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Integral Representations of Bargmann Type for the β-Modified Bergman Space on Punctured Unit Disc

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Abstract
We deal with the integral representation of Bargmann type of the functions belonging to the β-modified Bergman space on the punctured unit disc, by means of some special kernel-distribution involving the confluent hypergeometric functions and generalizing the classical second Bargmann transform. As application, we derive integral formula on the unit disc for the product of confluent hypergeometric functions, by considering the associated fractional Hankel transform.

Keywords Reproducing kernel · β-Modified Bargmann transform · β-Fractional Hankel transform

Mathematics Subject Classification Primary 30H20; Secondary 46E22 · 44A20

1 Introduction and Statement of Main Result

The so-called Bargmann transform was introduced by Bargmann [1] in his famous paper. It is defined as a unitary integral transform mapping the wave functions (Schrödinger) space to the classical Bargmann–Fock space on the complex plane, whose kernel function corresponds to the generating function of the Hermite functions. Since then, it has been investigated by many authors within the framework of

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quantum mechanics, signal processing and time frequency analysis. We refer to the nice book by Folland [5], the survey by Hall [8], and the rich list of references therein.

In contrast, the second transform introduced by Bargmann in the same paper [1, p. 203] does not gain such renewed interest. It can be realized as the coherent state transform on the quantum mechanical configuration space $\mathcal{H}_\gamma(\mathbb{R}^+)$, the Hilbert space of complex-valued measurable functions that are square integrable on the real half line $\mathbb{R}^+$ with respect to the measure $(x^\gamma/\Gamma(\gamma+1))\mathrm{d}x$; $\gamma > 0$, where $\Gamma(\cdot)$ is the gamma function. More explicitly, the transform

$$[\widetilde{\mathcal{B}}_\gamma \varphi](z) = \frac{(\gamma/\pi)^{1/2}}{(1-z)^{\gamma+1}} \int_0^{+\infty} \varphi(x) \exp\left(\frac{-x(1+z)}{2(1-z)}\right) \frac{x^\gamma}{\Gamma(\gamma+1)} \mathrm{d}x, \tag{1.1}$$

made $\mathcal{H}_\gamma(\mathbb{R}^+)$ unitary isomorphic to the classical weighted Bergman space $\mathcal{A}^{2,\gamma}(\mathbb{D})$ of holomorphic functions on the unit disc $\mathbb{D} = \{z = x + iy \in \mathbb{C}; |z| < 1\}$ belonging to the Hilbert space $L^2(\mathbb{D}, (1-|z|^2)^{\gamma-1}\mathrm{d}\lambda)$, $\mathrm{d}\lambda(z) = \mathrm{d}x\mathrm{d}y$ being the area measure. The phase space $\mathcal{A}^{2,\gamma}(\mathbb{D})$ may be realized as the null space of the densely defined second-order differential operator

$$\Delta_\gamma = -4\left(1-|z|^2\right)\left(\left(1-|z|^2\right)\frac{\partial^2}{\partial z \partial \bar{z}} - (\gamma+1)\bar{z}\frac{\partial}{\partial \bar{z}}\right), \tag{1.2}$$

which can be interpreted as a Hamiltonian of a charged particle in an external constant magnetic field. The involved kernel function is closely connected to the generating function for the generalized Laguerre polynomials

$$L_n^{(\tau)}(x) = \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\tau+n+1)}{\Gamma(n-k+1)\Gamma(\tau+k+1)} \frac{x^k}{k!}. \tag{1.3}$$

Namely, we have

$$A(z;x) = \mathrm{e}^{-x/2} \sum_{n=0}^{\infty} z^n L_n^{(\gamma)}(x) = (1-z)^{-\gamma-1} \exp\left(\frac{-x(1+z)}{2(1-z)}\right).$$

Other generating functions on the underlying Hilbert space $\mathcal{H}_\gamma(\mathbb{R}^+)$ have been considered by Bargmann in his paper [2] of 1967 (Part II), whose associated phase spaces are different of $\mathcal{A}^{2,\gamma}(\mathbb{D})$.

Notice for instance that equivalent forms of $\widetilde{\mathcal{B}}_\gamma$ in (1.1) can be given making use of canonical isometrics onto $\mathcal{H}_\gamma(\mathbb{R}^+)$. Thus, on the Hilbert space $L^2_\tau(\mathbb{R}^+) := L^2(\mathbb{R}^+, \mathrm{d}\nu_\tau)$; $\tau > 0$, of complex-valued functions that are square integrable on $\mathbb{R}^+$ with respect to the inner scalar product

$$\langle \varphi, \psi \rangle_\tau = \int_0^{+\infty} \varphi(x)\overline{\psi(x)}\mathrm{d}\nu_\tau(x); \quad \mathrm{d}\nu_\tau(x) := x^\tau \mathrm{e}^{-x}\mathrm{d}x,$$

the transform $\widetilde{\mathcal{B}}_\gamma$ reads equivalently

$$[\mathcal{A}_\tau f](z) = \left(\frac{1}{\pi\Gamma(\tau)}\right)^{1/2} \int_0^{+\infty} \frac{x^\tau e^{-x}}{(1-z)^{\tau+1}} \exp\left(\frac{-zx}{1-z}\right) f(x)\mathrm{d}x \qquad (1.4)$$

and maps $L^2_\tau(\mathbb{R}^+)$ unitarily onto $\mathcal{A}^{2,\tau}(\mathbb{D})$. Another variant on $L^2(\mathbb{R}^+, \mathrm{d}x/x)$ was considered in [3], and next extended to generalized Bargmann transforms giving rise to the integral representation of the weighted m-th Bergman space seen as the L^2-eigenspace of Δ_γ in (1.2) associated the m-th hyperbolic Landau level $4m(\gamma - m)$; $m = 0, 1, \dots, [\gamma/2]$, where $[\cdot]$ denotes the integer part.

