Analysis on a Diffusive SI Epidemic Model with Logistic Source and Saturation Infection Mechanism

Lingmin Dong[1] · Bo Li[1] · Guanghui Zhang[2]

Abstract

In this paper, we consider an SI epidemic reaction–diffusion model with logistic source and saturation infection mechanism. We first establish the uniform boundedness and the extinction and persistence of the infectious disease in terms of the basic reproductive number. We also discuss the global stability of the unique endemic equilibrium when the spatial environment is homogeneous. Then we investigate the asymptotic behavior of the endemic equilibria in the heterogeneous environment when the movement rate of the susceptible and infected populations is small. Our results, together with the other two related epidemic models , not only show that the logistic growth, the infection mechanism, and the population movement can play an important role in the transmission dynamics of disease, but also suggest that increasing the inhibitory effect of the susceptible individuals instead of reducing the mobility of the populations can control the epidemic disease modeled by the SI system under consideration.

✉ Guanghui Zhang
  guanghuizhang@hust.edu.cn

  Lingmin Dong
  dlmdlm389@163.com

  Bo Li
  libo5181923@163.com

[1] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, China

[2] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

1 Introduction

More and more epidemic models with nonlinear incidence have been studied by many authors to describe the spreading of disease transmission and predict the trend of epidemics. One of the earliest models was discussed by Kermack and Mckendrickc [25], who used an SIR (susceptible–infected–recovered) compartmental model to model the plague epidemic in Bombay. In this SIR model, they assumed that the infected and susceptible individuals were completely mixed so that the density-dependent transmission βSI was used, where $\beta > 0$ is the transmission coefficient. However, the frequency-dependent transmission function $\beta SI/N$ (N is the total population) is applied in the situation that the infected and susceptible populations are randomly mixed. For more information about these two kinds of incidence functions, one may further refer to [10, 11, 59]. Michaelis–Menten combined the incidence rate of the density-dependent and frequency-dependent to derive a more general form $\beta C(N)SI/N$, where the Michaelis–Menten contact rate can be taken as $C(N) = \frac{aN}{1+bN}$; see for example [63]. In particular, if we take $C(N) = 1$, then the general one becomes the frequency-dependent transmission, and if we choose $a = 1$ and $b = 0$, then it becomes the density-dependent transmission βSI.

Capasso and Serio [5] found that the number of effective contacts between infected and susceptible individuals did not always increase linearly by studying the cholera epidemic propagate in Bari. Then they proposed a saturated incidence rate $g(I)S$ into the model, especially, $g(I) = \frac{\beta I}{1+mI}$, which means that the effective contacts may saturate at the high infective. While the incidence rate is saturated by the susceptible, May and Anderson [2] introduced the other saturated incidence rate $\frac{\beta SI}{1+\alpha S}$ to study the dynamics of host-parasite. Here, the positive number α and m are the coefficients that measure the inhibitory effect. The more general saturated incidence rate form can be found in [24]. Related discussion on the epidemic models with nonlinear incidence can be referred to [6, 22, 23, 47, 52, 56, 57, 61, 62] and the references therein.

In recent decades, people have realized that environmental heterogeneity and individual mobility can play an important role in studying the transmission of infectious diseases. Allen et al. [1] proposed an SIS (susceptible–infected–susceptible) epidemic reaction–diffusion model

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = -\dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega, \\[2mm]
\displaystyle\int_\Omega (S_0(x) + I_0(x))\mathrm{d}x = N > 0,
\end{cases}
\tag{1}
$$

where the habitat Ω is a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$; $S(x,t)$ and $I(x,t)$ represent, respectively, the densities of susceptible and infected individuals at location $x \in \Omega$ and time $t > 0$; the positive constants d_S and d_I represent the diffusion coefficients of the susceptible individuals and infected individuals, respectively; the positive Hölder continuous function $\beta(x)$ and $\gamma(x)$ stand for, respectively, the rates of disease transmission and recovery at x. The homogeneous Neumann boundary conditions are imposed which means no flux can cross the boundary $\partial\Omega$. Various kinds of SIS (susceptible–infected–susceptible) epidemic reaction–diffusion systems with standard incidence rates have been extensively studied; one may refer to [7–9, 16, 18, 21, 26, 27, 29, 31–37, 44, 46, 48, 49, 51, 53–55, 58] and the references therein.

By direct calculation, we can find that in (1) that the total population

$$
\int_\Omega (S(x,t) + I(x,t))\mathrm{d}x = \int_\Omega (S_0(x) + I_0(x))\mathrm{d}x = N
$$

is fixed for all $t > 0$. In general, the total population number can not always keep a constant in the real world. On the other hand, the birth date of susceptible population and the death rate induced by disease are important factors in the evolution of disease transmission; see [19, 20]. Then Li, Peng and Wang [35] used the recruitment term to describe the growth of the susceptible population. Moreover, most mathematical models imply that the logistic source seems to be a suitable choice of describing the intrinsic growth of the susceptible individuals; see, for instance, [12, 15]. Therefore, Li et al. [32] introduced the logistic source $a(x)S - b(x)S^2$ in the first equation of (1) to model the susceptible population growth. More precisely, the model proposed in [32] reads as follows.

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = a(x)S - b(x)S^2 - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega,
\end{cases}
\tag{2}
$$

where $a(x)$ and $b(x)$ are positive Hölder continuous functions, $a(x)$ and $\frac{a(x)}{b(x)}$ represent, respectively, the birth of susceptible populations and the intrinsic carrying capacity.

On the other hand, Huo et al. [22] discussed the following SIS epidemic model with saturated incidence rate and the logistic sources

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = a(x)S - b(x)S^2 - \dfrac{\beta(x)SI}{1+mI} + \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)SI}{1+mI} - \gamma(x)I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial S}{\partial v} = \dfrac{\partial I}{\partial v} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega.
\end{cases}
\tag{3}
$$

The models with the above-mentioned saturated incidence rate have been investigated by many mathematicians, see [24, 52, 56, 60–62].

Motivated by the above works, in this paper, we consider the following SI epidemic model:

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = a(x)S - b(x)S^2 - \dfrac{\beta(x)SI}{1+\alpha S} - \mu(x)S, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)SI}{1+\alpha S} - (\mu(x)+\eta(x))\,I, & x \in \Omega,\ t > 0, \\[2mm]
\dfrac{\partial S}{\partial v} = \dfrac{\partial I}{\partial v} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm]
S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega,
\end{cases}
\tag{4}
$$

where the parameters $\mu(x)$ and $\eta(x)$ are the positive Hölder continuous functions on $\overline{\Omega}$, $\mu(x)$ accounts for the rate of natural death and $\eta(x)$ is the death rate caused by the disease; α is a positive constant and measures the inhibitory effect. It is worth mentioning that Zhang et. al [62] studied a time-delay SIR epidemic model with the same incidence rate and the logistic source. One may further refer to [41, Chapter 15] for the derivation of the ODE version of model (4).

As in [4, 28, 64], we introduce the following favorable set:

$$
F^+ = \{x \in \Omega \,|\, a(x) > \mu(x)\}
$$

in order to reflect the feature of the heterogeneous environment. We always assume that the favorable set F^+ is nonempty through this paper. Then one can apply the standard theory of semilinear parabolic systems to conclude that (4) has a classical solution provided that the initial functions $S_0(x)$ and $I_0(x)$ are nonnegative continuous functions. If additionally $\int_\Omega I_0 dx > 0$, it follows from the strong maximum principle and the Hopf boundary Lemma of parabolic equations that $S(x,t) > 0$ and $I(x,t) > 0$ for $x \in \overline{\Omega}$ and $t > 0$. The steady-state problem corresponding to (4) satisfies the following elliptic system:

$$
\begin{cases}
-d_S \Delta S = a(x)S - b(x)S^2 - \dfrac{\beta(x)SI}{1+\alpha S} - \mu(x)S, & x \in \Omega, \\[2mm]
-d_I \Delta I = \dfrac{\beta(x)SI}{1+\alpha S} - (\mu(x)+\eta(x))\,I, & x \in \Omega, \\[2mm]
\dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{5}
$$

Biologically, only nonnegative solutions of (5) are of our interest. The solution $(S(x), I(x))$ of (5) is called a disease-free equilibrium (DFE) if $I(x) \equiv 0$ for all $x \in \Omega$; and the solution $(S(x), I(x))$ of (5) is called an endemic equilibrium (EE) if $I(x) > 0$ for some $x \in \Omega$. Then we obtain that any EE satisfies $S(x) > 0$ and $I(x) > 0$ for $x \in \overline{\Omega}$ by the strong maximum principle and the Hopf boundary lemma of elliptic equations.

In the present paper, our first goal is to study the extinction and persistence of the disease via the basic reproduction number $\mathcal{R}_0$. Indeed, Theorem 2 tells us that the disease vanishes if $\mathcal{R}_0 < 1$, whereas if $\mathcal{R}_0 > 1$ and $I_0(x) \not\equiv 0$, the solutions of system (4) is uniformly persistent and so the EE exists. Furthermore, we establish the global stability of EE by constructing a suitable Lyapunov function; see Theorem 3. As our second goal, we study the asymptotic profile of EE as the immigration rate of susceptible or infected individuals tends to zero (see Theorems 4–5). Our results show that the infectious disease always persists though the movement rate of susceptible or infected populations is controlled to be sufficiently small. Similar conclusions still hold with systems (2) and (3). These theoretical results imply that the controlling of the mobility of susceptible or infected populations in the epidemic model with logistic sources is not an effective strategy to eradicate the disease infection. In the discussion section, we will compare our results with those for the other two related models (2) and (3), in order to understand the effect of the incidence rate, the logistic sources, and the mobility of the population; see the last section for more details.

The rest of this paper is organized as follows. In Sect. 2, the dynamics of the epidemic model (4) are analyzed in terms of the basic reproduction number. First, the uniform boundedness of solutions to (4) is established; then, the definition and properties of the basic reproduction number are studied, and finally, the long-time behavior of system (4) by $\mathcal{R}_0$ is obtained. Section 3 is devoted to studying the global stability of EE and exploring the spatial distribution of the disease if the movement of the susceptible or infected populations is small. In the last section, the discussion of our paper and the comparisons of the results between our problem (4) and the related systems (2) and (3) are given.

