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On M-Stationary Conditions and Duality for Multiobjective Mathematical Programs with Vanishing Constraints

Mohd Hassan[1] · **J. K. Maurya**[2] · **S. K. Mishra**[1]

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Abstract

Mathematical programs with vanishing constraints are the optimization problems that do not satisfy most of the constraint qualifications due to nonconvex feasible region. Hence, some weaker first-order conditions like M-stationary come into existence. In this paper, we establish necessary and sufficient M-stationary conditions for multiobjective mathematical problems with vanishing constraints. We formulate Wolfe and Mond–Weir type dual models for the treated problem and propose weak, strong and strict converse duality results for Wolfe and Mond–Wier dual models. Further, we provide some examples in the support of our theory.

Keywords Vanishing constraints · Multiobjective optimization problems · Optimality conditions · Duality · Constraint qualifications

Mathematics Subject Classification 90C29 · 90C30 · 90C46

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Mohd Hassan, J. K. Maurya and S. K. Mishra have contributed equally to this work.

✉ J. K. Maurya
jitendramourya150@gmail.com

Mohd Hassan
hassanbezee2608@gmail.com

S. K. Mishra
bhu.skmishra@gmail.com

1 Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, Uttar Pradesh 221005, India

2 Department of Mathematics, Kashi Naresh Government Postgraduate College, Gyanpur, Bhadohi, Uttar Pradesh 221304, India

1 Introduction

The multiobjective optimization problems play a crucial role in a real-life scenarios. In daily-life, we deal with various objectives simultaneously, which is considered as multiobjective optimization problem. The solution of multiobjective optimization problems is not a single point likewise single objective optimization problem but it consists of a set of points which is called efficient solution points or Pareto optimal points.

The multiobjective mathematical program with vanishing constraints (MMPVC) is one of the significant nonlinear optimization problem and it is one of the active area of research in recent years. Achtziger and Kanzow [1] constructed the mathematical program with vanishing constraints (MPVC) as:

Consider the functions $\mathfrak{f}_i,\ \mathfrak{g}_i,\ \mathfrak{h}_i,\ \mathscr{H}_i,\ \mathscr{G}_i : \mathbb{R}^n \to \mathbb{R}$ as continuously differentiable on $\mathbb{R}^n$ and

$$\begin{aligned} \min\ \mathfrak{f}(\mathfrak{z}) &= (\mathfrak{f}_1(\mathfrak{z}), \ldots, \mathfrak{f}_p(\mathfrak{z})) \\ \text{subject to } \mathfrak{g}_i(\mathfrak{z}) &\leqq 0,\ \forall\, i = 1, \ldots, q, \\ \mathfrak{h}_i(\mathfrak{z}) &= 0,\ \forall\, i = 1, \ldots, r, \\ \mathscr{H}_i(\mathfrak{z}) &\geqq 0,\ \forall\, i = 1, \ldots, m, \\ \mathscr{G}_i(\mathfrak{z})\mathscr{H}_i(\mathfrak{z}) &\leqq 0,\ \forall\, i = 1, \ldots, m. \end{aligned} \tag{1}$$

The mathematical programs with vanishing constraints (MPVC) have a large number of applications in mixed integer optimal control problems [2], pathfinding problems with logic communication constraints in robot motion planning [3], scheduling problems [4], and many more areas of research. The MPVC is similar to one of the other well-known nonlinear optimization problem, termed as mathematical programs with equlibrium constraints (MPEC). For more details on MPEC, we refer [5–8].

The constraint $\mathscr{G}_i(\mathfrak{z})\mathscr{H}_i(\mathfrak{z})$ present in the MPVC, constructs the feasible region nonconvex and disconnected. Due to this reason most of the conventional constraint qualifications as linearly independent constraint qualification (LICQ), Mangasarian-Fromovitz constraint qualification (MFCQ) do not satisfied. The Karush-Kuhn-Tucker optimality conditions which are used to solve most of the nonlinear optimization problems is not significant in this case.

Achtziger and Kanzow [1] established several constraint qualifications and necessary optimality conditions for MPVC. Recently, Huang and Ho [9] studied the optimality and duality for multiobjective fractional programming problems in complex spaces. Mishra et al. [10] discussed the optimality and duality conditions for nonsmooth multiobjective optimization involving generalized type-I functions. Hoheisel and Kanzow [11] proposed first-order sufficient optimality and second-order necessary as well as sufficient optimality conditions using generalized convexity. Hoheisel and Kanzow [12] discussed stationary conditions under weaker assumptions of constraint qualification. Further, Hoheisel and Kanzow [13] investigated necessary and sufficient optimality conditions through Abadie and Guignard type constraint qualifications for mathematical programs with vanishing constraint (MPVC). For more details on MPVC, we refer to [14–19] and the references therein.

Recently, M-stationary conditions for MPVC using Fritz John approach under weak constraint qualification has been established and further discussion on local error bound results have been done in [20]. Zhang et al. [21] generalized the existing constraint qualifications and established proper Pareto optimality conditions for multiobjective mathematical programs with equilibrium constraints (MMPEC). Further, Zhang et al. [22] introduced strong Pareto S-stationary conditions and established necessary and sufficient optimality conditions for multiobjective mathematical programs with equilibrium constraints.

Motivated by the above discussions, we introduce strong efficient M-stationary conditions and generalized Guignard constraint qualification for MMPVC (MMPVC-GGCQ), and establish necessary optimality conditions for the MMPVC. Further, we prove converse implication holds under generalized convexity assumptions. We formulate Wolfe type and Mond–Weir type dual models for the MMPVC and establish weak, strong and converse duality results for both dual models and illustrate our results via suitable examples. The organization of the paper is as follows: in Sect. 2, we recall some basic definitions and results needed in the sequel of the paper. In Sect. 3, we derive necessary and sufficient optimality conditions for the MMPVC. In Sect. 4, we provide a brief explanation on Wolfe and Mond–Weir type dual models as well as the interrelation between primal and dual solution through weak, strong and converse duality results. In Sect. 5, we provide concluding remarks and some future directions.

2 Preliminaries

We collect some notations, definitions and essential results. The notation $\langle \cdot, \cdot \rangle$ denotes the inner product. We denote $\mathscr{B}(\mathfrak{z}^*, \delta)$, as the open ball centered at $\mathfrak{z}^*$ with radius $\delta > 0$. For a given $\mathfrak{z}^*$, $\mathcal{N}(\mathfrak{z}^*)$ is the family of the neighborhoods of $\mathfrak{z}^*$. If vectors $y,\ \mathfrak{z} \in \mathbb{R}^n$, then we recall the inequalities:

$$
\begin{aligned}
&y \leqq \mathfrak{z} \iff y_i \leqq \mathfrak{z}_i,\ i = 1, \ldots, n,\\
&y \leq \mathfrak{z} \iff y \leqq \mathfrak{z} \ \text{ and } \ y \neq \mathfrak{z},\ y \nleq \mathfrak{z} \text{ is negation of } y \leq \mathfrak{z},\\
&y < \mathfrak{z} \iff y_i < \mathfrak{z}_i,\ i = 1, \ldots, n,\ y \nless \mathfrak{z} \text{ is negation of } y < \mathfrak{z}.
\end{aligned}
$$

The following index sets will be used in the sequel:

$$
\begin{aligned}
I_{\mathfrak{f}} &:= \{1, 2, \ldots, p\},\\
I_{\mathfrak{g}}(\mathfrak{z}^*) &:= \{i \in \{1, 2, \ldots, q\} | \mathfrak{g}_i(\mathfrak{z}^*) = 0\},\\
I_{\mathfrak{h}} &:= \{1, 2, \ldots, r\},\\
I_{+}(\mathfrak{z}^*) &:= \{i \in \{1, 2, \ldots, m\} | \mathscr{H}_i(\mathfrak{z}^*) > 0\},\\
I_{0}(\mathfrak{z}^*) &:= \{i \in \{1, 2, \ldots, m\} | \mathscr{H}_i(\mathfrak{z}^*) = 0\}.
\end{aligned} \tag{2}
$$

We divide the index set I_+ into the following subsets:

$$
I_{+0}(\mathfrak{z}^*) := \{i | \mathscr{H}_i(\mathfrak{z}^*) > 0,\ \mathscr{G}_i(\mathfrak{z}^*) = 0\},
$$

$$I_{+-}(\mathfrak{z}^*) := \{i|\mathscr{H}_i(\mathfrak{z}^*) > 0,\ \mathscr{G}_i(\mathfrak{z}^*) < 0\}. \tag{3}$$

Similarly, we divide the index set I_0 in the following subsets:

$$\begin{aligned} I_{0+}(\mathfrak{z}^*) &:= \{i|\mathscr{H}_i(\mathfrak{z}^*) = 0,\ \mathscr{G}_i(\mathfrak{z}^*) > 0\}, \\ I_{00}(\mathfrak{z}^*) &:= \{i|\mathscr{H}_i(\mathfrak{z}^*) = 0,\ \mathscr{G}_i(\mathfrak{z}^*) = 0\}, \\ I_{0-}(\mathfrak{z}^*) &:= \{i|\mathscr{H}_i(\mathfrak{z}^*) = 0,\ \mathscr{G}_i(\mathfrak{z}^*) < 0\}. \end{aligned} \tag{4}$$

Definition 2.1 On the basis of solutions of multiobjective optimization problems [23], we extend the following definitions:

1. A point $\mathfrak{z}^*$ from the feasible region, is called a weak efficient solution of the MMPVC (1), if there is no other feasible point $\mathfrak{z}$, such that
$$\mathfrak{f}(\mathfrak{z}) < \mathfrak{f}(\mathfrak{z}^*).$$
2. A point $\mathfrak{z}^*$ from the feasible region, is called an efficient solution of the MMPVC (1), if there is no other feasible point $\mathfrak{z}$, such that
$$\mathfrak{f}(\mathfrak{z}) \leq \mathfrak{f}(\mathfrak{z}^*).$$
3. A point $\mathfrak{z}^*$ from the feasible region, is called locally efficient solution of the MMPVC (1) , if there exists a neighborhood $\mathscr{U}$ of $\mathfrak{z}^*$ and there is no other feasible point $\mathfrak{z} \in \mathscr{U}$, such that
$$\mathfrak{f}(\mathfrak{z}) \leq \mathfrak{f}(\mathfrak{z}^*).$$

Definition 2.2 We recall the following definitions on generalized convexity from [24].

1. A differentiable function $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}$, is called convex at point $\mathfrak{z}^* \in \mathbb{R}^n$, if
$$\mathfrak{f}(\mathfrak{z}) - \mathfrak{f}(\mathfrak{z}^*) \geqq \langle \nabla f(\mathfrak{z}^*), \mathfrak{z} - \mathfrak{z}^* \rangle,\ \forall \mathfrak{z} \in \mathbb{R}^n.$$
2. A differentiable function $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}$, is called strictly convex at point $\mathfrak{z}^* \in \mathbb{R}^n$, if
$$\mathfrak{f}(\mathfrak{z}) - \mathfrak{f}(\mathfrak{z}^*) > \langle \nabla \mathfrak{f}(\mathfrak{z}^*), \mathfrak{z} - \mathfrak{z}^* \rangle,\ \forall \mathfrak{z} \in \mathbb{R}^n \textit{ and } \mathfrak{z} \neq \mathfrak{z}^*.$$
3. A differentiable function $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}$, is called pseudoconvex at point $\mathfrak{z}^* \in \mathbb{R}^n$, if
$$\langle \nabla \mathfrak{f}(\mathfrak{z}^*), \mathfrak{z} - \mathfrak{z}^* \rangle \geqq 0 \Rightarrow \mathfrak{f}(\mathfrak{z}) \geqq \mathfrak{f}(\mathfrak{z}^*),\quad \forall \mathfrak{z} \in \mathbb{R}^n.$$
4. A differentiable function $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}$, is called strictly pseudoconvex at point $\mathfrak{z}^* \in \mathbb{R}^n$, if
$$\langle \nabla \mathfrak{f}(\mathfrak{z}^*), \mathfrak{z} - \mathfrak{z}^* \rangle \geqq 0 \Rightarrow \mathfrak{f}(\mathfrak{z}) > \mathfrak{f}(\mathfrak{z}^*),\quad \forall \mathfrak{z} \in \mathbb{R}^n \textit{ and } \mathfrak{z} \neq \mathfrak{z}^*.$$

5. A differentiable function $\mathfrak{f} : \mathbb{R}^n \to \mathbb{R}$, is called quasiconvex at point $\mathfrak{z}^* \in \mathbb{R}^n$, if

$$\mathfrak{f}(\mathfrak{z}) \leqq \mathfrak{f}(\mathfrak{z}^*) \Rightarrow \langle \nabla \mathfrak{f}(\mathfrak{z}^*), \mathfrak{z} - \mathfrak{z}^* \rangle \leqq 0, \quad \forall \mathfrak{z} \in \mathbb{R}^n.$$

Some preliminaries about cones are taken from [25], which are as follows:

1. A set $\mathscr{P} \subset \mathbb{R}^n$, be such that

$$\lambda \mathfrak{z} \in \mathscr{P}, \ \forall \, \lambda \geqq 0, \ \forall \, \mathfrak{z} \in \mathscr{P},$$

then $\mathscr{P}$ is called cone.