Recently, the so-called β-modified Bergman space $\mathcal{A}^{2,\alpha}_\beta(\mathbb{D}) = \mathcal{H}ol(\mathbb{D}^*) \cap L^{2,\alpha}_\beta(\mathbb{D})$ was introduced in [7]. It is defined as the closed subspace of holomorphic functions on the punctured unit disc $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ embedded in the Hilbert space $L^{2,\alpha}_\beta(\mathbb{D}) := L^2(\mathbb{D}, \mathrm{d}\mu_{\alpha,\beta})$; $\alpha, \beta > -1$, of Borel measurable complex-valued functions f on $\mathbb{D}$ subject to

$$\|f\|^2_{\alpha,\beta} := \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}\mu_{\alpha,\beta}(z) < +\infty,$$

where

$$\mathrm{d}\mu_{\alpha,\beta}(z) := |z|^{2\beta}\left(1 - |z|^2\right)^\alpha \mathrm{d}\lambda(z).$$

The sequence characterization reads

$$\mathcal{A}^{2,\alpha}_\beta(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^{n-m}; \quad \sum_{n=0}^{\infty} B(\alpha+1, \beta+n-m+1)|a_n|^2 < +\infty \right\},$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ denotes the classical Beta function and $m = \min\{k \in \mathbb{Z};\ \beta \le k\}$ (i.e., $\beta = \beta_0 + m$ with $m \in \mathbb{N}$ and $-1 < \beta_0 \le 0$). We refer to [6, 7] for the motivation and additional information on this space. The nature and the distribution of the zeros of its reproducing kernel $K_{\alpha,\beta}$ is described in details in [6].

In the present paper, we are concerned with integral representation theorems for the considered β-modified Bergman space. We provide sufficient and necessary conditions to given holomorphic function on the punctured unit disc to be the image of some complex-valued function belonging to the configuration space $L^2_\tau(\mathbb{R}^+)$ by some second Bargmann-like transform. Namely, we deal with the integral transforms

$$\left[\mathcal{G}_{\alpha,\beta}\psi\right](z) := c_\alpha \int_0^{+\infty} \frac{x^{\alpha+1}}{z^\beta(1-z)^{\alpha+2}} \exp\left(-\frac{x}{1-z}\right) \psi(x)\mathrm{d}x \qquad (1.5)$$

and

$$\left[\widetilde{\mathcal{G}}_{\alpha,\beta}\psi\right](z) := \frac{c_\alpha}{\Gamma(-\alpha)} \int_0^{+\infty} \frac{e^{-x}}{x^{\alpha+1} z^{\alpha+\beta+1}(1-z)}\, {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\,\middle|\, -\frac{xz}{1-z}\right) \psi(x)\mathrm{d}x. \qquad (1.6)$$

Here,

$$c_\alpha = \left(\frac{1}{\pi\Gamma(\alpha+1)}\right)^{1/2} \tag{1.7}$$

and ${}_1F_1$ denotes the confluent hypergeometric function defined by the absolutely convergent series

$${}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| z\right) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}, \tag{1.8}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol for any nonnegative integer n. For the special case $\beta = 0$ (and then $m = 0$), we recover the conventional weighted Bergman space $\mathcal{A}^{2,\alpha+1}(\mathbb{D})$. Moreover, the corresponding transform $\mathcal{G}_{\alpha,0}$, given through 1.5, reduces to the second Bargmann transform in (1.4) with $\tau = \alpha + 1 = \gamma$.

More precisely, we aim to show that, under the assumption that $\tau = \alpha+1$ and β is in addition a nonnegative integer, a complex-valued function f belongs to $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D})$ if and only if there exists $\varphi \in L^2_\tau(\mathbb{R}^+)$ such that $f = \mathcal{G}_{\alpha,\beta}\varphi$ (Theorem 3.1). A similar statement holds true for the transform $\widetilde{\mathcal{G}}_{\alpha,\beta}$ when $-1 < \alpha < 0$ and $\alpha+\beta$ is a nonnegative integer with $\tau = -\alpha - 1$ (Theorem 3.2). This leads to the concrete description of the spectral and analytical properties of $\mathcal{G}_{\alpha,\beta}$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}$ when acting on the Hilbert spaces $L^2_{\alpha+1}(\mathbb{R}^+)$ and $L^2_{-\alpha-1}(\mathbb{R}^+)$, respectively. This explicit characterization is valid only for special values of α and β. The general case needs further investigations. Thus, we also show that the concerned kernel functions are closely connected to the reproducing kernel function and we give the explicit expression of the inversion formula for the integral transforms $\mathcal{G}_{\alpha,\beta}$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}$. As application, by combining the obtained results with the fractional Hankel transform, we derive original integral formulas on the unit disc, including the one for the product of the confluent hypergeometric function, in terms of the modified Bessel function (Theorem 4.1).

In Sect. 2, we give an equivalent closed formula for the reproducing kernel of the β-modified Bergman space $\mathcal{A}^{2,\alpha}_\beta(\mathbb{D})$ (Proposition 2.1). In Sect. 3, we establish the main properties of the transforms $\mathcal{G}_{\alpha,\beta}$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}$, including the explicit action on an orthonormal basis of $L^2_\tau(\mathbb{R}^+)$, leading to the proof of our first main results (Theorems 3.1 and 3.2). Section 4 is devoted to the proof of Theorem 4.1 concerning a direct application of the obtained result making use of associated fractional Hankel transform.