2 Threshold Dynamical Behaviors

In this section, we aim to establish the dynamical behaviors of (4) in terms of the basic reproduction number $\mathcal{R}_0$. First of all, we study the auxiliary parabolic problem

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = a(x)S - b(x)S^2 - \mu(x)S, & x \in \Omega,\ t > 0, \\
\dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\
S(x,0) = S_0(x), & x \in \Omega.
\end{cases}
\tag{6}
$$

The associated steady-state problem satisfies

$$
\begin{cases}
-d_S \Delta S = a(x)S - b(x)S^2 - \mu(x)S, & x \in \Omega, \\
\dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{7}
$$

Denote by $\tilde{S}(x)$ the unique positive solution of (7) if it exists. Then, we use [4, Theorem A.1] to derive the following conclusion

Lemma 1 *Suppose that F^+ is nonempty. Consider the eigenvalue problem with indefinite weight:*

$$
\begin{cases}
-d_S \Delta \varphi(x) = \Lambda \left(a(x) - \mu(x)\right)\varphi, & x \in \Omega, \\
\dfrac{\partial \varphi(x)}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{8}
$$

If $\int_\Omega (a(x) - \mu(x))\,\mathrm{d}x < 0$, let $\Lambda_1(a(x) - \mu(x))$ be the principal positive eigenvalue of (8), and if $\int_\Omega (a(x) - \mu(x))\,\mathrm{d}x \geq 0$, set $\Lambda_1(a(x) - \mu(x)) = 0$. Then the problem has a unique positive steady-state $\tilde{S}$, which is a global attractor for nonnegative solutions when $0 < d_S < 1/\Lambda_1(a(x) - \mu(x))$. When $d_S \geq 1/\Lambda_1(a(x) - \mu(x))$, there is no positive steady state for (6), and all nonnegative solutions to (6) decay to 0 as $t \to \infty$.

2.1 The Uniform Bound of Solutions to (4)

From now on, for the sake of simplicity, let us denote

$$
h^* = \max_{x \in \bar{\Omega}} h(x), \quad h_* = \min_{x \in \bar{\Omega}} h(x),
$$

where $h(x) = \beta(x), a(x), b(x), \mu(x), \eta(x)$.

The uniform bounds of solutions of (4) are given as follows.

Theorem 1 *There exists a positive constant C independent of initial data such that*

$$
\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t \geq T,
\tag{9}
$$

for some large time $T > 0$.

Proof It follows from the first equation of (4) that

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S \leq a(x)S - b(x)S^2, & x \in \Omega, \ t > 0, \\[2mm]
\dfrac{\partial S}{\partial v} = 0, & x \in \partial\Omega, \ t > 0, \\[2mm]
S(x,0) = S_0(x), & x \in \Omega.
\end{cases}
$$

Let W be the solution of the following problem

$$
\begin{cases}
\dfrac{\partial W}{\partial t} - d_S \Delta W = a(x)W - b(x)W^2, & x \in \Omega, \ t > 0, \\[2mm]
\dfrac{\partial W}{\partial v} = 0, & x \in \partial\Omega, \ t > 0, \\[2mm]
W(x,0) = S_0(x), & x \in \Omega.
\end{cases}
$$

Applying the standard comparison principal, we get

$$
S(x,t) \leq W(x,t), \quad \text{for all } x \in \Omega \text{ and } t \geq 0. \tag{10}
$$

Moreover, it is well known that

$$
\limsup_{t \to \infty} S(x,t) \leq \lim_{t \to \infty} W(x,t) = \frac{a^*}{b_*} \quad \text{uniformly for } x \in \overline{\Omega}.
$$

In what follows, we use C to represent a positive constant, which is independent of d_S but allows to vary from line to line. Thus, we can find a large time $T_1 > 0$ such that

$$
S(x,t) \leq 2C, \quad t \geq T_1, \ x \in \Omega. \tag{11}
$$

Denote

$$
U(t) = \int_\Omega [S(x,t) + I(x,t)] \, dx.
$$

Then, it follows from (4) that

$$
\begin{aligned}
\frac{dU(t)}{dt} &= \int_\Omega a(x)S(x,t) - b(x)S^2(x,t)dx - \int_\Omega \mu(x)S(x,t)dx \\
&\quad - \int_\Omega (\mu(x) + \eta(x)) I(x,t)dx \\
&\leq \int_\Omega a(x)S(x,t)dx - \int_\Omega \mu(x)[S(x,t) + I(x,t)]dx.
\end{aligned}
$$

Due to (10), we obtain

$$\frac{dU(t)}{dt} \leq k - mU(t), \ t \geq T_1,$$

where $k = Ca^*|\Omega|$ and $m = \mu_*$. This yields

$$U(t) \leq U(0)e^{-mt} + \frac{k}{m}[1 - e^{-mt}], \ t \geq T_1.$$

That is,

$$\int_{\Omega} [S(x, t) + I(x, t)] \, dx \leq e^{-mt} \int_{\Omega} [S_0(x) + I_0(x)] \, dx + \frac{k}{m}[1 - e^{-mt}], \ t \geq T_1. \tag{12}$$

Therefore, we have

$$\limsup_{t \to \infty} \int_{\Omega} I(x, t) dx \leq \limsup_{t \to \infty} \int_{\Omega} [S(x, t) + I(x, t)] \, dx \leq \frac{k}{m}.$$

By setting

$$V(x, S, I) = a(x)S - b(x)S^2 - \frac{\beta(x)SI}{1 + \alpha S},$$

from (11), for all $x \in \overline{\Omega}$, $t \geq T_1$, we have

$$|V(x, S, I)| \leq a^*|S| + b^*|S|^2 + \frac{\beta_*}{\alpha}|I| \leq C + C|I|.$$

Now, we use [14, Lemma 2.1] (due to [30]) with $\sigma = p_0 = 1$ to the system (4) conclude that there exists a constant $C > 0$ independent of initial data such that

$$S(x, t), \ \ I(x, t) \leq C, \ \ \forall t > T_2, \ x \in \overline{\Omega}$$

for some $T_2 \geq T_1$. This completes our proof. $\square$

2.2 The Basic Reproduction Number

In this subsection, we will define the basic reproduction number $\mathcal{R}_0$ and show the properties of $\mathcal{R}_0$. Linearizing the equation of I of system (4) at DFE $(\tilde{S}(x), 0)$, we have the following parabolic problem

$$\begin{cases} \dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)\tilde{S}(x)}{1 + \alpha\tilde{S}(x)} I - (\mu(x) + \eta(x))\, I, & x \in \Omega, \ t > 0, \\[3mm] \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\[3mm] I(x, 0) = I_0(x), & x \in \Omega. \end{cases}$$

As in [1], we define the basic reproduction number $\mathcal{R}_0$:

$$\mathcal{R}_0 = \sup_{0 \neq \varphi \in H^1(\Omega)} \left\{ \dfrac{\int_\Omega \frac{\beta(x)\tilde{S}(x)}{1+\alpha\tilde{S}(x)} \varphi^2 \mathrm{d}x}{\int_\Omega d_I |\nabla\varphi|^2 + (\mu(x) + \eta(x))\, \varphi^2 \mathrm{d}x} \right\}. \tag{13}$$

It should be noticed that the basic reproduction number $\mathcal{R}_0$ defined by (13) implicitly depends not only on the diffusion rate d_I of the infected population but also on the diffusion rate d_S of the susceptible population and saturation rate α. To stress the dependence of $\mathcal{R}_0$ on these parameters, we denote by $\mathcal{R}_0(d_I, d_S, \alpha)$ as the basic reproduction number of system (4).

Let (λ_1, ψ_1) be the principal eigenpair of the eigenvalue problem:

$$\begin{cases} d_I \Delta\psi + \dfrac{\beta(x)\tilde{S}(x)}{1 + \alpha\tilde{S}(x)} \psi - (\mu(x) + \eta(x))\, \psi + \lambda\psi = 0, & x \in \Omega, \\[3mm] \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{14}$$

Then, we have the following properties.

Proposition 1 *Then the following statements hold.*

(a) *For fixed $d_S, \alpha > 0$, then*

(a1) $\mathcal{R}_0(d_I, d_S, \alpha)$ *is a monotone decreasing function of d_I with*

$$\mathcal{R}_0(d_I, d_S, \alpha) \to \max_{x \in \Omega} \dfrac{\beta(x)\tilde{S}(x)}{[1 + \alpha\tilde{S}(x)](\mu(x) + \eta(x))}, \quad \text{as } d_I \to 0,$$

and

$$\mathcal{R}_0(d_I, d_S, \alpha) \to \int_\Omega \dfrac{\beta(x)\tilde{S}(x)}{1 + \alpha\tilde{S}(x)} \mathrm{d}x \Big/ \int_\Omega (\mu(x) + \eta(x)) \mathrm{d}x, \quad \text{as } d_I \to \infty.$$

(a2) *If $\int_\Omega \frac{\beta(x)\tilde{S}(x)}{1+\alpha\tilde{S}(x)} \mathrm{d}x < \int_\Omega (\mu(x) + \eta(x)) \mathrm{d}x$ and $\frac{\beta(x)\tilde{S}(x)}{1+\alpha\tilde{S}(x)} - (\mu(x) + \eta(x))$ changes sign for $x \in \Omega$. Then there exists a threshold value $d_I \in (0, \infty)$ so that $\mathcal{R}_0(d_I, d_S, \alpha) < 1$ for $d_I > d_{I*}$ and $\mathcal{R}_0(d_I, d_S, \alpha) > 1$ for $d_I < d_{I*}$.*

(a3) *If $\int_\Omega \frac{\beta(x)\tilde{S}}{1+\alpha\tilde{S}} \mathrm{d}x \geq \int_\Omega (\mu(x) + \eta(x)) \mathrm{d}x$, then $\mathcal{R}_0(d_I, d_S, \alpha) > 1$ for all d_I.*

(b) *For fixed $d_I, \alpha > 0$, then*

$$\mathcal{R}_0(d_I, d_S, \alpha) \to \mathcal{R}_0^* := \sup_{0 \neq \varphi \in H^1(\Omega)} \left\{ \frac{\int_\Omega \frac{\beta(x)(a(x) - \mu(x))_+}{b(x) + \alpha(a(x) - \mu(x))_+} \varphi^2 \mathrm{d}x}{\int_\Omega d_I |\nabla \varphi|^2 + (\mu(x) + \eta(x)) \varphi^2 \mathrm{d}x} \right\},$$

as $d_S \to 0$, where

$$(a(x) - \mu(x))_+ = \begin{cases} a(x) - \mu(x), & a(x) > \mu(x), \\ 0, & a(x) \le \mu(x). \end{cases}$$

(c) *For fixed $d_I, d_S > 0$, then $\mathcal{R}_0(d_I, d_S, \alpha) \to 0$, as $\alpha \to \infty$.*
(d) *$\mathcal{R}_0(d_I, d_S, \alpha) > 1$ when $\lambda_1 < 0$, $\mathcal{R}_0(d_I, d_S, \alpha) = 1$ when $\lambda_1 = 0$ and $\mathcal{R}_0(d_I, d_S, \alpha) < 1$ when $\lambda_1 > 0$.*

The proof of Proposition 1 is similar to that of [1, Lemma 2.3], and hence, the details are omitted.

