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Extremal Trees for the General Randić Index with a Given Domination Number

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Abstract

For a graph G with vertex set V_G and edge set E_G, the general Randić index $R_\alpha(G)$ is defined as the sum of items $(d_u d_v)^\alpha$ of all edges $uv \in E_G$, where d_u and d_v denote the degrees of vertices $u, v \in V_G$, respectively, and α is an arbitrary real number. In this paper, we determine the upper bound on $R_\alpha(G)$ with $\alpha \in [\alpha_1, 0)$ of trees in terms of domination number, where $\alpha_1 \approx -0.5287$ is the unique nonzero root of the equation $27^\alpha - 6^\alpha - 4^\alpha + 2^\alpha = 0$. Moreover, the lower bound on $R_\alpha(G)$ with $\alpha \in [\alpha_2, 0)$ of trees in terms of domination number is also obtained, where $\alpha_2 \approx -0.5696$ is the unique nonzero root of the equation $2 \cdot 4^\alpha + 2^{\alpha+1} - 3^\alpha - 2 \cdot 6^\alpha - 1 = 0$. Finally, the corresponding extremal trees are characterized.

Keywords The general Randić index · Domination number · Extremal trees

AMS Subject Classification 05C05 · 05C35 · 05C69

1 Introduction

The Randić index, also known as the connectivity index or the branching index, was introduced by the chemist Randić [22] in 1975, which has been extensively utilized in applications for chemistry, biology, and complex network [8,12,13,23]. In addition, various mathematical, physical, and chemical properties of the Randić index have been

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established in decades [7,14,18,24,27]. The formula of the Randić index for a graph $G = (V_G, E_G)$ is given by

$$R = R(G) = \sum_{uv \in E_G} (d_u d_v)^{-\frac{1}{2}},$$

where d_u and d_v denote the degrees of vertices $u, v \in V_G$, respectively.

Ballobás and Erdös [2] generalized the Randić index by replacing $-\frac{1}{2}$ with a real number α and named it the general Randić index, which is defined as

$$R_\alpha = R_\alpha(G) = \sum_{uv \in E_G} (d_u d_v)^\alpha.$$

A key turning point involving the mathematical investigation of the general Randić index appeared in the second half of the 1990s, when a significant and ever growing research on this matter started, causing numerous publications (see [4,9,10,14,20] and the reference therein).

Regarding the problem of extremal trees on the Randić index and the general Randić index, researchers have tried to find precise bounds of $R(G)$ and $R_\alpha(G)$ with some given graph parameters, such as the matching number [16,21], the number of pendent vertices [5,28], and the degree sequence [25]. Recently, the relationship between topological indices and domination number has become an interesting research topic and has attracted lots of attention [3,6,11,15,17,19,26]. Bermudo et al. [1] got the extremal trees of Randić index with a given domination number. Naturally, we intend to generalize their results by considering the general Randić index.

First, some terminology and notations need to be introduced. A set $D \subseteq V_G$ is called a dominating set in G if every vertex in $V_G \backslash D$ is adjacent to at least one vertex in D. The domination number of G, denoted by $\gamma(G)$, is defined as the minimum cardinality among all dominating sets D of G, i.e., $\gamma(G) = \min_{D \subseteq V_G}\{|D|\}$. A vertex $v \in V_G$ is called a pendent vertex if $d_v = 1$, while it is called a support vertex if v is a neighbor of a pendent vertex. The set of neighbors of v is $N(v) = \{u \in V_G | \ uv \in E_G\}$. The diameter of a tree is the longest path between two pendent vertices. Besides, $G - \{v\}$ denote the graph obtained from G by deleting the vertex v together with its incident edges.

In this paper, we determine the upper bound of $R_\alpha(G)$ with $\alpha \in [\alpha_1, 0)$ of trees in terms of domination number, where $\alpha_1 \approx -0.5287$ is the unique nonzero root of the equation $27^\alpha - 6^\alpha - 4^\alpha + 2^\alpha = 0$. Moreover, the lower bound of $R_\alpha(G)$ with $\alpha \in [\alpha_2, 0)$ of trees in terms of domination number is also obtained, where $\alpha_2 \approx -0.5696$ is the unique nonzero root of the equation $2 \cdot 4^\alpha + 2^{\alpha+1} - 3^\alpha - 2 \cdot 6^\alpha - 1 = 0$. Finally, we characterize the corresponding extremal trees.

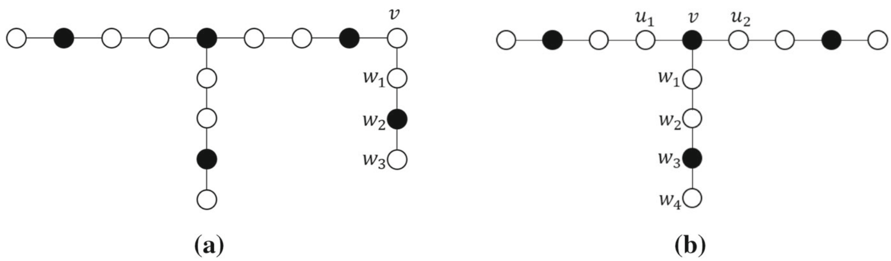


Fig. 1 Two trees in the graph family $\mathcal{T}_1$

2 The Upper Bound on the $R_\alpha(G)$ with $\alpha \in [\alpha_1, 0)$ of Trees in Terms of Domination Number

To characterize extremal n-vertex trees of the Randić index with a given domination number γ, Bermudo et al. [1] constructed two graph families $\mathcal{T}_1$ and $\mathcal{T}_2$, which are described in Definitions 2.1 and 3.1, respectively. In this section, we determined the upper bound on the general Randić index R_α of trees with a given domination number γ for $\alpha \in [\alpha_1, 0)$, where $\alpha_1 \approx -0.5287$ is the unique nonzero root of the equation $27^\alpha - 6^\alpha - 4^\alpha + 2^\alpha = 0$. Then, we prove that the corresponding extremal trees also belong to $\mathcal{T}_1$.

Definition 2.1 [1] Let $\mathcal{T}_1$ be a set of trees and the path with $3t$ vertices (P_{3t}) all belong to $\mathcal{T}_1$, where t is a positive integer number. The following are two ways to construct a new graph that belongs to the graph family $\mathcal{T}_1$:

(i) If there exist a tree $T' \in \mathcal{T}_1$ and a pendent vertex $v \in V_{T'}$, we take a path P_{3t}, whose consecutive vertices are $w_1, w_2, \ldots, w_{3t}$, then the tree T, such that $V_T = V_{T'} \cup V_{P_{3t}}$ and $E_T = E_{T'} \cup E_{P_{3t}} \cup \{vw_1\}$, belongs to $\mathcal{T}_1$ (see Fig. 1a).

(ii) If $T' \in \mathcal{T}_1$ satisfies that there exists a vertex $v \in D(T')$ such that $N(v) = \{u_1, u_2\}$ and $d_{u_1} = d_{u_2} = 2$, where $D(T')$ is a minimum dominating set in T', we take a path P_{3t+1}, whose consecutive vertices are $w_1, w_2, \ldots, w_{3t+1}$, then the tree T, such that $V_T = V_{T'} \cup V_{P_{3t+1}}$ and $E_T = E_{T'} \cup E_{P_{3t+1}} \cup \{vw_1\}$, belongs to $\mathcal{T}_1$ (see Fig. 1b).

To simplify the proof of the following theorem giving upper bound on $R_\alpha(T)$, we first give a lemma to show the monotonicity of two functions.

Lemma 2.2 *Let $\psi_1(z) = z^\alpha(z-1)$ and $\phi_1(z) = \psi_1(z) - \psi_1(z-1) = z^\alpha(z-1) - (z-1)^\alpha(z-2)$. If $\alpha \in (-1, 0)$, the function $\psi_1(z)$ increases monotonically for $z \in (0, \infty)$ and the function $\phi_1(z)$ decreases monotonically for $z \in (1, \infty)$.*

Proof Since $\psi_1'(z) = (\alpha+1)z^\alpha - \alpha z^{\alpha-1} > 0$ for $\alpha \in (-1, 0)$ and $z > 0$, we obtain that $\psi_1(z)$ is a monotonically increasing function for $z \in (0, \infty)$ if $\alpha \in (-1, 0)$. Note that $\psi_1''(z) = \alpha z^{\alpha-2}[(\alpha+1)z - (\alpha-1)] < 0$ for $\alpha \in (-1, 0)$ and $z > 0$. Suppose that $\alpha \in (-1, 0)$ and there exist z_1 and z_2 such that $z_2 > z_1 > 1$, then by Lagrange mean value theorem, we get

$$\begin{aligned}\phi_1(z_2)-\phi_1(z_1) &= \phi_1'(\xi_1)(z_2-z_1)\\ &= [\psi_1'(\xi_1)-\psi_1'(\xi_1-1)](z_2-z_1)\\ &= \psi_1''(\eta_1)(z_2-z_1)<0,\end{aligned}$$

where $1<z_1<\xi_1<z_2$ and $0<\xi_1<\eta_1<\xi_1$. Hence, if $\alpha\in(-1,0)$, $\phi_1(z)$ is a monotonically decreasing function for $z\in(1,\infty)$.

This complete the proof. □

Theorem 2.3 *Let T be an n-vertex tree with domination number γ. Then,*

$$R_\alpha(T)\le \frac{4n+3\gamma-15}{5}\cdot 4^\alpha+2\cdot 2^\alpha+(n-3\gamma)\cdot\left(3\cdot 6^\alpha+2^\alpha-\frac{19}{5}\cdot 4^\alpha\right),\quad \text{for}\quad \alpha\in[\alpha_1,0), \tag{2.1}$$

with the equality holding if and only if $T\in\mathcal{T}_1$, where $\alpha_1\approx -0.5287$ is the unique nonzero root of the equation $27^\alpha-6^\alpha-4^\alpha+2^\alpha=0$.

For convenience, we denote $f(n,\gamma,\alpha)=\frac{4n+3\gamma-15}{5}\cdot 4^\alpha+2\cdot 2^\alpha+(n-3\gamma)\cdot(3\cdot 6^\alpha+2^\alpha-\frac{19}{5}\cdot 4^\alpha)$. Next, we introduce a necessary lemma.

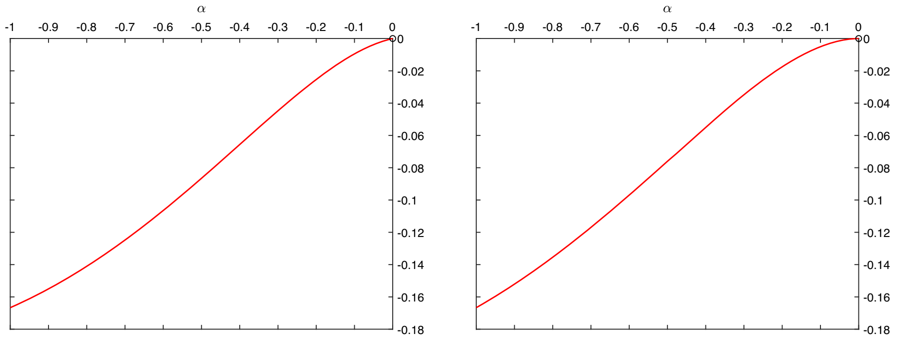
Lemma 2.4 *Let T be an n-vertex tree with domination number γ. Suppose that there exists a vertex $v\in V_T$ such that $d_v\ge 3$, $N(v)=\{u_1,u_2,\ldots,u_{d_v}\}$, $d_{u_1}\ge 2$ and $d_{u_i}=1$ for $i=2,3,\ldots,d_v$. Take $T_{-1}=T-\{u_{d_v}\}$, then*

(i) *If $d_v\ge 4$, $\alpha\in(-1,0)$ and $R_\alpha(T_{-1})\le f(n-1,\gamma,\alpha)$, then $R_\alpha(T)<f(n,\gamma,\alpha)$.*
(ii) *If $d_v=3$, $d_{u_1}=2$, $\alpha\in(-1,0)$ and $R_\alpha(T_{-1})\le f(n-1,\gamma,\alpha)$, then $R_\alpha(T)<f(n,\gamma,\alpha)$*

Proof (i) By calculation, we have

$$\begin{aligned}R_\alpha(T) &= R_\alpha(T_{-1})-(d_v-2)\cdot(d_v-1)^\alpha-[d_{u_1}\cdot(d_v-1)]^\alpha+(d_v-1)\cdot(d_v)^\alpha\\ &\quad+(d_{u_1}\cdot d_v)^\alpha\\ &= R_\alpha(T_{-1})+\left[(d_v)^\alpha-(d_v-1)^\alpha\right]\cdot[(d_v-2)+(d_{u_1})^\alpha]+(d_v)^\alpha\\ &\le \frac{4(n-1)+3\gamma-15}{5}\cdot 4^\alpha+2\cdot 2^\alpha+(n-3\gamma-1)\\ &\quad\cdot\left(3\cdot 6^\alpha+2^\alpha-\frac{19}{5}\cdot 4^\alpha\right)\\ &\quad+\left[(d_v)^\alpha-(d_v-1)^\alpha\right]\cdot[(d_v-2)+(d_{u_1})^\alpha]+(d_v)^\alpha\\ &= f(n,\gamma,\alpha)+3\cdot 4^\alpha-3\cdot 6^\alpha-2^\alpha+\left[(d_v)^\alpha-(d_v-1)^\alpha\right]\\ &\quad\cdot[(d_v-2)+(d_{u_1})^\alpha]+(d_v)^\alpha\\ &\le f(n,\gamma,\alpha)+3\cdot 4^\alpha-3\cdot 6^\alpha-2^\alpha+(d_v)^\alpha(d_v-1)-(d_v-1)^\alpha(d_v-2).\end{aligned}$$

From Lemma 2.2, we get $3\cdot 4^\alpha-3\cdot 6^\alpha-2^\alpha+(d_v)^\alpha(d_v-1)-(d_v-1)^\alpha(d_v-2)=3\cdot 4^\alpha-3\cdot 6^\alpha-2^\alpha+\phi_1(d_v)\le 3\cdot 4^\alpha-3\cdot 6^\alpha-2^\alpha+\phi_1(4)=6\cdot 4^\alpha-3\cdot 6^\alpha-2\cdot 3^\alpha-2^\alpha$ for any $d_v\ge 4$.



