} \right\}^{(s)} &= M_k^{(s)}(1,\ldots,n) = \sum_{\substack{J\subseteq[n] \\ |J|\equiv r \pmod{s+1}}} \sum_{\substack{a_1+\cdots+a_n=k \\ a_j\equiv 1 \pmod{s+1} \\ \text{for } j\in J}} 1^{a_1}\cdots n^{a_n} \\
&= \sum_{\substack{J\subseteq[n] \\ |J|\equiv r \pmod{s+1}}} \prod_{j\in J} j \sum_{(b_1+\cdots+b_n+\lfloor\frac{|J|}{s+1}\rfloor)+\frac{r}{s+1}=\frac{k}{s+1}} 1^{(s+1)b_1}\cdots n^{(s+1)b_n} \\
&= \sum_{\substack{J\subseteq[n] \\ |J|\equiv r \pmod{s+1}}} \prod_{j\in J} j \sum_{b_1+\cdots+b_n=\lfloor\frac{k}{s+1}\rfloor-\lfloor\frac{|J|}{s+1}\rfloor} 1^{(s+1)b_1}\cdots n^{(s+1)b_n} \\
&= \sum_{i=0}^{\lfloor\frac{n-r}{s+1}\rfloor} \left(\sum_{\substack{|J|=i(s+1)+r \\ J\subseteq[n]}} \prod_{j\in J} j \right) h_{\lfloor\frac{k}{s+1}\rfloor-i}(1^{s+1},\ldots,n^{s+1}).
\end{aligned}$$

From (9), the equality follows. □

For example, for $n=4$, $k=8$, and $s=3$ we have $\left\{ {4+8 \atop 4} \right\}^{(s)} = 107331$. On the other hand,

$$\begin{aligned}
&\sum_{i=0}^{1} h_{2-i}(1^4,2^4,3^4,4^4) \left[{5 \atop 5-4i} \right] \\
&= (1^4 2^4 + 1^4 3^4 + 2^4 3^4 + 1^4 4^4 + 2^4 4^4 + 3^4 4^4 + 1^8 + 2^8 + 3^8 + 4^8)\cdot 1 \\
&\quad + (1^4+2^4+3^4+4^4)\cdot 24 = 107331.
\end{aligned}$$

From Theorem 4.2 and by the little Fermat's theorem, i.e., $a^p \equiv a \pmod{p}$ with $a \in \mathbb{Z}$, we conclude the following interesting congruence.

Corollary 4.3 *Let n and k be nonnegative integers and p a prime number. If r is the remainder of k when divided by p, then the following congruence holds*

$$\left\{ {n+k \atop n} \right\}^{(p-1)} \equiv \sum_{i=0}^{\min\{\lfloor\frac{n-r}{p}\rfloor,\lfloor\frac{k}{p}\rfloor\}} \left\{ {n+\lfloor\frac{k}{p}\rfloor-i \atop n} \right\} \left[{n+1 \atop n+1-(r+i\cdot p)} \right] \pmod{p}.$$

4.1 The ℓ-Modular Symmetric Function

Given $0 \le \ell < s+1$, we can define the *ℓ-modular symmetric function* by

$$M_k^{(s,\ell)}(x_1,\ldots,x_n) := \sum_{\substack{j_1+\cdots+j_n=k \\ j_1,\ldots,j_n\equiv 0,\ell \pmod{s+1}}} x_1^{j_1}\cdots x_n^{j_n}.$$

Using the definition above, we can extend Theorem 4.2 using the *Stirling numbers of the first kind with higher level*, defined in [13, 14]. Some applications of the Stirling numbers of higher level in special polynomials can be found in [11, 12].

Let $\mathfrak{S}_n$ denote the set of permutations of the set $[n]$. We will assume that permutations are expressed in *standard cycle form*, i.e., minimal elements first within each cycle, with cycles arranged left-to-right in ascending order of minimal elements. If $n, k \geq 0$, then let $\mathfrak{S}_{(n,k)}$ denote the set of permutations of $\mathfrak{S}_n$ having exactly k cycles. It is clear that $\mathfrak{S}_n = \cup_{k=0}^{n} \mathfrak{S}_{(n,k)}$ and $|\mathfrak{S}_{(n,k)}| = \left[{n \atop k} \right]$. Given a permutation σ in $\mathfrak{S}_n$, let $\min(\sigma)$ denote the set of the minimal elements in each cycle of σ. For example, if $\sigma = (1\,4\,5)(2\,3)(6)(7\,9)(8)$, then we have that $\min(\sigma) = \{1, 2, 6, 7, 8\}$.