Remark 1 In view of [43, Lemma 3.2], the unique positive solution of (7) satisfies

$$\tilde{S}(x) \to \frac{(a(x) - \mu(x))_+}{b(x)} \quad \text{uniformly on } \overline{\Omega}, \text{ as } d_S \to 0.$$

Then the principal eigenvalue λ_1 of the eigenvalue problem (14) converges to λ_1^* as $d_S \to 0$, where λ_1^* is the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -d_I \Delta\phi = \dfrac{\beta(x)(a(x) - \mu(x))_+}{b(x) + \alpha(a(x) - \mu(x))_+} \phi - (\mu(x) + \eta(x)) \phi + \lambda\phi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{15}$$

Furthermore, $\mathcal{R}_0^* > 1$ when $\lambda_1^* < 0$, $\mathcal{R}_0^* = 1$ when $\lambda_1^* = 0$ and $\mathcal{R}_0^* < 1$ when $\lambda_1^* > 0$.

2.3 The Extinction and Persistence of Solutions to (4)

It turns out that the long-time dynamics of solutions of (4) is completely determined by $\mathcal{R}_0$. More precisely, we have

Theorem 2 *Let (S, I) be the unique solution of (4). Then the following statements hold.*

(i) *If $\mathcal{R}_0 < 1$, then*

$$\lim_{t \to \infty} (S(x, t) - \tilde{S}(x)) = 0 \quad \text{and} \quad \lim_{t \to \infty} I(x, t) = 0$$

uniformly for $x \in \overline{\Omega}$, where $\tilde{S}(x)$ is the unique positive solution of (6).

(ii) *If $\mathcal{R}_0 > 1$, then system (4) is uniformly persistent in the sense that for $I(\cdot, 0) \not\equiv 0$, there exists a constant $\epsilon_0 > 0$ independent of the initial data, such that any solution (S, I) satisfies*

$$\liminf_{t \to \infty} S(x, t) \geq \epsilon_0, \quad \liminf_{t \to \infty} I(x, t) \geq \epsilon_0$$

uniformly for $x \in \overline{\Omega}$. Furthermore, (4) admits at least one EE.

Proof By the first equation of (4), we see that

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S \leq a(x)S - b(x)S^2 - \mu(x)S, & x \in \Omega, \ t > 0, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ S(x, 0) = S_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{16}$$

Then, using the standard comparison principle for parabolic equation, we obtain

$$S(x, t) \leq \overline{S}(x, t), \quad \forall x \in \overline{\Omega}, \ t \geq 0,$$

where $\overline{S}$ is the unique solution of (6). Moreover, it follows from Lemma 1 that

$$\limsup_{t \to \infty} S(x, t) \leq \limsup_{t \to \infty} \overline{S}(x, t) = \tilde{S}(x) \ \text{ uniformly for all } \ x \in \overline{\Omega}. \tag{17}$$

For any given small $\varepsilon \geq 0$, there exists a large time $T > 0$ such that

$$S(x, t) \leq \tilde{S} + \varepsilon, \quad \forall x \in \overline{\Omega}, \ t \geq T. \tag{18}$$

Note that $\mathcal{R}_0 < 1$. Then, we use Proposition 1 to conclude that $\lambda_1 > 0$. Let $\lambda_1(\varepsilon)$ be the principal eigenvalue of (14) with $\tilde{S}(x)$ replaced by $(\tilde{S}(x) + \varepsilon)$ and let $\psi_1(x) > 0$ be the corresponding eigenfunction. Thus, we can choose small $\varepsilon > 0$ such that $\lambda_1(\varepsilon) > 0$ by the continuous dependence the principal eigenvalue on the parameters. For such ε, we apply (18) and the second equation of (4) to see that I satisfies

$$\begin{cases} \dfrac{\partial I}{\partial t} - d_I \Delta I \leq \dfrac{\beta(x)(\tilde{S}(x) + \varepsilon)}{1 + \alpha(\tilde{S}(x) + \varepsilon)} I - (\mu(x) + \eta(x)) I, & x \in \Omega, \ t > T, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, \ t > T. \end{cases}$$

Define $u(x, t) = M^* e^{-\lambda_1(\varepsilon)t} \psi_1(x)$, where the positive constant M_* is chosen such that $M_* \psi_1(x) \geq I(x, T)$ for all $x \in \Omega$. It is easily seen that $u(x, t)$ satisfies the following auxiliary system

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_I \Delta u = \dfrac{\beta(x)(\tilde{S}(x) + \varepsilon)}{1 + \alpha(\tilde{S}(x) + \varepsilon)} u - (\mu(x) + \eta(x))\, u, & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm] u(x, 0) = M_* \psi_1(x), & x \in \Omega. \end{cases}$$

It follows from the parabolic comparison principle that $I(x, t + T) \le u(x, t)$ for $x \in \Omega, t > 0$. Therefore, it holds that

$$I(x, t) \to 0 \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty.$$

Using this fact, for any small $\varepsilon > 0$, we can find $\tilde{T}$ such that $I(x, t) \le \varepsilon$ for $x \in \Omega$, $t \ge \tilde{T}$ and $a(x) - \varepsilon\beta^* - \mu(x) > 0$ for some $x \in \Omega$. It is easily seen from the first equation in (4) that

$$S_t - d_S \Delta S \ge a(x)S - b(x)S^2 - \varepsilon\beta^* S - \mu(x)S, \quad x \in \Omega,\ t \ge \tilde{T}.$$

Denote by $\tilde{W}$ the unique positive solution of the following problem

$$\begin{cases} \dfrac{\partial \tilde{W}}{\partial t} - d_S \Delta \tilde{W} = a(x)\tilde{W} - b(x)\tilde{W}^2 - \varepsilon\beta^* \tilde{W} - \mu(x)\tilde{W}, & x \in \Omega,\ t > 0, \\[2mm] \dfrac{\partial \tilde{W}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\[2mm] \tilde{W}(x, 0) = S(x, \tilde{T}) > 0, & x \in \Omega. \end{cases}$$

Then, we use the standard comparison principle for parabolic equations to infer that

$$S(x, t + \tilde{T}) \ge \tilde{W}(x, t), \quad \forall x \in \overline{\Omega},\ t \ge 0,$$

which yields that

$$\liminf_{t \to \infty} S(x, t) \ge \tilde{S}_\varepsilon(x) \text{ uniformly for all } x \in \overline{\Omega},$$

where $\tilde{S}_\varepsilon(x)$ is the unique positive solution of

$$\begin{cases} -d_S \Delta v = a(x)v - b(x)v^2 - \varepsilon\beta^* v - \mu(x)v, & x \in \Omega, \\[2mm] \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Here we used the fact that $a(x) - \varepsilon\beta^* - \mu(x) > 0$ for some $x \in \Omega$, which ensures the existence of the positive solution $\tilde{S}_\varepsilon(x)$. By the arbitrariness of ε, letting $\varepsilon \to 0$, we easily obtain

$$\liminf_{t \to \infty} S(x, t) \ge \tilde{S}(x) \text{ uniformly for all } x \in \overline{\Omega}.$$

Therefore, in light of (17), one can observe that

$$\lim_{t \to \infty} S(x, t) = \tilde{S}(x) \quad \text{uniformly for all } x \in \overline{\Omega}.$$

This completes the proof of (i).

Next, we will claim that (ii) holds. Assume that $\mathcal{R}_0 > 1$. We make use of the arguments of [40, Theorem 2.1]. Let $X = C(\overline{\Omega}, \mathbb{R}_+^2)$,

$$X_0 = \{(S_0, I_0) \in X, \ I_0 \not\equiv 0\}, \quad \partial X_0 = X \setminus X_0 = \{(S_0, I_0) \in X, \ I_0 \equiv 0\},$$

and $X = X_0 \cup \partial X_0$. For a given $(S_0, I_0) \in X$, the system (4) has a semiflow, denoted by $\Psi(t)$, and

$$\Psi(t)(S_0, I_0) = (S(\cdot, t), I(\cdot, t)), \ \forall t \geq 0,$$

where the $(S(\cdot, t), I(\cdot, t))$ is the unique solution of (4). In light of Theorem 1, $\Psi(t)$ is point dissipative. It also follows from the standard parabolic L^p-theory and embedding theorems that $\Psi(t)$ is compact from X to X for any fixed $t > 0$.

By the uniqueness of solutions, we observe that $I(x, t) \equiv 0$ for all $t \geq 0$. Then, by the similar process as in the proof of assertion (i), we can get

$$S(x, t) \to \tilde{S}(x) \quad \text{uniformly on } \overline{\Omega} \text{ as } t \to \infty.$$

This proves that $(\tilde{S}(x), 0)$ attracts $(S_0, I_0) \in \partial X_0$.