(a) $6 \cdot 4^\alpha - 3 \cdot 6^\alpha - 2 \cdot 3^\alpha - 2^\alpha$ **(b)** $2(2^\alpha - 1)(2^\alpha - 3^\alpha)$

Fig. 2 The two functions of α in Lemma 2.4

It can be checked, using computer program, that $6 \cdot 4^\alpha - 3 \cdot 6^\alpha - 2 \cdot 3^\alpha - 2^\alpha < 0$ for every $\alpha \in (-1, 0)$ (see Fig. 2a), i.e., $R_\alpha(T) < f(n, \gamma, \alpha)$.

(ii) If $d_v = 3, d_{u_1} = 2$, we have $3 \cdot 4^\alpha - 3 \cdot 6^\alpha - 2^\alpha + [(d_v)^\alpha - (d_v - 1)^\alpha] \cdot [(d_v - 2) + (d_{u_1})^\alpha] + (d_v)^\alpha = 3 \cdot 4^\alpha - 3 \cdot 6^\alpha - 2^\alpha + (3^\alpha - 2^\alpha) \cdot (1 + 2^\alpha) + 3^\alpha = 2 \cdot (2^\alpha - 1) \cdot (2^\alpha - 3^\alpha)$. It can be checked that $2^\alpha - 1 < 0$ and $2^\alpha - 3^\alpha > 0$ for $\alpha \in (-1, 0)$, which implies that $2(2^\alpha - 1)(2^\alpha - 3^\alpha) < 0$ for every $\alpha \in (-1, 0)$ (see Fig. 2b). Thus, we get $R_\alpha(T) < f(n, \gamma, \alpha)$.

This proves the lemma. □

Proof of Theorem 2.3 For any tree with 4 vertices (S_4 or P_4), we have $R_\alpha(S_4) = 3 \cdot 3^\alpha < f(4, 1, \alpha)$, and $R_\alpha(P_4) = 4^\alpha + 2 \cdot 2^\alpha < f(4, 2, \alpha)$ if $\alpha \in (-1, 0)$. Suppose that the inequality (2.1) holds for any $(n-1)$-vertex trees and then we consider a tree with n vertices. We take a diameter path of T, denoted by $v_1 v_2 \cdots v_d$. By Lemma 2.4, we can suppose that $d_{v_2} \leq 3$. Then, we discuss the following two cases.

Case 1 $d_{v_2} = 3$.

From Lemma 2.4, one can assume that $d_{v_3} \geq 3$. Denote $N(v_2) = \{v_1, v_3, u_1\}$ and $N(v_3) = \{v_2, v_4, w_1, w_2, \ldots, w_{d_{v_3}-2}\}$. Then, we take $T_{-3} = T - \{v_1, v_2, u_1\}$. There exists a dominating set $D(T)$ such that $v_2 \in D(T)$ and $v_3 \in N[D \backslash \{v_2\}]$, implying $\gamma(T) = \gamma(T_{-3}) + 1$. Note that $d_{w_i} \leq 3$ for $i = 1, 2, \ldots, d_{v_3} - 2$. By the above consideration, we have

$$
\begin{aligned}
R_\alpha(T) = {} & R_\alpha(T_{-3}) - [d_{v_4} \cdot (d_{v_3} - 1)]^\alpha - (d_{v_3} - 1)^\alpha \cdot \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha + (d_{v_3} \cdot d_{v_4})^\alpha \\
& + (d_{v_3})^\alpha \cdot \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha + (3 \cdot d_{v_3})^\alpha + 2 \cdot 3^\alpha \\
\leq {} & \frac{4(n-3) + 3(\gamma - 1) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 3 - 3(\gamma - 1)] \\
& \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
&- [(d_{v_3}-1)^\alpha - (d_{v_3})^\alpha] \cdot [(d_{v_4})^\alpha + \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha] + [(d_{v_3})^\alpha + 2] \cdot 3^\alpha \\
&\leq f(n,\gamma,\alpha) - [(d_{v_3}-1)^\alpha - (d_{v_3})^\alpha] \cdot [(d_{v_3}-2) \cdot 3^\alpha] + [(d_{v_3})^\alpha + 2] \\
&\quad \cdot 3^\alpha - 3 \cdot 4^\alpha \\
&= f(n,\gamma,\alpha) + [(d_{v_3}-1) \cdot (d_{v_3})^\alpha - (d_{v_3}-2) \cdot (d_{v_3}-1)^\alpha + 2] \\
&\quad \cdot 3^\alpha - 3 \cdot 4^\alpha.
\end{aligned}
$$

By Lemma 2.2, we calculate that $[(d_{v_3}-1)\cdot(d_{v_3})^\alpha - (d_{v_3}-2)\cdot(d_{v_3}-1)^\alpha + 2]\cdot 3^\alpha - 3\cdot 4^\alpha = \left[\phi_1(d_{v_3}) + 2\right]\cdot 3^\alpha - 3\cdot 4^\alpha \leq [\phi_1(3)+2]\cdot 3^\alpha - 3\cdot 4^\alpha = 2\cdot 9^\alpha - 6^\alpha + 2\cdot 3^\alpha - 3\cdot 4^\alpha < 0$ for any $d_{v_3} \geq 3$ and $\alpha \in (-1,0)$ (see Fig. 3a). Thus, for $\alpha \in (-1,0)$, $R_\alpha(T) < f(n,\gamma,\alpha)$.

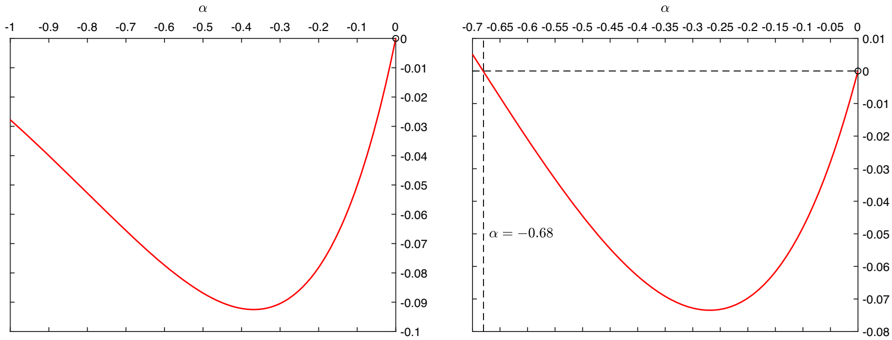
Case 2 For any diameter path of T, $d_{v_2} = 2$.

Case 2.1 $d_{v_3} \geq 3$. Denote $N(v_3) = \{v_2, v_4, w_1, w_2, \ldots, w_{d_{v_3}-2}\}$. Then, we take $T_{-2} = T - \{v_1, v_2\}$. Obviously, there exists a dominating set $D(T)$ such that $v_2 \in D(T)$ and $v_3 \in N[D\backslash\{v_2\}]$, implying $\gamma(T) = \gamma(T_{-2}) + 1$. By calculation, we get

$$
\begin{aligned}
R_\alpha(T) &= R_\alpha(T_{-2}) - [d_{v_4} \cdot (d_{v_3}-1)]^\alpha - (d_{v_3}-1)^\alpha \cdot \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha + (d_{v_4} \cdot d_{v_3})^\alpha \\
&\quad + (d_{v_3})^\alpha \cdot \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha + (2 \cdot d_{v_3})^\alpha + 2^\alpha \\
&\leq \frac{4(n-2) + 3(\gamma-1) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 2 - 3(\gamma-1)] \\
&\quad \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \\
&\quad - [(d_{v_3}-1)^\alpha - (d_{v_3})^\alpha] \cdot \left[(d_{v_4})^\alpha + \sum_{i=1}^{d_{v_3}-2} (d_{w_i})^\alpha\right] + [(d_{v_3})^\alpha + 1] \cdot 2^\alpha \\
&\leq f(n,\gamma,\alpha) - [(d_{v_3}-1)^\alpha - (d_{v_3})^\alpha] \cdot [(d_{v_3}-2) \cdot 2^\alpha] + [(d_{v_3})^\alpha + 1] \cdot 2^\alpha \\
&\quad - 6 \cdot 4^\alpha + 3 \cdot 6^\alpha + 2^\alpha \\
&= f(n,\gamma,\alpha) + [(d_{v_3}-1)(d_{v_3})^\alpha - (d_{v_3}-2)(d_{v_3}-1)^\alpha] \cdot 2^\alpha \\
&\quad - 6 \cdot 4^\alpha + 3 \cdot 6^\alpha + 2 \cdot 2^\alpha
\end{aligned}
$$

Combining Lemma 2.2, we calculate that $[(d_{v_3}-1)(d_{v_3})^\alpha - (d_{v_3}-2)(d_{v_3}-1)^\alpha] \cdot 2^\alpha - 6 \cdot 4^\alpha + 3 \cdot 6^\alpha + 2^{\alpha+1} = \phi_1(d_{v_3}) \cdot 2^\alpha - 6 \cdot 4^\alpha + 3 \cdot 6^\alpha + 2^{\alpha+1} \leq \phi_1(3) \cdot 2^\alpha - 6 \cdot 4^\alpha + 3 \cdot 6^\alpha + 2^{\alpha+1} = 5 \cdot 6^\alpha - 7 \cdot 4^\alpha + 2^{\alpha+1} < 0$ for $d_{v_3} \geq 3$ and $\alpha \in [-0.68, 0)$ (see Fig. 3b). Hence, for $\alpha \in [-0.68, 0)$, $R_\alpha(T) < f(n,\gamma,\alpha)$.

Case 2.2 $d_{v_3} = 2$. Denote $N(v_4) = \{v_3, v_5, x_1, x_2, \ldots, x_{d_{v_4}-2}\}$. For any x_i ($i = 1, 2, \ldots, d_{v_4}-2$), if there exist two vertices y_1, y_2 such that $y_2 \in N(x_i)$ and $y_1 \in N(y_2)$, then we get a new diameter path of T, that is $y_1, y_2, x_i, v_4, v_5, \ldots, v_d$, then



(a) $2 \cdot 9^\alpha - 6^\alpha + 2 \cdot 3^\alpha - 3 \cdot 4^\alpha$ **(b)** $5 \cdot 6^\alpha - 7 \cdot 4^\alpha + 2^{\alpha+1}$

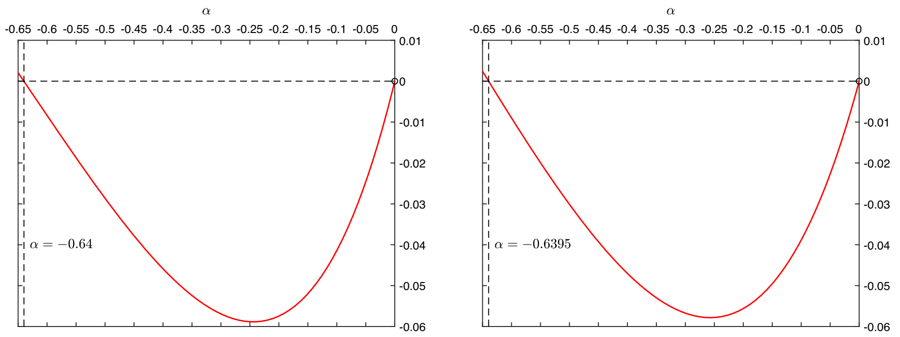
Fig. 3 The two functions of α in Case 1 and Case 2.1 of Theorem 2.3

by the above cases, we can assume that $d_{x_i} = d_{y_2} = 2$. Moreover, if x_i is a support vertex, from Lemma 2.4, we can suppose that $d_{x_i} = 2$ or $d_{x_i} = 3$.