2. The negative polar cone of cone $\mathscr{P}$, defined by

$$\mathscr{P}^\circ := \{d \in \mathbb{R}^n : d^T \mathfrak{z} \leqq 0, \forall \, \mathfrak{z} \in \mathscr{P}\},$$

is closed and convex cone.

3. The dual cone of $\mathscr{P}$, is defined as

$$\mathscr{P}^* := \{d \in \mathbb{R}^n : d^T \mathfrak{z} \geqq 0 \, \forall \mathfrak{z} \in \mathscr{P}\}.$$

4. The tangent cone of a set $\mathscr{P}$ at a point $\mathfrak{z}^* \in cl\mathscr{P}$, defined by

$$T\left(\mathfrak{z}^*; \mathscr{P}\right) := \{d \in \mathbb{R}^n : \exists \{\mathfrak{z}^k\} \subset \mathscr{P}, t_k \downarrow 0 \ \text{ such that } \ \mathfrak{z}^k \to \mathfrak{z}^* \text{ and } \frac{\mathfrak{z}^k - \mathfrak{z}^*}{t_k} \to d\},$$

is closed cone.

5. The Fréchet normal cone at point $\mathfrak{z}^* \in cl\mathscr{P}$, is defined as

$$\hat{\mathscr{N}}(\mathfrak{z}^*; \mathscr{P}) := T(\mathfrak{z}^* : \mathscr{P})^\circ.$$

6. The limiting normal cone at $\mathfrak{z}^* \in cl\mathscr{P}$, is defined by

$$\mathscr{N}(\mathfrak{z}^*; \mathscr{P}) := \{\lim_{k\to\infty} w^k | \exists \, \mathfrak{z}^k \subseteq \mathscr{P}, \ \mathfrak{z}^k \to \mathfrak{z}^*, \ w^k \in \hat{\mathscr{N}}(\mathfrak{z}^*; \mathscr{P})\}.$$

7. A multifunction $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be locally upper Lipschitzian at a point $\mathfrak{z}^*$ with modulus κ, if for some neighbourhood $\mathscr{N}$ of $\mathfrak{z}^*$ and all $\mathfrak{z} \in \mathscr{N}$,

$$\mathscr{F}(\mathfrak{z}) \subset \mathscr{F}(\mathfrak{z}^*) + \kappa \|\mathfrak{z} - \mathfrak{z}^*\| \mathbb{B},$$

where $\mathbb{B} = \{\mathfrak{z} \in \mathbb{R}^n : \|\mathfrak{z}\| \leqq 1\}$.

Proposition 2.3 [26,Proposition 1] *Let F be a polyhedral multifunction from $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Then there exist $\kappa \in \mathbb{R}_+$ such that $\mathscr{F}$ is locally upper Lipschitzian with modulus κ at each point $\mathfrak{z}^* \in \mathbb{R}^n$.*

Theorem 2.4 [27,Corollary 4.2] *Let* $\mathscr{M}:\mathbb{R}^m \rightrightarrows \mathbb{R}^m$, $\mathscr{C}$, $\mathscr{D} \subseteq \mathbb{R}^m$ *and* $\bar{\mathfrak{z}} \in \mathscr{C} \cap \mathscr{D}$. *Assume that the map*

$$\mathscr{M}(y) = \{\mathfrak{z} \in \mathscr{C} : \mathfrak{z} + y \in \mathscr{D}\},$$

is calm at $(0, \bar{\mathfrak{z}}) \in Gph(\mathscr{M})$, *then one has*

$$\mathscr{N}(\bar{\mathfrak{z}}, \mathscr{C} \cap \mathscr{D}) \subseteq \mathscr{N}(\bar{\mathfrak{z}}, \mathscr{C}) + \mathscr{N}(\bar{\mathfrak{z}}, \mathscr{D}),$$

where $Gph(\mathscr{M})$ *represents the graph of* $\mathscr{M}$.

We also use the following Lagrangian function:

$$\begin{aligned}\varphi(\mathfrak{z}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) = (\eta^{\mathfrak{f}})^T \mathfrak{f}(\mathfrak{z}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{z}) + (\eta^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{z}) \\ - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{z}) + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{z}),\end{aligned}$$

and

$$\begin{aligned}\nabla\varphi(\mathfrak{z}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) = \nabla\mathfrak{f}(\mathfrak{z})\eta^{\mathfrak{f}} + \nabla\mathfrak{g}(\mathfrak{z})\eta^{\mathfrak{g}} + \nabla\mathfrak{h}(\mathfrak{z})\eta^{\mathfrak{h}} \\ - \nabla\mathscr{H}(\mathfrak{z})\eta^{\mathscr{H}} + \nabla\mathscr{G}(\mathfrak{z})\eta^{\mathscr{G}}.\end{aligned}$$

Lemma 2.5 [12] *Let the set* $\mathscr{Q} = \{(\alpha, \beta) \in \mathbb{R}^m \times \mathbb{R}^m | \alpha_i \geqq 0,\ \alpha_i\beta_i \leqq 0\ \forall i = 1, \ldots, m\}$ *be given. Then, the following statement hold:*

1. $\hat{\mathscr{N}}((0, 0); \mathscr{Q}) = \{(a, b) | a = 0,\ b \leqq 0\}$,
2. $\mathscr{N}((0, 0); \mathscr{Q}) = \{(a, b) | a_i \geqq 0,\ a_i b_i = 0\ \forall i = 1, \ldots, m\}$.

3 Optimality Conditions for the MMPVC (1)

Consider the following scalar optimization problem, which is formulated by motivation of Hybrid method [28,Section 4.2] used for solution of multiobjective programming. We name it scalarized multiobjective mathematical programs with vanishing constraints (SMMPVC):

$$\begin{aligned}&\min \sum_{i=1}^{p} \lambda_i \mathfrak{f}_i(\mathfrak{z}),\ \text{subject to } \mathfrak{f}_i(\mathfrak{z}) \leqq \mathfrak{f}_i(\mathfrak{z}^*),\ i \in I_{\mathfrak{f}}, \lambda_i > 0, \\ &\mathfrak{z} \in S = \{\mathfrak{z} \in \mathbb{R}^n : \mathfrak{g}(\mathfrak{z}) \leqq 0,\ h(\mathfrak{z}) = 0,\ \mathscr{H}_i(\mathfrak{z}) \geqq 0,\ \mathscr{G}_i(\mathfrak{z})\mathscr{H}_i(\mathfrak{z}) \leqq 0\},\end{aligned}$$

where $\mathfrak{z}^*$ is an arbitrary feasible point of MMPVC (1).

Subsequent result interlinks the solutions of SMMPVC and MMPVC (1).

Theorem 3.1 [28,Theorem 4.7] *A point* $\mathfrak{z}^* \in S$, *is an optimal solution of the SMMPVC problem if and only if* $\mathfrak{z}^*$ *is an efficient solution of MMPVC* (1).

In [29], several constraint qualifications introduced and shows that generalized Guignard constraint qualification is the weakest constraint qualification of them. Therefore, inspired by [12, 29] and formulation of SMMPVC, we define linearizing cone and MMPVC-GGCQ. Now, we define set $\mathscr{P}$, which plays crucial role in the development of the important results of this paper.

$$\mathscr{P} = \{\mathfrak{z} \in \mathbb{R}^n : \mathfrak{f}_i(\mathfrak{z}) \leqq \mathfrak{f}_i(\mathfrak{z}^*)\ (i \in I_{\mathfrak{f}}),\ g(\mathfrak{z}) \leqq 0,\ \mathfrak{h}(\mathfrak{z}) = 0,\ \mathscr{H}_i(\mathfrak{z}) \geqq 0,\ \mathscr{G}_i(\mathfrak{z})\mathscr{H}_i(\mathfrak{z}) \leqq 0\},$$

where $\mathfrak{z}^*$ is an arbitrary feasible point of MMPVC (1).

Definition 3.2 Linearizing cone of $\mathscr{P}$ at point $\mathfrak{z}^*$ is denoted and defined as

$$\begin{aligned}\mathscr{L}(\mathscr{P};\mathfrak{z}^*) := \{d \in \mathbb{R}^n | \nabla \mathfrak{f}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{\mathfrak{f}},\\ \nabla \mathfrak{g}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{\mathfrak{g}}(\mathfrak{z}^*),\\ \nabla \mathfrak{h}_i(\mathfrak{z}^*)^T d &= 0\ \forall\, i \in I_{\mathfrak{h}},\\ \nabla \mathscr{H}_i(\mathfrak{z}^*)^T d &= 0\ \forall\ i \in I_{0+}(\mathfrak{z}^*),\\ \nabla \mathscr{H}_i(\mathfrak{z}^*)^T d &\geqq 0\ \forall\, i \in I_{00}(\mathfrak{z}^*) \cup I_{0-}(\mathfrak{z}^*),\\ \nabla \mathscr{G}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{+0}(\mathfrak{z}^*),\\ (\nabla \mathscr{H}_i(\mathfrak{z}^*)^T d)(\nabla \mathscr{G}_i(\mathfrak{z}^*)^T d) &\leqq 0\ \forall\ i \in I_{00}(\mathfrak{z}^*)\}.\end{aligned}$$

Definition 3.3 MMPVC Guignard constraint qualification (MMPVC-GGCQ) holds at a feasible point $\mathfrak{z}^* \in \mathscr{P}$ of MMPVC (1) if

$$T(\mathscr{P};\mathfrak{z}^*)^* \subseteq \mathscr{L}(\mathscr{P};\mathfrak{z}^*)^*.$$

Now, we extend the concept of stationary conditions of [12] from scalar to multiobjective case. To do so we define the sets $\mathscr{W}_1$ and $\mathscr{W}_2$, as follows:

$$\begin{aligned}\mathscr{W}_1 := \{(d,\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m | \nabla \mathfrak{f}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{\mathfrak{f}},\\ \nabla \mathfrak{g}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{\mathfrak{g}}(\mathfrak{z}^*),\\ \nabla \mathfrak{h}_i(\mathfrak{z}^*)^T d &= 0\ \forall\, i \in I_{\mathfrak{h}},\\ \nabla \mathscr{H}_i(\mathfrak{z}^*)^T d &= 0\ \forall\, i \in I_{0+}(\mathfrak{z}^*),\\ \nabla \mathscr{H}_i(\mathfrak{z}^*)^T d &\geqq 0\ \forall\, i \in I_{0-}(\mathfrak{z}^*),\\ \nabla \mathscr{G}_i(\mathfrak{z}^*)^T d &\leqq 0\ \forall\, i \in I_{+0}(\mathfrak{z}^*),\\ \nabla \mathscr{H}_i(\mathfrak{z}^*)^T d - \alpha_i &\leqq 0\ \forall\, i \in I_{00}(\mathfrak{z}^*),\\ \nabla \mathscr{G}_i(\mathfrak{z}^*)^T d - \beta_i &\leqq 0\ \forall\, i \in I_{00}(\mathfrak{z}^*)\},\end{aligned} \tag{5}$$

and

$$\mathscr{W}_2 = \{(d,\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m | \beta_i \geqq 0,\ \alpha_i\beta_i \leqq 0\ \forall\, i = 1,\ldots,m\}.$$

Now, we present an important lemma which will be used in main result.

Lemma 3.4 *Let the multifunction* $\Phi : \mathbb{R}^{n+2m} \rightrightarrows \mathbb{R}^{n+2m}$ *be given by*

$$\Phi(x) = \{y \in \mathscr{W}_1 | x + y \in \mathscr{W}_2\}. \tag{6}$$

Then, Φ *is polyhedral multifunction.*

Proof Proof is direct consequence of [25,Example 9.57]. □

Theorem 3.5 *Let* $\mathfrak{z}^*$ *be a locally efficient solution of the MMPVC* (1) *such that MMPVC-GGCQ holds at* $\mathfrak{z}^*$. *Then, there exist multipliers* $\eta = (\eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{G}}, \eta^{\mathscr{H}}) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ *such that*

$$\begin{aligned}
&\nabla\varphi(\mathfrak{z}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) = 0,\\
&\eta^{\mathfrak{f}} > 0,\ \mathfrak{g}(\mathfrak{z}^*) \leqq 0,\ \eta^{\mathfrak{g}} \geqq 0,\ \mathfrak{g}(\mathfrak{z}^*)^T \eta^{\mathfrak{g}} = 0,\\
&\eta_i^{\mathscr{H}} = 0\ (\ i \in I_+(\mathfrak{z}^*)),\ \eta_i^{\mathscr{H}} \geqq 0\ (\ i \in I_{0-}(\mathfrak{z}^*)),\ \ \eta_i^{\mathscr{H}}\ \textit{free}\ (\ i \in I_{0+}(\mathfrak{z}^*)),\\
&\eta_i^{\mathscr{G}} = 0\ (\ i \in I_{0-}(\mathfrak{z}^*) \cup I_{+-}(\mathfrak{z}^*) \cup I_{0+}(\mathfrak{z}^*)),\ \eta_i^{\mathscr{G}} \geqq 0\ (\ i \in I_{+0}(\mathfrak{z}^*) \cup I_{00}(\mathfrak{z}^*)),\\
&\eta_i^{\mathscr{G}} \eta_i^{\mathscr{H}} = 0\ (\ i \in I_{00}(\mathfrak{z}^*)).
\end{aligned}$$