2 Closed Formula for $K_{\alpha,\beta}$

We give below a closed formula for the reproducing kernel $K_{\alpha,\beta}$ in terms of the Gauss hypergeometric function ${}_2F_1$ defined by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{2.1}$$

Proposition 2.1 *The reproducing kernel of $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ is given by*

$$K_{\alpha,\beta}(z,w) := \frac{(z\overline{w})^{-m}}{\pi B(\alpha+1,\beta-m+1)} {}_2F_1\left(\begin{matrix} 1, \alpha+\beta-m+2 \\ \beta-m+1 \end{matrix} \middle| z\overline{w} \right). \tag{2.2}$$

Proof The closed formula (2.2) for the reproducing kernel $K_{\alpha,\beta}(z,w)$ can be handled making use its expansion reads

$$K_{\alpha,\beta}(z,w) := \sum_{n=0}^{\infty} \mathrm{e}^{\alpha,\beta}_n(z)\overline{\mathrm{e}^{\alpha,\beta}_n(w)},$$

for orthogonal basis $\mathrm{e}^{\alpha,\beta}_n$ of $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$. Thus, using the orthonormal basis

$$\mathrm{e}^{\alpha,\beta}_n(z) := \gamma^{\alpha,\beta}_n z^{n-m}; \quad n = 0, 1, 2, \dots,$$

with

$$(\gamma^{\alpha,\beta}_n)^2 := \frac{1}{\pi B(\alpha+1,\beta-m+1)} \frac{(\alpha+\beta-m+2)_n}{(\beta-m+1)_n} \tag{2.3}$$

we easily deduce the closed expression in (2.2). The existence of $K_{\alpha,\beta}$ is due to the Riesz representation theorem since the evaluation mapping is continuous. In fact, we have

$$\begin{aligned} |f(z)| &\le \left(\sum_{n=0}^{\infty} |\gamma^{\alpha,\beta}_n|^2 \right)^{1/2} \|f\|_{\alpha,\beta} \\ &\le \frac{1}{\pi\Gamma(\alpha+1)} \left(\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\beta-m+n+2)}{\Gamma(n+\beta-m+1)} \right)^{1/2} \|f\|_{\alpha,\beta} \end{aligned}$$

for every $f \in \mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$. The convergence of the involved series can be handled using the Euler asymptotic estimate

$$\frac{\Gamma(\alpha+\beta-m+n+2)}{\Gamma(n+\beta-m+1)} \sim \frac{1}{n^{\alpha+1}}$$

valid for n large enough. □

Remark 2.1 Some interesting estimates for $f \in \mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ are obtained in [7] and can be used to prove the continuity of $f \mapsto f(z)$ for fixed $z \in \mathbb{D}^*$.

For $\beta = 0$ (and then $m = 0$), we recover the reproducing kernel of the weighted Bergman space $\mathcal{A}^{2,\alpha+1}(\mathbb{D})$, to wit

$$K_{\alpha,0}(z,w) := \frac{1}{\pi B(\alpha+1,1)} {}_2F_1\left(\begin{matrix} 1, \alpha+2 \\ 1 \end{matrix} \middle| z\overline{w} \right) = \frac{\alpha+1}{\pi} \frac{1}{(1-z\overline{w})^{\alpha+2}}.$$

The following result discusses the particular case $\alpha = 0$.

Corollary 2.2 *The reproducing kernel of $\mathcal{A}^{2,0}_{\beta}(\mathbb{D})$ is given by the closed formula*

$$K_{0,\beta}(z,w) := \frac{(\beta-m+1)-(\beta-m)z\overline{w}}{\pi(1-z\overline{w})^2(z\overline{w})^m}. \tag{2.4}$$

Proof The proof is immediate by direct computation since for $|\xi|<1$, we have

$${}_2F_1\left(\begin{matrix}1, c+1\\ c\end{matrix}\middle|\xi\right) = \left(Id + \frac{\xi}{c}\frac{d}{\mathrm{d}\xi}\right)\left(\frac{1}{1-\xi}\right) = \frac{c+(1-c)\xi}{c(1-\xi)^2}.$$

□

3 Integral Representation of $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$

The construction of bounded integral transforms on separable Hilbert space with prescribed kernel function can be done following a general scheme. In fact, for our underlying Hilbert spaces $L^2_\tau(\mathbb{R}^+)$ with the orthonormal basis φ^τ_n and $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ with the orthonormal basis $e^{\alpha,\beta}_n$, the condition

$$\left\|\sum_{n=0}^{N} a_n e^{\alpha,\beta}_n\right\|_{\alpha,\beta} \le c\left\|\sum_{n=0}^{N} a_n \varphi^\tau_n\right\|_{\tau} \tag{3.1}$$

(for certain $c\ge 0$) for every N and $a_n\in\mathbb{C}$; $0\le n\le N$, shows the existence of a uniquely defined bounded operator $\mathcal{G}$ on $L^2_\tau(\mathbb{R}^+)$ such that $\mathcal{G}\varphi^\tau_n = e^{\alpha,\beta}_n$ (see [1, p. 195]). Moreover, the corresponding kernel function is given by the bilinear generating function

$$\sum_{n=0}^{\infty}\overline{\varphi^\tau_n}(x)e^{\alpha,\beta}_n(z).$$

We aim to prove the following main results

Theorem 3.1 *The integral transform $\mathcal{G}_{\alpha,\beta}$ is well bounded defined operator on the Hilbert space $L^2_{\alpha+1}(\mathbb{R}^+)$ when β is a nonnegative integer and maps it isometrically onto the β-modified Bergman space $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$. Its inverse $\mathcal{G}^{-1}_{\alpha,\beta}:\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})\longrightarrow L^2_{\alpha+1}(\mathbb{R}^+)$ is given by*

$$\mathcal{G}^{-1}_{\alpha,\beta}[f](x) = c_\alpha\int_{\mathbb{D}}\frac{1}{\overline{z}^\beta(1-\overline{z})^{\alpha+2}}\exp\left(-\frac{x\overline{z}}{1-\overline{z}}\right)f(z)\mathrm{d}\mu_{\alpha,\beta}(z).$$