Given a positive integer s, let $\left[\!\left[{n \atop k} \right]\!\right]_s$ denote the number of ordered s-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_s) \in \mathfrak{S}_{(n,k)} \times \mathfrak{S}_{(n,k)} \times \cdots \times \mathfrak{S}_{(n,k)} = \mathfrak{S}_{(n,k)}^s$, such that $\min(\sigma_1) = \min(\sigma_2) = \cdots = \min(\sigma_s)$.

The sequence $\left[\!\left[{n \atop k} \right]\!\right]_s$ satisfies the following recurrence relation

$$\left[\!\left[{n \atop k} \right]\!\right]_s = \left[\!\left[{n-1 \atop k-1} \right]\!\right]_s + (n-1)^s \left[\!\left[{n-1 \atop k} \right]\!\right]_s, \tag{10}$$

with the initial conditions $\left[\!\left[{0 \atop 0} \right]\!\right]_s = 1$ and $\left[\!\left[{n \atop 0} \right]\!\right]_s = \left[\!\left[{0 \atop n} \right]\!\right]_s = 0$ hold for $n \geq 1$.

Given integers $n \geq 0$ and $s \geq 1$, let $\Omega_{n,s}(x)$ denote the polynomials

$$\Omega_{n,s}(x) := x(x + 1^s)(x + 2^s) \cdots (x + (n-1)^s), \quad \text{with} \quad \Omega_{0,s}(x) = 1.$$

The Stirling numbers of the first kind with higher level are the connection constants between the polynomials $(\Omega_{n,s}(x))_{n \geq 0}$ and the canonical basis $(x^n)_{n \geq 0}$. Indeed, if $n \geq 0$, then

$$\Omega_{n,s}(x) = x(x + 1^s)(x + 2^s) \cdots (x + (n-1)^s) = \sum_{k=0}^{n} \left[\!\left[{n \atop k} \right]\!\right]_s x^k. \tag{11}$$

From a similar argument as in Theorem 4.2, in combination with (11), we can obtain the following theorem.

Theorem 4.4 *Let n and k be nonnegative integers and $s + 1 > \ell \geq 0$, such that $\gcd(\ell, s+1) = 1$. If r is the remainder of $k\ell^{-1}$ when divided by $s + 1$, then the following equation holds*

$$M_k^{(s,\ell)}(1, \ldots, n) = \sum_{i=0}^{\min\{\lfloor \frac{n-r}{s+1} \rfloor, \lfloor \frac{k}{s+1} \rfloor - \lfloor \frac{r\ell}{s+1} \rfloor\}} h_{\lfloor \frac{k}{s+1} \rfloor - \lfloor \frac{r\ell}{s+1} \rfloor - i\ell}(1^{s+1}, \ldots, n^{s+1}) \left[\!\left[{n+1 \atop n+1-r-i(s+1)} \right]\!\right]_\ell.$$

5 The *s*-Elementary Symmetric Function

The *s-elementary symmetric polynomial* is defined by the expression

$$E_k^{(s)}(n) = \sum_{\substack{a_1+\cdots+a_n=k \\ a_i \le s}} x_1^{a_1} \cdots x_n^{a_n}. \tag{12}$$

An equivalent definition of this symmetric polynomial already exists in a paper by Bazeniar et al. [3]. For further properties of this symmetric function, see [1].

Theorem 5.1 *If* $s \equiv 1 \pmod 2$, *then for every positive integers n and k the following identity holds*

$$\sum_{i=0}^{k} (-1)^i E_i^{(s)}(n) \cdot M_{k-i}^{(s)}(n) = 0.$$

Proof The inverse of the generating function in Theorem 2.1 is given by

$$\prod_{i=1}^{n} \frac{1-(x_i t)^{s+1}}{1+x_i t} = \prod_{i=1}^{n} \frac{1-(-x_i t)^{s+1}}{1-(-x_i t)} = \prod_{i=1}^{n} \left(1 - x_i t + (-x_i t)^2 + \cdots + (-x_i t)^s\right).$$

In each product, we can create any number in between 1 and s. Hence

$$\sum_{k=0}^{n\cdot s} E_k^{(s)}(n)(-t)^k = \prod_{i=1}^{n} \frac{1-(x_i t)^{s+1}}{1+x_i t},$$

and the desired identity follows. □

We can express the modular symmetric function $M_k^{(s)}$ as convolutions involving the complete and elementary symmetric functions.