Set $M_0 = (\tilde{S}(x), 0)$. For any given $\epsilon >$, we are going to show that

$$\limsup_{t \to \infty} d(\Psi(t)(S_0, I_0), M_0) = \limsup_{t \to \infty} \|\Psi(t)(S_0, I_0) - M_0\| \geq \epsilon, \quad \forall (S_0, I_0) \in X_0. \tag{19}$$

Suppose that

$$\limsup_{t \to \infty} d(\Psi(t)(S_0, I_0), M_0) < \epsilon,$$

for some $(S_0, I_0) \in X_0$. Without loss of generality, there exists $T_0 > 0$ such that $d(\Psi(t)(S_0, I_0), M_0) < \epsilon$. Then, it is clearly that

$$\tilde{S}(x) - \epsilon < S(x, t) < \tilde{S}(x) + \epsilon, \quad \text{for } x \in \overline{\Omega}, \ t > T_0. \tag{20}$$

Due to $\mathcal{R}_0 > 1$, it follows from Proposition 1(d) that $\lambda_1 < 0$. We can choose a positive constant ϵ small enough such that $\lambda_1(\epsilon) < 0$, and $(\lambda_1(\epsilon), \Phi_1)$ is the eigenpair of the eigenvalue problem

$$\begin{cases} -d_I \Delta \Phi = \dfrac{\beta(x)(\tilde{S}(x) - \epsilon)}{1 + \alpha(\tilde{S}(x) - \epsilon)} \Phi - (\mu(x) + \eta(x)) \Phi + \lambda \Phi, & x \in \Omega, \\ \dfrac{\partial \Phi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Let $\omega(x, t) = \delta e^{-\lambda(\epsilon)t} \Phi_1(x)$, where the positive constant δ will be chosen below. Then ω satisfies

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - d_I \Delta \omega = \dfrac{\beta(x)(\tilde{S}(x) - \epsilon)}{1 + \alpha(\tilde{S}(x) - \epsilon)} \omega - (\mu(x) + \eta(x)) \omega, & x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ \omega(x, 0) = \delta \Phi_1(x), & x \in \Omega. \end{cases} \tag{21}$$

It follows from (20) that the second equation of (4) satisfies

$$\frac{\partial I}{\partial t} - d_I \Delta I \leq \frac{\beta(x)(\tilde{S}(x) - \epsilon)}{1 + \alpha(\tilde{S}(x) - \epsilon)} I - (\mu(x) + \eta(x)) I,$$

for $x \in \Omega$, $t > T_0$. If we choose δ small enough such that $\delta \Phi_1(x) < I(x, T_0)$ for $x \in \Omega$, we obtain that I is an upper solution to the problem (21), that is, $\omega(x, t) \leq I(x, t + T_0)$ for $x \in \Omega$ and $t > 0$. It follows from $\lambda_1(\epsilon) < 0$ that $I(x, t) \to \infty$ uniformly on $\overline{\Omega}$ as $t \to \infty$, which contradicts Theorem 1.

Finally, we can use the argument of [40, 50] to derive the desired conclusion of (ii). The proof is complete. □

2.4 The Global Stability of EE

In this subsection, we study the global stability of the EE of problem (4) in the spatially homogeneous environment. That is, we consider

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = aS - bS^2 - \dfrac{\beta SI}{1 + \alpha S} - \mu S, & x \in \Omega, \ t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta SI}{1 + \alpha S} - (\mu + \eta) I, & x \in \Omega, \ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ S(x, 0) = S_0(x), \ I(x, 0) = I_0(x), & x \in \Omega, \end{cases} \tag{22}$$

where $\gamma, \alpha, \beta, \mu, \eta$ are positive constants and $a > \mu$. Clearly, (22) has a unique EE, denoted by $(\hat{S}, \hat{I})$ if and only if $\mathcal{R}_0 = \frac{\beta(a - \mu)}{[b + \alpha(a - \mu)](\mu + \eta)} > 1$.

By elementary calculation, we have from the two equations of (22) that

$$\hat{S} = \frac{\mu + \eta}{\beta - \alpha(\mu + \eta)} = \frac{(\mathcal{R}_0 - 1)[\gamma + \alpha(a - \mu)] + b}{a - \mu},$$

$$\hat{I} = \frac{[a - b\hat{S}) - \mu](1 + \alpha\hat{S})}{\beta} = \frac{(\mathcal{R}_0 - 1)[(b + \alpha(a - \mu))(\mu + \eta)]}{[\beta - \alpha(\eta + \mu)]^2}.$$

Now, we consider the global stability of the EE under certain conditions as follows.

Theorem 3 *Assume that $\mathcal{R}_0 > 1$ holds, then the EE is globally asymptotically stable if*

$$\frac{\alpha\beta\hat{I}}{1 + \alpha\hat{S}} \leq b. \tag{23}$$

Proof We choose the following Lyapunov functional

$$V(t) := \int_{\Omega} L(S(x, t), I(x, t))dx, \quad t > 0,$$

where

$$L(S, I) = \int_{\hat{S}}^{S} \frac{\xi - \hat{S}}{\xi} d\xi + \kappa \int_{\hat{I}}^{I} \frac{\xi - \hat{I}}{\xi} d\xi,$$

with $\kappa = 1 + \alpha\hat{S}$.

For convenience, let us denote

$$f(S, I) = aS - bS^2 - \frac{\beta SI}{1 + \alpha S} - \mu S, \quad g(S, I) = \frac{\beta SI}{1 + \alpha S} - (\mu + \eta)I.$$

Then we have

$$L_S(S, I)f(S, I) = (S - \hat{S})\left[\frac{f(S, I)}{S} - \frac{f(\hat{S}, \hat{I})}{\hat{S}}\right]$$

$$= \left(-b + \frac{\alpha\beta\hat{I}}{(1 + \alpha S)(1 + \alpha\hat{S})}\right)(S - \hat{S})^2 - \frac{\beta}{1 + \alpha S}(S - \hat{S})(I - \hat{I}),$$

$$L_I(S, I)g(S, I) = (I - \hat{I})\left[\frac{g(S, I)}{I} - \frac{g(\hat{S}, \hat{I})}{\hat{I}}\right] = \frac{\kappa\beta}{(1 + \alpha S)(1 + \alpha\hat{S})}(S - \hat{S})(I - \hat{I}),$$

where we used the fact that $f(\hat{S}, \hat{I}) = g(\hat{S}, \hat{I}) = 0$. It follows that

$$
\begin{aligned}
\frac{d}{dt} V(t) &= \int_{\Omega} L_S(S, I) S_t + L_I(S, I) I_t \, dx \\
&= \int_{\Omega} \left(\frac{S - S^*}{S} d_S \Delta S + \kappa \frac{I - I^*}{I} d_I \Delta I \right) dx \\
&\quad + \int_{\Omega} \left(-b + \frac{\alpha \beta \hat{I}}{(1 + \alpha S)(1 + \alpha \hat{S})} \right) (S - \hat{S})^2 dx \\
&\quad - \int_{\Omega} \left[\frac{\beta}{1 + \alpha S} - \frac{\kappa \beta}{(1 + \alpha S)(1 + \alpha \hat{S})} \right] (S - \hat{S})(I - \hat{I}) dx \\
&= - \int_{\Omega} \left(\frac{S^*}{S^2} d_S |\nabla S|^2 + \kappa \frac{I^*}{I^2} d_I |\nabla I|^2 \right) dx \\
&\quad + \int_{\Omega} \left(-b + \frac{\alpha \beta \hat{I}}{(1 + \alpha S)(1 + \alpha \hat{S})} \right) (S - \hat{S})^2 dx.
\end{aligned}
$$

By (23), we have

$$
\begin{aligned}
V'(t) &< \int_{\Omega} \left(-b + \frac{\alpha \beta \hat{I}}{(1 + \alpha S)(1 + \alpha \hat{S})} \right) (S - \hat{S})^2 dx \\
&< \int_{\Omega} \left(-b + \frac{\alpha \beta \hat{I}}{(1 + \alpha \hat{S})} \right) (S - \hat{S})^2 dx \leq 0.
\end{aligned}
$$

Moreover, $V'(t) = 0$ if and only if $S = \hat{S}$ and $| \nabla I | = 0$. Denote

$$
E := \{ x \in H^1(\Omega) | V'(x) = 0 \}.
$$

Then it is easy to see that the maximal invariant subset of E is $(\hat{S}, \hat{I})$. By some standard arguments, we see that

$$
(S, I)(\cdot, t) \to (\hat{S}, \hat{I}) \quad \text{in } [H^1(\Omega)]^2, \quad \text{as } t \to \infty.
$$

Moreover, since we have the L^∞ estimates of S and I in Theorem 1, by some standard arguments, we know

$$
\| S(x, t) \|_{C^2(\bar{\Omega})} + \| I(x, t) \|_{C^2(\bar{\Omega})} \leq C_0, \quad \text{for all large } t,
$$

for some positive constant C_0. Therefore, the Sobolev embedding theorem allows one to assert

$$
(S, I)(\cdot, t) \to (\hat{S}, \hat{I}) \quad \text{in } [L^\infty(\Omega)]^2, \quad \text{as } t \to \infty,
$$

that is, $(\hat{S}, \hat{I})$ attracts all solutions of (4). Furthermore, using a similar process as in [43, Lemma 3.1], we see that the EE is globally asymptotically stable. This completes the proof.   □

Remark 2 If $\alpha = 0$, condition (23) always holds, and thus $(\hat{S}, \hat{I})$ is globally asymptotically stable as long as it exists. However, due to the appearance of the saturated incidence rate ($\alpha > 0$), $(\hat{S}, \hat{I})$ may be unstable. Let us denote $f(S, I)$, $g(S, I)$ as in the proof of Theorem 3. Then the Jacobian of the system (22) evaluated at $(\hat{S}, \hat{I})$ can be obtained easily as

$$\begin{pmatrix} f_S & f_I \\ g_S & g_I \end{pmatrix} = \begin{pmatrix} a - 2b\hat{S} - \frac{\beta\hat{I}}{(1+\alpha\hat{S})^2} - \mu & -\frac{\beta\hat{S}}{1+\alpha\hat{S}} \\ \frac{\beta\hat{I}}{(1+\alpha\hat{S})^2} & 0 \end{pmatrix}.$$

By checking the conditions of Turing instability [41], $(\hat{S}, \hat{I})$ is unstable if

$$d_I f_S + d_S g_I > 0 \quad \text{and} \quad d_I f_S + d_S g_I > 2\sqrt{d_I d_S (f_S g_I - g_S f_I)},$$

which hold provided that

$$a - 2b\hat{S} - \frac{\beta\hat{I}}{(1+\alpha\hat{S})^2} - \mu > 0,$$

$$d_I \left[a - 2b\hat{S} - \frac{\beta\hat{I}}{(1+\alpha\hat{S})^2} - \mu \right] > 2\sqrt{\frac{d_I d_S \beta^2 \hat{S}\hat{I}}{(1+\alpha\hat{S})^3}}.$$

However, the above conditions fail when $\alpha = 0$, because we have

$$a - 2b\hat{S} - \frac{\beta\hat{I}}{(1+\alpha\hat{S})^2} - \mu = a - 2b\hat{S} - \beta\hat{I} - \mu = -b\hat{S} < 0.$$

3 The Asymptotic Behavior of EE

This section is devoted to the investigation of the asymptotic behavior of the EE of (4) for the small mobility of susceptible or infected individuals in the spatially heterogeneous environment.