Proof Since $\mathfrak{z}^*$ is a local efficient solution of MMPVC (1), therefore from Theorem 3.1, $\mathfrak{z}^*$ is a local optimal solution of SMMPVC problem. Then, from basic optimality conditions, we have

$$\left(\sum_{i \in I_{\mathfrak{f}}} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)\right)^T d \geqq 0\ \forall\, d \in T(\mathscr{P}; \mathfrak{z}^*),\ \lambda_i^{\mathfrak{f}} > 0.$$

From MMPVC-GGCQ

$$\sum_{i=1}^{p} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*) \in T(\mathscr{P}; \mathfrak{z}^*)^* \subseteq \mathscr{L}(\mathscr{P}; \mathfrak{z}^*)^*,$$

then

$$\left(\sum_{i=1}^{p} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)\right)^T d \geqq 0\ \forall\, d \in \mathscr{L}(\mathscr{P}; \mathfrak{z}^*),\ \lambda_i^{\mathfrak{f}} > 0.$$

Equivalently, $d^* = 0$ being a minimizer of

$$\min_d \left(\sum_{i=1}^{p} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)\right)^T d \geqq 0,\ \text{subject to}\ d \in \mathscr{L}(\mathscr{P}; \mathfrak{z}^*),\ \lambda_i^{\mathfrak{f}} > 0. \tag{7}$$

Now, $d^* = 0$ being a minimizer of (7) is equivalent to $(d^*, \alpha^*, \beta^*) = (0, 0, 0)$ is minimizer of

$$\min_{d,\alpha,\beta} \left(\sum_{i=1}^{p} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*) \right)^T d \geqq 0, \text{ subject to } (d, \alpha, \beta) \in \mathscr{W} = \mathscr{W}_1 \cap \mathscr{W}_2. \tag{8}$$

Making use of [25,Proposition 6.5 and Theorem 6.12] in (8), we get

$$(-\sum_{i \in I_{\mathfrak{f}}} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*), 0, 0) \in T((0,0,0), \mathscr{W})^0 = \hat{\mathscr{N}}((0,0,0), \mathscr{W}) \subseteq \mathscr{N}((0,0,0), \mathscr{W}). \tag{9}$$

Now, applying the results of Lemma 3.4, Proposition 2.3 and Theorem 2.4 in (9), we get

$$\left(-\sum_{i \in I_{\mathfrak{f}}} \lambda_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*), 0, 0) \in \mathscr{N}((0,0,0), \mathscr{W}_1) + \mathscr{N}((0,0,0), \mathscr{W}_2 \right). \tag{10}$$

Hence, there exist $(\tau, \eta^{\mathfrak{g}}, \eta^{h}, \eta^{\mathscr{G}}, \eta^{\mathscr{H}}) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$, such that

$$\begin{aligned}
\begin{pmatrix} -\sum\limits_{i \in I_{\mathfrak{f}}} \lambda_i \nabla \mathfrak{f}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} &\in \sum_{i \in I_{\mathfrak{f}}} \tau_i \begin{pmatrix} \nabla \mathfrak{f}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} + \sum_{i \in I_{\mathfrak{g}}(\mathfrak{z}^*)} \eta_i^{\mathfrak{g}} \begin{pmatrix} \nabla \mathfrak{g}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} \\
&+ \sum_{i \in I_{\mathfrak{h}}} \eta_i^{\mathfrak{h}} \begin{pmatrix} \nabla \mathfrak{h}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} - \sum_{i \in I_{0+}(\mathfrak{z}^*) \cup I_{0-}(\mathfrak{z}^*)} \eta_i^{\mathscr{H}} \begin{pmatrix} \nabla \mathscr{H}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} \\
&+ \sum_{i \in I_{+0}(\mathfrak{z}^*)} \eta_i^{\mathscr{G}} \begin{pmatrix} \nabla \mathscr{G}_i(\mathfrak{z}^*) \\ 0 \\ 0 \end{pmatrix} - \sum_{i \in I_{00}(\mathfrak{z}^*)} \eta_i^{\mathscr{H}} \begin{pmatrix} \nabla \mathscr{H}_i(\mathfrak{z}^*) \\ 0 \\ -e^i \end{pmatrix} \\
&+ \sum_{i \in I_{00}(\mathfrak{z}^*)} \eta_i^{\mathscr{G}} \begin{pmatrix} \nabla \mathscr{G}_i(\mathfrak{z}^*) \\ -e^i \\ 0 \end{pmatrix} + \mathscr{N}((0,0,0); \mathscr{W}_2),
\end{aligned} \tag{11}$$

with conditions

$$\eta_i^{\mathfrak{g}} \geqq 0 \ (i \in I_{\mathfrak{g}}(\mathfrak{z}^*)), \ \eta_i^{\mathscr{H}} \geqq 0 \ (i \in I_{0-}(\mathfrak{z}^*)), \ \eta_i^{\mathscr{G}} \geqq 0 \ (i \in I_{+0}(\mathfrak{z}^*)),$$

where $e^i \in \mathbb{R}^m$, in which i^{th} component is 1 and rest all are zero. Since,

$$\begin{aligned}
\mathscr{N}((0,0,0); \mathscr{W}_2) &= \mathscr{N}(0; \mathbb{R}^n) \times \mathscr{N}((0,0); \{(\alpha, \beta) | \beta_i \geqq 0, \ \beta_i \alpha_i \leqq 0 \ \forall i = 1, \ldots, m\}) \\
&= \{0\}^n \times \{(a, b) | a_i \geqq 0, \ a_i b_i = 0 \ \forall i = 1, \ldots, m\}.
\end{aligned}$$

Then, from (11) we have

$$\eta_i^{\mathscr{G}} \geqq 0,\ \eta_i^{\mathscr{G}}\eta_i^{\mathscr{H}} = 0\ \forall i \in I_{00}(\mathfrak{z}^*). \tag{12}$$

Now, substituting $\eta_i^{\mathfrak{f}} = \lambda_i + \tau_i > 0\ (i \in I_{\mathfrak{f}}),\ \eta_i^{\mathfrak{g}} = 0\ (i \notin I_{\mathfrak{g}}(\mathfrak{z}^*)),\ \eta_i^{\mathscr{H}} = 0\ (i \in I_+(\mathfrak{z}^*)),\ \eta_i^{\mathscr{G}} = 0\ (i \in I_{0+}(\mathfrak{z}^*) \cup I_{0-}(\mathfrak{z}^*) \cup I_{+-}(\mathfrak{z}^*)),$ we get the required result. □

From Theorem 3.5, we propose following definition.

Definition 3.6 A feasible point $\mathfrak{z}^*$ is said to be strong efficient M-stationary point of the MMPVC (1) if there exist multipliers $\eta = (\eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{G}}, \eta^{\mathscr{H}}) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ which holds the following conditions

$$\begin{aligned}
\nabla\varphi(\mathfrak{z}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) = 0,\ \eta^{\mathfrak{f}} > 0,\ \eta^{\mathfrak{g}} \geqq 0,\ \mathfrak{g}(\mathfrak{z}^*)^T\eta^{\mathfrak{g}} = 0,\\
\eta_i^{\mathscr{H}} = 0\ (\ i \in I_+(\mathfrak{z}^*)),\ \eta_i^{\mathscr{H}} \geqq 0\ (\ i \in I_{0-}(\mathfrak{z}^*)),\ \ \eta_i^{\mathscr{H}} \text{ free } (\ i \in I_{0+}(\mathfrak{z}^*)),\\
\eta_i^{\mathscr{G}} = 0\ (\ i \in I_{0-}(\mathfrak{z}^*) \cup I_{+-}(\mathfrak{z}^*) \cup I_{0+}(\mathfrak{z}^*)),\ \eta_i^{\mathscr{G}} \geqq 0\ (\ i \in I_{+0}(\mathfrak{z}^*) \cup I_{00}(\mathfrak{z}^*)),\\
\eta_i^{\mathscr{G}}\eta_i^{\mathscr{H}} = 0\ (i \in I_{00}(\mathfrak{z}^*)).
\end{aligned}$$

Example 3.1 Consider the problem

$$\begin{aligned}
&\min\ f(\mathfrak{z}) = (\mathfrak{f}_1(\mathfrak{z}), \mathfrak{f}_2(\mathfrak{z}))\\
&\text{subject to }\ \mathscr{H}(\mathfrak{z}) \geqq 0,\ \mathscr{G}(\mathfrak{z})\mathscr{H}(\mathfrak{z}) \leqq 0,\\
&\text{where } \mathfrak{f}_1(\mathfrak{z}) = \mathfrak{z}_1^2,\ \mathfrak{f}_2(\mathfrak{z}) = \mathfrak{z}_2^2,\ \mathscr{H}(\mathfrak{z}) = \mathfrak{z}_1^2 + \mathfrak{z}_2^2 - 1,\\
&\mathscr{G}(\mathfrak{z}) = -\mathfrak{z}_1\mathfrak{z}_2,\ \ \text{and } \mathfrak{z} \in \mathbb{R}^2,
\end{aligned}$$

at point $\mathfrak{z}^* = (1, 0)$. Then,

$$\begin{aligned}
\mathscr{L}(\mathscr{P}; \mathfrak{z}^*) = \{d \in \mathbb{R}^2 : &\nabla\mathfrak{f}_1(\mathfrak{z}^*)^T d \leqq 0,\\
&\nabla\mathfrak{f}_2(\mathfrak{z}^*)^T d \leqq 0,\\
&\nabla\mathscr{H}(\mathfrak{z}^*)^T d \geqq 0,\\
&(\nabla\mathscr{H}(\mathfrak{z}^*)^T d)(\nabla\mathscr{G}(\mathfrak{z}^*)^T d) = 0\},
\end{aligned}$$

$$\mathscr{L}(\mathscr{P}; \mathfrak{z}^*) = \{d \in \mathbb{R}^2 : d_1 = 0,\ d_1 d_2 = 0\},$$

$$\mathscr{L}(\mathscr{P}; \mathfrak{z}^*)^* = \{d \in \mathbb{R}^2 : d_2 = 0,\ d_1 d_2 = 0\}.$$

And

$$T(\mathscr{P}; \mathfrak{z}^*) = \{\mathfrak{z} \in \mathbb{R}^2 : \mathfrak{z}_1 \geqq 0,\ \mathfrak{z}_1\mathfrak{z}_2 \geqq 0\},$$

$$T(\mathscr{P}; \mathfrak{z}^*)^* = \{\mathfrak{z} \in \mathbb{R}^2 : \mathfrak{z}_1 \geqq 0,\ \mathfrak{z}_2 = 0\}.$$

Now,

$$T(\mathscr{P}; \mathfrak{z}^*)^* \subseteq \mathscr{L}(\mathscr{P}; \mathfrak{z}^*)^*,$$

implies MMPVC-GGCQ hold at point $\mathfrak{z}^*$. Therefore, for $\eta_1^{\mathfrak{f}} = \eta^{\mathscr{H}},\ \eta^{\mathscr{G}} = 0$, the expression

$$\begin{aligned} &\eta_1^{\mathfrak{f}} \nabla \mathfrak{f}_1(\mathfrak{z}^*) + \eta_2^{\mathfrak{f}} \nabla \mathfrak{f}_2(\mathfrak{z}^*) - \eta^{\mathscr{H}} \nabla \mathscr{H}(\mathfrak{z}^*) + \eta^{\mathscr{G}} \nabla \mathscr{G}(\mathfrak{z}^*) \\ &= \eta_1^{\mathfrak{f}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \eta_2^{\mathfrak{f}} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \eta^{\mathscr{H}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned}$$

shows that $\mathfrak{z}^* = (0, 1)$ is a strong efficient M-stationary point.

Next example shows that non efficient point may also satisfied MMPVC-GGCQ, but fails to become a strong efficient M-stationary point.