Theorem 5.2 *Let k, n, and s be positive integers and let* $x_1, x_2, \ldots, x_n$ *be independent variables. Then,*

$$M_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{\lfloor k/s+1 \rfloor} h_j(x_1^{s+1}, x_2^{s+1}, \ldots, x_n^{s+1}) e_{k-(s+1)j}(x_1, x_2, \ldots, x_n).$$

Proof According to (2), we have

$$\begin{aligned} &\sum_{k=0}^{\infty} M_k^{(s)}(x_1, x_2, \ldots, x_n) t^k \\ &= \left(\prod_{i=1}^{n} \frac{1}{1-(x_i t)^{s+1}}\right)\left(\prod_{i=1}^{n} (1+x_i t)\right) \end{aligned}$$

$$= \left(\sum_{j=0}^{\infty} h_j(x_1^{s+1}, x_2^{s+1}, \ldots, x_n^{s+1})(t)^{(s+1)j}\right)\left(\sum_{j=0}^{\infty} e_j(x_1, x_2, \ldots, x_n)t^j\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\lfloor k/s+1\rfloor} h_j(x_1^{s+1}, x_2^{s+1}, \ldots, x_n^{s+1})e_{k-(s+1)j}(x_1, x_2, \ldots, x_n)\right)t^k.$$

As required. □

Inspired by Theorem 5.2, we provide the following generalization.

Theorem 5.3 *Let k, n and s be three positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then,*

$$h_k(x_1^s, x_2^s, \ldots, x_n^s) = \sum_{j=0}^{k(s+1)} (-1)^j h_j(x_1, x_2, \ldots, x_n) M_{k(s+1)-j}^{(s)}(x_1, x_2, \ldots, x_n)$$

and

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{\lfloor k/s+1\rfloor} (-1)^j e_j(x_1, x_2, \ldots, x_n) M_{k-j(s+1)}^{(s)}(x_1, x_2, \ldots, x_n).$$

If k is not congruent to 0 modulo $s+1$, then

$$\sum_{j=0}^{k}(-1)^j h_j(x_1, x_2, \ldots, x_n) M_{k-j}^{(s)}(x_1, x_2, \ldots, x_n) = 0.$$

Proof The relation given in Theorem 2.1 can be rewritten as

$$\prod_{i=1}^{n} \frac{1}{1+x_i t} \sum_{k=0}^{\infty} M_k^{(s)}(x_1, x_2, \ldots, x_n)t^k = \prod_{i=1}^{n} \frac{1}{1-(x_i t)^{s+1}}$$

or

$$\prod_{i=1}^{n}(1-(x_i t)^{s+1}) \sum_{k=0}^{\infty} M_k^{(s)}(x_1, x_2, \ldots, x_n)t^k = \prod_{i=1}^{n}\big(1+(x_i t)\big).$$

Thus, we deduce that

$$\sum_{k=0}^{\infty} h_k(x_1^{s+1}, x_2^{s+1}, \ldots, x_n^{s+1})t^{k(s+1)}$$
$$= \left(\sum_{k=0}^{\infty}(-1)^k h_k(x_1, x_2, \ldots, x_n)t^k\right)\left(\sum_{k=0}^{\infty} M_k^{(s)}(x_1, x_2, \ldots, x_n)t^k\right)$$

and

$$\sum_{k=0}^{\infty} e_k(x_1, x_2, \ldots, x_n)t^k$$
$$= \left(\sum_{k=0}^{\infty}(-1)^k e_k(x_1^{s+1}, x_2^{s+1}, \ldots, x_n^{s+1})t^{k(s+1)}\right)\left(\sum_{k=0}^{\infty} M_k^{(s)}(x_1, x_2, \ldots, x_n)t^k\right).$$

The proof follows easily by comparing the coefficients of t^{ks} on both sides of these equations. □

The following result allows us to express a convolution of the modular symmetric function $M_k^{(s)}$ as convolutions involving the complete and elementary symmetric functions.

Theorem 5.4 *Let k, n and s be three positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then,*

$$\sum_{j=0}^{k} e_j(x_1, x_2, \ldots, x_n)h_{k-j}(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{k} M_j^{(s)}(x_1, x_2, \ldots, x_n)E_{k-j}^{(s)}(x_1, x_2, \ldots, x_n).$$

We can now define the modular s-Stirling numbers of the first kind by the following equality

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}^{(s)} = (n!)^s E_k^{(s)}\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right).$$

These numbers were introduced by Bazeniar et al. [2]. They are interpreted $\left[{n \atop k}\right]^{(s)}$ as the number of s-tuple permutations of $[n]$ having together k cycles. Inspired by the combinatorial interpretation given in Theorem 4.1 for the modular s-Stirling numbers of the second kind, we give in the following theorem another combinatorial interpretation for $\left[{n \atop k}\right]^{(s)}$.