3.1 The Case of $d_S \to 0$

In this subsection, we aim to establish the asymptotic profiles of any positive solution of (4) as $d_S \to 0$ while $d_I > 0$ is fixed. Our main result can be stated as follows.

Theorem 4 *Assume that fix $d_I > 0$ and $\mathcal{R}_0^* > 1$, let $d_S \to 0$, then every positive solution (S, I) of (5) satisfies (up to a subsequence of $d_S \to 0$)*

$$(S, I) \to (S^{**}, I^{**}) \quad \text{uniformly on } \overline{\Omega},$$

where

$$S^{**} = G(x, I^{**}(x)) := \frac{1}{2\alpha b(x)} \Big[(\alpha a(x) - b(x) - \alpha \mu(x))$$
$$+ \sqrt{(\alpha a(x) - b(x) - \alpha \mu(x))^2 + 4\alpha b(a - \beta(x), I^{**}(x) - \mu(x))} \Big]$$

where I^{**} is a positive solution to

$$\begin{cases} -d_I \Delta I^{**} = \dfrac{\beta(x) G(x, I^{**}(x))}{1 + \alpha G(x, I^{**}(x))} I^{**}(x) - (\mu(x) + \eta(x)) I^{**}(x), & x \in \Omega, \\ \dfrac{\partial I^{**}}{\partial \nu} = 0, & x \in \partial \Omega. \end{cases} \tag{24}$$

Proof We divide our proof into three steps as follows.

Step 1. The estimates of $(S(x), I(x))$ of (5). Let $S(x_1) = \max\limits_{\overline{\Omega}} S(x)$. As in [39, Proposition 2.2] (or see [45]), we have $\Delta S(x) \le 0$. By the first equation of (5), it follows that

$$\frac{a^2(x_1)}{4b(x_1)} \ge \frac{\beta(x_1) S(x_1) I(x_1)}{1 + \alpha S(x_1)} + \mu(x_1) S(x_1) \ge \mu(x_1) S(x_1).$$

Thus, it holds that

$$S(x) \le S(x_1) \le \frac{(a^*)^2}{4b_* \mu_*} := C. \tag{25}$$

Here and in what follows, the positive constant C does not depend on $d_S > 0$ which varies from place to place.

We rewrite the second equation of (5) as follows

$$\begin{cases} \Delta I + \Big[\dfrac{\beta(x) S}{d_I(1 + \alpha S)} - \dfrac{\mu(x) + \eta(x)}{d_I} \Big] I = 0, & x \in \Omega, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega. \end{cases}$$

As

$$\Big\| \frac{\beta(x) S}{d_I(1 + \alpha S)} - \frac{\mu(x) + \eta(x)}{d_I} \Big\|_{L^\infty(\Omega)} \le \frac{\beta^*}{\alpha d_I} + \frac{\mu^* + \eta^*}{d_I},$$

we use the Harnack-type inequality (see, e.g., [38] or [43, Lemma 2.2]) to conclude that

$$\max_{\overline{\Omega}} I \le C \min_{\overline{\Omega}} I. \tag{26}$$

In view of (5), it follows that

$$\int_{\Omega} (\mu(x) + \eta(x)) \, I(x) \mathrm{d}x = \int_{\Omega} \frac{\beta(x) SI}{1 + \alpha S} \mathrm{d}x = \int_{\Omega} [a(x)S - b(x)S^2 - \mu(x)S] \mathrm{d}x$$
$$\leq \int_{\Omega} a(x) S(x) \mathrm{d}x,$$

which implies that

$$\int_{\Omega} I(x) \mathrm{d}x \leq \frac{1}{(\mu_* + \eta_*)} \int_{\Omega} (\mu(x) + \eta(x)) \, I(x) \mathrm{d}x$$
$$\leq \frac{a^*}{(\mu_* + \eta_*)} \int_{\Omega} S(x) \mathrm{d}x \leq C. \tag{27}$$

Now, we use (27) and integrate the second equation of system (5) over Ω to get

$$\beta_* \int_{\Omega} \frac{SI}{1 + \alpha S} \mathrm{d}x \leq (\eta^* + \mu^*) \int_{\Omega} I \mathrm{d}x \leq C. \tag{28}$$

In view of (26) and (27), it follows that

$$I(x) \leq C \min_{\overline{\Omega}} I \leq \frac{C}{|\Omega|} \int_{\Omega} I \mathrm{d}x \leq C, \quad \forall x \in \overline{\Omega}. \tag{29}$$

Suppose that I has no positive lower bound, we can find a subsequence of $d_S \to 0$, say $d_n := d_{S,n}$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n, I_n) := (S_{d_{S,n}}, I_{d_{S,n}})$ of (5) with $d_S = d_n$, such that $\min_{\overline{\Omega}} I_n \to 0$. Then, we apply (26) to obtain that

$$I_n \to 0 \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to 0.$$

We may choose arbitrarily small $\epsilon > 0$ such that

$$0 \leq I_n(\cdot) \leq \epsilon \quad \text{for all large } n.$$

This fact, together with the first equation of (5), implies that for all large n, (S_n, I_n) satisfies

$$-d_n \Delta S_n \leq a(x) S_n - b(x) S_n^2 - \mu(x) S_n, \ x \in \Omega; \quad \frac{\partial S_n}{\partial \nu} = 0, \ x \in \partial\Omega$$

and

$$-d_n \Delta S_n \geq a(x) S_n - b(x) S_n^2 - \epsilon \beta^* S_n - \mu(x) S_n, \ x \in \Omega; \quad \frac{\partial S_n}{\partial \nu} = 0, \ x \in \partial\Omega.$$

Given any large n, consider the following two auxiliary systems:

$$- d_n \Delta w = a(x)w - b(x)w^2 - \mu(x)w, \ x \in \Omega; \quad \frac{\partial w}{\partial \nu} = 0, \ x \in \partial\Omega \qquad (30)$$

and

$$- d_n \Delta v = a(x)v - b(x)v^2 - \epsilon\beta^* v - \mu(x)v, \ x \in \Omega; \quad \frac{\partial v}{\partial \nu} = 0, \ x \in \partial\Omega. \ (31)$$

Denote by w_n and v_n the unique positive solution of (30) and (31), respectively. Using a simple comparison argument, we deduce that

$$w_n \leq S_n \leq u_n \quad \text{on } \overline{\Omega}, \quad \text{for all large } n. \qquad (32)$$

By the similar argument to those in [13, Lemma 2.2], it is not hard to show that

$$w_n \to \frac{(a(x) - \mu(x))_+}{b(x)} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty$$

and

$$v_n \to \frac{(a(x) - \mu(x) - \epsilon\beta^*)_+}{b(x)} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty.$$

Letting $n \to \infty$ in (32) gives

$$\frac{(a(x) - \mu(x) - \epsilon\beta^*)_+}{b(x)} \leq \liminf_{n\to\infty} S_n(x)$$
$$\leq \limsup_{n\to\infty} S_n(x) \leq \frac{(a(x) - \mu(x))_+}{b(x)} \quad \text{on } \overline{\Omega}. \qquad (33)$$

Due to the arbitrary choice of small ϵ, one immediately gets

$$S_n \to \frac{(a(x) - \mu(x))_+}{b(x)} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty. \qquad (34)$$

From the second equation of (5), I_n satisfies

$$- d_I \Delta I_n = \frac{\beta(x)S_n I_n}{1 + \alpha S_n} - (\mu(x) + \eta(x)) I_n, \ x \in \Omega; \quad \frac{\partial I_n}{\partial \nu} = 0, \ x \in \partial\Omega. \ (35)$$

Define

$$\tilde{I}_n := \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}.$$

Then $\|\tilde{I}_n\|_{L^\infty(\Omega)} = 1$ for all $n \geq 1$, and $\tilde{I}_n$ solves

$$- d_I \Delta \tilde{I}_n = \left[\frac{\beta(x) S_n}{1 + \alpha S_n} - (\mu(x) + \eta(x)) \right] \tilde{I}_n, \ x \in \Omega; \quad \frac{\partial \tilde{I}_n}{\partial \nu} = 0, \ x \in \partial\Omega. \tag{36}$$

By a standard compactness argument for elliptic equations and passing to a further subsequence (if necessary), we may assume that

$$\tilde{I}_n \to \tilde{I} \ \text{in} \ C^1(\overline{\Omega}), \quad \text{as } n \to \infty, \tag{37}$$

where $\tilde{I} \in C^1(\overline{\Omega})$ with $\tilde{I} \geq 0$ on $\overline{\Omega}$ and $\|\tilde{I}\|_{L^\infty(\Omega)} = 1$. In view of (34), (36) and (37), it follows that $\tilde{I}$ satisfies

$$\begin{cases} -d_I \Delta \tilde{I} = \left[\dfrac{\beta(x)(a(x) - \mu(x))_+}{b(x) + \alpha(a(x) - \mu(x))_+} - (\mu(x) + \eta(x)) \right] \tilde{I}, \ x \in \Omega, \\ \dfrac{\partial \tilde{I}}{\partial \nu} = 0, \ x \in \partial\Omega. \end{cases}$$

From the Harnack-type inequality (see, [38] or [43, Lemma 2.2]), it follows that $\tilde{I} > 0$ on $\overline{\Omega}$. For the uniqueness of the principal eigenvalue, it follows that $\lambda_1^* = 0$. This is a contradiction with our assumption that $\lambda_1^* < 0$. Thus, I has a positive lower bound C, which is independent of $0 < d_S \leq 1$.