Example 3.2 Consider the problem

$$\begin{aligned} &\min \mathfrak{f}(\mathfrak{z}) = (\mathfrak{f}_1(\mathfrak{z}),\ \mathfrak{f}_2(\mathfrak{z})) \\ &\text{subject to } \mathscr{H}(\mathfrak{z}) \geqq 0,\ \mathscr{G}(\mathfrak{z})\mathscr{H}(\mathfrak{z}) \leqq 0, \\ &\text{where } \mathfrak{f}_1(\mathfrak{z}) = \mathfrak{z}_1\mathfrak{z}_2,\ \mathfrak{f}_2(\mathfrak{z}) = \mathfrak{z}_2^2,\ \mathscr{H}(\mathfrak{z}) = \mathfrak{z}_1\mathfrak{z}_2 - 1, \\ &\mathscr{G}(\mathfrak{z}) = -(\mathfrak{z}_1 + \mathfrak{z}_2 - 2),\ \mathfrak{z} \in \mathbb{R}^2, \end{aligned}$$

at a feasible point $\mathfrak{z}^* = (1, 1)$. Then,

$$\begin{aligned} \mathscr{L}(\mathscr{P}; \mathfrak{z}^*) = \{d \in \mathbb{R}^2 :& \nabla \mathfrak{f}_1(\mathfrak{z}^*)^T d \leqq 0,\ \nabla \mathfrak{f}_2(\mathfrak{z}^*)^T d \leqq 0, \\ & \nabla \mathscr{H}(\mathfrak{z}^*)^T d \geqq 0,\ (\nabla \mathscr{H}(\mathfrak{z}^*)^T d)(\nabla \mathscr{G}(\mathfrak{z}^*)^T d) \leqq 0\}, \\ =& \{(d_1, d_2) \in \mathbb{R}^2 : d_1 + d_2 = 0,\ d_2 \leqq 0\}, \end{aligned}$$

and

$$T(\mathscr{P}; \mathfrak{z}^*) = \{(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{R}^2 : \mathfrak{z}_1 + \mathfrak{z}_2 \geqq 0\}.$$

Since,

$$\mathscr{L}(\mathscr{P}; \mathfrak{z}^*) \subseteq T(\mathscr{P}; \mathfrak{z}^*),$$

that is MMPVC-GGCQ hold at point $\mathfrak{z}^*$. Therefore, for $\eta_1^{\mathfrak{f}} > 0, \eta_2^{\mathfrak{f}} > 0,\ \eta^{\mathscr{G}} \geqq 0,\ \eta^{\mathscr{H}} \eta^{\mathscr{G}} = 0$, the expression

$$\eta_1^{\mathfrak{f}} \nabla \mathfrak{f}_1(\mathfrak{z}^*) + \eta_2^{\mathfrak{f}} \nabla \mathfrak{f}_2(\mathfrak{z}^*) - \eta^{\mathscr{H}} \nabla \mathscr{H}(\mathfrak{z}^*) + \eta^{\mathscr{G}} \nabla \mathscr{G}(\mathfrak{z}^*)$$

$$= \eta_1^{\mathfrak{f}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \eta_2^{\mathfrak{f}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \eta^{\mathscr{H}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \eta^{\mathscr{G}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

shows that $\mathfrak{z}^* = (1, 1)$ is not a strong efficient M-stationary point. Note that point $(1, 1)$, is dominated by $(2, \frac{1}{2})$. Hence, $(1, 1)$ is not an efficient point, but it is weak efficient point.

Consider the following index set:

$$\begin{aligned}
J^+(\mathfrak{z}^*) &:= \{i \in I_{\mathfrak{h}} |\ \eta_i^{\mathfrak{h}} > 0\}, \\
J^-(\mathfrak{z}^*) &:= \{i \in I_{\mathfrak{h}} |\ \eta_i^{\mathfrak{h}} < 0\}, \\
I_{00}^+(\mathfrak{z}^*) &:= \{i \in I_{00}(\mathfrak{z}^*) | \eta_i^{\mathscr{H}} > 0,\ \eta_i^{\mathscr{G}} = 0\}, \\
I_{00}^-(\mathfrak{z}^*) &:= \{i \in I_{00}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{H}} < 0,\ \eta_i^{\mathscr{G}} = 0\}, \\
I_{00}^{0+}(\mathfrak{z}^*) &:= \{i \in I_{00}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{H}} = 0,\ \eta_i^{\mathscr{G}} > 0\}, \\
I_{0+}^+(\mathfrak{z}^*) &:= \{i \in I_{0-}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{H}} > 0\}, \\
I_{0+}^-(\mathfrak{z}^*) &:= \{i \in I_{0+}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{H}} < 0\}, \\
I_{0-}^+(\mathfrak{z}^*) &:= \{i \in I_{0-}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{H}} > 0\}, \\
I_{+0}^+(\mathfrak{z}^*) &:= \{i \in I_{+0}(\mathfrak{z}^*) |\ \eta_i^{\mathscr{G}} > 0\}.
\end{aligned}$$

In the following theorem under the certain generalized convexity assumptions a strong M-stationary point of MMPVC (1) will be efficient solutions.

Theorem 3.7 *Let $\mathfrak{z}^* \in S$ be a strong efficient M-stationary point for the MMPVC (1) Suppose that $\mathfrak{f}_i$ $(i \in I_{\mathfrak{f}})$ or $(\eta^{\mathfrak{f}})^T \mathfrak{f}$ $(\eta^{\mathfrak{f}} > 0)$ are pseudoconvex at $\mathfrak{z}^*$, $\mathfrak{g}_i$ $(i \in I_{\mathfrak{g}}(\mathfrak{z}^*))$, $\mathfrak{h}_i (i \in I_{\mathfrak{h}})$ are affine and $-\mathscr{H}_i$ $(i \in I_{00}^+(\mathfrak{z}^*) \cup I_{0+}^+(\mathfrak{z}^*) \cup I_{0-}^+(\mathfrak{z}^*))$, $\mathscr{H}_i (i \in I_{00}^{-0}(\mathfrak{z}^*) \cup I_{0+}^-(\mathfrak{z}^*))$, $\mathscr{G}_i$ $(i \in I_{00}^{0+}(\mathfrak{z}^*) \cup I_{+0}^+(\mathfrak{z}^*))$ are quasiconvex at $\mathfrak{z}^*$, then*

(a) *$\mathfrak{z}^*$ is a local weakly efficient solution for MMPVC, if $I_{00}^-(\mathfrak{z}^*) \cup I_{0-}^+(\mathfrak{z}^*) = \emptyset$,*
(b) *$\mathfrak{z}^*$ is a global weakly efficient solution for MMPVC, if $I_{0+}^-(\mathfrak{z}^*) \cup I_{00}^-(\mathfrak{z}^*) \cup I_{+0}^{0+}(\mathfrak{z}^*) \cup I_{00}^{0+}(\mathfrak{z}^*) = \emptyset$.*

Proof (*b*) Since $\mathfrak{z}^* \in S$ is a strong efficient M-stationary point MMPVC with the multipliers $(\eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$, such that

$$\Bigg\langle \sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \nabla \mathfrak{g}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{r} \eta_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{z}^*)^T - \sum_{i=1}^{m} \eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{m} \eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{z}^*)^T \Bigg\rangle = 0.$$

We prove this result by contradiction. Suppose that $\mathfrak{z}^*$ is not globally weak efficient solution of MMPVC (1). Then, there exists a feasible point $\mathfrak{z}$ such that $\mathfrak{f}(\mathfrak{z}) < \mathfrak{f}(\mathfrak{z}^*)$.

Since each $\mathfrak{f}_i$ or $(\eta^{\mathfrak{f}})^T\mathfrak{f}$ $(\eta^{\mathfrak{f}} > 0)$ are pseudoconvex at $\mathfrak{z}^*$, then we have

$$\sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) < 0. \tag{13}$$

Now, for any $i \in I_{\mathfrak{g}}(\mathfrak{z}^*)$, we have

$$\mathfrak{g}_i(\mathfrak{z}) \leqq \mathfrak{g}_i(\mathfrak{z}^*) = 0,$$

then from quasiconvexity of $\mathfrak{g}_i$ $(i \in I_{\mathfrak{g}}(\mathfrak{z}^*))$, we get

$$\nabla \mathfrak{g}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0. \tag{14}$$

Similarly, we get

$$\nabla \mathfrak{h}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0 \ \ \forall\, i \in J^- \tag{15}$$

and,

$$\nabla \mathfrak{h}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \geqq 0 \ \ \forall\, i \in J^+. \tag{16}$$

Again since $\mathfrak{z}$ be any feasible point for problem(MMPEC) similarly, we have

$$-\nabla \mathscr{H}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0\, \forall\, i \in I_{00}^{+}(\mathfrak{z}^*) \cup I_{0+}^{+}(\mathfrak{z}^*) \cup I_{0-}^{+}(\mathfrak{z}^*), \tag{17}$$

$$\nabla \mathscr{G}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0\, \forall\, i \in I_{00}^{0+}(\mathfrak{z}^*) \cup I_{+0}^{+}(\mathfrak{z}^*). \tag{18}$$

Now, we verify statement (b), in this case when $I_{0+}^{-}(\mathfrak{z}^*) \cup I_{00}^{-}(\mathfrak{z}^*) \cup I_{+0}^{0+}(\mathfrak{z}^*) \cup I_{00}^{0+}(\mathfrak{z}^*) = \emptyset$. Multiplying (14)-(18) by $\eta_i^{\mathfrak{g}} \geq 0$ $(i \in I_{\mathfrak{g}}(\mathfrak{z}^*))$, $\eta_i^{\mathfrak{h}} > 0$ $(i \in J^+)$, $\eta_i^{\mathfrak{h}} < 0\,(i \in J^-)$, $\eta_i^{\mathscr{H}} > 0\,(i \in I_{00}^{+}(\mathfrak{z}^*) \cup I_{0+}^{+}(\mathfrak{z}^*) \cup I_{0-}^{+}(\mathfrak{z}^*))$, $\eta_i^{\mathscr{G}} > 0\,(i \in I_{00}^{0+}(\mathfrak{z}^*) \cup I_{+0}^{+}(\mathfrak{z}^*))$ respectively and adding to (13) , we have

$$0 = \left\langle \sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \nabla \mathfrak{g}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{r} \eta_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{z}^*)^T \right.$$
$$\left. - \sum_{i=1}^{m} \eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{z}^*)^T + \sum_{i=1}^{m} \eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{z}^*)^T,\ \mathfrak{z} - \mathfrak{z}^* \right\rangle < 0,$$

which is a contradiction. This completes the proof.

To establish statement (a), we only need to prove the following two conditions for any feasible $\mathfrak{z}$ sufficiently close to $\mathfrak{z}^*$,

$$-\eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0\, \forall\, i \in I_{0+}^{-}(\mathfrak{z}^*), \tag{19}$$

and

$$\eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0\, \forall\, i \in I_{+0}^{0+}(\mathfrak{z}^*). \tag{20}$$

To do this, we observe that (14) to (18) have already satisfied as in case (b), so we obtain the required result for all feasible point $\mathfrak{z}$ sufficiently close to $\mathfrak{z}^*$.

First, let $i \in I_{0+}^{-}(\mathfrak{z}^*)$, by the continuity it follows that $\mathscr{G}_i(\mathfrak{z}) > 0$ and thus $\mathscr{H}_i(\mathfrak{z}) = 0$ for any $\mathfrak{z} \in S$ sufficiently close to $\mathfrak{z}^*$. Using the quasiconvexity of $\mathscr{H}_i$ $(i \in I_{0+}^{-}(\mathfrak{z}^*))$, this implies $\nabla \mathscr{H}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0$, and since we have $\eta_i^{\mathscr{H}} < 0$ $(i \in I_{0+}^{-}(\mathfrak{z}^*))$, we got (19).

Second, let $i \in I_{+0}^{0+}(\mathfrak{z}^*)$, from continuity it follows that $\mathscr{H}_i(\mathfrak{z}) > 0$ and thus $\mathscr{G}_i(\mathfrak{z}) \leqq 0$ for any $\mathfrak{z} \in S$ sufficiently close to $\mathfrak{z}^*$. Using the quasiconvexity of $\mathscr{G}_i$ $(i \in I_{+0}^{0+}(\mathfrak{z}^*))$, this implies $\nabla \mathscr{G}_i(\mathfrak{z}^*)^T(\mathfrak{z} - \mathfrak{z}^*) \leqq 0$, which gives (20), since we have $\eta_i^{\mathscr{G}} > 0$ $(i \in I_{+0}^{0+}(\mathfrak{z}^*))$. □

4 Duality

In this section, we propose Wolfe type and Mond–Weir type dual model to the MMPVC (1) and establish weak, strong and strict converse duality results using convexity, quasiconvexity, pseudoconvexity and strict pseudoconvexity assumptions. The Wolfe type dual model to the MMPVC (1) is defined by WDMMPVC as follows:

$$\max \mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u})e + (\eta^h)^T h(\mathfrak{u})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{u})e \tag{21}$$

$$\begin{aligned}
\text{subject to } S_{WD} = \{ & (\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^h, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) : \nabla\varphi(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^h, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) = 0, \\
& \eta^{\mathfrak{f}} > 0, (\eta^{\mathfrak{f}})^T e = 1, \ \eta^{\mathfrak{g}} \geqq 0, \ \eta_i^{\mathscr{H}} = 0 \ (i \in I_+(\mathfrak{u})), \\
& \eta_i^{\mathscr{H}} \text{ free } (i \in I_{0+}(\mathfrak{u})), \ \eta_i^{\mathscr{H}} \geqq 0 \ (i \in I_{0-}(\mathfrak{u})), \\
& \eta_i^{\mathscr{G}} = 0 \ (i \in I_{0-}(\mathfrak{u}) \cup I_{+-}(\mathfrak{u}) \cup I_{0+}(\mathfrak{u})), \\
& \eta_i^{\mathscr{G}} \geqq 0 \ (i \in I_{+0}(\mathfrak{u}) \cup I_{00}(\mathfrak{u})), \ \eta_i^{\mathscr{G}} \eta_i^{\mathscr{H}} = 0 \ (i \in I_{00}(\mathfrak{u})), \\
& \eta_i^{\mathscr{H}} = 0 \ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{-0}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})), \\
& \eta_i^{\mathscr{G}} = 0 \ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})), \\
& \eta_i^{\mathscr{G}} \geqq 0 \ (i \in I_{-0}(\mathfrak{u})), \ \ e = (1, \ldots, 1) \in \mathbb{R}^p \},
\end{aligned}$$