Case 2.2.1 $d_{v_4} \geq 4$. Take $T_{-3} = T - \{v_1, v_2, v_3\}$, then we have

$$
\begin{aligned}
R_\alpha(T) = R_\alpha(T_{-3}) - [(d_{v_4} - 1) \cdot d_{v_5}]^\alpha - (d_{v_4} - 1)^\alpha \cdot \sum_{i=1}^{d_{v_4}-2} (d_{x_i})^\alpha \\
+ (d_{v_4} \cdot d_{v_5})^\alpha + (d_{v_4})^\alpha \cdot \sum_{i=1}^{d_{v_4}-2} (d_{x_i})^\alpha + (2 \cdot d_{v_4})^\alpha + 4^\alpha + 2^\alpha \\
\leq \frac{4(n-3) + 3(\gamma - 1) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 3 - 3(\gamma - 1)] \\
\cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \\
- [(d_{v_4} - 1)^\alpha - (d_{v_4})^\alpha] \cdot \left[(d_{v_5})^\alpha + \sum_{i=1}^{d_{v_4}-2} (d_{x_i})^\alpha\right] + (2 \cdot d_{v_4})^\alpha + 4^\alpha + 2^\alpha \\
\leq f(n, \gamma, \alpha) - [(d_{v_4} - 1)^\alpha - (d_{v_4})^\alpha] \cdot [(d_{v_4} - 2) \cdot 3^\alpha] \\
+ (2 \cdot d_{v_4})^\alpha - 2 \cdot 4^\alpha + 2^\alpha \\
= f(n, \gamma, \alpha) + [(d_{v_4} - 1)(d_{v_4})^\alpha - (d_{v_4} - 2)(d_{v_4} - 1)^\alpha] \cdot 3^\alpha \\
+ (2^\alpha - 3^\alpha)(d_{v_4})^\alpha - 2 \cdot 4^\alpha + 2^\alpha.
\end{aligned}
$$

If $\alpha < 0$, the function $(2^\alpha - 3^\alpha)(d_{v_4})^\alpha$ monotonically decreases for $d_{v_4} \geq 0$. Combining Lemma 2.2, we calculate that $[(d_{v_4} - 1)(d_{v_4})^\alpha - (d_{v_4} - 2)(d_{v_4} - 1)^\alpha] \cdot 3^\alpha + (2^\alpha - 3^\alpha)(d_{v_4})^\alpha - 2 \cdot 4^\alpha + 2^\alpha = \phi_1(d_{v_4}) \cdot 3^\alpha + (2^\alpha - 3^\alpha)(d_{v_4})^\alpha - 2 \cdot 4^\alpha + 2^\alpha \leq \phi_1(4) \cdot 3^\alpha + (2^\alpha - 3^\alpha) \cdot 4^\alpha - 2 \cdot 4^\alpha + 2^\alpha = 2 \cdot 12^\alpha - 2 \cdot 9^\alpha + 8^\alpha - 2 \cdot 4^\alpha + 2^\alpha < 0$ for $d_{v_4} \geq 4$ and $\alpha \in [-0.64, 0)$ (see Fig. 4a). Hence, for $\alpha \in [-0.64, 0)$, $R_\alpha(T) < f(n, \gamma, \alpha)$.



(a) $2 \cdot 12^{\alpha} - 2 \cdot 9^{\alpha} + 8^{\alpha} - 2 \cdot 4^{\alpha} + 2^{\alpha}$ **(b)** $-5 \cdot 4^{\alpha} + 2 \cdot 2^{\alpha} + 4 \cdot 6^{\alpha} - 3^{\alpha}$

Fig. 4 The two functions of α in Case 2.2.1 and Case 2.2.2 of Theorem 2.3

Case 2.2.2 $d_{v_4} \leq 3$. If there exists a minimum dominating set $D(T)$ such that $v_4 \in D(T)$, then we take $T_{-2} = T - \{v_1, v_2\}$. By calculation, we have

$$\begin{aligned} R_\alpha(T) &= R_\alpha(T_{-2}) - (d_{v_4})^\alpha + (2 \cdot d_{v_4})^\alpha + 4^\alpha + 2^\alpha \\ &\leq \frac{4(n-2) + 3(\gamma - 1) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 2 - 3(\gamma - 1)] \\ &\quad \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) - (d_{v_4})^\alpha + (2 \cdot d_{v_4})^\alpha + 4^\alpha + 2^\alpha \\ &= f(n, \gamma, \alpha) - (d_{v_4})^\alpha + (2 \cdot d_{v_4})^\alpha - 5 \cdot 4^\alpha + 2 \cdot 2^\alpha + 3 \cdot 6^\alpha. \end{aligned}$$

Note that $-(d_{v_4})^\alpha + (2 \cdot d_{v_4})^\alpha - 5 \cdot 4^\alpha + 2 \cdot 2^\alpha + 3 \cdot 6^\alpha \leq -5 \cdot 4^\alpha + 2 \cdot 2^\alpha + 4 \cdot 6^\alpha - 3^\alpha < 0$ for any $d_{v_4} \leq 3$ and $\alpha \in [-0.6395, 0)$ (see Fig. 4b), then we have $R_\alpha(T) < f(n, \gamma, \alpha)$ for $\alpha \in [-0.6395, 0)$.

Next, we suppose that $v_4 \notin D(T)$ for any minimum dominating set D in T.

Case 2.2.2.1 $d_{v_4} = 3$. Denote $N(v_5) = \{v_4, a_1, \ldots, a_{d_{v_5}-1}\}$, then we study the following three cases, which are depicted in Fig. 5.

(I) If $d_{v_5} \geq 9$, we take $T_{-7} = T - \{v_1, v_2, v_3, v_4, x_1, y_1, y_2\}$. One can check that $\gamma(T_{-7}) = \gamma(T) - 2$ and

$$\begin{aligned} R_\alpha(T) = R_\alpha(T_{-7}) &- [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha \\ &+ 2 \cdot 6^\alpha + 2 \cdot 4^\alpha + 2 \cdot 2^\alpha \\ &\leq \frac{4(n-7) + 3(\gamma - 2) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 7 - 3(\gamma - 2)] \\ &\quad \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \end{aligned}$$

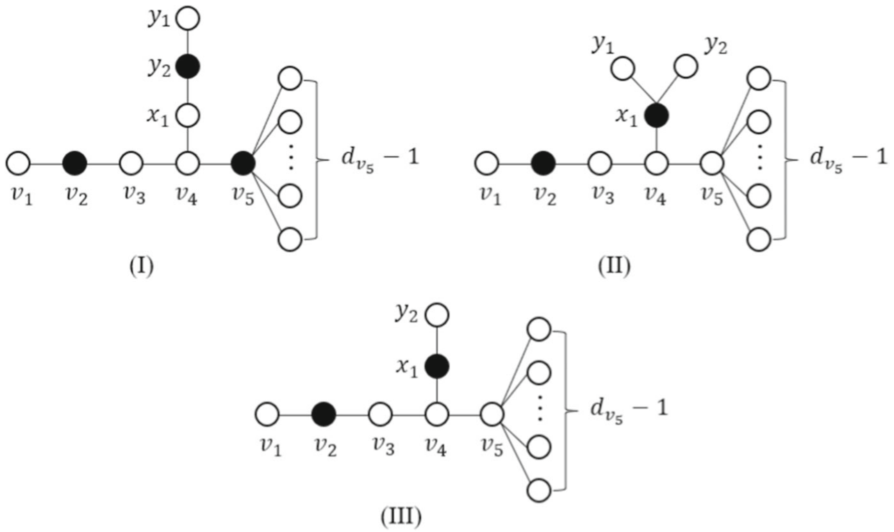


Fig. 5 Three trees for Case 2.2.2.1

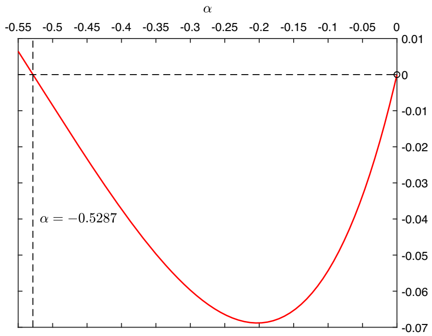
$$- [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha + 2 \cdot 6^\alpha + 2 \cdot 4^\alpha + 2 \cdot 2^\alpha$$

$$= f(n, \gamma, \alpha) - [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha - 6^\alpha - 4^\alpha + 2^\alpha.$$

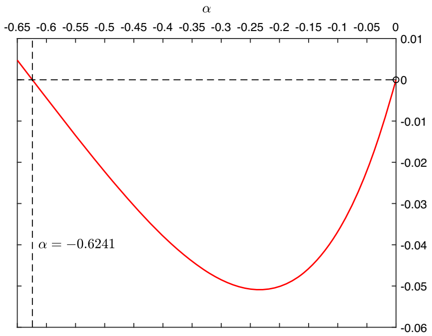
Easily, it can be checked that $(3 \cdot d_{v_5})^\alpha - 6^\alpha - 4^\alpha + 2^\alpha \le 27^\alpha - 6^\alpha - 4^\alpha + 2^\alpha < 0$ for any $d_{v_5} \ge 9$ and $\alpha \in [-0.5287, 0)$ (see Fig. 6a). Hence, for $\alpha \in [-0.5287, 0)$, $R_\alpha(T) < f(n, \gamma, \alpha)$. For $d_{v_5} \le 8$, we take $T_{-3} = T - \{v_1, v_2, v_3\}$. One can see $\gamma(T_{-3}) = \gamma(T) - 1$ and

$$\begin{aligned} R_\alpha(T) &= R_\alpha(T_{-3}) - 4^\alpha - (2 \cdot d_{v_5})^\alpha + 2 \cdot 6^\alpha + (3 \cdot d_{v_5})^\alpha + 4^\alpha + 2^\alpha \\ &\le \frac{4(n-3) + 3(\gamma - 1) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 3 - 3(\gamma - 1)] \\ &\quad \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \\ &\quad - 4^\alpha - (2 \cdot d_{v_5})^\alpha + 2 \cdot 6^\alpha + (3 \cdot d_{v_5})^\alpha + 4^\alpha + 2^\alpha \\ &= f(n, \gamma, \alpha) - 3 \cdot 4^\alpha + (3^\alpha - 2^\alpha) \cdot (d_{v_5})^\alpha + 2 \cdot 6^\alpha + 2^\alpha. \end{aligned}$$

Since the inequality $-3 \cdot 4^\alpha + (3^\alpha - 2^\alpha) \cdot (d_{v_5})^\alpha + 2 \cdot 6^\alpha + 2^\alpha \le 24^\alpha - 16^\alpha - 3 \cdot 4^\alpha + 2 \cdot 6^\alpha + 2^\alpha < 0$ holds for any $d_{v_5} \le 8$ and $\alpha \in [-0.6241, 0)$ (see Fig. 6b), we have $R_\alpha(T) < f(n, \gamma, \alpha)$ for $\alpha \in [-0.6241, 0)$.



(a) $27^{\alpha} - 6^{\alpha} - 4^{\alpha} + 2^{\alpha}$



(b) $24^{\alpha} - 16^{\alpha} - 3 \cdot 4^{\alpha} + 2 \cdot 6^{\alpha} + 2^{\alpha}$

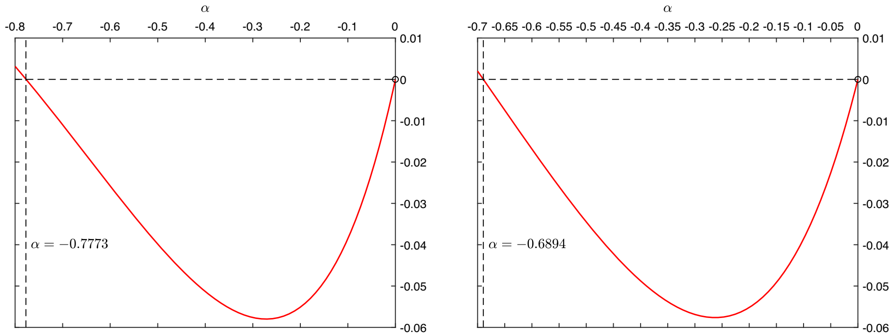
Fig. 6 The two functions of α in Case 2.2.2.1 (I) of Theorem 2.3

(II) Assume that $d_{v_5} \geq 4$, then we take $T_{-7} = T - \{v_1, v_2, v_3, v_4, x_1, y_1, y_2\}$. Similarly, we get

$$\begin{aligned} R_{\alpha}(T) &= R_{\alpha}(T_{-7}) - [(d_{v_5} - 1)^{\alpha} - (d_{v_5})^{\alpha}] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^{\alpha} + (3 \cdot d_{v_5})^{\alpha} + 9^{\alpha} \\ &\quad + 2 \cdot 3^{\alpha} + 6^{\alpha} + 4^{\alpha} + 2^{\alpha} \\ &\leq \frac{4(n-7) + 3(\gamma - 2) - 15}{5} \cdot 4^{\alpha} + 2 \cdot 2^{\alpha} + [n - 7 - 3(\gamma - 2)] \\ &\quad \cdot \left(3 \cdot 6^{\alpha} + 2^{\alpha} - \frac{19}{5} \cdot 4^{\alpha}\right) - [(d_{v_5} - 1)^{\alpha} - (d_{v_5})^{\alpha}] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^{\alpha} \\ &\quad + (3 \cdot d_{v_5})^{\alpha} + 9^{\alpha} + 2 \cdot 3^{\alpha} + 6^{\alpha} + 4^{\alpha} + 2^{\alpha} \\ &= f(n, \gamma, \alpha) - [(d_{v_5} - 1)^{\alpha} - (d_{v_5})^{\alpha}] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^{\alpha} + (3 \cdot d_{v_5})^{\alpha} + 9^{\alpha} \\ &\quad + 2 \cdot 3^{\alpha} - 2 \cdot 6^{\alpha} - 2 \cdot 4^{\alpha}. \end{aligned}$$

It can be compute that $(3 \cdot d_{v_5})^{\alpha} + 9^{\alpha} + 2 \cdot 3^{\alpha} - 2 \cdot 6^{\alpha} - 2 \cdot 4^{\alpha} \leq 12^{\alpha} + 9^{\alpha} + 2 \cdot 3^{\alpha} - 2 \cdot 6^{\alpha} - 2 \cdot 4^{\alpha}$ is always negative for $d_{v_5} \geq 4$ and $\alpha \in [-0.7773, 0)$ (see Fig. 7a). Thus, for $\alpha \in [-0.7773, 0)$, $R_{\alpha}(T) < f(n, \gamma, \alpha)$. If $d_{v_5} \leq 3$, we take $T_{-3} = T - \{v_1, v_2, v_3\}$. Then,

$$\begin{aligned} R_{\alpha}(T) &= R_{\alpha}(T_{-3}) - (2 \cdot d_{v_5})^{\alpha} - 6^{\alpha} + (3 \cdot d_{v_5})^{\alpha} + 9^{\alpha} + 6^{\alpha} + 4^{\alpha} + 2^{\alpha} \\ &\leq \frac{4(n-3) + 3(\gamma - 1) - 15}{5} \cdot 4^{\alpha} + 2 \cdot 2^{\alpha} + [n - 3 - 3(\gamma - 1)] \\ &\quad \cdot \left(3 \cdot 6^{\alpha} + 2^{\alpha} - \frac{19}{5} \cdot 4^{\alpha}\right) \end{aligned}$$