Step 2. Convergence of I. Obviously, I satisfies

$$\begin{cases} -d_I \Delta I + (\mu(x) + \eta(x)) I = \dfrac{\beta(x) S}{1 + \alpha S} I, & x \in \Omega, \\ \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{38}$$

In view of (25) and (29), we have

$$\left\| \frac{\beta(x) S}{1 + \alpha S} I \right\|_{L^p(\Omega)} \leq \|\beta(x) S I\|_{L^p(\Omega)} \leq C, \quad \forall \, p > 1.$$

By the standard L^p-estimate for elliptic equations (see, e.g., [17]), we see that

$$\|I\|_{W^{2,p}(\Omega)} \leq C \ \text{for any given } p > 1.$$

Taking p to be sufficiently large, we see from the embedding theorem (see, e.g., [17]) that

$$\|I\|_{C^{1+\theta}(\overline{\Omega})} \leq C \ \text{for some } 0 < \theta < 1.$$

Therefore, there exists a subsequence of $d_S \to 0$, say $d_n := d_{S,n}$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n, I_n) := (S_{d_{S,n}}, I_{d_{S,n}})$ of (5) with

$d_S = d_n$, such that

$$I_n \to I^{**} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty, \tag{39}$$

where $I^{**} \in C^1(\overline{\Omega})$ and $I^{**} > 0$.

Step 3. Convergence of S. From the first equation of (5), S_n satisfies

$$\begin{cases} -d_n \Delta S_n = a(x) S_n - b(x) S_n^2 - \dfrac{\beta(x) S_n I_n}{1 + \alpha S_n} - \mu(x) S_n, & x \in \Omega, \\ \dfrac{\partial S_n}{\partial \nu} = 0, & x \in \partial \Omega. \end{cases}$$

Due to (39), given any small $\epsilon > 0$, we have for all large n that

$$\begin{aligned} a(x)S_n - b(x)S_n^2 &- \frac{\beta(x)S_n I_n}{1 + \alpha S_n} - \mu(x)S_n \\ &\leq a(x)S_n - b(x)S_n^2 - \frac{\beta(x)S_n(I^{**} - \epsilon)}{1 + \alpha S_n} - \mu(x)S_n \\ &= S_n \frac{(a(x) - b(x)S_n)(1 + \alpha S_n) - \beta(x)(I^{**} - \epsilon) - \mu(x)(1 + \alpha S_n)}{1 + \alpha S_n} \\ &=: S_n \frac{(g_+^{1,\epsilon}(x, I^{**}(x)) - S_n)(S_n - g_-^{1,\epsilon}(x, I^{**}(x)))}{1 + \alpha S_n}. \end{aligned}$$

Here, $g_+^{1,\epsilon}(x, I^{**}(x))$ and $g_-^{1,\epsilon}(x, I^{**}(x))$ are the root of the following equation with respect to the unknown function g:

$$(a(x) - b(x)g)(1 + \alpha g) - \beta(x)(I^{**} - \epsilon) - \mu(x)(1 + \alpha g) = 0.$$

For large enough n, we consider the following auxiliary problem

$$\begin{cases} -d_n \Delta W = W \dfrac{(g_+^{1,\epsilon}(x, I^{**}(x)) - W)(W - g_-^{1,\epsilon}(x, I^{**}(x)))}{1 + \alpha W}, & x \in \Omega \\ \dfrac{\partial W}{\partial \nu} = 0, & x \in \partial \Omega. \end{cases} \tag{40}$$

In view of the bounds of S and I in the proof of step 1, we can further assume that $g_+^{1,\epsilon} > g_-^{1,\epsilon} \geq 0$ on $\overline{\Omega}$. In addition, S_n is a subsolution to (40) and a sufficiently large positive constant C is a supersolution. Hence, (40) has at least one positive solution denoted by W_n which satisfies $S_n \leq W_n \leq C$.

By the similar argument as in proof of [13, Lemma 2.2] (or [32, Lemma 5.1]), we find that

$$W_n \to g_+^{1,\epsilon} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty.$$

Therefore, we have

$$\limsup_{n\to\infty} S_n \le g_+^{1,\epsilon}, \quad \text{uniformly on } \overline{\Omega}. \tag{41}$$

On the other hand, by (39), for all large n we have

$$a(x)S_n - b(x)S_n^2 - \frac{\beta(x)S_n I_n}{1+\alpha S_n} - \mu(x)S_n$$

$$\ge a(x)S_n - b(x)S_n^2 - \frac{\beta(x)S_n(I^{**}+\epsilon)}{1+\alpha S_n} - \mu(x)S_n$$

$$= S_n \frac{(a(x)-b(x)S_n)(1+\alpha S_n) - \beta(x)(I^{**}+\epsilon) - \mu(x)(1+\alpha S_n)}{1+\alpha S_n}$$

$$=: S_n \frac{(g_+^{2,\epsilon}(x, I^{**}(x)) - S_n)(S_n - g_-^{2,\epsilon}(x, I^{**}(x)))}{1+\alpha S_n}.$$

Here, $g_+^{2,\epsilon}(x, I^{**}(x))$ and $g_-^{2,\epsilon}(x, I^{**}(x))$ are the root of the following equation with respect to the unknown function g:

$$(a(x)-b(x)g)(1+\alpha g) - \beta(x)(I^{**}+\epsilon) - \mu(x)(1+\alpha g) = 0.$$

Consider the following auxiliary problem:

$$\begin{cases} -d_n \Delta W = W \dfrac{(g_+^{2,\epsilon}(x, I^{**}(x)) - W)(W - g_-^{2,\epsilon}(x, I^{**}(x)))}{1+\alpha W}, & x \in \Omega \\ \dfrac{\partial W}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{42}$$

In the same fashion, we also get $g_+^{2,\epsilon} > g_-^{2,\epsilon} \ge 0$. Observe that S_n and 0 is a pair of upper and lower solution of (42). Hence, one can assert that (42) admits at least one positive solution, and further get

$$\liminf_{n\to\infty} S_n(x) \ge g_+^{2,\epsilon}(x, I^{**}(x)) \quad \text{uniformly on } \overline{\Omega}. \tag{43}$$

As

$$g_+^{1,0}(x, I^{**}(x)) = g_+^{2,0}(x, I^{**}(x)) := G(x, I^{**}(x)),$$

by the arbitrariness of ϵ, it immediately follows from (41) and (43) that

$$S_n(x) \to G(x, I^{**}(x)) \quad \text{uniformly for } x \in \overline{\Omega}, \quad \text{as } n \to \infty.$$

Furthermore, it is easily seen that I^{**} satisfies (24) by (35). $\qquad\square$

3.2 The Case of $d_I \to 0$

We now fix $d_S > 0$ and analyze the asymptotic behavior of positive solution of (5) as $d_I \to 0$. Due to mathematical difficulty, we will consider one space dimension case by taking $\Omega = (0, 1)$. By Proposition 1 (a1) and Theorem 2 (ii), to ensure the existence of positive solutions of (5) for all small d_I, it is necessary to assume that $\{\beta(x)\tilde{S}/(1 + \alpha\tilde{S}) > (\eta(x) + \mu(x)) : x \in [0, 1]\}$ is nonempty.

Theorem 5 *Assume that the set $\{\beta(x)\tilde{S}/(1 + \alpha\tilde{S}) > (\eta(x) + \mu(x)) : x \in [0, 1]\}$ is nonempty and fix $d_S > 0$, let $d_I \to 0$ then every positive solution (S, I) of (5) satisfies (up to a subsequence of $d_I \to 0$) that*

$$S \to S^0 \quad uniformly \ on \ [0, 1],$$

where $S^0 \in C([0, 1])$ and $S^0 > 0$ on $[0, 1]$, and $\int_0^1 I dx \to I^0$ for some positive constant I^0.

Proof It is easy to check that (25), (27) and (28) remain true for all $d_I > 0$. Note that S satisfies

$$\begin{cases} -d_S S'' + b(x)S^2 + \mu(x)S = a(x)S - \dfrac{\beta(x)SI}{1 + \alpha S}, & x \in (0, 1), \\ S'(0) = S'(1) = 0. \end{cases} \tag{44}$$

One can use the well-known elliptic L^1-theory [3] (or see [42, Lemma 2.2]) to (44) to find that

$$\|S\|_{W^{1,p}(0,1)} \le C \quad \text{for any given } p > 1.$$

By taking a properly large p and using the Sobolev embedding theorem, we have

$$\|S\|_{C^\theta([0,1])} \le C \quad \text{for some } 0 < \theta < 1.$$

This tells us that there exists a subsequence of $d_I \to 0$, say $d_n := d_{I,n}$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n, I_n) := (S_{d_{I,n}}, I_{d_{I,n}})$ of (5) with $d_I = d_n$, such that

$$S_n \to S^0 \ \text{ in } \ C([0, 1]) \ \text{ as } \ n \to \infty.$$

On the other hand, by (27), up to a further subsequence of d_n if necessary, it follows that $\int_0^1 I_n dx \to I^0$ as $n \to \infty$.