where

$$\begin{aligned}
I_{\mathfrak{g}}(\mathfrak{u}) &:= \{i \in \{1, 2, \ldots, p\} : \ \mathfrak{g}_i(\mathfrak{u}) = 0\}, \ I_{\mathfrak{h}}(\mathfrak{u}) := \{i \in \{1, 2, \ldots, r\} : \ \mathfrak{h}_i(\mathfrak{u}) = 0\}, \\
I_+(\mathfrak{u}) &:= \{i \in \{1, 2, \ldots, m\} : \ \mathscr{H}_i(\mathfrak{u}) > 0\}, \ I_0(\mathfrak{u}) := \{i \in \{1, 2, \ldots, m\} : \ \mathscr{H}_i(\mathfrak{u}) = 0\}, \\
I_{+0}(\mathfrak{u}) &:= \{i : \ \mathscr{H}_i(\mathfrak{u}) > 0, \ \mathscr{G}_i(\mathfrak{u}) = 0\}, \ I_{+-}(\mathfrak{u}) := \{i : \ \mathscr{H}_i(\mathfrak{u}) > 0, \ \mathscr{G}_i(\mathfrak{u}) < 0\}, \\
I_{0+}(\mathfrak{u}) &:= \{i : \ \mathscr{H}_i(\mathfrak{u}) = 0, \ \mathscr{G}_i(\mathfrak{u}) > 0\}, \ I_{00}(\mathfrak{u}) := \{i : \ \mathscr{H}_i(\mathfrak{u}) = 0, \ \mathscr{G}_i(\mathfrak{u}) = 0\}, \\
I_{0-}(\mathfrak{u}) &:= \{i : \ \mathscr{H}_i(\mathfrak{u}) = 0, \ \mathscr{G}_i(\mathfrak{u}) < 0\}, \ I_{++}(\mathfrak{u}) := \{i : \ \mathscr{H}_i(\mathfrak{u}) > 0, \ \mathscr{G}_i(\mathfrak{u}) > 0\}, \\
I_{-+}(\mathfrak{u}) &:= \{i : \ \mathscr{H}_i(\mathfrak{u}) < 0, \ \mathscr{G}_i(\mathfrak{u}) > 0\}, \ I_{-0}(\mathfrak{u}) := \{i : \ \mathscr{H}_i(\mathfrak{u}) < 0, \ \mathscr{G}_i(\mathfrak{u}) = 0\}, \\
& \qquad I_{--}(\mathfrak{u}) := \{i : \ \mathscr{H}_i(\mathfrak{u}) < 0, \ \mathscr{G}_i(\mathfrak{u}) < 0\},
\end{aligned}$$

and

$$
\begin{aligned}
&I_{00}^{+}(\mathfrak{u}) := \{i \in I_{00}(\mathfrak{u}) : \ \eta_i^{\mathscr{H}} > 0, \ \eta_i^{\mathscr{G}} = 0\}, \ I_{00}^{0+}(\mathfrak{u}) := \{i \in I_{00}(\mathfrak{u}) : \ \eta_i^{\mathscr{H}} = 0, \ \eta_i^{\mathscr{G}} > 0\}, \\
&I_{00}^{-}(\mathfrak{u}) := \{i \in I_{00}(\mathfrak{u}) \ : \ \eta_i^{\mathscr{H}} < 0, \ \eta_i^{\mathscr{G}} = 0\}, \ I_{0+}^{+}(\mathfrak{u}) := \{i \in I_{0+}(\mathfrak{u}) : \eta_i^{\mathscr{H}} > 0\}, \\
&I_{0+}^{-}(\mathfrak{u}) := \{i \in I_{0+}(\mathfrak{u}) : \eta_i^{\mathscr{H}} < 0\}, \ I_{+0}^{0+}(\mathfrak{u}) := \{i \in I_{+0}(\mathfrak{u}) : \ \eta_i^{\mathscr{H}} = 0, \ \eta_i^{\mathscr{G}} > 0\}, \\
&I_{0-}^{+}(\mathfrak{u}) := \{i \in I_{0-}(\mathfrak{u}) : \eta_i^{\mathscr{H}} > 0\}.
\end{aligned} \tag{22}
$$

Now, we collect all $\mathfrak{u}$ from S_{WD}, which will be used later in this paper

$$proj S_{WD}^{\mathfrak{u}} := \{\mathfrak{u} : (\mathfrak{u}, \ \eta^{\mathfrak{g}}, \ \eta^{\mathfrak{h}}, \ \eta^{\mathscr{H}}, \ \eta^{\mathscr{G}}) \in S_{WD}\}.$$

Definition 4.1 Let $\bar{\mathfrak{u}} \in proj S_{WD}^{\mathfrak{u}}$. Then,

(i) $(\bar{\mathfrak{u}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \ \bar{\eta}^{\mathfrak{h}}, \ \bar{\eta}^{\mathscr{H}}, \ \bar{\eta}^{\mathscr{G}}) \in S_{WD}$ is said to be a locally efficient solution of WDMMPVC, if there exists $V \in \mathcal{N}(\bar{\mathfrak{u}})$, such that there is no $\mathfrak{u} \in proj S_{WD}^{\mathfrak{u}} \cap V$ satisfying

$$\mathscr{L}(\bar{\mathfrak{u}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) \ \leq \ \mathscr{L}(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \ \eta^{\mathscr{G}})$$

(ii) $(\bar{\mathfrak{u}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) \in S_{WD}$ is said to be a locally weak efficient solution of WDMMPVC, if there exists $V \in \mathcal{N}(\bar{\mathfrak{u}})$, such that there is no $\mathfrak{u} \in proj S_{WD}^{\mathfrak{u}} \cap V$ satisfying

$$\mathscr{L}(\bar{\mathfrak{u}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) < \mathscr{L}(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}).$$

Theorem 4.2 (Weak duality) *Let $\mathfrak{z}$ be a feasible point of the MMPVC (1) and $(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}})$ be feasible point of WDMMPVC (21). Suppose that*

(i) *$\mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u})e + (\eta^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{u})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{u})e$, are convex at $\mathfrak{u} \in proj S_{WD}^{\mathfrak{u}} \cup S$, or*

(ii) *$\mathfrak{f}_i, \ \mathfrak{g}_i, \ -\eta_i^{\mathscr{H}} \mathscr{H}_i, \ \mathscr{G}_i$ are convex and $\mathfrak{h}_i$ are affine at $\mathfrak{u} \in proj S_{WD}^{\mathfrak{u}} \cup S$.*

Then,

$$\mathfrak{f}(\mathfrak{z}) \not< \mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u})e + (\eta^{h})^T \mathfrak{h}(\mathfrak{u})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{u})e.$$

Proof Assume that

$$\mathfrak{f}(\mathfrak{z}) < \mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u})e + (\eta^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{u})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{u})e. \tag{23}$$

Since $\mathfrak{z} \in S$, then

$$
\begin{aligned}
&\mathfrak{f}(\mathfrak{z}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{z})e + (\eta^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{z})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{z})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{z})e \\
&< \mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u})e + (\eta^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{u})e - (\eta^{\mathscr{H}})^T \mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T \mathscr{G}(\mathfrak{u})e.
\end{aligned}
$$

Therefore, from convexity hypothesis

$$\sum_{i=1}^{p}\eta_i^{\mathfrak{f}}\nabla\mathfrak{f}_i(\mathfrak{u})+\sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\nabla\mathfrak{g}_i(\mathfrak{u})+\sum_{i=1}^{r}\eta_i^{\mathfrak{h}}\nabla\mathfrak{h}_i(\mathfrak{u})-\sum_{i=1}^{m}\eta_i^{\mathscr{H}}\nabla\mathscr{H}_i(\mathfrak{u})+\sum_{i=1}^{m}\eta_i^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})<0,$$

which contradicts the feasibility of $\mathfrak{u}$. Hence, we obtain the expected result.

Another way, multiplying (23) by $\eta^{\mathfrak{f}}>0$ with conditions $\sum_{i=1}^{p}\eta_i^{\mathfrak{f}}=1$, we have

$$\begin{aligned}&(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z})-(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{u})-(\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{u})\\&\quad-(\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{u})+(\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{u})-(\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{u})<0.\end{aligned}\tag{24}$$

Using convexity assumptions of the given functions, we have

$$\mathfrak{f}_i(\mathfrak{z})-\mathfrak{f}_i(\mathfrak{u})\geqq\langle\nabla\mathfrak{f}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle,\ \forall\, i=1,\ldots,p,\tag{25}$$

$$\mathfrak{g}_i(\mathfrak{z})-\mathfrak{g}_i(\mathfrak{u})\geqq\langle\nabla\mathfrak{g}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle,\ \forall\, i\in I_{\mathfrak{g}}(\mathfrak{u}),\tag{26}$$

$$\mathfrak{h}_i(\mathfrak{z})-\mathfrak{h}_i(\mathfrak{u})=\langle\nabla\mathfrak{h}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle\ \forall\, i,\tag{27}$$

$$-\mathscr{H}_i(\mathfrak{z})+\mathscr{H}_i(\mathfrak{u})\geqq\langle-\nabla\mathscr{H}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle,\ i\in I_{00}^{+}(\mathfrak{u})\cup I_{0+}^{+}(\mathfrak{u})\cup I_{0-}^{+}(\mathfrak{u}),\tag{28}$$

$$\mathscr{H}_i(\mathfrak{z})-\mathscr{H}_i(\mathfrak{u})\geqq\langle\nabla\mathscr{H}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle\ \forall\, i\in I_{00}^{-}(\mathfrak{u})\cup I_{0+}^{-}(\mathfrak{u}),\tag{29}$$

$$\mathscr{G}_i(\mathfrak{z})-\mathscr{G}_i(\mathfrak{u})\geqq\langle\nabla\mathscr{G}_i(\mathfrak{u}),\mathfrak{z}-\mathfrak{u}\rangle\ \forall\, i\in I_{00}^{0+}(\mathfrak{u})\cup I_{0+}^{0+}(\mathfrak{u}).\tag{30}$$

Multiplying (25)–(30) by $\eta_i^{\mathfrak{f}}>0,\ \eta_i^{\mathfrak{g}}\geqq 0,\ \eta_i^{\mathfrak{h}}$ free, $\eta_i^{\mathscr{H}}>0,\ -\eta_i^{\mathscr{H}}>0,\ \eta_i^{\mathscr{G}}>0$, respectively, setting remaining multipliers are 0 and adding, we get

$$\begin{aligned}&(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z})-(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{u})+(\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{z})-(\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{u})+(\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{z})-(\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{u})\\&\quad-(\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{z})+(\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{u})+(\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{z})-(\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{u})\geqq\Bigg\langle\sum_{i=1}^{p}\eta_i^{\mathfrak{f}}\nabla\mathfrak{f}_i(\mathfrak{u})\\&\quad+\sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\nabla\mathfrak{g}_i(\mathfrak{u})+\sum_{i=1}^{r}\eta_i^{\mathfrak{h}}\nabla\mathfrak{h}_i(\mathfrak{u})-\sum_{i=1}^{m}\eta_i^{\mathscr{H}}\nabla\mathscr{H}_i(\mathfrak{u})+\sum_{i=1}^{m}\eta_i^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}),\ \mathfrak{z}-\mathfrak{u}\Bigg\rangle.\end{aligned}$$

Since, $\mathfrak{z}\in S$, therefore

$$(\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{z})\leqq 0,\ (\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{z})=0,\ (\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{z})\leqq 0,\ -(\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{z})\leqq 0$$

and using conditions of (21), we get

$$(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z})-(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{u})-(\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{u})-(\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{u})+(\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{u})-(\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{u})\geqq 0,$$

which contradicts (24). Hence, we get the required result. □

Theorem 4.3 (Strong duality) *Let $\mathfrak{z}^*$ be a locally efficient solution of MMPVC (1) and satisfies the MMPVC-GGCQ at $\mathfrak{z}^*$. If assumptions of weak duality Theorem 4.2 satisfied. Then, there exist $(\bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ such that $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}})$ is an efficient solution of the WDMMPVC (21) and respective values are equal.*

Proof As $\mathfrak{z}^*$ is a locally efficient solution of the MMPVC (1) and the MMPVC-GGCQ is satisfied at $\mathfrak{z}^*$, then from Theorem 3.5, there exist $(\bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}}) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ such that strong efficient M-stationary conditions are satisfied. That is,

$$\begin{aligned}
&\nabla\mathfrak{f}(\mathfrak{z}^*)\bar{\eta}^{\mathfrak{f}} + \nabla\mathfrak{g}(\mathfrak{z}^*)\bar{\eta}^{\mathfrak{g}} + \nabla\mathfrak{h}(\mathfrak{z}^*)\bar{\eta}^{\mathfrak{h}} - \nabla\mathscr{H}(\mathfrak{z}^*)\bar{\eta}^{\mathscr{H}} + \nabla\mathscr{G}(\mathfrak{z}^*)\bar{\eta}^{\mathscr{G}} = 0,\\
&\quad \bar{\eta}^{\mathfrak{f}} > 0,\ \mathfrak{g}(\mathfrak{z}^*) \leqq 0,\ \ \bar{\eta}^{\mathfrak{g}} \geqq 0,\ \mathfrak{g}(\mathfrak{z}^*)^T\bar{\eta}^{\mathfrak{g}} = 0,\\
&\quad \bar{\eta}^{\mathscr{H}}_i = 0\ (\ i \in I_+),\ \bar{\eta}^{\mathscr{H}}_i \geqq 0\ (\ i \in I_{0-}),\ \ \bar{\eta}^{\mathscr{H}}_i \text{ free } (\ i \in I_{0+}),\\
&\quad \bar{\eta}^{\mathscr{G}}_i = 0\ (\ i \in I_{0-} \cup I_{+-} \cup I_{0+}),\ \bar{\eta}^{\mathscr{G}}_i \geqq 0\ (\ i \in I_{+0} \cup I_{00}),\ \bar{\eta}^{\mathscr{G}}_i\bar{\eta}^{\mathscr{H}}_i = 0\ (\ i \in I_{00}).
\end{aligned}$$