(a) $12^\alpha + 9^\alpha + 2 \cdot 3^\alpha - 2 \cdot 6^\alpha - 2 \cdot 4^\alpha$ **(b)** $2 \cdot 9^\alpha - 6^\alpha - 2 \cdot 4^\alpha + 2^\alpha$

Fig. 7 The two functions of α in Case 2.2.2.1 (II) of Theorem 2.3

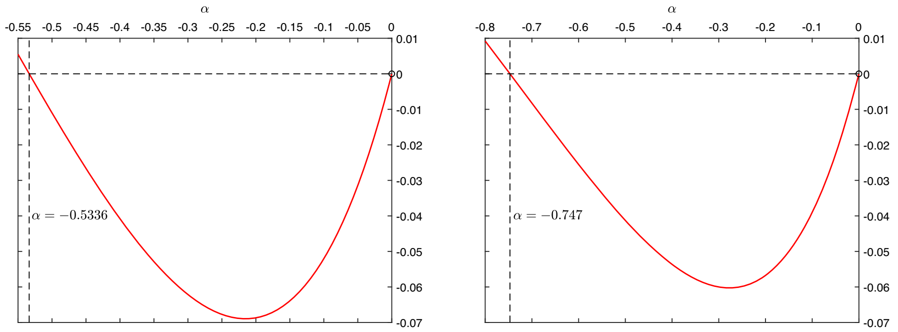
$$+ (3^\alpha - 2^\alpha) \cdot (d_{v_5})^\alpha + 9^\alpha + 4^\alpha + 2^\alpha$$
$$= f(n, \gamma, \alpha) - 2 \cdot 4^\alpha + (3^\alpha - 2^\alpha) \cdot (d_{v_5})^\alpha + 9^\alpha + 2^\alpha.$$

Since $-2 \cdot 4^\alpha + (3^\alpha - 2^\alpha) \cdot (d_{v_5})^\alpha + 9^\alpha + 2^\alpha \leq 2 \cdot 9^\alpha - 6^\alpha - 2 \cdot 4^\alpha + 2^\alpha < 0$ for any $d_{v_5} \leq 3$ and $\alpha \in [-0.6894, 0)$ (see Fig. 7b), we have for $\alpha \in [-0.6894, 0)$, $R_\alpha(T) < f(n, \gamma, \alpha)$.

(III) In this case, we assume that $d_{v_5} \geq 5$. Let $T_{-6} = T - \{v_1, v_2, v_3, v_4, x_1, y_1\}$, then we have

$$R_\alpha(T) = R_\alpha(T_{-6}) - [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha$$
$$+ 2 \cdot 6^\alpha + 4^\alpha + 2 \cdot 2^\alpha$$
$$\leq \frac{4(n-6) + 3(\gamma - 2) - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n - 6 - 3(\gamma - 2)]$$
$$\cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right)$$
$$- [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha + 2 \cdot 6^\alpha + 4^\alpha + 2 \cdot 2^\alpha$$
$$= f(n, \gamma, \alpha) - [(d_{v_5} - 1)^\alpha - (d_{v_5})^\alpha] \cdot \sum_{i=1}^{d_{v_5}-1} (d_{a_i})^\alpha + (3 \cdot d_{v_5})^\alpha$$
$$+ 2 \cdot 6^\alpha - 5 \cdot 4^\alpha + 2 \cdot 2^\alpha.$$

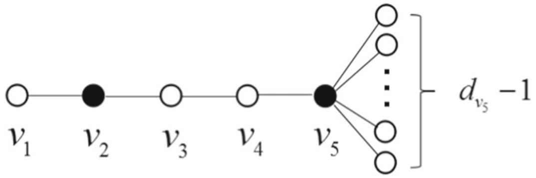
Since the inequality $(3 \cdot d_{v_5})^\alpha + 2 \cdot 6^\alpha - 5 \cdot 4^\alpha + 2 \cdot 2^\alpha \leq 15^\alpha + 2 \cdot 6^\alpha - 5 \cdot 4^\alpha + 2 \cdot 2^\alpha <$ 0 holds for any $d_{v_5} \geq 5$ and $\alpha \in [-0.5336, 0)$ (see Fig. 8a), we have $R_\alpha(T) <$



(a) $15^\alpha + 2\cdot 6^\alpha - 5\cdot 4^\alpha + 2\cdot 2^\alpha$ **(b)** $12^\alpha - 8^\alpha + 2\cdot 6^\alpha + 2^\alpha - 3\cdot 4^\alpha$

Fig. 8 The two functions of α in Case 2.2.2.1 (III) of Theorem 2.3

Fig. 9 The tree for Case 2.2.2.2



$f(n,\gamma,\alpha)$ for $\alpha \in [-0.5336, 0)$. For $d_{v_5} \le 4$, we take $T_{-3} = T - \{v_1, v_2, v_3\}$. Then,

$$\begin{aligned}
R_\alpha(T) &= R_\alpha(T_{-3}) - (2\cdot d_{v_5})^\alpha - 4^\alpha + (3\cdot d_{v_5})^\alpha + 2\cdot 6^\alpha + 4^\alpha + 2^\alpha \\
&\le \frac{4(n-3)+3(\gamma-1)-15}{5}\cdot 4^\alpha + 2\cdot 2^\alpha + [n-3-3(\gamma-1)] \\
&\quad \cdot \left(3\cdot 6^\alpha + 2^\alpha - \frac{19}{5}\cdot 4^\alpha\right) \\
&\quad + (3^\alpha - 2^\alpha)\cdot (d_{v_5})^\alpha + 2\cdot 6^\alpha + 2^\alpha \\
&= f(n,\gamma,\alpha) + (3^\alpha - 2^\alpha)\cdot (d_{v_5})^\alpha + 2\cdot 6^\alpha + 2^\alpha - 3\cdot 4^\alpha.
\end{aligned}$$

Since $(3^\alpha - 2^\alpha)\cdot (d_{v_5})^\alpha + 2\cdot 6^\alpha + 2^\alpha - 3\cdot 4^\alpha \le 12^\alpha - 8^\alpha + 2\cdot 6^\alpha + 2^\alpha - 3\cdot 4^\alpha$ is negative for $d_{v_5} \le 4$ and $\alpha \in [-0.747, 0)$ (see Fig. 8b), we get $R_\alpha(T) < f(n,\gamma,\alpha)$ for $\alpha \in [-0.747, 0)$.

Case 2.2.2.2 $d_{v_4} = 2$. The corresponding tree is depicted in Fig. 9. We denote $N(v_5) = \{v_4, a_1, a_2, \ldots, a_{d_{v_5}-1}\}$ and consider the following two cases:

Case 2.2.2.2.1 $d_{v_5} = 2$. Let $T_{-3} = T - \{v_1, v_2, v_3\}$, then we have

$$\begin{aligned}
R_\alpha(T) &= R_\alpha(T_{-3}) - 2^\alpha + 3\cdot 4^\alpha + 2^\alpha \\
&\le \frac{4(n-3)+3(\gamma-1)-15}{5}\cdot 4^\alpha + 2\cdot 2^\alpha + [n-3-3(\gamma-1)] \\
&\quad \cdot \left(3\cdot 6^\alpha + 2^\alpha - \frac{19}{5}\cdot 4^\alpha\right) + 3\cdot 4^\alpha
\end{aligned}$$

$$= \frac{4n+3\gamma-15}{5}\cdot 4^{\alpha}+2\cdot 2^{\alpha}+(n-3\gamma)\cdot\left(3\cdot 6^{\alpha}+2^{\alpha}-\frac{19}{5}\cdot 4^{\alpha}\right)$$
$$= f(n,\gamma,\alpha).$$

Case 2.2.2.2.2 $d_{v_5}\geq 3$. If $d_{a_i}\leq 2$ with $i=1,2,\ldots,d_{v_5}-1$, we take $T_{-4}=T-\{v_1,v_2,v_3,v_4\}$, then

$$\begin{aligned}
R_\alpha(T) &= R_\alpha(T_{-4})-[(d_{v_5}-1)^{\alpha}-(d_{v_5})^{\alpha}]\cdot\sum_{i=1}^{d_{v_5}-1}(d_{a_i})^{\alpha}+(2\cdot d_{v_5})^{\alpha}+2\cdot 4^{\alpha}+2^{\alpha}\\
&\leq \frac{4(n-4)+3(\gamma-1)-15}{5}\cdot 4^{\alpha}+2\cdot 2^{\alpha}+[n-4-3(\gamma-1)]\\
&\quad\cdot\left(3\cdot 6^{\alpha}+2^{\alpha}-\frac{19}{5}\cdot 4^{\alpha}\right)-[(d_{v_5}-1)^{\alpha}-(d_{v_5})^{\alpha}]\cdot\big(d_{v_5}-1\big)\\
&\quad\cdot 2^{\alpha}+2\cdot 4^{\alpha}+2^{\alpha}\\
&= f(n,\gamma,\alpha)+[(d_{v_5})^{\alpha+1}-(d_{v_5}-1)^{\alpha+1}]\cdot 2^{\alpha}+2\cdot 4^{\alpha}-3\cdot 6^{\alpha}.
\end{aligned}$$

One can check that $[(d_{v_5})^{\alpha+1}-(d_{v_5}-1)^{\alpha+1}]\cdot 2^{\alpha}+2\cdot 4^{\alpha}-3\cdot 6^{\alpha}\leq\big(3^{\alpha+1}-2^{\alpha+1}\big)\cdot 2^{\alpha}+2\cdot 4^{\alpha}-3\cdot 6^{\alpha}=0$ for any $d_{v_5}\geq 3$. Hence, $R_\alpha(T)\leq f(n,\gamma,\alpha)$.

Based on the above discussion, we suppose that $d_{a_1}=\max_{1\leq i\leq d_{v_5}-1}d_{a_i}\geq 3$. Denote $N(a_1)=\{v_5,b_1,b_2,\ldots,b_{d_{a_1}-1}\}$, then by all above cases and Lemma 2.4, one can see that $1\leq d_{b_j}\leq 3$ and all vertex in $N(b_j)\backslash\{a_1\}$ are pendent vertices. For convenience, the number of vertices in $\{b_1,b_2,\ldots,b_{d_{a_1}-1}\}$ with degree i is written by n_i; then, we obtain $n_1+n_2+n_3=d_{a_1}-1$. Next, we construct two subgraphs T', T'' of T such that $V_T=V_{T'}\cup V_{T''}$ and $E_T=E_{T'}\cup E_{T''}\cup\{v_5a_1\}$. By calculation, we have

$$\begin{aligned}
R_\alpha(T) &= R_\alpha(T')+R_\alpha(T'')-[(d_{v_5}-1)^{\alpha}-(d_{v_5})^{\alpha}]\\
&\quad\cdot\left[2^{\alpha}+\sum_{i=2}^{d_{v_5}-1}(d_{a_i})^{\alpha}\right]-n_1\cdot(d_{a_1}-1)^{\alpha}-n_2\cdot[2(d_{a_1}-1)]^{\alpha}\\
&\quad-n_3\cdot[3(d_{a_1}-1)]^{\alpha}+n_1\cdot(d_{a_1})^{\alpha}+n_2\cdot(2\cdot d_{a_1})^{\alpha}+n_3\cdot(3\cdot d_{a_1})^{\alpha}\\
&\quad+(d_{a_1}\cdot d_{v_5})^{\alpha}\\
&\leq \frac{4n+3\gamma-15}{5}\cdot 4^{\alpha}+2\cdot 2^{\alpha}+(n-3\gamma)\cdot\left(3\cdot 6^{\alpha}+2^{\alpha}-\frac{19}{5}\cdot 4^{\alpha}\right)\\
&\quad+2\cdot 2^{\alpha}-3\cdot 4^{\alpha}+(d_{a_1}\cdot d_{v_5})^{\alpha}\\
&\quad-[(d_{v_5}-1)^{\alpha}-(d_{v_5})^{\alpha}]\cdot[2^{\alpha}+(d_{v_5}-2)\cdot(d_{a_1})^{\alpha}]\\
&\quad-[(d_{a_1}-1)^{\alpha}-(d_{a_1})^{\alpha}]\cdot(n_1+n_2+n_3)\cdot 3^{\alpha}\\
&\leq f(n,\gamma,\alpha)-[(d_{v_5}-1)^{\alpha}-(d_{v_5})^{\alpha}]\cdot[2^{\alpha}+(d_{a_1})^{\alpha}]\\
&\quad-[(d_{a_1}-1)^{\alpha}-(d_{a_1})^{\alpha}]\cdot(d_{a_1}-1)\cdot 3^{\alpha}\\
&\quad+2\cdot 2^{\alpha}-3\cdot 4^{\alpha}+(d_{a_1}\cdot d_{v_5})^{\alpha}
\end{aligned}$$