Theorem 3.2 *The transform $\widetilde{\mathcal{G}}_{\alpha,\beta}$ is well bounded defined operator on the Hilbert space $L^2_{-\alpha-1}(\mathbb{R}^+)$ with $-1<\alpha<0$, and $\alpha+\beta$ is a nonnegative integer. Its range* *coincides with the β-modified Bergman space $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$. The inverse is given by*

$$\widetilde{\mathcal{G}}^{-1}_{\alpha,\beta}[f](x) = \frac{c_\alpha}{\Gamma(-\alpha)} \int_{\mathbb{D}} \frac{1}{\overline{z}^{\alpha+\beta+1}(1-\overline{z})} {}_1F_1\left(\begin{matrix} 1 \\ -\alpha \end{matrix} \middle| -\frac{x\overline{z}}{1-\overline{z}} \right) f(z) \mathrm{d}\mu_{\alpha,\beta}(z).$$

To this end, we begin by providing the expansion series of the involved kernel functions

$$g_{\alpha,\beta}(x,z) = c_\alpha \frac{x^{\alpha+1}}{z^\beta (1-z)^{\alpha+2}} \exp\left(-\frac{x}{1-z}\right)$$

and

$$\tilde{g}_{\alpha,\beta}(x,z) = \frac{c_\alpha}{\Gamma(-\alpha)} \frac{x^{-\alpha-1}\mathrm{e}^{-x}}{z^{\alpha+\beta+1}(1-z)} {}_1F_1\left(\begin{matrix} 1 \\ -\alpha \end{matrix} \middle| -\frac{xz}{1-z} \right) \tag{3.2}$$

of $\mathcal{G}_{\alpha,\beta}(x;z)$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}(x;z)$. Thus, let consider the functions

$$\varphi^\tau_n(x) = \rho^\tau_n L^{(\tau)}_n(x); \quad \rho^\tau_n := \left(\frac{n!}{\Gamma(\tau+n+1)}\right)^{1/2}, \tag{3.3}$$

associated with the generalized Laguerre polynomials $L^{(\tau)}_n(x)$ in (1.3), and constituting an orthonormal basis of $L^2_\tau(\mathbb{R}^+)$.

Proposition 3.3 *Let c_α be the constant in (1.7). The distribution-kernels for $\mathcal{G}_{\alpha,\beta}(x;z)$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}(x;z)$ are given, respectively, by*

$$g_{\alpha,\beta}(x,z) = c_\alpha x^{\alpha+1}\mathrm{e}^{-x} \sum_{n=0}^{\infty} z^{n-m} L^{(\tau)}_n(x) \tag{3.4}$$

when $\beta = m$ is a nonnegative integer and $\alpha > -1$, and by

$$\tilde{g}_{\alpha,\beta}(x,z) = \frac{c_\alpha}{\Gamma(-\alpha)} x^{-\alpha-1}\mathrm{e}^{-x} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\tau+n+1)} z^{n-\alpha-\beta-1} L^{(\tau)}_n(x). \tag{3.5}$$

when $\alpha+\beta$ is a nonnegative integer and $-1 < \alpha < 0$.

Proof Let

$$S^\tau_{\alpha,\beta}(x;z) := \sum_{n=0}^{\infty} \varphi^\tau_n \mathrm{e}^{\alpha,\beta}_n(z) = \sum_{n=0}^{\infty} \gamma^{\alpha,\beta}_n \rho^\tau_n \, z^{n-m} L^{(\tau)}_n(x)$$

For $\tau = \alpha+1$ and $\beta = m$ (β nonnegative integer), we get $\gamma^{\alpha,\beta}_n \rho^\tau_n = c_\alpha$ for any $n = 0, 1, 2, \ldots$. Therefore, the previous series reduces further to

$$S^{\alpha+1}_{\alpha,\beta}(x;z) = c_\alpha z^{-m} \sum_{n=0}^{\infty} z^n L^{(\tau)}_n(x) = c_\alpha \frac{\exp\left(-\frac{xz}{1-z}\right)}{z^m(1-z)^{\alpha+2}}.$$

The last equality readily follows by means of the classical generating function for the Laguerre polynomials [10, p.242]. This shows that $x^{\alpha+1}\mathrm{e}^{-x}S^{\alpha+1}_{\alpha,\beta}(x;z) = g_{\alpha,\beta}(x;z)$ is exactly the kernel function of the integral transform given through (1.5).

Now, for the case of $\tau = -\alpha - 1$ with $-1 < \alpha < 0$ and under the assumption that $\alpha + \beta$ is a nonnegative integer, we obtain $\gamma_j^{\alpha,\beta} = c_\alpha \rho_j^{\alpha,\beta}$ and hence

$$S^{-\alpha-1}_{\alpha,\beta}(x;z) = c_\alpha \sum_{j=0}^{\infty} (\rho_j^{\alpha,\beta})^2 z^{j-m} L_j^{(\tau)}(x) = \frac{c_\alpha}{\Gamma(\tau+1)} \sum_{j=0}^{\infty} \frac{j!}{(\tau+1)_j} z^{j-m} L_j^{(\tau)}(x).$$

Thus, we recognize the generating function [10, p. 242]

$$\sum_{j=0}^{\infty} \frac{(\lambda)_j}{(\tau+1)_j} z^j L_j^{(\tau)}(x) = (1-z)^{-\lambda} {}_1F_1\left(\begin{matrix} \lambda \\ \tau+1 \end{matrix} \middle| -\frac{xz}{1-z} \right).$$

This shows that

$$S^{-\alpha-1}_{\alpha,\beta}(x;z) = \frac{c_\alpha}{\Gamma(-\alpha)} \frac{1}{z^m(1-z)} {}_1F_1\left(\begin{matrix} 1 \\ \tau+1 \end{matrix} \middle| -\frac{xz}{1-z} \right) = x^{\alpha+1}\mathrm{e}^{x}\widetilde{g}_{\alpha,\beta}(x,z).$$

□

Remark 3.1 It is interesting to point out that the conditions $-1 < \alpha < 0$ and $\alpha + \beta$ are nonnegative integer imply that $\alpha + \beta + 1 = m = \min\{k = 0, 1, 2, \ldots; k \geq \beta\}$, and then, the corresponding integral transform $\widetilde{G}_{\alpha,\beta}$ is very specific and depends only in β.