Therefore, $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ is feasible point of the WDMMPVC (21). Then, from feasibility and weak duality Theorem 4.2, we have

$$\begin{aligned}
\mathfrak{f}(\mathfrak{z}^*) &= \mathfrak{f}(\mathfrak{z}^*) + (\bar{\eta}^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{z}^*)e + (\bar{\eta}^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{z}^*)e + (\bar{\eta}^{\mathscr{H}})^T\mathscr{H}(\mathfrak{z}^*)e - (\bar{\eta}^{\mathscr{G}})^T\mathscr{G}(\mathfrak{z}^*)e\\
&\nleq \mathfrak{f}(\mathfrak{u}) + (\eta^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{u})e + (\eta^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{u})e - (\eta^{\mathscr{H}})^T\mathscr{H}(\mathfrak{u})e + (\eta^{\mathscr{G}})^T\mathscr{G}(\mathfrak{u})e,
\end{aligned}$$

for any feasible solution $(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{G}}, \eta^{\mathscr{H}}) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ of the WDMMPVC (21). Hence $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ is an efficient solution of the WDMMPVC (21) and values of both objective are same. □

Example 4.1 Consider the following MMPVC problem:

$$\begin{aligned}
&\min\ \mathfrak{f}(\mathfrak{z}) = (\mathfrak{f}_1(\mathfrak{z}), \mathfrak{f}_2(\mathfrak{z})),\ \text{where } \mathfrak{f}_1(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_1^2,\ \mathfrak{f}_2(\mathfrak{z}) = \mathfrak{z}_2^2,\\
&\text{subject to}\ \mathscr{H}(\mathfrak{z}) = \mathfrak{z}_2 \geqq 0,\\
&\qquad \mathscr{G}(\mathfrak{z})^T\mathscr{H}(\mathfrak{z}) = -(\mathfrak{z}_1 + \mathfrak{z}_2)\mathfrak{z}_2 \leqq 0.
\end{aligned}$$

Feasible set $S = \{\mathfrak{z} \in \mathbb{R}^2 : \mathfrak{z}_2 \geqq 0,\ -(\mathfrak{z}_1 + \mathfrak{z}_2)\mathfrak{z}_2 \leqq 0\}$. The WDMMPEC is as follows:

$$\begin{aligned}
&\max\ \mathfrak{f}(\mathfrak{u}) - \eta^{\mathscr{H}}\mathscr{H}(\mathfrak{u})e + \eta^{\mathscr{G}}\mathscr{G}(\mathfrak{u})e,\\
&\text{subject to}\ \eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\\
&= \eta_1^{\mathfrak{f}}\begin{bmatrix}2\mathfrak{u}_1\\0\end{bmatrix} + \eta_2^{\mathfrak{f}}\begin{bmatrix}0\\2\mathfrak{u}_2\end{bmatrix} - \eta^{\mathscr{H}}\begin{bmatrix}0\\1\end{bmatrix} + \eta^{\mathscr{G}}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},\ \eta^{\mathfrak{f}} > 0,\ \eta_1^{\mathfrak{f}} + \eta_2^{\mathfrak{f}} = 1,\\
&\text{remaining multipliers follows the Wolfe type dual conditions},\ e = (1,\ 1) \in \mathbb{R}^2.
\end{aligned}$$

Consider the following cases

Case 1. $\mathfrak{u}_1 = 0,\ \mathfrak{u}_2 = 0 \Rightarrow \mathscr{H}(\mathfrak{u}) = 0,\ \mathscr{G}(\mathfrak{u}) = 0,$ then for $\eta^{\mathscr{H}} = 0, \eta^{\mathscr{G}} = 0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})=(0,0),$$

$\Rightarrow \mathfrak{u}_1=0,\ \mathfrak{u}_2=0$ is feasible solution.

Case 2. $\mathfrak{u}_1=0,\ \mathfrak{u}_2>0 \Rightarrow \mathscr{H}(\mathfrak{u})>0,\ \mathscr{G}(\mathfrak{u})<0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}=0,$
and $\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1=0,\ \mathfrak{u}_2>0$ is not feasible solution.

Case 3. $\mathfrak{u}_1=0,\ \mathfrak{u}_2<0 \Rightarrow \mathscr{H}(\mathfrak{u})<0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}=0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1=0,\ \mathfrak{u}_2<0$ is not feasible solution.

Case 4. $\mathfrak{u}_1>0,\ \mathfrak{u}_2=0 \Rightarrow \mathscr{H}(\mathfrak{u})=0,\ \mathscr{G}(\mathfrak{u})<0,$ then $\eta^{\mathscr{H}}\geqq 0,\ \eta^{\mathscr{G}}=0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1>0,\ \mathfrak{u}_2=0$ is not feasible solution.

Case 5. $\mathfrak{u}_1>0,\ \mathfrak{u}_2>0 \Rightarrow \mathscr{H}(\mathfrak{u})>0,\ \mathscr{G}(\mathfrak{u})<0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}=0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1>0,\ \mathfrak{u}_2>0$ is not feasible solution.

Case 6. $\mathfrak{u}_1>0,\ \mathfrak{u}_2<0 \Rightarrow \mathscr{H}(\mathfrak{u})<0,\ \mathscr{G}(\mathfrak{u})>$ or $=$ or $<0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}\geqq 0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1>0,\ \mathfrak{u}_2<0$ is not feasible solution.

Case 7. $\mathfrak{u}_1<0,\ \mathfrak{u}_2=0 \Rightarrow \mathscr{H}(\mathfrak{u})=0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}$ free , $\eta^{\mathscr{G}}=0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1<0,\ \mathfrak{u}_2=0$ is not feasible solution.

Case 8. $\mathfrak{u}_1<0,\ \mathfrak{u}_2>0 \Rightarrow \mathscr{H}(\mathfrak{u})>0,\ \mathscr{G}(\mathfrak{u})>$ or $=$ or $<0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}\geqq 0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1<0,\ \mathfrak{u}_2>0$ is not feasible solution.

Case 9. $\mathfrak{u}_1<0,\ \mathfrak{u}_2<0 \implies \mathscr{H}(\mathfrak{u})<0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}=0, \eta^{\mathscr{G}}=0,$
$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$

$\Rightarrow \mathfrak{u}_1 < 0,\ \mathfrak{u}_2 < 0$ is not feasible solution. Hence, $\mathfrak{u} = (0, 0)$ is only feasible point of Wolfe type dual of MMPVC (1). Since $\mathfrak{f}(\mathfrak{z}) \geqq 0$, then

$$\mathfrak{f}(\mathfrak{z}) \not< \mathfrak{f}(\mathfrak{u}) + \eta^{\mathfrak{g}}\mathfrak{g}(\mathfrak{u})e - \eta^{\mathscr{G}}\mathscr{G}(\mathfrak{u})e - \eta^{\mathscr{H}}\mathscr{H}(\mathfrak{u})e.$$

Hence the weak duality Theorem 4.2 is verified. Strong duality Theorem 4.3 can be verified at point $\mathfrak{z}^* = (0,\ 0)$ easily.

Theorem 4.4 (Strict converse duality) *Let $\mathfrak{z}^*$ be a feasible point of the MMPVC (1) and $(\bar{\mathfrak{z}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ be a feasible point of the WDMMPVC (21). If the assumptions of strong duality Theorem 4.3 hold and at least one $\mathfrak{f}_i$ be strictly convex and remaining convex at $\bar{\mathfrak{z}}$, then $\mathfrak{z}^* = \bar{\mathfrak{z}}$.*

Proof On contrary, assume that $\mathfrak{z}^* \neq \bar{\mathfrak{z}}$. From strong duality Theorem 4.3 there exist $\bar{\eta} = (\bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}}) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$, such that $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{h}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ is an efficient solution of the WDMMPVC (21) and

$$\mathfrak{f}(\mathfrak{z}^*) = \mathfrak{f}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathfrak{g}})^T\mathfrak{g}(\bar{\mathfrak{z}})e + (\bar{\eta}^{\mathfrak{h}})^T\mathfrak{h}(\bar{\mathfrak{z}})e - (\bar{\eta}^{\mathscr{H}})^T\mathscr{H}(\bar{\mathfrak{z}})e + (\bar{\eta}^{\mathscr{G}})^T\mathscr{G}(\bar{\mathfrak{z}})e. \tag{31}$$

Multiplying (31) by $\bar{\eta}^{\mathfrak{f}} > 0$ $(\sum_{i=1}^{p} \bar{\eta}_i^{\mathfrak{f}} = 1)$, we have

$$(\bar{\eta}^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z}^*) - (\bar{\eta}^{\mathfrak{f}})^T\mathfrak{f}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathfrak{g}})^T\mathfrak{g}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathfrak{h}})^T\mathfrak{h}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathscr{H}})^T\mathscr{H}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathscr{G}})^T\mathscr{G}(\bar{\mathfrak{z}}) = 0. \tag{32}$$

Using the strict convexity and convexity assumptions, we have

$$\mathfrak{f}_k(\mathfrak{z}^*) - \mathfrak{f}_k(\bar{\mathfrak{z}}) > \langle \nabla\mathfrak{f}_k(\bar{\mathfrak{z}}), \mathfrak{z}^* - \bar{\mathfrak{z}}\rangle,\ \mathfrak{f}_i(\mathfrak{z}^*) - \mathfrak{f}_i(\bar{\mathfrak{z}}) \geqq \langle \nabla\mathfrak{f}_i(\bar{\mathfrak{z}}), \mathfrak{z}^* - \bar{\mathfrak{z}}\rangle,\ i \in I_{\mathfrak{f}} \setminus \{k\}, \tag{33}$$

$$\mathfrak{g}_i(\mathfrak{z}) - \mathfrak{g}_i(\mathfrak{u}) \geqq \langle \nabla\mathfrak{g}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u}\rangle,\ i \in I^*_{\mathfrak{g}}(\mathfrak{u}), \tag{34}$$

$$\mathfrak{h}_i(\mathfrak{z}) - \mathfrak{h}_i(\mathfrak{u}) = \langle \nabla\mathfrak{h}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u}\rangle,\ \forall\, i, \tag{35}$$

$$-\mathscr{H}_i(\mathfrak{z}) + \mathscr{H}_i(\mathfrak{u}) \geqq \langle -\nabla\mathscr{H}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u}\rangle,\ i \in I^{+0}_{00}(\mathfrak{z}^*) \cup I^{+}_{0+}(\mathfrak{z}^*) \cup I^{+}_{0-}(\mathfrak{z}^*), \tag{36}$$

$$\mathscr{H}_i(\mathfrak{z}) - \mathscr{H}_i(\mathfrak{u}) \geqq \langle \nabla\mathscr{H}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u}\rangle\ \forall\, i \in I^{-0}_{00}(\mathfrak{z}^*) \cup I^{-}_{0+}(\mathfrak{z}^*), \tag{37}$$

$$\mathscr{G}_i(\mathfrak{z}) - \mathscr{G}_i(\mathfrak{u}) \geqq \langle \nabla\mathscr{G}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u}\rangle\ \forall\, i \in I^{0+}_{00}(\mathfrak{z}^*) \cup I^{0+}_{+0}(\mathfrak{z}^*). \tag{38}$$

Multiplying (33)–(38) by $\eta_i^{\mathfrak{f}} > 0,\ \eta_i^{\mathfrak{g}} \geqq 0,\ \eta_i^{\mathfrak{h}}$ free, $\eta_i^{\mathscr{H}} > 0, -\eta_i^{\mathscr{H}} > 0,\ \eta_i^{\mathscr{G}} > 0$, setting remaining multipliers 0, respectively and adding, we get

$$\begin{aligned}
&(\bar{\eta}^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z}^*) - (\bar{\eta}^{\mathfrak{f}})^T\mathfrak{f}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathfrak{g}})^T\mathfrak{g}(\mathfrak{z}^*) - (\bar{\eta}^{\mathfrak{g}})^T\mathfrak{g}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathfrak{h}})^T\mathfrak{h}(\mathfrak{z}^*) - (\bar{\eta}^{\mathfrak{h}})^T\mathfrak{h}(\bar{\mathfrak{z}})\\
&\quad -(\bar{\eta}^{\mathscr{H}})^T\mathscr{H}(\mathfrak{z}) + (\bar{\eta}^{\mathscr{H}})^T\mathscr{H}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathscr{G}})^T\mathscr{G}(\mathfrak{z}^*) - (\bar{\eta}^{\mathscr{G}})^T\mathscr{G}(\bar{\mathfrak{z}}) > \Bigg\langle \sum_{i=1}^{p} \bar{\eta}_i^{\mathfrak{f}}\nabla\mathfrak{f}_i(\bar{\mathfrak{z}})\\
&\quad + \sum_{i=1}^{q} \bar{\eta}_i^{\mathfrak{g}}\nabla\mathfrak{g}_i(\bar{\mathfrak{z}}) - \sum_{i=1}^{m} \bar{\eta}_i^{\mathscr{H}}\nabla\mathscr{H}_i(\bar{\mathfrak{z}}) + \sum_{i=1}^{r} \bar{\eta}_i^{\mathscr{G}}\nabla\mathscr{G}_i(\bar{\mathfrak{z}}), \mathfrak{z}^* - \bar{\mathfrak{z}}\Bigg\rangle.
\end{aligned}$$