Now, we consider the function $g(z) = -[(z-1)^\alpha - z^\alpha]\cdot[2^\alpha + (d_{a_1})^\alpha] + (d_{a_1}\cdot z)^\alpha$. Since

$$\begin{aligned}
g'(z) &= \alpha\left\{\left[z^{\alpha-1} - (z-1)^{\alpha-1}\right]\left[2^\alpha + (d_{a_1})^\alpha\right] + (d_{a_1})^\alpha\cdot z^{\alpha-1}\right\} \geq 0\\
&\Leftrightarrow \left[z^{\alpha-1} - (z-1)^{\alpha-1}\right]\left[2^\alpha + (d_{a_1})^\alpha\right] + (d_{a_1})^\alpha\cdot z^{\alpha-1} \leq 0\\
&\Leftrightarrow z^{\alpha-1}\left[2^\alpha + 2\cdot(d_{a_1})^\alpha\right] \leq (z-1)^{\alpha-1}\left[2^\alpha + (d_{a_1})^\alpha\right]\\
&\Leftrightarrow \left(\frac{z}{z-1}\right)^{\alpha-1} \leq \frac{2^\alpha + (d_{a_1})^\alpha}{2^\alpha + 2\cdot(d_{a_1})^\alpha}\\
&\Leftrightarrow \left(\frac{z}{z-1}\right)^{1-\alpha} \geq \frac{2^\alpha + 2\cdot(d_{a_1})^\alpha}{2^\alpha + (d_{a_1})^\alpha}\\
&\Leftrightarrow \frac{z}{z-1} \geq \left[\frac{2^\alpha + 2\cdot(d_{a_1})^\alpha}{2^\alpha + (d_{a_1})^\alpha}\right]^{\frac{1}{1-\alpha}}\\
&\Leftrightarrow z \leq \frac{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1},
\end{aligned}$$

we have that $g(z)$ attains its maximum value if $z = \frac{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1}$. Hence,

$$\begin{aligned}
&-[(d_{v_5}-1)^\alpha - (d_{v_5})^\alpha]\cdot[2^\alpha + (d_{a_1})^\alpha] - [(d_{a_1}-1)^\alpha - (d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha + 2\cdot 2^\alpha - 3\cdot 4^\alpha\\
&\quad + (d_{a_1}\cdot d_{v_5})^\alpha\\
&\leq 2\cdot 2^\alpha - 3\cdot 4^\alpha - \left(\left\{\frac{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1} - 1\right\}^\alpha - \left\{\frac{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1}\right\}^\alpha\right)\\
&\quad \cdot[2^\alpha + (d_{a_1})^\alpha]\\
&\quad - [(d_{a_1}-1)^\alpha - (d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha + (d_{a_1})^\alpha\cdot\left\{\frac{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}}}{\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1}\right\}^\alpha\\
&= -[(d_{a_1}-1)^\alpha - (d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha + 2\cdot 2^\alpha - 3\cdot 4^\alpha\\
&\quad - \frac{1}{\left(\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{1}{1-\alpha}} - 1\right)^\alpha}\left\{[2^\alpha + (d_{a_1})^\alpha] - \left[2^\alpha + 2\cdot(d_{a_1})^\alpha\right]\cdot\left[\frac{2\cdot(d_{a_1})^\alpha + 2^\alpha}{(d_{a_1})^\alpha + 2^\alpha}\right]^{\frac{\alpha}{1-\alpha}}\right\}\\
&= -[(d_{a_1}-1)^\alpha - (d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha + 2\cdot 2^\alpha - 3\cdot 4^\alpha
\end{aligned}$$

$$+\frac{2^\alpha+(d_{a_1})^\alpha}{\left(\left[\frac{2\cdot(d_{a_1})^\alpha+2^\alpha}{(d_{a_1})^\alpha+2^\alpha}\right]^{\frac{1}{1-\alpha}}-1\right)^\alpha}\left(\left[\frac{2\cdot(d_{a_1})^\alpha+2^\alpha}{(d_{a_1})^\alpha+2^\alpha}\right]^{\frac{1}{1-\alpha}}-1\right)$$

$$=-[(d_{a_1}-1)^\alpha-(d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha+2\cdot 2^\alpha-3\cdot 4^\alpha$$

$$+[2^\alpha+(d_{a_1})^\alpha]\left(\left[\frac{2\cdot(d_{a_1})^\alpha+2^\alpha}{(d_{a_1})^\alpha+2^\alpha}\right]^{\frac{1}{1-\alpha}}-1\right)^{1-\alpha}$$

$$=-[(d_{a_1}-1)^\alpha-(d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha+2\cdot 2^\alpha-3\cdot 4^\alpha$$

$$+[2^\alpha+(d_{a_1})^\alpha]\left(\left[1+\frac{2^{-\alpha}}{(d_{a_1})^{-\alpha}+2^{-\alpha}}\right]^{\frac{1}{1-\alpha}}-1\right)^{1-\alpha}.$$

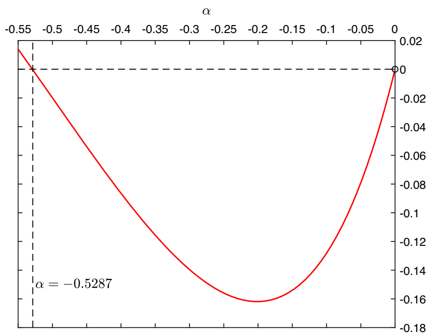
Note that the function $\varphi(z)=2\cdot 2^\alpha-3\cdot 4^\alpha+(2^\alpha+z^\alpha)\left(\left[1+\frac{2^{-\alpha}}{z^{-\alpha}+2^{-\alpha}}\right]^{\frac{1}{1-\alpha}}-1\right)^{1-\alpha}$ always decreases for $z>0$ if $\alpha<0$. By calculation, we obtain $\varphi(16)<0$ for $\alpha\in[-0.5287,0)$ (see Fig. 10a), which implies that the function $\omega(d_{a_1})=-[(d_{a_1}-1)^\alpha-(d_{a_1})^\alpha]\cdot(d_{a_1}-1)\cdot 3^\alpha+2\cdot 2^\alpha-3\cdot 4^\alpha+[2^\alpha+(d_{a_1})^\alpha]\left(\left[1+\frac{2^{-\alpha}}{(d_{a_1})^{-\alpha}+2^{-\alpha}}\right]^{\frac{1}{1-\alpha}}-1\right)^{1-\alpha}$ is always negative for any $d_{a_1}\geq 16$ and $\alpha\in[-0.5287,0)$. Therefore, we only check the value of $\omega(d_{a_1})$ for $3\leq d_{a_1}\leq 15$ (see Fig. 10b, c). So, we have $R_\alpha(T)<f(n,\gamma,\alpha)$ for $\alpha\in[-0.5287,0)$.

Next, we will prove that $R_\alpha(T)=f(n(T),\gamma(T),\alpha)$ if and only if $T\in\mathcal{T}_1$. For any path P_{3t}, we have $R_\alpha(P_{3t})=f(3k,k,\alpha)$. If $T'\in\mathcal{T}_1$ such that $R_\alpha(T')=f(n(T'),\gamma(T'),\alpha)$, we take a pendent vertex $v\in T'$ and a path $P_{3t}=w_1w_2\cdots w_{3t}$, and consider a new tree T such that $V_T=V_{T'}\cup V_{P_{3t}}$ and $E_T=E_{T'}\cup E_{P_{3t}}\cup\{vw_1\}$ (Definition 2.1 (i)). By calculation, we have

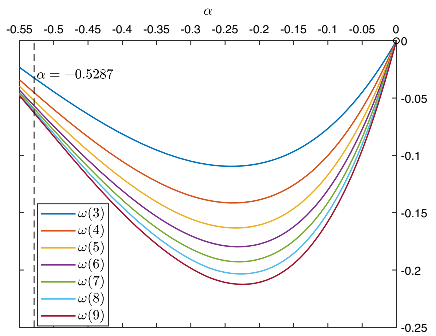
$$\begin{aligned}R_\alpha(T)&=R_\alpha(T')+R_\alpha(P_{3t})-2\cdot 2^\alpha+3\cdot 4^\alpha\\&=\frac{4n(T')+3\gamma(T')-15}{5}\cdot 4^\alpha+2\cdot 2^\alpha+[n(T')-3\gamma(T')]\\&\quad\cdot\left(3\cdot 6^\alpha+2^\alpha-\frac{19}{5}\cdot 4^\alpha\right)\\&\quad+2\cdot 2^\alpha+(3t-3)\cdot 4^\alpha-2\cdot 2^\alpha+3\cdot 4^\alpha\\&=\frac{4n+3\gamma-15}{5}\cdot 4^\alpha+2\cdot 2^\alpha+(n(T)-3\gamma(T))\cdot\left(3\cdot 6^\alpha+2^\alpha-\frac{19}{5}\cdot 4^\alpha\right)\\&=f(n(T),\gamma(T),\alpha).\end{aligned}$$

Suppose that there exists a tree $T'\in\mathcal{T}_1$ such that $R_\alpha(T')=f(n(T'),\gamma(T'),\alpha)$ and a vertex $v\in V_{T'}$ satisfies that $N(v)=\{u_1,u_2\}$, $d_{u_1}=d_{u_2}=2$, and $v\in D(T')$, where D is a minimum dominating set in T'. Then, we construct a new tree $T\in\mathcal{T}_1$. Let $P_{3t+1}=w_1w_2\cdots w_{3t+1}$, $V_T=V_{T'}\cup V_{P_{3t+1}}$ and $E_T=E_{T'}\cup E_{P_{3t+1}}\cup\{vw_1\}$ (Definition 2.1 (ii)). Hence,

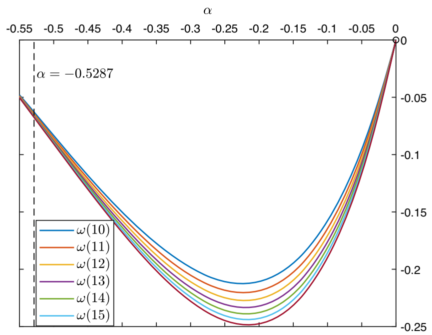
$$R_\alpha(T)=R_\alpha(T')+R_\alpha(P_{3t+1})-2\cdot 4^\alpha-2^\alpha+3\cdot 6^\alpha+4^\alpha$$



(a) $\varphi(16)$



(b) $\omega(d_{a_1})$ for $d_{a_1} = 3, 4, \cdots, 9$.



(c) $\omega(d_{a_1})$ for $d_{a_1} = 10, 11, \cdots, 15$.

Fig. 10 Several functions of α in Case 2.2.2.2.2 of Theorem 2.3

$$
\begin{aligned}
&= \frac{4n(T') + 3\gamma(T') - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + [n(T') - 3\gamma(T')] \\
&\quad \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \\
&\quad - 2 \cdot 2^\alpha + (3t - 2) \cdot 4^\alpha - 2 \cdot 4^\alpha - 2^\alpha + 3 \cdot 6^\alpha + 4^\alpha \\
&= \frac{4n + 3\gamma - 15}{5} \cdot 4^\alpha + 2 \cdot 2^\alpha + (n(T) - 3\gamma(T)) \cdot \left(3 \cdot 6^\alpha + 2^\alpha - \frac{19}{5} \cdot 4^\alpha\right) \\
&= f(n(T), \gamma(T), \alpha).
\end{aligned}
$$

Finally, we need to prove that any tree T will belong to the family $\mathcal{T}_1$ if $R_\alpha(T) = f(n(T), \gamma(T), \alpha)$. Suppose that there exists a tree T such that $R_\alpha(T) = f(n(T), \gamma(T), \alpha)$, and $T \notin \mathcal{T}_1$. Let T be the tree with the minimum number of vertices, which satisfies the above assumptions. If $v_1 v_2 \cdots v_d$ is a diameter path of T, by above proofs, we suppose that $d_{v_2} = d_{v_3} = d_{v_4} = 2$, $2 \leq d_{v_5} \leq 3$, $d_{a_1} \leq 2$ and $d_{a_2} \leq 2$. Then, we investigate the following cases.

If $d_{v_5} = 3$ and $d_{a_2} \leq 2$, we let $T_{-4} = T - \{v_1, v_2, v_3, v_4\}$. Then,

$$\begin{aligned}R_\alpha(T) &= R_\alpha(T_{-4}) - (2^\alpha - 3^\alpha)\cdot[(d_{a_1})^\alpha + (d_{a_2})^\alpha] + 6^\alpha + 2\cdot 4^\alpha + 2^\alpha\\ &\leq \frac{4(n-4)+3(\gamma-1)-15}{5}\cdot 4^\alpha + 2\cdot 2^\alpha + (n-3\gamma-1)\\ &\quad \cdot\left(3\cdot 6^\alpha + 2^\alpha - \frac{19}{5}\cdot 4^\alpha\right)\\ &\quad - (2^\alpha - 3^\alpha)\cdot[(d_{a_1})^\alpha + (d_{a_2})^\alpha] + 6^\alpha + 2\cdot 4^\alpha + 2^\alpha\\ &= f(n,\gamma,\alpha) - (2^\alpha - 3^\alpha)\cdot[(d_{a_1})^\alpha + (d_{a_2})^\alpha] - 2\cdot 6^\alpha + 2\cdot 4^\alpha.\end{aligned}$$

If $d_{a_1} = 1$ or $d_{a_2} = 1$, then we get $-(2^\alpha - 3^\alpha)\cdot[(d_{a_1})^\alpha + (d_{a_2})^\alpha] - 2\cdot 6^\alpha + 2\cdot 4^\alpha < 0$ for $\alpha < 0$, a contradiction. If $d_{a_1} = d_{a_2} = 2$, then $-(2^\alpha - 3^\alpha)\cdot[(d_{a_1})^\alpha + (d_{a_2})^\alpha] - 2\cdot 6^\alpha + 2\cdot 4^\alpha = 0$ and $R_\alpha(T_{-4}) = f(n(T_{-4}), \gamma(T_{-4}), \alpha)$. If $T_{-4} \in \mathcal{T}_1$, based on the definition of T_{-4}, one can check that $T \in \mathcal{T}_1$, a contradiction. If $T_{-4} \notin \mathcal{T}_1$, we have $|V_{T_{-4}}| < |V_T|$, a contradiction.