The obtained kernel functions are closely connected to the reproducing kernel function discussed in the previous section. Namely, we prove the following key result.

Proposition 3.4 *Under the assumptions of Proposition 3.3, we have*

$$\left\langle S^\tau_{\alpha,\beta}(.,z), S^\tau_{\alpha,\beta}(.,w) \right\rangle_\tau = K_{\alpha,\beta}(z,w). \tag{3.6}$$

Proof The result can be checked at least formally using the expansion series of the involved series provided by Proposition 3.3. However, a rigorous proof can be given by direct computation. Below, we prove only the identity involving the second transform with $\alpha + \beta + 1 = m$. Thus, we have

$$\left\langle S^\tau_{\alpha,\beta}(.,z), S^\tau_{\alpha,\beta}(.,w) \right\rangle_\tau = \frac{c_\alpha^2}{\Gamma^2(-\alpha)} \frac{(z\overline{w})^{-m}}{(1-z)(1-\overline{w})} N_\alpha(z), \tag{3.7}$$

where $N_\alpha(z)$ stands for

$$N_\alpha(z) := \int_0^{+\infty} \mathrm{e}^{-x} x^{-(\alpha+1)} {}_1F_1\left(\begin{matrix} 1 \\ -\alpha \end{matrix} \middle| -\frac{xz}{1-z} \right) {}_1F_1\left(\begin{matrix} 1 \\ -\alpha \end{matrix} \middle| -\frac{x\overline{w}}{1-\overline{w}} \right) \mathrm{d}x.$$

The explicit computation of the integral $N_\alpha(z)$ can be handled making the change of variable $y = -xz/(1-z)$ for $z \in]-1, 0[$ and using again the identity [10, p. 293]. Indeed, we get

$$N_\alpha(z) = \int_0^{+\infty} y^{-(\alpha+1)} e^{-\frac{y(z-1)}{z}} \left(\frac{z}{z-1}\right)^\alpha {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| y\right) {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| \frac{\overline{w}(z-1)}{z(\overline{w}-1)} y\right) \mathrm{d}y$$
$$= \Gamma(-\alpha)\,(1-z)\,(1-\overline{w})\,{}_2F_1\left(\begin{matrix}1, 1\\-\alpha\end{matrix}\middle| \frac{\overline{w}(z-1)}{z(\overline{w}-1)}\frac{z^2(\overline{w}-1)}{(z-1)}\right). \quad (3.8)$$

Using the principle of isolated zeros, the Eq. (3.8) remains true for all $z \in \mathbb{D}^*$. Therefore, from (3.7) and (3.8) we obtain

$$\left\langle S_{\alpha,\beta}(., z), S^\tau_{\alpha,\beta}(., w)\right\rangle_\tau = \frac{(z\overline{w})^{-m}}{\pi B(\alpha+1, \beta-m+1)} {}_2F_1\left(\begin{matrix}1, \alpha+\beta-m+2\\ \beta-m+1\end{matrix}\middle| z\overline{w}\right).$$

The right hand-side is exactly the closed expression of the reproducing kernel function given in Proposition 2.1. □

Proposition 3.5 *The integral transform $\mathcal{G}_{\alpha,\beta}$ is well defined on the Hilbert space $L^2_\tau(\mathbb{R}^+)$ for $\tau = \alpha + 1$ and β being a nonnegative integer, while $\widetilde{\mathcal{G}}_{\alpha,\beta}$ is well defined on the Hilbert space $L^2_{\tau,\sigma}(\mathbb{R}^+)$ for $\tau = -(\alpha + 1)$, $-1 < \alpha < 0$, and $\alpha + \beta$ being a nonnegative integer.*

Proof Set $K_{\alpha,\beta_z}(w) := K_{\alpha,\beta}(w, z)$. Then, by making use of the Cauchy–Schwarz inequality, we obtain

$$\left|\left(\mathcal{G}_{\alpha,\beta}\varphi\right)(z)\right| \le \left(\int_0^{+\infty} |S^\tau_{\alpha,\beta}(x, z)|^2 \mathrm{d}\nu(x)\right)^{1/2} \|\varphi\|_\tau$$
$$\le \left(K_{\alpha,\beta}(z, z)\right)^{1/2} \|\varphi\|_\tau$$
$$\le \left\|K_{\alpha,\beta_z}\right\|_{\alpha,\beta} \|\varphi\|_\tau.$$

These inequalities readily follow making use of Propositions 3.3 and 3.6. The fact $K_{\alpha,\beta}(z, z) = \left\|K_{\alpha,\beta_z}\right\|^2_{\alpha,\beta}$ is well-known from general theory of reproducing kernels and can be handled at least formally using the reproducing property applied to K_{α,β_z} (belonging to $\mathcal{A}^{2,\alpha}_\beta(\mathbb{D})$). The proof for the second transform $\widetilde{\mathcal{G}}_{\alpha,\beta}$ is quite similar. □

The following result gives the explicit action of the considered integral transforms on the orthogonal basis of $L^2_\tau(\mathbb{R}^+)$ generated by the generalized Laguerre polynomials.