Since $\mathfrak{z}^*$ is feasible point, therefore $(\bar{\eta}^{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{z}^*) \leqq 0,\ (\bar{\eta}^{\mathfrak{h}})^T \mathfrak{h}(\mathfrak{z}^*) = 0, -(\bar{\eta}^{\mathscr{H}})^T \mathscr{H}(\mathfrak{z}^*) \leqq 0,\ (\bar{\eta}^{\mathscr{G}})^T \mathscr{G}(\mathfrak{z}^*) \leqq 0$ and using duality conditions, we get

$$(\bar{\eta}^{\mathfrak{f}})^T \mathfrak{f}(\mathfrak{z}^*) - (\bar{\eta}^{\mathfrak{f}})^T \mathfrak{f}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathfrak{g}})^T \mathfrak{g}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathfrak{h}})^T \mathfrak{h}(\bar{\mathfrak{z}}) + (\bar{\eta}^{\mathscr{H}})^T \mathscr{H}(\bar{\mathfrak{z}}) - (\bar{\eta}^{\mathscr{G}})^T \mathscr{G}(\bar{\mathfrak{z}}) > 0,$$

which contradicts (32). Hence $\mathfrak{z}^* = \bar{\mathfrak{z}}$. □

We now propose Mond–Weir type dual model to the MMPVC (1) and establish weak duality, strong duality and strict converse duality results using quasiconvexity, pseudoconvexity and strict pseudoconvexity assumptions. The Mond–Weir type dual model to the MMPVC (1) with respect to feasible point $\mathfrak{z}^*$, denoted by MWDMMPVC, as follows:

$$\begin{aligned}
&\max \mathfrak{f}(\mathfrak{u}),\\
&\text{subject to } (\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) \in S_{MWD} = \{(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) :\\
&\sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{u}) + \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \nabla \mathfrak{g}_i(\mathfrak{u}) + \sum_{i=1}^{r} \eta_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{u}) - \sum_{i=1}^{m} \eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{u}) + \sum_{i=1}^{m} \eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{u}) = 0,
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\eta^{\mathfrak{f}} > 0,\ \eta_i^{\mathscr{H}} = 0\ (\, i \in I_+(\mathfrak{u})),\ \eta_i^{\mathscr{H}} \geqq 0\ (\, i \in I_{0-}(\mathfrak{u})),\ \ \eta_i^{\mathscr{H}} \text{ free } (\, i \in I_{0+}(\mathfrak{u})),\\
&\eta_i^{\mathscr{G}} = 0\ (\, i \in I_{0-}(\mathfrak{u}) \cup I_{+-}(\mathfrak{u}) \cup I_{0+}(\mathfrak{u})),\ \eta_i^{\mathscr{G}} \geqq 0\ (\, i \in I_{+0}(\mathfrak{u}) \cup I_{00}(\mathfrak{u})),\\
&\eta^{\mathfrak{g}} \geqq 0,\ \eta_i^{\mathscr{G}} \eta_i^{\mathscr{H}} = 0\ (i \in I_{00}(\mathfrak{u})),\ \eta_i^{\mathscr{H}} = 0\ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{-0}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})),\\
&\eta_i^{\mathscr{G}} = 0\ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})),\ \eta_i^{\mathscr{G}} \geqq 0\ (i \in I_{-0}(\mathfrak{u}))\}.
\end{aligned}$$

Other indexing are same as of WDMMPVC (21). Consider the following projection set

$$proj\, S^{\mathfrak{u}}_{MWD} = \{\mathfrak{u} : (\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}}) \in S_{MWD}\}.$$

Definition 4.5 Let $(\bar{\mathfrak{u}}, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) \in S_{MWD}$

(i) is said to be a locally efficient solution of MWDMMPVC, if there exists $V \in \mathcal{N}(\bar{\mathfrak{u}})$, such that there is no $\mathfrak{u} \in proj\, S^{\mathfrak{u}}_{MWD} \cap V$ satisfying

$$\mathfrak{f}(\bar{\mathfrak{u}}) \ \leq\ \mathfrak{f}(\mathfrak{u})$$

(ii) is said to be a locally weak efficient solution of MWDMMPVC, if there exists $V \in \mathcal{N}(\mathfrak{z}^*)$, such that there is no $\mathfrak{u} \in proj\, S^{\mathfrak{u}}_{MWD} \cap V$ satisfying

$$\mathfrak{f}(\bar{\mathfrak{u}}) < \mathfrak{f}(\mathfrak{u}).$$

Theorem 4.6 (Weak duality) *Let $\mathfrak{z}$ be a feasible point of the MMPVC (1) and $(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{H}}, \eta^{\mathscr{G}})$ be a feasible point of the MWDMMPVC (39). Suppose that the given functions $\mathfrak{g}(\mathfrak{u})^T\eta^{\mathfrak{g}}$, $-\mathscr{H}^T\eta^{\mathscr{H}}$, $\mathscr{G}^T\eta^{\mathscr{G}}$, are quasiconvex and $\mathfrak{h}_i$ $(i \in I_{\mathfrak{h}})$, are affine at $\mathfrak{u}$. If any of the following holds:*

(a) $\eta_i^{\mathfrak{f}} \geqq 1$ *and* $\mathfrak{f}_i\,(\forall i \in I_{\mathfrak{f}})$ *are pseudoconvex at* $\mathfrak{u}$;

(b) $\eta_i^{\mathfrak{f}} \geqq 1\ (\forall i \in I_{\mathfrak{f}})$ *and* $\sum_{i=1}^{p} \eta_i^{\mathfrak{f}}\mathfrak{f}_i(\cdot)$ *is pseudoconvex at* $\mathfrak{u}$.

Then,

$$\mathfrak{f}(\mathfrak{z}) \nleq \mathfrak{f}(\mathfrak{u}). \tag{40}$$

Proof Assume that

$$\mathfrak{f}(\mathfrak{z}) \leq \mathfrak{f}(\mathfrak{u}),$$

Then,

$$\mathfrak{f}_i(\mathfrak{z}) \leqq \mathfrak{f}_i(\mathfrak{u}),\ \forall\, i \in I_{\mathfrak{f}},\ \ \text{except at least one } k,\ \text{such that}$$

$$\mathfrak{f}_k(\mathfrak{z}) < \mathfrak{f}_k(\mathfrak{u}).$$

Multiplying by $\eta_i^{\mathfrak{f}} \geqq 1$ and adding, we get

$$(\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{z}) < (\eta^{\mathfrak{f}})^T\mathfrak{f}(\mathfrak{u}). \tag{41}$$

Using the quasiconvexity assumptions, we get

$$\sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\mathfrak{g}_i(\mathfrak{z}) \leqq \sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\mathfrak{g}_i(\mathfrak{u}) \Rightarrow \left\langle \sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\nabla\mathfrak{g}_i(\mathfrak{u}), \mathfrak{z}-\mathfrak{u}\right\rangle \leqq 0, \tag{42}$$

$$\sum\eta_i^{\mathfrak{h}}\mathfrak{h}_i(\mathfrak{z}) = \sum\eta_i^{\mathfrak{h}}\mathfrak{h}_i(\mathfrak{u}) \Rightarrow \left\langle \sum\eta_i^{\mathfrak{h}}\nabla\mathfrak{h}_i(\mathfrak{u}), \mathfrak{z}-\mathfrak{u}\right\rangle = 0,\ i \in I_{\mathfrak{h}}, \tag{43}$$

$$-\sum\eta_i^{\mathscr{H}}\mathscr{H}_i(\mathfrak{z}) \leqq -\sum\eta_i^{\mathscr{H}}\mathscr{H}_i(\mathfrak{u}) \Rightarrow \left\langle -\sum\eta_i^{\mathscr{H}}\nabla\mathscr{H}_i(\mathfrak{u}), \mathfrak{z}-\mathfrak{u}\right\rangle \leqq 0, \tag{44}$$

$$\sum\eta_i^{\mathscr{G}}\mathscr{G}_i(\mathfrak{z}) \leqq \sum\eta_i^{\mathscr{G}}\mathscr{G}_i(\mathfrak{u}) \Rightarrow \left\langle \sum\eta_i^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}), \mathfrak{z}-\mathfrak{u}\right\rangle \leqq 0, \tag{45}$$

Adding (42)–(45), we get

$$\left\langle \sum_{i=1}^{q}\eta_i^{\mathfrak{g}}\nabla\mathfrak{g}_i(\mathfrak{u}) + \sum_{i=1}^{r}\eta_i^{\mathfrak{h}}\nabla\mathfrak{h}_i(\mathfrak{u}) - \sum_{i=1}^{m}\eta_i^{\mathscr{H}}\nabla\mathscr{H}_i(\mathfrak{u}) + \sum_{i=1}^{m}\eta_i^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}),\ \mathfrak{z}-\mathfrak{u}\right\rangle \leqq 0. \tag{46}$$

From (39) and (46), we have

$$\left\langle \sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\mathfrak{u}),\ \mathfrak{z} - \mathfrak{u} \right\rangle \geqq 0. \tag{47}$$

Using assumptions (a) or (b) in (47), we get

$$(\eta^{\mathfrak{f}})^T \mathfrak{f}(\mathfrak{z}) \geqq (\eta^{\mathfrak{f}})^T \mathfrak{f}(\mathfrak{u}),$$

which contradicts (41). Hence, we get the required result. □

Theorem 4.7 (Strong duality) *Suppose $\mathfrak{z}^*$ is a locally efficient solution of the MMPVC (1) and satisfies the MMPVC-GGCQ at $\mathfrak{z}^*$. If assumptions of weak duality Theorem 4.6 holds. Then, there exist $(\bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}}) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ such that $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{H}}, \bar{\eta}^{\mathscr{G}})$ is an efficient solution of the MWDMMPVC (39) and respective values are equal.*

Proof As $\mathfrak{z}^*$ is a locally efficient solution of the MMPVC (1) and the MMPVC-GCQ satisfied at $\mathfrak{z}^*$, then from Theorem 3.5, there exist $(\bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}}) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ such that $\mathfrak{z}^*$ is a strong efficient M-stationary point. That is,

$$\sum_{i=1}^{p} \bar{\eta}_i^{\mathfrak{f}} \nabla f_i(\mathfrak{z}^*) + \sum_{i=1}^{q} \bar{\eta}_i^{\mathfrak{g}} \nabla \mathfrak{g}_i(\mathfrak{z}^*) + \sum_{i=1}^{r} \bar{\eta}_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{z}^*) - \sum_{i=1}^{m} \bar{\eta}_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{z}^*)$$
$$+ \sum_{i=1}^{m} \bar{\eta}_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{z}^*) = 0,$$

$$\eta^{\mathfrak{f}} > 0,\ \eta_i^{\mathscr{H}} = 0\ (\ i \in I_+(\mathfrak{u})),\ \eta_i^{\mathscr{H}} \geqq 0\ (\ i \in I_{0-}(\mathfrak{u})),\quad \eta_i^{\mathscr{H}} \text{ free } (\ i \in I_{0+}(\mathfrak{u})),$$
$$\eta_i^{\mathscr{G}} = 0\ (\ i \in I_{0-}(\mathfrak{u}) \cup I_{+-}(\mathfrak{u}) \cup I_{0+}(\mathfrak{u})),\ \eta_i^{\mathscr{G}} \geqq 0\ (\ i \in I_{+0}(\mathfrak{u}) \cup I_{00}(\mathfrak{u})),$$
$$(\eta_{\mathfrak{g}})^T \mathfrak{g}(\mathfrak{u}) = 0,\ \eta^{\mathfrak{g}} \geqq 0,\ \eta_i^{\mathscr{G}} \eta_i^{\mathscr{H}} = 0\ (\ i \in I_{00}(\mathfrak{u})).$$

Therefore, $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ is feasible of the MWDMMPVC (39). Then, from feasibility and weak duality Theorem 4.6, we have

$$\mathfrak{f}(\mathfrak{z}^*) \nleq \mathfrak{f}(\mathfrak{u}),$$

for any feasible solution $(\mathfrak{u}, \eta^{\mathfrak{f}}, \eta^{\mathfrak{g}}, \eta^{\mathfrak{h}}, \eta^{\mathscr{G}}, \eta^{\mathscr{H}}) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^m$ of the MWDMMPVC (39). Hence $(\mathfrak{z}^*, \bar{\eta}^{\mathfrak{f}}, \bar{\eta}^{\mathfrak{g}}, \bar{\eta}^{\mathfrak{h}}, \bar{\eta}^{\mathscr{G}}, \bar{\eta}^{\mathscr{H}})$ is an efficient solution of the MWDMMPVC (39) and the values of both objective are equal. □

Next example verify the Mond–Weir type dual model and duality results for MMPVC.