In what follows, we are going to show I^0, $S^0 > 0$. We first prove $I^0 > 0$. To this end, we use a contradiction argument and suppose that $I^0 = 0$. By integrating (44) from 0 to x, we have

$$(S^0)'(x) = -\frac{1}{d_S} \int_0^x \left[a(y)S_n(y) - b(y)S_n^2(y) - \frac{\beta(y)S_n(y)I_n(y)}{1 + \alpha S_n(y)} - \mu(y)S_n(y) \right] dy$$

uniformly on [0, 1]. Letting $n \to \infty$, due to $\int_0^1 I_n dx \to 0$, one can infer that

$$S_n'(x) = -\frac{1}{d_S} \int_0^x \left[a(y)S^0(y) - b(y)(S^0)^2(y) - \mu(y)S^0(y) \right] dy \text{ uniformly on } [0, 1].$$

By means of $S_n(x) - S_n(0) = \int_0^x S_n'(y) dy$ for any $n \geq 1$, it is easily seen that S_0 satisfies

$$S^0(x) - S^0(0) = -\frac{1}{d_S} \int_0^x \left\{ \int_0^y \left[a(z)S^0(z) - b(z)(S^0)^2(z) - \mu(z)S^0(z) \right] dz \right\} dy,$$

which in turn gives

$$\begin{cases} -d_S (S^0)'' = a(x)S^0(x) - b(x)(S^0)^2(x) - \mu(x)S^0, & x \in (0, 1), \\ (S^0)'(0) = 0. \end{cases} \tag{45}$$

Again, one can integrate (44) from x to 1 and apply a similar process as before to deduce that $(S^0)'(1) = 0$. Hence, by virtue of (45), we can conclude that $S^0 = \tilde{S}$, which means that $S_n \to \tilde{S}$ uniformly on [0, 1] as $n \to \infty$.

One can easily observe that $\lambda_1^n = 0$, $\forall n \geq 1$, where λ_1^n is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} d_n \Delta \psi + \dfrac{\beta(x)S_n(x)}{1 + \alpha S_n(x)} \psi - (\mu(x) + \eta(x)) \psi + \lambda \psi = 0, & x \in (0, 1), \\ \psi'(1) = \psi'(0) = 0. \end{cases}$$

Applying the same analysis as in [1, Lemma 2.3], it follows that

$$0 = \lambda_1^n \to$$
$$\min_{x \in [0,1]} \left(\mu(x) + \eta(x) \right) - \frac{\beta(x)\tilde{S}}{1 + \alpha\tilde{S}}, \text{ as } n \to \infty.$$

Clearly, this leads to a contradiction because $\{\beta(x)\tilde{S}/(1 + \alpha\tilde{S}) > (\eta(x) + \mu(x)) : x \in (0, 1)\}$ is non-empty by our assumption. Thus, we must have $I^0 > 0$.

To show $S^0 > 0$ on [0, 1], we proceed indirectly again and suppose that $S^0(x) = 0$ for some $x \in [0, 1]$. Then, applying the Harnack inequality to the S-equation, one will see immediately that $S^0(x) = 0$ for all $x \in [0, 1]$. As a result, we have $\int_0^1 S_n dx \to 0$ as $n \to \infty$. Integrating the second equation of (5) from 0 to 1, one gets

$$(\eta_* + \mu_*) \int_0^1 I_n dx \leq \beta^* \int_0^1 \frac{S_n I_n}{1 + \alpha S_n} dx \to 0, \text{ uniformly as } n \to \infty,$$

which yields $\int_0^1 I_n dx \to 0$. This arrives at a contradiction with $\int_0^1 I_n dx \to I^0 > 0$. The proof is complete. □

4 Discussion

In this paper, we have studied the SI epidemic model (4) with logistic source and saturation infection mechanism. For the parabolic problem (4), we have established the uniform boundedness and the extinction and persistence of the infectious disease in terms of the basic reproductive number $\mathcal{R}_0$. We also obtained the global stability of the unique endemic equilibrium when the spatial environment is homogeneous. For the steady-state solution problem (5), we have investigated the asymptotic behavior of the endemic equilibria in the heterogeneous environment when the movement rate of the susceptible and infected populations is small.

In what follows, we first want to compare the influence of immigration rate, logistic sources, and incidence rate on the basic reproduction number of models (1)–(4).

Allen et al. [1] introduced the epidemic model (1) with standard incidence rate $\frac{SI}{S+I}$ and defined the basic reproduction number

$$\mathcal{R}_0 = \sup_{0 \neq \varphi \in H^1(\Omega)} \left\{ \frac{\int_\Omega \beta(x) \varphi^2 \mathrm{d}x}{\int_\Omega (d_I |\nabla \varphi|^2 + \gamma(x) \varphi^2) \mathrm{d}x} \right\}.$$

It is clear that the $\mathcal{R}_0$ defined here depends on the immigration rate of infected individuals d_I, the transmission rate $\beta(x)$, and the recovery rate $\gamma(x)$. Then, Li et al. [32] added the logistic source $a(x)S - b(x)S^2$ to system (1); but the basic reproduction number is the same as the one that without logistic source. In other words, the logistic sources have no influence on the definition of the basic number of the epidemic model with standard incidence rate.

As the infection mechanism changes to the saturated incidence rate $\frac{SI}{1+mI}$, Huo and Cui in [22] defined a basic reproduction number of system (3) as

$$\mathcal{R}_0 = \sup_{0 \neq \varphi \in H^1(\Omega)} \left\{ \frac{\int_\Omega \beta(x) \hat{S}(x) \varphi^2 \mathrm{d}x}{\int_\Omega (d_I |\nabla \varphi|^2 + \gamma(x) \varphi^2) \mathrm{d}x} \right\},$$

where $\hat{S}$ is the unique solution of (7) with $\mu(x) \equiv 0$. It is easily seen that such $\mathcal{R}_0$ depends on $\hat{S}$, which is continuously dependent on the logistic sources and parameter d_S, except the coefficients d_I, $\beta(x)$ and $\gamma(x)$. However, the basic reproduction number $\mathcal{R}_0$ of our model (4) not only depends on the parameters d_S, d_I, $a(x)$, $b(x)$, $\beta(x)$ and $\gamma(x)$ but also depends on the death rate $\mu(x)$ and the saturated coefficient α.

In the spatially heterogeneous environment, we have obtained that threshold dynamics in terms of the basic reproduction number, that is, the uniform persistence property holds if $\mathcal{R}_0 > 1$, and the disease extinction occurs if $\mathcal{R}_0 < 1$. Based on the ultimately uniform boundedness in Theorem 1, we established the uniform persistence property of (4), that is, there existed at least one EE when $\mathcal{R}_0 > 1$. Moreover, the disease will die out if the saturation factor α is large enough. Biologically, the epidemic will be extinct in the long run provided that the effective prevention measures (α) are taken. For example, people disinfected, washed, and blocked the market of infected food during the outbreak of the COVID-19. Therefore, we know how important the effective

prevention and control strategy is in the absence of sufficient medical treatment and vaccines.

Finally, we discussed that the global stability and asymptotic profiles of the endemic equilibrium. When the environment is spatially homogeneous, that is, all the parameters in (4) are positive constants, the global stability of endemic equilibrium has been shown by establishing suitable Lyapunov function for the basic reproduction number $\mathcal{R}_0 > 1$; see Theorem 3. By Remark 2, when $\alpha = 0$, condition (23) always holds. Hence, $(\hat{S}, \hat{I})$ is globally asymptotically stable as long as it exists. However, if the saturated incidence rate $\alpha > 0$, $(\hat{S}, \hat{I})$ may be unstable.

Furthermore, in the case of $d_S \to 0$, Theorem 4 shows that the disease exists in the entire habitat. On the other hand, Theorem 5 suggests that the susceptible population is positive while the total infected population tends to a positive constant as $d_I \to 0$ in the one-dimensional interval. The results have suggested that the density of the infected population will not vanish when the mobility of the susceptible or infected population goes to zero. The above discussion reveals that more effective measures α should be taken to control the sources of infection and cut off the channels of transmission so as to eradicate the disease.

Indeed, [22, Theorems 4.2 and 4.3], [32, Theorems 4.1–4.2] and Theorems 4–5 here have shown that the infectious disease does not die out for the low diffusion rate of susceptible or infected individuals, and thus the epidemic disease cannot be eliminated by controlling the mobility of individuals. Combined with the discussion in [22, 32], we can conclude that the logistic source enhances the persistence of the disease, and the infectious disease will become threatening and harder to control. Our results here, together with the other two related epidemic models [22, 32], show that the logistic growth, the infection mechanism, and the population movement play an important role in the transmission dynamics of disease.

In summary, our discussion above shows that, in order to eradicate the disease modeled of the susceptible individuals, instead of reducing the mobility of the populations.

References

1. Allen, L.J.S., Bolker, B.M., Lou, Y., Nevai, A.L.: Asymptotic profiles of the steady states for an SIS epidemic reaction–diffusion model. Discrete Contin. Dyn. Syst. **21**, 1–20 (2008)
2. Anderson, R.M., May, R.M.: Population biology of infectious diseases. Nature **280**, 361–367 (1979)
3. Brezis, H., Strauss, W.A.: Semi-linear second-order elliptic equations in L^1. J. Math. Soc. Japan **25**, 565–590 (1973)
4. Cantrell, R.S., Cosner, C., Hutson, V.: Ecological models, permanence and spatial heterogeneity. Rocky Mountain J. Math. **26**, 1–35 (1996)
5. Capasso, V., Serio, G.: A generalization of the Kermack–McKendrick deterministic epidemic model. Math. Biosci. **42**, 41–61 (1978)
6. Cui, R.: Asymptotic profiles of the endemic equilibrium of a reaction–diffusion-advection SIS epidemic model with saturated incidence rate. Discrete Contin. Dyn. Syst. Ser. B **26**, 2997–3022 (2021)
7. Cui, R., Lam, K.-Y., Lou, Y.: Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments. J. Differ. Equ. **263**, 2343–2373 (2017)
8. Cui, R., Li, H., Peng, R., Zhou, M.: Concentration behavior of endemic equilibrium for a reaction–diffusion-advection SIS epidemic model with mass action infection mechanism, Calc. Var. Partial Differential Equations, 60, Paper No. 184, 38 pp, (2021)