Proposition 3.6 *Keep assumptions on the reals α and β as in Proposition 3.3. Then, we have*

$$\mathcal{G}_{\alpha,\beta}\left(\varphi^\tau_n\right)(z) = \mathrm{e}^{\alpha,\beta}_n(z)$$

and

$$\widetilde{\mathcal{G}}_{\alpha,\beta}(\varphi_n^\tau)(z) = \mathrm{e}_n^{\alpha,\beta}(z).$$

Proof The formal proof is immediate using the expansions of the involved kernels in Proposition 3.3. However, we present below a direct proof by making explicit computation. Indeed, for $\tau = \alpha + 1$, we have

$$\begin{aligned}\left[\mathcal{G}_{\alpha,\beta}\varphi_n^{\alpha+1}\right](z) &= \frac{c_\alpha \rho_n^{\alpha+1}}{z^\beta (1-z)^{\alpha+2}} \int_0^{+\infty} x^{\alpha+1} \exp\left(-\frac{x}{1-z}\right) L_n^{(\alpha+1)}(x)\mathrm{d}x \\ &= \left(\frac{1}{\pi B(\alpha+1, n+1)}\right)^{1/2} z^{n-m} = e_n^{\alpha,\beta}(z)\end{aligned}$$

by means of [10, p. 244]. For the second case of $\tau = -(\alpha+1)$ with $-1 < \alpha < 0$ and $\alpha + \beta$ being a nonnegative integer (and then $\alpha + \beta + 1 = m$), we make use of [10, p. 293] to derive the following

$$\begin{aligned}\left[\mathcal{G}_{\alpha,\beta}\varphi_n^{\tau}\right](z) &= \frac{c_\alpha \rho_n^{\tau}}{\Gamma(-\alpha) z^m (1-z)} \int_0^{+\infty} \frac{\mathrm{e}^{-x}}{x^{\alpha+1}} {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{xz}{1-z}\right) L_n^{(\tau)}(x)\mathrm{d}x \\ &= \frac{c_\alpha \rho_n^{\tau}}{\Gamma(-\alpha) z^m (1-z)} \frac{(-\alpha)_n}{n!} \int_0^{+\infty} \frac{\mathrm{e}^{-x}}{x^{\alpha+1}} {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{xz}{1-z}\right) {}_1F_1\left(\begin{matrix}-n-m\\-\alpha\end{matrix}\middle| x\right) \mathrm{d}x \\ &= \left(\frac{1}{\pi B(\alpha+1, \beta-m+n+1)}\right)^{1/2} z^{n-m} = e_n^{\alpha,\beta}(z).\end{aligned}$$

□

Proposition 3.7 *The range of the configuration space $L^2_{\alpha+1}(\mathbb{R}^+)$ (resp. $L^2_{-\alpha-1}(\mathbb{R}^+)$) by the transform $\mathcal{G}_{\alpha,\beta}$ (resp. $\widetilde{\mathcal{G}}_{\alpha,\beta}$), under the assumptions of Proposition 3.3, is exactly the β-modified Bergman space $\mathcal{A}^{2,\alpha}_\beta(\mathbb{D})$. Moreover, the transforms $\mathcal{G}_{\alpha,\beta}$ and $\widetilde{\mathcal{G}}_{\alpha,\beta}$ define unitary bounded operators.*

Proof From Proposition 3.6, the condition (3.1) holds since for every $\varphi = \sum_{n=0}^N a_n \varphi_n^\tau \in Span\{\varphi_n^\tau; n = 0, 1, 2, \ldots\}$, we have $\mathcal{G}_{\alpha,\beta}\varphi = \sum_{n=0}^N a_n \mathrm{e}_n^{\alpha,\beta}$ and then

$$\|\varphi\|_\tau^2 = \sum_{n=0}^N |a_n|^2 = \left\|\mathcal{G}_{\alpha,\beta}\varphi\right\|^2_{\alpha,\beta}.$$

Therefore, $\mathcal{G}_{\alpha,\beta}$ defines a bounded operator from $L^2_\tau(\mathbb{R}^+)$ onto its range

$$\mathcal{G}_{\alpha,\beta}\left(L^2_\tau(\mathbb{R}^+)\right) = \overline{Span\left\{\mathrm{e}_n^{\alpha,\beta}; n \geq 0\right\}}^{L^{2,\alpha}_\beta(\mathbb{D})} = \mathcal{A}^{2,\alpha}_\beta(\mathbb{D}).$$

□

Corollary 3.8 *The inverse integral transforms from $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ onto $L^2_\tau(\mathbb{R}^+)$ are given by*

$$\mathcal{G}^{-1}_{\alpha,\beta}[f](x) = c_\alpha \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha z^\beta}{(1-\overline{z})^{\alpha+2}} \exp\left(-\frac{x\overline{z}}{1-\overline{z}}\right) f(z)\mathrm{d}\lambda(z) \tag{3.9}$$

and

$$\widetilde{\mathcal{G}}^{-1}_{\alpha,\beta}[f](x) = \frac{c_\alpha}{\Gamma(-\alpha)} \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha z^\beta}{\overline{z}^{\alpha+1}(1-\overline{z})} {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{x\overline{z}}{1-\overline{z}}\right) f(z)\mathrm{d}\lambda(z). \tag{3.10}$$