If $d_{v_5} = 2$, we take $T_{-3} = T - \{v_1, v_2, v_3\}$. Then, we get

$$\begin{aligned}R_\alpha(T) &= R_\alpha(T_{-3}) - 2^\alpha + 3\cdot 4^\alpha + 2^\alpha\\ &\leq \frac{4(n-3)+3(\gamma-1)-15}{5}\cdot 4^\alpha + 2\cdot 2^\alpha + (n-3\gamma)\\ &\quad \cdot\left(3\cdot 6^\alpha + 2^\alpha - \frac{19}{5}\cdot 4^\alpha\right) + 3\cdot 4^\alpha\\ &= \frac{4n+3\gamma-15}{5}\cdot 4^\alpha + 2\cdot 2^\alpha + (n-3\gamma)\cdot\left(3\cdot 6^\alpha + 2^\alpha - \frac{19}{5}\cdot 4^\alpha\right)\\ &= f(n,\gamma,\alpha).\end{aligned}$$

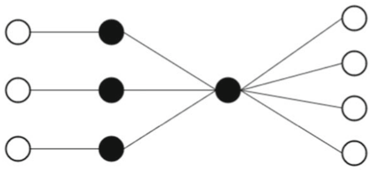
Note that $R_\alpha(T_{-3}) = f(n(T_{-3}), \gamma(T_{-3}), \alpha)$. Analogously, we can assume that $T_{-3} \in \mathcal{T}_1$. Since v_4 is a pendent vertex in T_{-3}, we can conclude that $T \in \mathcal{T}_1$. If $T_{-3} \notin \mathcal{T}_1$, we obtain a contradiction with minimality of T.

This completes the proof. □

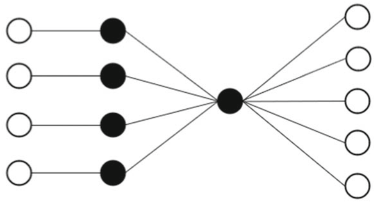
3 The Lower Bound on the R_α with $\alpha \in [\alpha_2, 0)$ of Trees in Terms of Domination Number

In this section, for $\alpha \in [\alpha_2, 0)$, the lower bound on the general Randić index R_α of trees using not only the order, but also the order and the domination number, is obtained, where $\alpha_2 \approx -0.5696$ is the unique nonzero root of the equation $2\cdot 4^\alpha + 2^{\alpha+1} - 3^\alpha - 2\cdot 6^\alpha - 1 = 0$. Moreover, we prove the corresponding extremal trees belong to $\mathcal{T}_2$.

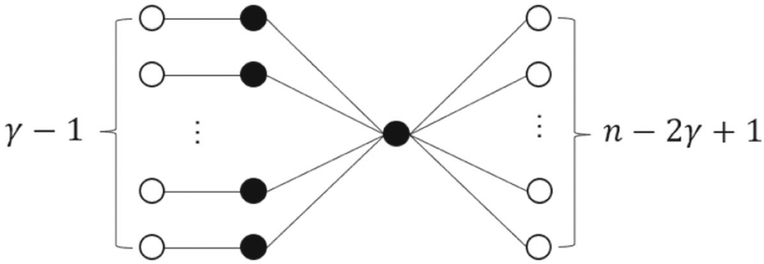
Definition 3.1 [1] Let $\mathcal{T}_2$ be a set of trees with order n and domination number γ, which are obtained from the star $S_{n-\gamma+1}$ by attaching a pendent edge to its $\gamma - 1$ pendent vertices (see Fig. 11).



(a) The extremal tree in $\mathcal{T}_2(11,4)$ with $R_\alpha(T) = 3 \cdot 14^\alpha + 4 \cdot 7^\alpha + 3 \cdot 2^\alpha$.



(b) The extremal tree in $\mathcal{T}_2(14,5)$ with $R_\alpha(T) = 4 \cdot 22^\alpha + 2 \cdot 9^\alpha + 4 \cdot 2^\alpha$.



(c) The extremal tree in $\mathcal{T}_2(n,\gamma)$ with $R_\alpha(T) = (n-2\gamma+1)(n-\gamma)^\alpha + 2^\alpha(\gamma-1)(1+(n-\gamma)^\alpha)$.

Fig. 11 Several extremal trees in graph family $\mathcal{T}_2$

Theorem 3.2 *Let T be an n-vertex tree with a domination number γ. Then,*

$$R_\alpha(T) \geq (n-2\gamma+1)(n-\gamma)^\alpha + 2^\alpha(\gamma-1)[1+(n-\gamma)^\alpha], \quad \text{for} \quad \alpha \in [\alpha_2, 0), \tag{3.1}$$

with the equality holding if and only if $T \in \mathcal{T}_2$, where $\alpha_2 \approx -0.5696$ is the unique nonzero root of the equation $2 \cdot 4^\alpha + 2^{\alpha+1} - 3^\alpha - 2 \cdot 6^\alpha - 1 = 0$.

Proof To simplify the computations, we denote $h(n,\gamma,\alpha) = (n-2\gamma+1)(n-\gamma)^\alpha + 2^\alpha(\gamma-1)[1+(n-\gamma)^\alpha]$. For any 4-vertex tree, one can see that $R_\alpha(S_4) = 3 \cdot 3^\alpha = h(4,1,\alpha)$, and $R_\alpha(P_4) = 4^\alpha + 2 \cdot 2^\alpha = h(4,2,\alpha)$. We suppose that the inequality shown in (3.1) is always true for any tree with $n-1$ vertices. Next, we will consider an n-vertex trees. We take a diameter path $v_1 v_2 \cdots v_d$ in T, and denote $N(v_2) = \{v_1, v_3, u_1, u_2, \ldots, u_{d_{v_2}-2}\}$, $N(v_3) = \{v_2, v_4, w_1, w_2, \ldots, w_{d_{v_3}-2}\}$. Let $T_{-1} = T - \{v_1\}$, then we investigate the following two cases.

Case 1 If $\gamma(T_{-1}) = \gamma(T)$, we obtain

$$\begin{aligned} R_\alpha(T) &= R_\alpha(T_{-1}) + (d_{v_2}-1) \cdot (d_{v_2})^\alpha + (d_{v_2} \cdot d_{v_3})^\alpha - (d_{v_2}-2) \cdot (d_{v_2}-1)^\alpha \\ &\quad - [(d_{v_2}-1) \cdot d_{v_3}]^\alpha \\ &\geq (n-2\gamma) \cdot (n-\gamma-1)^\alpha + (\gamma-1) \cdot [2 \cdot (n-\gamma-1)]^\alpha + (\gamma-1) \cdot 2^\alpha \\ &\quad + (d_{v_2}-1) \cdot (d_{v_2})^\alpha + (d_{v_2} \cdot d_{v_3})^\alpha - (d_{v_2}-2) \cdot (d_{v_2}-1)^\alpha \end{aligned}$$

$$
\begin{aligned}
&- [(d_{v_2} - 1) \cdot d_{v_3}]^{\alpha} \\
&= h(n, \gamma, \alpha) + [(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] \cdot [(n - \gamma - 1)^{\alpha} \\
&- (n - \gamma)^{\alpha}] - (n - \gamma - 1)^{\alpha} \\
&- [(d_{v_2} - 1)^{\alpha} - (d_{v_2})^{\alpha}] \cdot [(d_{v_2} - 1) + (d_{v_3})^{\alpha}] + (d_{v_2} - 1)^{\alpha}.
\end{aligned}
$$

Since $T \cong S_n$ if $n = d_{v_2} + 1$ and $T \in \mathcal{T}_2$ if $n = d_{v_2} + 2$, we only need to study the case that $n \geq d_{v_2} + 3$. Consider the following function:

$$
\begin{aligned}
\kappa(\gamma) &= [(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] \cdot [(n - \gamma - 1)^{\alpha} - (n - \gamma)^{\alpha}] - (n - \gamma - 1)^{\alpha} \\
&= [(n - 2\gamma) + (\gamma - 1) \cdot 2^{\alpha}](n - \gamma - 1)^{\alpha} - [(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] \\
&\quad (n - \gamma)^{\alpha}.
\end{aligned}
$$

By calculation, we get

$$
\begin{aligned}
\kappa'(\gamma) &= -\alpha(n - \gamma - 1)^{\alpha - 1}[(n - 2\gamma) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma - 1)^{\alpha}(2^{\alpha} - 2) \\
&\quad + \alpha(n - \gamma)^{\alpha - 1}[(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] - (n - \gamma)^{\alpha}(2^{\alpha} - 2) \\
&= (n - \gamma - 1)^{\alpha - 1}\left\{-\alpha[(n - 2\gamma) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma - 1)(2^{\alpha} - 2)\right\} \\
&\quad - (n - \gamma)^{\alpha - 1}\left\{-\alpha[(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma)(2^{\alpha} - 2)\right\}.
\end{aligned}
$$

Let $\vartheta_1(\gamma) = -\alpha[(n - 2\gamma) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma - 1)(2^{\alpha} - 2)$ and $\vartheta_2(\gamma) = -\alpha[(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma)(2^{\alpha} - 2)$. It follows that $\vartheta_1'(\gamma) = \vartheta_2'(\gamma) = -\alpha(2^{\alpha} - 2) + (2 - 2^{\alpha}) = (1 + \alpha)(2 - 2^{\alpha}) > 0$ for $\alpha \in (-1, 0)$. Hence, $\vartheta_1(\gamma) \leq \vartheta_1(\frac{n}{2}) = -\alpha(\frac{n}{2} - 1)2^{\alpha} + (\frac{n}{2} - 1)(2^{\alpha} - 2) = (\frac{n}{2} - 1)[(1 - \alpha)2^{\alpha} - 2] < 0$ for $\alpha \in (-1, 0)$, and $\vartheta_2(\gamma) \leq \vartheta_2(\frac{n}{2}) = -\alpha[1 + (\frac{n}{2} - 1)2^{\alpha}] + \frac{n}{2}(2^{\alpha} - 2) = (\frac{n}{2} - 1)[(1 - \alpha)2^{\alpha} - 2] + (2^{\alpha} - 2 - \alpha) < 0$ for $\alpha \in (-1, 0)$.

Suppose that $\alpha \in (-1, 0)$, then we consider the inequality $\kappa'(\gamma) = (n - \gamma - 1)^{\alpha - 1}\vartheta_1(\gamma) - (n - \gamma)^{\alpha - 1}\vartheta_2(\gamma) < 0$. Since

$$
\begin{aligned}
&(n - \gamma - 1)^{\alpha - 1}\vartheta_1(\gamma) - (n - \gamma)^{\alpha - 1}\vartheta_2(\gamma) < 0 \\
&\Leftrightarrow \frac{\vartheta_1(\gamma)}{\vartheta_2(\gamma)} > \left(\frac{n - \gamma}{n - \gamma - 1}\right)^{\alpha - 1} \\
&\Leftrightarrow \frac{-\alpha[(n - 2\gamma) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma - 1)(2^{\alpha} - 2)}{-\alpha[(n - 2\gamma + 1) + (\gamma - 1) \cdot 2^{\alpha}] + (n - \gamma)(2^{\alpha} - 2)} > \left(\frac{n - \gamma - 1}{n - \gamma}\right)^{1 - \alpha} \\
&\Leftrightarrow 1 - \frac{2 - 2^{\alpha} + \alpha}{(n - \gamma)(2 - 2^{\alpha} + \alpha) + (1 - \gamma)(\alpha - \alpha \cdot 2^{\alpha})} > \left(1 - \frac{1}{n - \gamma}\right)^{1 - \alpha} \\
&\Leftrightarrow 1 - \frac{1}{(n - \gamma) + \frac{\alpha(1 - \gamma)(1 - 2^{\alpha})}{2 - 2^{\alpha} + \alpha}} > \left(1 - \frac{1}{n - \gamma}\right)^{1 - \alpha}. \qquad (3.2)
\end{aligned}
$$

Note that the inequality $\frac{\alpha(1-\gamma)(1-2^\alpha)}{2-2^\alpha+\alpha} \geq 0$ holds for $\alpha \in (-1, 0)$ and $\gamma > 1$; one can check that

$$1 - \frac{1}{(n-\gamma) + \frac{\alpha(1-\gamma)(1-2^\alpha)}{2-2^\alpha+\alpha}} > 1 - \frac{1}{n-\gamma} > \left(1 - \frac{1}{n-\gamma}\right)^{1-\alpha},$$

for $\alpha \in (-1, 0)$ and $\gamma > 1$. So the function $\kappa'(\gamma)$ is always negative if $\alpha \in (-1, 0)$ and $\gamma > 1$, which implies that if $\alpha \in (-1, 0)$, the function $\kappa(\gamma)$ monotonically decreases for $\gamma > 1$.