9. Cui, R., Lou, Y.: A spatial SIS model in advective heterogeneous environments. J. Differ. Equ. **261**, 3305–3343 (2016)
10. de Jong, M. C. M., Diekmann, O., Heesterbeek, H.: How does transmission of infection depend on population size?, in Epidemic Models: Their Structure and Relation to Data, Cambridge University Press. New York, pp. 84-94, (1995)
11. Deng, K., Wu, Y.: Asymptotic behavior of an SIR reaction–diffusion model with a linear source. Discrete Contin. Dyn. Syst. Ser. B **24**, 5945–5957 (2018)
12. Du, Y., Lin, Z.: Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary. SIAM J. Math. Anal. **42**, 377–405 (2010)
13. Du, Y., Peng, R., Wang, M.: Effect of a protection zone in the diffusive Leslie predator-prey model. J. Differ. Equ. **246**, 3932–3956 (2009)
14. Du, Z., Peng, R.: A priori L^∞ estimates for solutions of a class of reaction–diffusion systems. J. Math. Biol. **72**, 1429–1439 (2016)
15. Fisher, R.: The wave of advance of advantageous genes. Ann. Eugen. **7**, 335–369 (1937)
16. Gao, D., Ruan, S.: An SIS patch model with variable transmission coefficients. Math. Biosci. **232**, 110–115 (2011)
17. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equation of Second Order. Springer, Berlin (2001)
18. Han, S., Lei, C., Zhang, X.: Qualitative analysis on a diffusive SIRS epidemic model with standard incidence infection mechanism. Z. Angew. Math. Phys. 71 , no. 6, Paper No. 190, 23 pp, (2020)
19. Hethcote, H.W.: The mathematics of infectious diseases. SIAM Rev. **42**, 599–653 (2000)
20. Hethcote, H.W.: Epidemiology models with variable population size, Mathematical understanding of infectious disease dynamics, 63-89, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. 16, World Sci. Publ. Hackensack, NJ, (2009)
21. Huang, W., Han, M., Liu, K.: Dynamics of an SIS reaction–diffusion epidemic model for disease transmission. Math. Biosci. Eng. **7**, 51–66 (2010)
22. Huo, X., Cui, R.: A reactiondiffusion SIS epidemic model with saturated incidence rate and logistic source, Applicable Analysis, (2020)
23. Jiang, Z., Wei, J.: Stability and bifurcation analysis in a delayed SIR model. Chaos Solitons Fractals **35**, 609–61 (2008)
24. Kaddar, A.: On the dynamics of a delayed SIR epidemic model with a modified saturated incidence rate. Electron. J. Differ. Equ. **133**, 7 (2009)
25. Kermack, W.O., McKendrick, A.G.: A contribution to the mathematical theory of epidemics. R. Soc. Lond. Proc. Ser. A **115**, 700–721 (1927)
26. Kuto, K., Matsuzawa, H., Peng, R.: Concentration profile of the endemic equilibria of a reaction–diffusion-advection SIS epidemic model, Calc. Var. Partial Differential Equations, 56, no. 4, Art. 112, 28 pp, (2017)
27. Lei, C., Li, F., Liu, J.: Theoretical analysis on a diffusive SIR epidemic model with nonlinear incidence in a heterogeneous environment. Discrete Contin. Dyn. Syst. Ser. B **23**, 4499–4517 (2018)
28. Lei, C., Lin, Z., Zhang, Q.: The spreading front of invasive species in favorable habitat or unfavorable habitat. J. Differ. Equ. **257**, 145–166 (2014)
29. Lei, C., Xiong, J., Zhou, X.: Qualitative analysis on an SIS epidemic reaction–diffusion model with mass action infection mechanism and spontaneous infection in a heterogeneous environment. Discrete Contin. Dyn. Syst. Ser. B **25**, 81–98 (2020)
30. Le, D.: Dissipativity and global attractors for a class of quasilinear parabolic systems. Commun. Partial. Diff. Eqns. **22**, 413–433 (1997)
31. Li, B., Bie, Q.: Long-time dynamics of an SIRS reaction–diffusion epidemic model. J. Math. Anal. Appl. **475**, 1910–1926 (2019)
32. Li, B., Li, H., Tong, Y.: Analysis on a diffusive SIS epidemic model with logistic source, Z. Angew. Math. Phys., 68, no. 4, Art. 96, 25 pp, (2017)
33. Li, B., Zhou, J., Zhou, X.: Asymptotic profiles of endemic equilibrium of a diffusive SIS epidemic system with nonlinear incidence function in a heterogeneous environment. Proc. Amer. Math. Soc. **148**, 4445–4453 (2020)
34. Li, H., Peng, R.: Dynamics and asymptotic profiles of endemic equilibrium for SIS epidemic patch models. J. Math. Biol. **79**, 1279–1317 (2019)
35. Li, H., Peng, R., Wang, F.-B.: Vary total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model. J. Differ. Equ. **262**, 885–913 (2017)

36. Li, H., Peng, R., Wang, Z.-A.: On a diffusive susceptible-infected-susceptible epidemic model with mass action mechanism and birth-death effect: analysis, simulations, and comparison with other mechanisms. SIAM J. Appl. Math. **78**, 2129–2153 (2018)
37. Li, H., Peng, R., Xiang, T.: Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion. Eur. J. Appl. Math. **31**, 26–56 (2018)
38. Lieberman, G.M.: Bounds for the steady-state Sel'kov model for arbitrary p in any number of dimensions. SIAM J. Math. Anal. **36**, 1400–1406 (2005)
39. Lou, Y., Ni, W.-M.: Diffusion, self-diffusion and cross-diffusion. J. Differ. Equ. **131**, 79–131 (1996)
40. Magal, P., Zhao, X.-Q.: Global attractive and steady states for uniformly persistent dynamical systems. SIAM. J. Math. Anal. **37**, 251–275 (2005)
41. Martcheva, M.: An Introduction to Mathematical Epidemiology, Texts in Applied Mathematics. Springer, New York (2015)
42. Ni, W.-M., Takagi, I.: On the Neumann problem for some semilinear elliptic equations and systems of activator-inhibitor type. Trans. Amer. Math. Soc. **297**, 351–368 (1986)
43. Peng, R., Shi, J., Wang, M.: On stationary patterns of a reaction–diffusion model with autocatalysis and saturation law. Nonlinearity **21**, 1471–1488 (2008)
44. Peng, R.: Asymptotic profiles of the positive steady state for an SIS epidemic reaction–diffusion model. Part I J. Differ. Equ. **247**, 1096–1119 (2009)
45. Peng, R.: Qualitative analysis on a diffusive and ratio-dependent predator-prey model. IMA J. Appl. Math. **78**, 566–586 (2013)
46. Peng, R., Liu, S.: Global stability of the steady states of an SIS epidemic reaction–diffusion model. Nonlinear Anal. **71**, 239–247 (2009)
47. Peng, R., Wu, Y.: Global L^∞-bounds and long-time behavior of a diffusive epidemic system in a heterogeneous environment. SIAM J. Math. Anal. **53**, 2776–2810 (2021)
48. Peng, R., Yi, F.: Asymptotic profile of the positive steady state for an SIS epidemic reaction–diffusion model: effects of epidemic risk and population movement. Phys. D **259**, 8–25 (2013)
49. Peng, R., Zhao, X.: A reaction–diffusion SIS epidemic model in a time-periodic environment. Nonlinearity **25**, 1451–1471 (2012)
50. Smith, H.L., Zhao, X.-Q.: Robust persistence for semidynamical systems. Nonlinear Anal. **47**, 6169–6179 (2001)
51. Song, P., Lou, Y., Xiao, Y.: A spatial SEIRS reaction–diffusion model in heterogeneous environment. J. Differ. Equ. **267**, 5084–5114 (2019)
52. Sun, X., Cui, R.: Analysis on a diffusive SIS epidemic model with saturated incidence rate and linear source in a heterogeneous environment. J. Math. Anal. Appl. **490**, 124212 (2020)
53. Suo, J., Li, B.: Analysis on a diffusive SIS epidemic system with linear source and frequency-dependent incidence function in a heterogeneous environment. Math. Biosci. Eng. **17**, 418–441 (2019)
54. Tong, Y., Lei, C.: An SIS epidemic reaction–diffusion model with spontaneous infection in a spatially heterogeneous environment. Nonlinear Anal. Real World Appl. **41**, 443–460 (2018)
55. Vargas-De-León, C.: On the global stability of SIS, SIR and SIRS epidemic models with standard incidence. Chaos Solitons Fractals **44**, 1106–1110 (2011)
56. Wang, Y., Wang, Z., Lei, C.: Asymptotic profile of endemic equilibrium to a diffusive epidemic model with saturated incidence rate. Math. Biosci. Eng. **16**, 3885–3913 (2019)
57. Wei, C., Chen, L.: A delayed epidemic model with pulse vaccination. Discrete Dyn. Nat. Soc. **746951**, 12 (2008)
58. Wen, X., Ji, J., Li, B.: Asymptotic profiles of the endemic equilibrium to a diffusive SIS epidemic model with mass action infection mechanism. J. Math. Anal. Appl. **458**, 715–729 (2018)
59. Wu, Y., Zou, X.: Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism. J. Differ. Equ. **261**, 4424–4447 (2016)
60. Zhang, J., Cui, R.: Qualitative analysis on a diffusive SIS epidemic system with logistic source and spontaneous infection in a heterogeneous environment. Nonlinear Anal. Real World Appl. **55**, 103115 (2020)
61. Zhang, J., Cui, R.: Asymptotic profiles of the endemic equilibrium of a diffusive SIS epidemic system with saturated incidence rate and spontaneous infection. Math. Methods Appl. Sci. **44**, 517–532 (2021)
62. Zhang, J., Jin, Z., Liu, Q., Zhang, Z.: Analysis of a delayed SIR model with nonlinear incidence rate. Discrete Dyn. Nat. Soc. **636153**, 16 (2008)
63. Zhang, T., Teng, Z.: Pulse vaccination delayed SEIRS epidemic model with saturation incidence. Appl. Math. Model. **32**, 1403–1416 (2008)

64. Zhou, P., Xiao, D.: The diffusive logistic model with a free boundary in heterogeneous environment. J. Differ. Equ. **256**, 1927–1954 (2014)

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