Proof Using Proposition 3.3, it is not hard to see that the kernel functions for the inverse integral transforms $\mathcal{G}^{-1}_{\alpha,\beta} : \mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D}) \longrightarrow L^2_{\alpha+1}(\mathbb{R}^+)$ and $\widetilde{\mathcal{G}}^{-1}_{\alpha,\beta} : \mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D}) \longrightarrow L^2_{-\alpha-1}(\mathbb{R}^+)$ are given by $(1-|z|^2)^\alpha |z|^{2\beta} S^\tau_{\alpha,\beta}(x;z)$ (with $\tau = \alpha+1$ and $\tau = -\alpha-1$, respectively). This can also be recovered starting from Proposition 3.6. More explicitly, we have

$$\mathcal{G}^{-1}_{\alpha,\beta}[f](x) = \int_{\mathbb{D}} \overline{x^{-\alpha-1}\mathrm{e}^x g_{\alpha,\beta}(x;z)}\, f(z)\mathrm{d}\mu_{\alpha,\beta}(z)$$

and

$$\widetilde{\mathcal{G}}^{-1}_{\alpha,\beta}[f](x) = \int_{\mathbb{D}} \overline{x^{\alpha+1}\mathrm{e}^x \widetilde{g}_{\alpha,\beta}(x;z)}\, f(z)\mathrm{d}\mu_{\alpha,\beta}(z).$$

which gives rise to (3.9) and (3.10), respectively. □

4 Application

Let I_η be the modified Bessel function

$$I_\eta(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\eta+n+1)} \left(\frac{\xi}{2}\right)^{2n+\eta}. \tag{4.1}$$

For $m = 0, 1, 2, \ldots$ and $\tau = \alpha+1$ or $\tau = -\alpha-1$, we consider

$$V(x, y|\theta) := \frac{\left(\sqrt{xy\theta}\right)^{-\tau}}{(1-\theta)\theta^m} \exp\left(\frac{-(x+y)\theta}{1-\theta}\right) I_\tau\left(\frac{2\sqrt{xy\theta}}{1-\theta}\right) \tag{4.2}$$

for $x, y > 0$ and $|\theta| < 1$, as well as

$$J^\alpha_1(\theta) = \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha}{\left(1-\theta z-\overline{z}+\theta|z|^2\right)^{\alpha+2}} \exp\left(-\frac{x\overline{z}+y\theta z-(x+y)\theta|z|^2}{1-\theta z-\overline{z}+\theta|z|^2}\right) \mathrm{d}\lambda(z) \tag{4.3}$$

and

$$J_2^\alpha(\theta) = \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha |z|^{-2(\alpha+1)}}{\left(1-\theta z-\overline{z}+\theta|z|^2\right)} {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{x\overline{z}}{1-\overline{z}}\right) \times {}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{y\theta z}{1-\theta z}\right) \mathrm{d}\lambda(z). \tag{4.4}$$

Theorem 4.1 *The integrals $J_1^\alpha(\theta)$ in (4.3) and $J_2^\alpha(\theta)$ in (4.4) are closely connected to $V(x, y|\theta)$ in (4.2). More precisely, we have*

$$c_\alpha^2 J_1^\alpha(\theta) = \theta^m V(x, y|\theta) \quad \textit{and} \quad c_\alpha^2 J_2^\alpha(\theta) = \theta^m \Gamma^2(-\alpha) V(x, y|\theta).$$

Proof In order to prove Theorem 4.1, we adopt the formalism presented recently in [3] that we adapt to the complex setting. Thus, let $\mathcal{S}_{\alpha,\beta}$ denote the integral transform $\mathcal{G}_{\alpha,\beta}$ or $\widetilde{\mathcal{G}}_{\alpha,\beta}$, and set

$$\mathcal{S}_{\alpha,\beta}(\varphi)(z) = \int_{\mathbb{R}^+} S_{\alpha,\beta}^\tau(x, z)\varphi(x)\mathrm{d}\nu_\tau(x)$$

with kernel function given by $S_{\alpha,\beta}^\tau(x, z) = x^{-\alpha-1}\mathrm{e}^x g_{\alpha,\beta}(x, z)$ in (3.4) when $\mathcal{S}_{\alpha,\beta} = \mathcal{G}_{\alpha,\beta}$ and $S_{\alpha,\beta}^\tau(x, z) = x^{\alpha+1}\mathrm{e}^x \widetilde{g}_{\alpha,\beta}(x, z)$ in (3.5) when $\mathcal{S}_{\alpha,\beta} = \widetilde{\mathcal{G}}_{\alpha,\beta}$. Thus, by combining $\mathcal{S}_{\alpha,\beta}$ and its inverse given by

$$\mathcal{S}_{\alpha,\beta}^{-1}(f)(x) = \int_{\mathbb{D}} S^\tau(x, \overline{z}) f(z)\mathrm{d}\mu_{\alpha,\beta}(z),$$

we perform the operator $\mathcal{H}_\theta = \mathcal{S}_{\alpha,\beta}^{-1} \circ \Phi_\theta \circ \mathcal{S}_{\alpha,\beta}$, where $\Phi_\theta(f)(z) = f(\theta z)$. Such operator is an integral transform with kernel given by the expansion series (converging absolutely and uniformly)

$$\begin{aligned} V(x, y|\theta) &= \sum_{j=0}^\infty \theta^{j-m} \left(\rho_{j+m}^\tau\right)^2 L_j^{(\tau)}(x) L_j^{(\tau)}(y) \\ &= \frac{1}{\theta^m} \sum_{j=0}^\infty \frac{j!}{\Gamma(\tau+j+1)} \theta^j L_j^{(\tau)}(x) L_j^{(\tau)}(y) \\ &= \frac{1}{\theta^m(1-\theta)} \exp\left(\frac{\theta(x+y)}{1-\theta}\right) \sqrt{\theta x y}^{-\tau} I_\tau\left(\frac{2\sqrt{\theta x y}}{1-\theta}\right). \end{aligned} \tag{4.5}$$

The last identity is obtained thanks to the Hardy–Hille formula [10, p. 242]. In the formula above, I_τ is the modified Bessel function in (4.1).