Theorem 5.5 *Let n and k be nonnegative integers and $s > 0$. The s-modular Stirling numbers of the first kind $\left[{n+1 \atop k+1}\right]^{(s)}$ count the number of set partitions $\pi \in \Pi(n(s+1)-k, n)$ such that $d(\pi) = (d_1, \ldots, d_n)$ has the property that $d_i \leq s$ for every $1 \leq i \leq n$.*

Proof From (12), we have the equality

$$(n!)^s E_k^{(s)}\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right) = E_{n\cdot s-k}^{(s)}(1, \ldots, n).$$

Using the same argument as in Theorem 4.1, we obtain the desired result. □

From Theorem 5.5 (taking $s = 1$), we obtain a probably new combinatorial interpretation for the Stirling numbers of the first kind in terms of set partitions. Indeed, $\left[{n+1 \atop k+1}\right]$ enumerates the set partitions in $\Pi(2n-k, n)$, such that the vector $d(\pi) = (d_1, \ldots, d_n)$ has the property that $d_i \leq 1$ for every $1 \leq i \leq n$. For example, $\left[{3+1 \atop 1+1}\right] = 11$, the partitions being

$$1/23/45, \quad 1/235/4, \quad 12/3/45, \quad 13/2/45, \quad 12/34/5, \quad 12/35/4,$$
$$135/2/4, \quad 15/23/4, \quad 124/3/5, \quad 125/3/4, \quad 13/25/4.$$

The modular s-Stirling numbers of the first kind satisfy the following recurrence relation

$$\left[{n \atop k}\right]^{(s)} = \sum_{\ell=0}^{s} \left[{n-1 \atop k-(s-\ell)}\right]^{(s)} \cdot (n-1)^{\ell}, \tag{13}$$

where $\left[{0 \atop 1-s}\right]^{(s)} = 1$, $\left[{n \atop k}\right]^{(s)} = 0$ if $k < 1-s$. See also, Bazeniar et al. [2]. From this relation, we can give the following combinatorial interpretation.

Theorem 5.6 *Let n and k be nonnegative integers and $s > 0$. Consider the set of s-tuples of permutations $(\sigma_1, \sigma_2, \ldots, \sigma_s)$ such that $\min(\sigma_i) \subseteq \min(\sigma_{i-1})$ for all $i > 1$ and $\sum_{j=1}^{s} |\min(\sigma_j)| = k + s - 1$. Then the number of such elements equals $\left[{n \atop k}\right]^{(s)}$.*

Proof By the recursion given in (13), one has that

$$\left[{n \atop k}\right]^{(s)} = \sum_{\ell=0}^{s} \left[{n-1 \atop k-(s-\ell)}\right]^{(s)} (n-1)^{\ell} = \sum_{\ell=0}^{s} \left[{n-1 \atop k-\ell}\right]^{(s)} (n-1)^{s-\ell}.$$

This recurrence corresponds to choosing if the last element of each permutation, i.e., n is going to be fixed or not. Call ℓ the number of permutations where n is going to be a fixed point. By the condition we imposed in the tuple, these have to be the first ℓ elements of the s-tuple. For the remaining $s - \ell$ elements of the tuple, we have to choose an element from the remaining $n-1$ to have n as a preimage in each one of the permutations. We can do this in $(n-1)^{s-\ell}$ ways. This shows the claim because the initial condition is in $1-s$, meaning we need $k-(1-s) = k+s-1$ cycles to fill. □

For example, take $n = 3$ and $k = 4$, the following correspond to the 15 tuples counted by $\left[{3 \atop 4}\right]^{(3)}$ having in total $3 + 4 - 1 = 6$ cycles.

$$((1)(2)(3), (1)(2,3), (1,2,3)), \quad ((1)(2)(3), (1)(2,3), (1,3,2)),$$
$$((1)(2)(3), (1,2)(3), (1,2,3)), \quad ((1)(2)(3), (1,2)(3), (1,3,2)),$$

$$((1)(2)(3),(1,3)(2),(1,2,3)),\quad ((1)(2)(3),(1,3)(2),(1,3,2)),$$
$$((1)(2,3),(1)(2,3),(1)(2,3)),\quad ((1)(2,3),(1)(2,3),(1,3)(2)),$$
$$((1)(2,3),(1,3)(2),(1)(2,3)),\quad ((1)(2,3),(1,3)(2),(1,3)(2)),$$
$$((1,2)(3),(1,2)(3),(1,2)(3)),\quad ((1,3)(2),(1)(2,3),(1)(2,3)),$$
$$((1,3)(2),(1)(2,3),(1,3)(2)),\quad ((1,3)(2),(1,3)(2),(1)(2,3)),$$
$$((1,3)(2),(1,3)(2),(1,3)(2)).$$

6 Concluding Remarks

In this paper, we have discussed several combinatorial properties for a new family of symmetric functions. As a consequence, we introduce new combinatorial sequences related to the counting of some restricted set partitions. We also establish interesting congruences satisfied by these sequences.

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