Note that $\gamma \leq \frac{n-d_{v_2}+2}{2}$ and $d_{v_3} \geq 2$, then we have $\kappa(\gamma) \geq \kappa\left(\frac{n-d_{v_2}+2}{2}\right)$ and

$$\begin{aligned}
&[(n-2\gamma+1) + (\gamma-1)\cdot 2^\alpha]\cdot[(n-\gamma-1)^\alpha - (n-\gamma)^\alpha] - (n-\gamma-1)^\alpha \\
&\quad - [(d_{v_2}-1)^\alpha - (d_{v_2})^\alpha]\cdot[(d_{v_2}-1) + (d_{v_3})^\alpha] + (d_{v_2}-1)^\alpha \\
&\geq \left[\left(d_{v_2} - \frac{1}{2}\right)^\alpha - \left(d_{v_2} + \frac{1}{2}\right)^\alpha\right]\cdot\left(d_{v_2} - 1 + \frac{3}{2}\cdot 2^\alpha\right) \\
&\quad + [(d_{v_2})^\alpha - (d_{v_2}-1)^\alpha]\cdot(d_{v_2} + 2^\alpha - 1) + (d_{v_2}-1)^\alpha - \left(d_{v_2} - \frac{1}{2}\right)^\alpha \\
&= \left\{\left[\left(d_{v_2} - \frac{1}{2}\right)^\alpha - \left(d_{v_2} + \frac{1}{2}\right)^\alpha\right] - [(d_{v_2}-1)^\alpha - (d_{v_2})^\alpha]\right\}(d_{v_2} + 2^\alpha - 1) \\
&\quad + 2^{\alpha-1}\left[\left(d_{v_2} - \frac{1}{2}\right)^\alpha - \left(d_{v_2} + \frac{1}{2}\right)^\alpha\right] + (d_{v_2}-1)^\alpha - \left(d_{v_2} - \frac{1}{2}\right)^\alpha.
\end{aligned}$$

Consider functions $\psi_2(z) = z^\alpha$ and $\phi_2(z) = \psi_2(z-1) - \psi_2(z) = (z-1)^\alpha - z^\alpha$.

Note that $\psi_2'(z) = \alpha z^{\alpha-1} < 0$ and $\psi_2''(z) = \alpha(\alpha-1)z^{\alpha-2} > 0$. Suppose that $\alpha < 0$ and there exist $z_2 > z_1 > 1$, then we obtain

$$\begin{aligned}
\phi_2(z_2) - \phi_2(z_1) &= \phi_2'(\xi_2)(z_2 - z_1) \\
&= [\psi_2'(\xi_2 - 1) - \psi_2'(\xi_2)](z_2 - z_1) \\
&= -\psi_2''(\eta_2)(z_2 - z_1) < 0,
\end{aligned}$$

where $1 < z_1 < \xi_2 < z_2$ and $0 < \xi_2 - 1 < \eta_2 < \xi_2$. Thus, if $\alpha < 0$, $\phi_2(z)$ is a monotonically decreasing function for $z \in (1, \infty)$. One can see that for any $d_{v_2} \geq 2$ and $\alpha \in (-1, 0)$, $\phi_2(d_{v_2} + \frac{1}{2}) - \phi_2(d_{v_2} + 1) > 0$, $\psi_2(d_{v_2} - \frac{1}{2}) - \psi_2(d_{v_2} + \frac{1}{2}) > 0$ and $\psi_2(d_{v_2} - 1) - \psi_2(d_{v_2} - \frac{1}{2}) > 0$. Obviously, if $d_{v_2} \geq 2$ and $\alpha \in (-1, 0)$, the inequality $\left\{[(d_{v_2} - \frac{1}{2})^\alpha - (d_{v_2} + \frac{1}{2})^\alpha] - [(d_{v_2}-1)^\alpha - (d_{v_2})^\alpha]\right\}(d_{v_2} + 2^\alpha - 1) + 2^{\alpha-1}[(d_{v_2} - \frac{1}{2})^\alpha - (d_{v_2} + \frac{1}{2})^\alpha] + (d_{v_2}-1)^\alpha - (d_{v_2} - \frac{1}{2})^\alpha > 0$ holds, i.e., $R_\alpha(T) > h(n, \gamma, \alpha)$.

Case 2 Suppose that $\gamma(T_{-1}) = \gamma(T) - 1$. Obviously, there exists a minimum dominating set $D(T)$ such that $v_3 \in D(T)$ and $d_{v_2} = 2$. Then,

$$\begin{aligned}
R_\alpha(T) &= R_\alpha(T_{-1}) - (d_{v_3})^\alpha + (2\cdot d_{v_3})^\alpha + 2^\alpha \\
&\geq [n - 1 - 2(\gamma-1) + 1]\cdot(n - 1 - \gamma + 1)^\alpha + (\gamma-2)\cdot 2^\alpha \\
&\quad \cdot (n - 1 - \gamma + 1)^\alpha + (\gamma-2)\cdot 2^\alpha
\end{aligned}$$

$$
\begin{aligned}
&- (d_{v_3})^{\alpha} + (2d_{v_3})^{\alpha} + 2^{\alpha} \\
&= h(n, \gamma, \alpha) + (n-\gamma)^{\alpha} - [2 \cdot (n-\gamma)]^{\alpha} - (d_{v_3})^{\alpha} + (2 \cdot d_{v_3})^{\alpha}.
\end{aligned}
$$

It is easy to see that $(n-\gamma)^{\alpha} - [2(n-\gamma)]^{\alpha} - (d_{v_3})^{\alpha} + (2d_{v_3})^{\alpha} \geq 0$ if $d_{v_3} \geq n-\gamma$. However, this only happens when $d_{v_3} = n-\gamma$ which implies that $T \in \mathcal{T}_2$.

If $d_{v_3} \leq n-\gamma-1$, we denote $N(v_3) = \{v_2, v_4, w_1, w_2, \ldots, w_{d_{v_3}-2}\}$. By the above investigations, we suppose that w_i $(i = 1, 2, \ldots, d_{v_3}-2)$ is a pendent vertex or a support vertex with degree 2. Besides, if v_4 is a pendent vertex or a support vertex with degree 2, we can conclude that $T \in \mathcal{T}_2$. Without loss of generality, we let $d_{w_1} = d_{w_2} = \cdots = d_{w_{l_1}} = 1$ and $d_{w_{l_1+1}} = \cdots = d_{w_{l_1+l_2}} = 2$, where $l_1 + l_2 = d_{v_3} - 2$. Then, we discuss the following two cases.

Case 2.1 $l_1 \geq 2$. We take $T_{-1} = T - \{w_1\}$, implying $\gamma(T_{-1}) = \gamma(T)$. Note that $\gamma - (2 + l_2) \leq \frac{n-(d_{v_3}+1+l_2)}{2}$, i.e., $\gamma \leq \frac{n-l_1+1}{2}$, we obtain

$$
\begin{aligned}
R_{\alpha}(T) &= R_{\alpha}(T_{-1}) - (d_{v_3}-1)^{\alpha} \cdot [(d_{v_4})^{\alpha} + (l_1-1) + (l_2+1) \cdot 2^{\alpha}] + (d_{v_3})^{\alpha} \\
&\quad \cdot [(d_{v_4})^{\alpha} + l_1 + (l_2+1) \cdot 2^{\alpha}] \\
&\geq (n-2\gamma) \cdot (n-\gamma-1)^{\alpha} + (\gamma-1) \cdot 2^{\alpha} \cdot (n-\gamma-1)^{\alpha} + (\gamma-1) \cdot 2^{\alpha} \\
&\quad - [(d_{v_3}-1)^{\alpha} - (d_{v_3})^{\alpha}] \cdot [(d_{v_4})^{\alpha} + l_1 + (l_2+1) \cdot 2^{\alpha}] + (d_{v_3}-1)^{\alpha} \\
&= h(n, \gamma, \alpha) + [n - 2\gamma + 1 + (\gamma-1) \cdot 2^{\alpha}] \cdot [(n-\gamma-1)^{\alpha} \\
&\quad - (n-\gamma)^{\alpha}] - (n-\gamma-1)^{\alpha} \\
&\quad - [(d_{v_3}-1)^{\alpha} - (d_{v_3})^{\alpha}] \cdot [l_1 + (l_2+2) \cdot 2^{\alpha}] + (d_{v_3}-1)^{\alpha}
\end{aligned}
$$

There exists $l \geq 0$ such that $n = l_1 + 2l_2 + 5 + l$, then we have $\gamma \leq \frac{l_1+2l_2+5+l-l_1+1}{2} = l_2 + 3 + \frac{l}{2}$. By the above consideration, we get

$$
\begin{aligned}
&[n - 2\gamma + 1 + (\gamma-1) \cdot 2^{\alpha}] \cdot [(n-\gamma-1)^{\alpha} - (n-\gamma)^{\alpha}] - (n-\gamma-1)^{\alpha} \\
&\quad - [(d_{v_3}-1)^{\alpha} - (d_{v_3})^{\alpha}] \cdot [l_1 + (l_2+2) \cdot 2^{\alpha}] + (d_{v_3}-1)^{\alpha} \\
&\geq \left[l_1 + \left(l_2 + 2 + \frac{l}{2}\right) \cdot 2^{\alpha}\right] \cdot \left[\left(d_{v_3} + \frac{l}{2} - 1\right)^{\alpha} - \left(d_{v_3} + \frac{l}{2}\right)^{\alpha}\right] \\
&\quad - \left(d_{v_3} + \frac{l}{2} - 1\right)^{\alpha} \\
&\quad - [(d_{v_3}-1)^{\alpha} - (d_{v_3})^{\alpha}] \cdot [l_1 + (l_2+2) \cdot 2^{\alpha}] + (d_{v_3}-1)^{\alpha} \\
&= \left[\left(d_{v_3} + \frac{l}{2}\right) \cdot 2^{\alpha} + (1-2^{\alpha})l_1\right] \cdot \left[\left(d_{v_3} + \frac{l}{2} - 1\right)^{\alpha} - \left(d_{v_3} + \frac{l}{2}\right)^{\alpha}\right] \\
&\quad - \left(d_{v_3} + \frac{l}{2} - 1\right)^{\alpha} \\
&\quad - [d_{v_3} \cdot 2^{\alpha} + (1-2^{\alpha})l_1] \cdot [(d_{v_3}-1)^{\alpha} - (d_{v_3})^{\alpha}] + (d_{v_3}-1)^{\alpha}.
\end{aligned}
$$

Next, we consider the function $r(z) = [2^\alpha z + (1-2^\alpha)l_1]\cdot[(z-1)^\alpha - z^\alpha] - (z-1)^\alpha$. If $(1-2^\alpha)l_1 \le 2^{\alpha-1}z$ and $\alpha \in (-1, 0)$, we have

$$\begin{aligned}
r'(z) &= (-\alpha)\cdot\left\{-\frac{2^\alpha}{\alpha}\cdot[(z-1)^\alpha - z^\alpha] + (z-1)^{\alpha-1} - [(z-1)^{\alpha-1} - z^{\alpha-1}]\right.\\
&\quad\left.\cdot[2^\alpha z - (2^\alpha - 1)l_1]\right\}\\
&\ge (-\alpha)\cdot\left\{-\frac{2^\alpha}{\alpha}\cdot[(z-1)^\alpha - z^\alpha] + (z-1)^{\alpha-1} - [(z-1)^{\alpha-1} - z^{\alpha-1}]\right.\\
&\quad\left.\cdot\left(\frac{3}{2}\cdot 2^\alpha z\right)\right\}\\
&= (-\alpha)\cdot\left\{\left(3\cdot 2^{\alpha-1} + \frac{2^\alpha}{\alpha}\right)[z^\alpha - (z-1)^\alpha] + (1 - 3\cdot 2^{\alpha-1})(z-1)^{\alpha-1}\right\}\\
&= (-\alpha)\cdot\left\{\left(3\cdot 2^{\alpha-1} + \frac{2^\alpha}{\alpha}\right)[\psi_2(z) - \psi_2(z-1)] + (1 - 3\cdot 2^{\alpha-1})\right.\\
&\quad\left.\cdot\frac{1}{\alpha}\psi_2'(z-1)\right\}
\end{aligned}$$

Suppose that $z \ge 2$ and $\alpha \in (-\frac{2}{3}, 0)$, we consider the following inequality:

$$\begin{aligned}
&\left(3\cdot 2^{\alpha-1} + \frac{2^\alpha}{\alpha}\right)[\psi_2(z) - \psi_2(z-1)] + \frac{1 - 3\cdot 2^{\alpha-1}}{\alpha}\psi_2'(z-1) > 0\\
&\quad\Leftrightarrow \left(3\cdot 2^{\alpha-1} + \frac{2^\alpha}{\alpha}\right)\cdot\psi_2'(z-1)\cdot\frac{\psi_2(z) - \psi_2(z-1)}{\psi_2'(z-1)}\\
&\qquad + \frac{1 - 3\cdot 2^{\alpha-1}}{\alpha}\psi_2'(z-1) > 0\\
&\quad\Leftrightarrow \frac{\psi_2(z) - \psi_2(z-1)}{\psi_2'(z-1)} > \frac{3\cdot 2^{\alpha-1} - 1}{3\alpha\cdot 2^{\alpha-1} + 2^\alpha}.
\end{aligned}$$

Let $t(z) = \frac{\psi_2(z) - \psi_2(z-1)}{\psi_2'(z-1)} = \frac{z^\alpha - (z-1)^\alpha}{\alpha(z-1)^{\alpha-1}}$. Consider the inequality $t'(z) = \left(\frac{z}{z-1}\right)^{\alpha-1} + \frac{1-\alpha}{\alpha}\left(\frac{z}{z-1}\right)^\alpha - \frac{1}{\alpha} > 0$, then we get

$$\begin{aligned}
&\left(\frac{z}{z-1}\right)^{\alpha-1} + \frac{1-\alpha}{\alpha}\left(\frac{z}{z-1}\right)^\alpha - \frac{1}{\alpha} > 0\\
&\quad\Leftrightarrow \alpha\left(\frac{z}{z-1}\right)^{\alpha-1} + (1-\alpha)\left(\frac{z}{z-1}\right)^\alpha - 1 < 0\\
&\quad\Leftrightarrow \alpha + (1-\alpha)\frac{z}{z-1} < \left(\frac{z}{z-1}\right)^{1-\alpha}\\
&\quad\Leftrightarrow \left(1 + \frac{1}{z-1}\right)^{1-\alpha} - (1-\alpha)\frac{1}{z-1} > 1.
\end{aligned}$$

Let $\rho(s) = (1+s)^{1-\alpha} - (1-\alpha)s$, where $s \in (0, 1]$ and $\alpha \in (-\frac{2}{3}, 0)$. Since $\rho'(s) = (1-\alpha)\left[(1+s)^{-\alpha} - 1\right] > 0$ for $s \in (0, 1]$ and $\rho'(0) = 0$, we have $\rho(s) > \rho(0) = 1$ for $s \in (0, 1]$, which implies that the inequality $\left(1 + \frac{1}{z-1}\right)^{1-\alpha} - (1-\alpha)\frac{1}{z-1} > 1$ holds for $z \geq 2$.