On the other hand, by Fubini theorem we get

$$\begin{aligned}\mathcal{H}_\theta(\varphi)(x) &= \int_{\mathbb{R}^+}\left(\int_{\mathbb{D}} S^\tau(x,\overline{z})S^\tau(y,\theta z)\mathrm{d}\mu_{\alpha,\beta}(z)\right)\varphi(y)\mathrm{d}\nu_\tau(y)\\ &= \int_{\mathbb{D}} S^\tau(x,\overline{z})\left(\int_{\mathbb{R}^+} S^\tau(y,\theta z)\varphi(y)\mathrm{d}\nu_\tau(y)\right)\mathrm{d}\mu_{\alpha,\beta}(z)\end{aligned}$$

for every $\varphi \in L^2_\tau(\mathbb{R}^+)$. Thus, the kernel function $V(x, y|\theta)$ reads

$$V(x,y|\theta) := \int_{\mathbb{D}} S^\tau(x,\overline{z})S^\tau(y,\theta z)\mathrm{d}\mu_{\alpha,\beta}(z). \tag{4.6}$$

More exactly, keeping in mind the expression of the kernel given through (3.4) and (3.5), we get

$$V(x,y|\theta) = \frac{c_\alpha^2}{\theta^\beta}\int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha \exp\left(-\frac{x\overline{z}+y\theta z-x\theta|z|^2-y\theta|z|^2}{1-\theta z-\overline{z}+\theta|z|^2}\right)}{\left(1-\theta z-\overline{z}+\theta|z|^2\right)^{\alpha+2}}\mathrm{d}\lambda(z)$$

and

$$\begin{aligned}V(x,y|\theta) = \frac{c_\alpha^2}{\Gamma^2(-\alpha)\theta^{\alpha+\beta+1}}\int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha}{|z|^{2\alpha+2}(1-\theta z)(1-\overline{z})}\,{}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{x\overline{z}}{1-\overline{z}}\right)\\ \times\,{}_1F_1\left(\begin{matrix}1\\-\alpha\end{matrix}\middle| -\frac{y\theta z}{1-\theta z}\right)\mathrm{d}\lambda(z)\end{aligned}$$

Subsequently, the assertion of Theorem 4.1 follows by comparing (4.6) and (4.5). This completes the proof of Theorem 4.1. □

Remark 4.1 The transform $\mathcal{H}_\theta$ in the previous proof is exactly the fractional Hankel transform considered by Namias in [11] (see also [3, 9]).

5 Concluding Remarks

The previous formalism can be extended to deal with a family of bounded integral transforms on truncated Hilbert subspace of $L^2_\tau(\mathbb{R}^+)$. Thus, if we denote by $L^2_{\tau,\sigma}(\mathbb{R}^+)$ the closed subspace spanned by the generalized Laguerre polynomials $L^{(\tau)}_{n+\sigma}$; $n = 0, 1, \ldots$. Their images $\mathcal{A}^{2,\alpha}_{\beta,\sigma}(\mathbb{D})$ form a decreasing sequence of closed subspaces of the β-modified Bergman space $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$, and so that for the limit case $\sigma = m$, we recover the transforms described in Sect. 2. In fact, by performing

$$S^{\tau,\sigma}_{\alpha,\beta}(x;z) := \sum_{n=a_0}^{\infty} \varphi^\tau_{n+\sigma}(x)\mathrm{e}^{\alpha,\beta}_n(z) = \sum_{j=a_0+\sigma}^{\infty} \gamma^{\alpha,\beta}_{j-\sigma}\rho^\tau_j\, z^{j-\sigma} L^{(\tau)}_j(x)$$

and specifying $\tau = \varepsilon(\alpha+1)$ with $\varepsilon = \pm 1$, and $\sigma = \beta + \frac{1-\varepsilon}{2}(\alpha+1)$ we get

$$\gamma^{\alpha,\beta}_{j-\sigma}\rho^{\tau}_{j} = \frac{c_\alpha}{2}\left((1+\varepsilon) + (1-\varepsilon)\frac{(1)_j}{\Gamma(-\alpha)(\tau+1)_j}\right)$$

Subsequently, we claim that

$$S^{\tau,\sigma}_{\alpha,\beta}(x;z) = \frac{(1+\varepsilon)}{2} S^{\alpha+1}_{\alpha,\beta}(x;z) + \frac{(1-\varepsilon)}{2} S^{-\alpha-1}_{\alpha,\beta}(x;z)$$
$$- \left(\frac{(1+\varepsilon)c_\alpha}{2} \sum_{j=0}^{a_0+\sigma-1} z^{j-\sigma} L^{(\tau)}_j(x) + \frac{(1-\varepsilon)c_\alpha}{2\Gamma(-\alpha)} \sum_{j=0}^{a_0+\sigma-1} \frac{(1)_j}{(\tau+1)_j} z^{j-\sigma} L^{(\tau)}_j(x) \right).$$

Here, $S^{\alpha+1}_{\alpha,\beta}(x;z)$ and $S^{-\alpha-1}_{\alpha,\beta}(x;z)$ can be computed in a similar way as those involved in the previous section. Thus, we claim

$$S^{\alpha+1}_{\alpha,\beta}(x;z) = \frac{c_\alpha}{z^\sigma(1-z)^{\tau+1}} \exp\left(-\frac{xz}{1-z}\right)$$

and

$$S^{-\alpha-1}_{\alpha,\beta}(x;z) = \frac{c_\alpha}{\Gamma(-\alpha) z^\sigma (1-z)} {}_1F_1\left(\begin{matrix} 1 \\ \tau+1 \end{matrix} \middle| -\frac{xz}{1-z} \right)$$

hold trues for some specific values of α and β. In a separate forthcoming paper, we emphasize to extend the results to other values of α and β, and to provide a concrete description of the ranges of the corresponding integral transforms.

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