Example 4.2 Consider the following MMPVC problem:

$$\min\ \mathfrak{f}(\mathfrak{z}) = (\mathfrak{f}_1(\mathfrak{z}), \mathfrak{f}_2(\mathfrak{z})),\ \text{where } \mathfrak{f}_1(\mathfrak{z}) = \mathfrak{z}_1^2 + \mathfrak{z}_2,\ \mathfrak{f}_2(\mathfrak{z}) = \mathfrak{z}_1^2,$$
$$\text{subject to } \mathscr{H}(\mathfrak{z}) = \mathfrak{z}_2 \geqq 0,\ \mathscr{G}(\mathfrak{z})^T\mathscr{H}(\mathfrak{z}) = -\mathfrak{z}_1\mathfrak{z}_2 \leqq 0.$$

Feasible region $S = \{\mathfrak{z} \in \mathbb{R}^2 :\ \mathfrak{z}_2 \geqq 0,\ -\mathfrak{z}_1\mathfrak{z}_2 \leqq 0\}$. Now, we formulate MWD-MMPVC dual model according as above discussion.

$$\max\ \mathfrak{f}(\mathfrak{u}) = (\mathfrak{u}_1^2 + \mathfrak{u}_2,\ \mathfrak{u}_1^2),$$
$$\text{subject to } \eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}(\mathfrak{u})$$
$$= \eta_1^{\mathfrak{f}}\begin{bmatrix}2\mathfrak{u}_1\\1\end{bmatrix} + \eta_2^{\mathfrak{f}}\begin{bmatrix}2\mathfrak{u}_1\\0\end{bmatrix} - \eta^{\mathscr{H}}\begin{bmatrix}0\\1\end{bmatrix} + \eta^{\mathscr{G}}\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},\ \eta_1^{\mathfrak{f}} > 0,\ \eta_2^{\mathfrak{f}} > 0.$$

$$\eta^{\mathfrak{f}} > 0,\ \eta_i^{\mathscr{H}} = 0\ (\ i \in I_+(\mathfrak{u})),\ \eta_i^{\mathscr{H}} \geqq 0\ (\ i \in I_{0-}(\mathfrak{u})),\ \ \eta_i^{\mathscr{H}}\ \text{free}\ (\ i \in I_{0+}(\mathfrak{u})),$$
$$\eta_i^{\mathscr{G}} = 0\ (\ i \in I_{0-}(\mathfrak{u}) \cup I_{+-}(\mathfrak{u}) \cup I_{0+}(\mathfrak{u})),\ \eta_i^{\mathscr{G}} \geqq 0\ (\ i \in I_{+0}(\mathfrak{u}) \cup I_{00}(\mathfrak{u})),$$
$$\eta^{\mathfrak{g}} \geqq 0,\ \eta_i^{\mathscr{G}}\eta_i^{\mathscr{H}} = 0\ (i \in I_{00}(\mathfrak{u})),\ \eta_i^{\mathscr{H}} = 0\ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{-0}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})),$$
$$\eta_i^{\mathscr{G}} = 0\ (i \in I_{++}(\mathfrak{u}) \cup I_{-+}(\mathfrak{u}) \cup I_{--}(\mathfrak{u})),\ \eta_i^{\mathscr{G}} \geqq 0\ (i \in I_{-0}(\mathfrak{u}))\}.$$

Consider the following cases

Case 1. $\mathfrak{u}_1 = 0,\ \mathfrak{u}_2 = 0 \Rightarrow \mathscr{H}(\mathfrak{u}) = 0,\ \mathscr{G}(\mathfrak{u}) = 0,$ then for $\eta^{\mathscr{H}} = \eta_1^{\mathfrak{f}}, \eta^{\mathscr{G}} = 0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}) = (0, 0),$$

$\Rightarrow \mathfrak{u}_1 = 0,\ \mathfrak{u}_2 = 0$ is feasible solution.

Case 2. $\mathfrak{u}_1 = 0,\ \mathfrak{u}_2 > 0 \Rightarrow \mathscr{H}(\mathfrak{u}) > 0,\ \mathscr{G}(\mathfrak{u}) = 0,$ then $\eta^{\mathscr{H}} = 0,\ \eta^{\mathscr{G}} \geqq 0,$

and $\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}) \neq (0, 0),$

$\Rightarrow \mathfrak{u}_1 = 0,\ \mathfrak{u}_2 > 0$ is not feasible solution.

Case 3. $\mathfrak{u}_1 = 0,\ \mathfrak{u}_2 < 0 \Rightarrow \mathscr{H}(\mathfrak{u}) < 0,\ \mathscr{G}(\mathfrak{u}) = 0,$ then $\eta^{\mathscr{H}} = 0, \eta^{\mathscr{G}} \geqq 0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}) \neq (0, 0),$$

$\Rightarrow \mathfrak{u}_1 = 0,\ \mathfrak{u}_2 < 0$ is not feasible solution.

Case 4. $\mathfrak{u}_1 > 0,\ \mathfrak{u}_2 = 0 \Rightarrow \mathscr{H}(\mathfrak{u}) = 0,\ \mathscr{G}(\mathfrak{u}) < 0,$ then $\eta^{\mathscr{H}} \geqq 0,\ \eta^{\mathscr{G}} = 0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u}) + \eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u}) - \eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u}) + \eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u}) \neq (0, 0),$$

$\Rightarrow \mathfrak{u}_1 > 0,\ \mathfrak{u}_2 = 0$ is not feasible solution.

Case 5. $\mathfrak{u}_1 > 0,\ \mathfrak{u}_2 > 0 \Rightarrow \mathscr{H}(\mathfrak{u}) > 0,\ \mathscr{G}(\mathfrak{u}) < 0,$ then $\eta^{\mathscr{H}} = 0, \eta^{\mathscr{G}} = 0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$$

$\Rightarrow \mathfrak{u}_1>0,\ \mathfrak{u}_2>0$ is not feasible solution.

Case 6. $\mathfrak{u}_1>0,\ \mathfrak{u}_2<0\Rightarrow\mathscr{H}(\mathfrak{u})<0,\ \mathscr{G}(\mathfrak{u})<0,$ then $\eta^{\mathscr{H}}=0,\eta^{\mathscr{G}}=0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$$

$\Rightarrow \mathfrak{u}_1>0,\ \mathfrak{u}_2<0$ is not feasible solution.

Case 7. $\mathfrak{u}_1<0,\ \mathfrak{u}_2=0\Rightarrow\mathscr{H}(\mathfrak{u})=0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}$ free , $\eta^{\mathscr{G}}=0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$$

$\Rightarrow \mathfrak{u}_1<0,\ \mathfrak{u}_2=0$ is not feasible solution.

Case 8. $\mathfrak{u}_1<0,\ \mathfrak{u}_2>0\Rightarrow\mathscr{H}(\mathfrak{u})>0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}=0,\eta^{\mathscr{G}}=0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$$

$\Rightarrow \mathfrak{u}_1<0,\ \mathfrak{u}_2>0$ is not feasible solution.

Case 9. $\mathfrak{u}_1<0,\ \mathfrak{u}_2<0\implies\mathscr{H}(\mathfrak{u})<0,\ \mathscr{G}(\mathfrak{u})>0,$ then $\eta^{\mathscr{H}}=0,\eta^{\mathscr{G}}=0,$

$$\eta_1^{\mathfrak{f}}\nabla\mathfrak{f}_1(\mathfrak{u})+\eta_2^{\mathfrak{f}}\nabla\mathfrak{f}_2(\mathfrak{u})-\eta^{\mathscr{H}}\nabla\mathscr{H}(\mathfrak{u})+\eta^{\mathscr{G}}\nabla\mathscr{G}_i(\mathfrak{u})\neq(0,0),$$

$\Rightarrow \mathfrak{u}_1<0,\ \mathfrak{u}_2<0$ is not feasible solution. Hence, $\mathfrak{u}=(0,0)$ is only feasible solution for above dual model and

$$\mathfrak{f}(\mathfrak{z})\nleq\mathfrak{f}(\mathfrak{u}).$$

Hence the weak duality Theorem 4.6 is verified and it is very simple to verify Strong duality Theorem 4.7 at point $\mathfrak{z}^*=(0,\ 0)$ easily.

Theorem 4.8 (Strict converse duality) *Let $\mathfrak{z}^*$ be a feasible point of the MMPVC (1) and $(\bar{\mathfrak{u}},\bar{\eta}^{\mathfrak{f}},\bar{\eta}^{\mathfrak{g}},\bar{\eta}^{\mathfrak{h}},\bar{\eta}^{\mathscr{G}},\bar{\eta}^{\mathscr{H}})$ be a feasible point of the MWDMMPVC (39), such that*

$$\sum_{i=1}^{p}\bar{\eta}_i^{\mathfrak{f}}\mathfrak{f}_i(\mathfrak{z}^*)\leqq\sum_{i=1}^{p}\bar{\eta}_i^{\mathfrak{f}}\mathfrak{f}_i(\bar{\mathfrak{u}}).$$

If $\mathfrak{g}^T\eta^{\mathfrak{g}},\ \mathfrak{h}^T\eta^{\mathfrak{h}},\ -\mathscr{H}^T\eta^{\mathscr{H}},\ \mathscr{G}^T\eta^{\mathscr{G}}$ are quasiconvex at $\bar{\mathfrak{u}}$ and any one of the following holds:

(a) $\eta_i^{\mathfrak{f}}>0$ and $\mathfrak{f}_i(\forall i\in I_{\mathfrak{f}})$ are pseudoconvex except at least one $\mathfrak{f}_i$ strictly pseudoconvex at $\bar{\mathfrak{u}}$,

$$(b)\ \eta_i^{\mathfrak{f}} > 0\ (\forall i \in I_{\mathfrak{f}})\ \ and\ \sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \mathfrak{f}_i(\cdot)\ strictly\ pseudoconvex\ at\ \bar{\mathfrak{u}}.$$

Then,

$$\mathfrak{z}^* = \bar{\mathfrak{u}}. \tag{48}$$

Proof Let $\mathfrak{z}^* \neq \bar{\mathfrak{u}}$. Then, from assumptions we have

$$\sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \mathfrak{g}_i(\mathfrak{z}) \leqq \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \mathfrak{g}_i(\mathfrak{u}) \Rightarrow \left\langle \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \nabla \mathfrak{g}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u} \right\rangle \leqq 0, \tag{49}$$

$$\sum \eta_i^{\mathfrak{h}} \mathfrak{h}_i(\mathfrak{z}) = \sum \eta_i^{\mathfrak{h}} \mathfrak{h}_i(\mathfrak{u}) \Rightarrow \left\langle \sum \eta_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u} \right\rangle = 0,\ i \in I_{\mathfrak{h}}, \tag{50}$$

$$-\sum \eta_i^{\mathscr{H}} \mathscr{H}_i(\mathfrak{z}) \leqq -\sum \eta_i^{\mathscr{H}} \mathscr{H}_i(\mathfrak{u}) \Rightarrow \left\langle -\sum \eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u} \right\rangle \leqq 0, \tag{51}$$

$$\sum \eta_i^{\mathscr{G}} \mathscr{G}_i(\mathfrak{z}) \leqq \sum \eta_i^{\mathscr{G}} \mathscr{G}_i(\mathfrak{u}) \Rightarrow \left\langle \sum \eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u} \right\rangle \leqq 0, \tag{52}$$

Adding (49)–(52), we get

$$\left\langle \sum_{i=1}^{q} \eta_i^{\mathfrak{g}} \nabla \mathfrak{g}_j(\mathfrak{u}) + \sum_{i=1}^{r} \eta_i^{\mathfrak{h}} \nabla \mathfrak{h}_i(\mathfrak{u}) - \sum_{i=1}^{m} \eta_i^{\mathscr{H}} \nabla \mathscr{H}_i(\mathfrak{u}) + \sum_{i=1}^{m} \eta_i^{\mathscr{G}} \nabla \mathscr{G}_i(\mathfrak{u}), \mathfrak{z} - \mathfrak{u} \right\rangle \leqq 0. \tag{53}$$

Since $\bar{\mathfrak{u}}$ is feasible point of the MWDMMPVC (39), then

$$\left\langle \sum_{i=1}^{p} \eta_i^{\mathfrak{f}} \nabla \mathfrak{f}_i(\bar{\mathfrak{u}}), \mathfrak{z}^* - \bar{\mathfrak{u}} \right\rangle \geqq 0. \tag{54}$$

Using assumptions (a) or (b) at $\bar{\mathfrak{u}}$, and from (54), we get

$$(\eta^{\mathfrak{f}})^T \mathfrak{f}(\mathfrak{z}^*) > (\eta^{\mathfrak{f}})^T \mathfrak{f}(\bar{\mathfrak{u}}),$$

which contradict the hypothesis. Hence, $\mathfrak{z}^* = \bar{\mathfrak{u}}$. □

5 Conclusion

In this article, we have established necessary optimality conditions for multiobjective mathematical programs with vanishing constraints under smooth assumptions using generalized Guignard constraint qualification and established sufficient optimality conditions using quasiconvexity and pseudoconvexity hypothesis. Moreover, we have formulated Wolfe type and Mond–Weir type dual models and established usual duality results under generalized convexity for multiobjective mathematical programs with vanishing constraints. We illustrated our results with help of some suitable examples. In future, the results of this paper can be extended for nonsmooth cases. we can generalize these results for an important mathematical program known as mathematical program with switching constraints (MPSC) motivated by recent work of Liang and Ye [30].

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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