Based on the above discussion, one can see that $t(z)$ is a monotonically increasing function for $z \geq 2$ if $\alpha \in (-\frac{2}{3}, 0)$. We calculate that $t(2) = \frac{2^\alpha - 1}{\alpha} > \frac{3\cdot 2^{\alpha-1} - 1}{3\alpha\cdot 2^{\alpha-1} + 2^\alpha}$ for any $\alpha \in (-\frac{2}{3}, 0)$ (the function $\frac{2^\alpha - 1}{\alpha} - \frac{3\cdot 2^{\alpha-1} - 1}{3\alpha\cdot 2^{\alpha-1} + 2^\alpha}$ is depicted in Fig. 12a), what implies that $r'(z) > 0$ for any $z \geq 2$ and $\alpha \in (-\frac{2}{3}, 0)$. If $l = 0$, one can check that $T \in \mathcal{T}_2$. Moreover, $d_{v_3} + \frac{l}{2} \geq d_{v_3} \geq l_1 + 2 \geq l_1(2^{1-\alpha} - 2)$ with $l > 0$, and thus, we get $r(d_{v_3} + \frac{l}{2}) > r(d_{v_3})$, which implies that

$$\begin{aligned}&[n - 2\gamma + 1 + (\gamma - 1)\cdot 2^\alpha]\cdot[(n-\gamma-1)^\alpha - (n-\gamma)^\alpha] - (n-\gamma-1)^\alpha \\ &\quad > [(d_{v_3} - 1)^\alpha - (d_{v_3})^\alpha]\cdot[l_1 + (l_2 + 2)\cdot 2^\alpha] + (d_{v_3} - 1)^\alpha.\end{aligned}$$

It is easy to conclude that $R_\alpha(T) > h(n, \gamma, \alpha)$ for $\alpha \in \left(-\frac{2}{3}, 0\right)$.

Case 2.2 $l_1 \leq 1$. Let $T_{-2} = T - \{v_1, v_2\}$, then we have

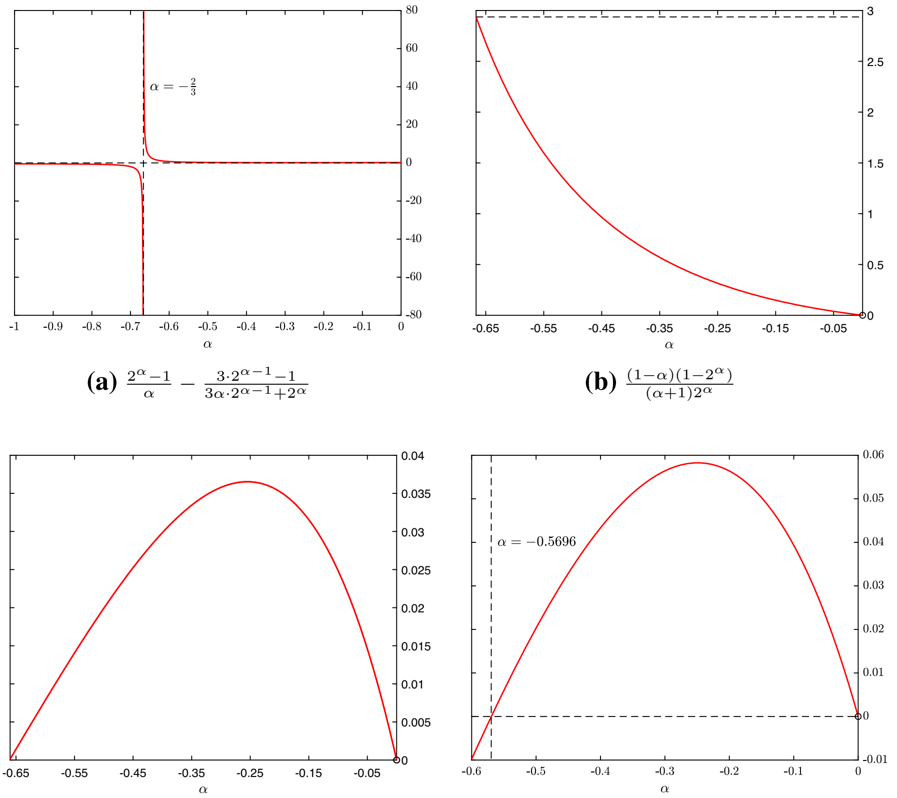
$$\begin{aligned}R_\alpha(T) &= R_\alpha(T_{-2}) - [(d_{v_3} - 1)^\alpha - (d_{v_3})^\alpha]\cdot[(d_{v_4})^\alpha + 1 + (d_{v_3} - 3)\cdot 2^\alpha] \\ &\quad + (2\cdot d_{v_3})^\alpha + 2^\alpha \\ &\geq [n - 2 - 2(\gamma - 1) + 1]\cdot(n - 2 - \gamma + 1)^\alpha + (\gamma - 2)\cdot 2^\alpha \\ &\quad \cdot (n - 2 - \gamma + 1)^\alpha + (\gamma - 2)\cdot 2^\alpha \\ &\quad - [(d_{v_3} - 1)^\alpha - (d_{v_3})^\alpha]\cdot[(d_{v_4})^\alpha + 1 + (d_{v_3} - 3)\cdot 2^\alpha] + (2\cdot d_{v_3})^\alpha + 2^\alpha \\ &= h(n, \gamma, \alpha) + [n - 2\gamma + 1 + (\gamma - 1)\cdot 2^\alpha]\cdot[(n-\gamma-1)^\alpha - (n-\gamma)^\alpha] \\ &\quad - 2^\alpha\cdot(n-\gamma-1)^\alpha \\ &\quad - [(d_{v_3} - 1)^\alpha - (d_{v_3})^\alpha]\cdot[1 + (d_{v_3} - 2)\cdot 2^\alpha] + (2\cdot d_{v_3})^\alpha\end{aligned}$$

Note that $\gamma \leq \frac{n}{2}$ for any graph, we get

$$\begin{aligned}&[n - 2\gamma + 1 + (\gamma - 1)\cdot 2^\alpha]\cdot[(n-\gamma-1)^\alpha - (n-\gamma)^\alpha] - 2^\alpha\cdot(n-\gamma-1)^\alpha \\ &\quad \geq -2^\alpha\cdot\left(\frac{n}{2} - 1\right)^\alpha - \left[\left(\frac{n}{2}\right)^\alpha - \left(\frac{n}{2} - 1\right)^\alpha\right]\cdot\left[2^\alpha\cdot\left(\frac{n}{2} - 1\right) + 1\right] \\ &\quad = \left[2^\alpha\cdot\left(\frac{n}{2} - 1\right) + 1 - 2^\alpha\right]\cdot\left(\frac{n}{2} - 1\right)^\alpha - \left(2^\alpha\cdot\frac{n}{2} + 1 - 2^\alpha\right)\cdot\left(\frac{n}{2}\right)^\alpha.\end{aligned}$$

Consider functions $\psi_3(z) = (2^\alpha\cdot z + 1 - 2^\alpha)\cdot z^\alpha$ and $\phi_3(z) = \psi_3(z) - \psi_3(z+1)$, then we calculate that $\psi_3'(z) = [(\alpha+1)2^\alpha\cdot z + \alpha(1 - 2^\alpha)]\cdot z^{\alpha-1}$ and $\psi_3''(z) = 2^{\alpha-2}\alpha\cdot[(\alpha+1)2^\alpha\cdot z + (\alpha-1)(1-2^\alpha)]$.

Suppose that $\alpha \in (-\frac{2}{3}, 0)$ and $z > 0$. Since $\frac{(1-\alpha)(1-2^\alpha)}{(\alpha+1)2^\alpha} < 3$ holds for every $\alpha \in (-\frac{2}{3}, 0)$ (see Fig. 12b), it can be checked that if $z \geq 3 > \frac{(1-\alpha)(1-2^\alpha)}{(\alpha+1)2^\alpha}$, the inequality $(\alpha+1)2^\alpha\cdot z + \alpha(1 - 2^\alpha) > 0$ holds, implying $\psi_3'(z) > 0$ and $\psi_3''(z) < 0$ for $\alpha \in (-\frac{2}{3}, 0)$ and $z \geq 3$.



(a) $\frac{2^{\alpha}-1}{\alpha} - \frac{3\cdot 2^{\alpha-1}-1}{3\alpha\cdot 2^{\alpha-1}+2^{\alpha}}$ **(b)** $\frac{(1-\alpha)(1-2^{\alpha})}{(\alpha+1)2^{\alpha}}$

(c) $\phi_3(3) - \phi_3(2)$ **(d)** $\phi_3(2) - \phi_3(1)$

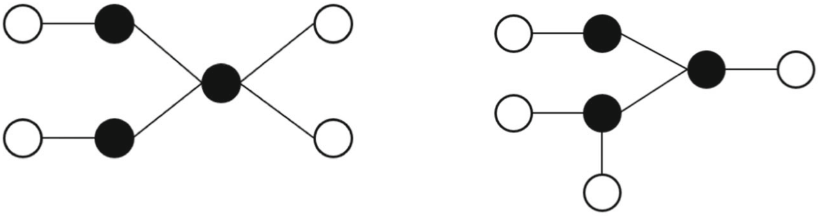
Fig. 12 The four functions of α in Theorem 3.2

Suppose that $\alpha \in (-\frac{2}{3}, 0)$ and there exist $z_2 > z_1 \geq 3$, we obtain

$$\begin{aligned}\phi_3(z_2) - \phi_3(z_1) &= \phi_3'(\xi_3)(z_2 - z_1)\\ &= -[\psi_3'(\xi_3 + 1) - \psi_3'(\xi_3)](z_2 - z_1)\\ &= -\psi_3''(\eta_3)(z_2 - z_1) > 0,\end{aligned}$$

where $3 \leq z_1 < \xi_3 < z_2$ and $3 < \xi_3 < \eta_3 < \xi_3 + 1$. In addition, we calculate that $\phi_3(3) - \phi_3(2) = 4 \cdot 6^{\alpha} + 2 \cdot 3^{\alpha} - 3 \cdot 8^{\alpha} - 2 \cdot 4^{\alpha} - 2^{\alpha} > 0$ for $\alpha \in [-0.65, 0)$ and $\phi_3(2) - \phi_3(1) = 2 \cdot 4^{\alpha} - 2^{\alpha+1} - 2 \cdot 6^{\alpha} - 3^{\alpha} - 1 > 0$ for $\alpha \in [-0.5696, 0)$ (see Fig. 12c, d). Based on the above discussions, we conclude that $\phi_3(z)$ is a monotonically increasing function if $\alpha \in [-0.5696, 0)$. Then, we get $\phi_3(\frac{n}{2} - 1) \geq \phi_3(d_{v_3} - 1)$ for $n \geq 2d_{v_3}$. The equality holds if and only if $n = 2d_{v_3}$, which implies that $T \in \mathcal{T}_2$. On the whole, the equality in (3.1) holds for every tree in $\mathcal{T}_2$, and if $T \notin \mathcal{T}_2$, the inequality (3.1) is strict.

This completes the proof. □



(a) The tree $\mathcal{T}_2(7,3)$ with $R_\alpha(T) = h(7,3,\alpha) = 2 \cdot (8^\alpha + 4^\alpha + 2^\alpha)$.

(b) The tree $T'(7,3)$ with $R_\alpha(T) = 9^\alpha + 6^\alpha + 3 \cdot 3^\alpha + 2^\alpha$.

Fig. 13 Two non-isomorphic 7-vertex trees with domination number 3

4 Conclusions

The extremal problem of the general Randić index of trees in terms of domination number is investigated in this paper. For any n-vertex tree T with domination number γ, we show that $R_\alpha(T) \leq f(n,\gamma,\alpha)$ if $\alpha \in [-0.5287, 0)$ and $R_\alpha(T) \geq h(n,\gamma,\alpha)$ if $\alpha \in [-0.5696, 0)$. In each case, extremal graphs are also characterized.

For P_4 and P_5, it can be calculated that $R_\alpha(P_4) > f(4,2,\alpha)$ and $R_\alpha(P_5) > f(5,2,\alpha)$ if $\alpha < -1$. Moreover, we construct a 7-vertex tree with domination number 3, written as $T'(7,3)$ (depicted in Fig. 13b), and show that $R_\alpha(T'(7,3)) = 9^\alpha + 6^\alpha + 3 \cdot 3^\alpha + 2^\alpha < h(7,3,\alpha)$ if $\alpha \leq -1.3568$. By observing the above simple examples, it can easily see that only when α is in a small interval, Theorems 2.3 and 3.2 hold.

The two ranges of the parameter α obtained in this paper may not be very accurate, and two lower bounds on α (α_1 and α_2) are quite near $-\frac{1}{2}$. It is hoped that in the future, one can discover new approaches to establish more extremal properties of the general Randić index and provide a more exact range of the parameter α about this issue.

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