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On Finsler Warped Product Metrics with Special Curvatures Properties

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Abstract
In this paper, we study a class of Finsler metrics called Finsler warped product metrics. We prove that every Finsler warped product metric is of isotropic E-curvature if and only if it is of isotropic S-curvature. Moreover, we prove that if the metric is of Douglas type and has isotropic S-curvature, then it must be Randers metric or Berwald metric.

Keywords Finsler metric · Warped product · Isotropic S-curvature

Mathematics Subject Classification 53B40 · 53C60

1 Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics [6]. For a Finsler metric $F = F(u, v)$, its geodesics curves are given by the system of differential equations $\ddot{c}^A + 2G^A(c, \dot{c}) = 0$, where the local functions $G^A = G^A(u, v)$ are called the spray coefficients. F is called a Berwald metric if G^A are quadratic in $v \in T_u M$ for any $u \in M$.

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Z. Shen, while studying volume comparison in Riemann–Finsler geometry, introduced the S-curvature, which is one of the most important non-Riemannian quantities in Finsler geometry, [20]. A Finsler metric F is said to be of isotropic S-curvature if

$$\mathbf{S} = (n+1)\kappa(u)F,$$

for some scalar function $\kappa = \kappa(u)$.

The Randers metric, which is the solution of Zermelo's navigation problem [1], has the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form with $||\beta||_\alpha < 1$. In [3], X. Cheng and Z. Shen investigated Randers metrics of scalar flag curvature and isotropic S-curvature. For more research on Finsler metrics with S-curvature and isotropic S-curvature, one can see [21,22,26].

The E-curvature is another the most important non-Riemannian quantity which has the following relation with the S-curvature [4]:

$$E_{AB} = \frac{1}{2} S_{v^A v^B}.$$

Moreover, F is of isotropic E-curvature if there is a scalar function $\kappa = \kappa(u)$ on an n-dimensional manifold M such that

$$\mathbf{E} = \frac{n+1}{2}\kappa F^{-1} h,$$

where h is the angular metric defined by $h_{AB} := F F_{v^A v^B}$. It has been known that if a Finsler metric F is of isotropic S-curvature, then it is an isotropic E-curvature. However, the converse is not true in general. Recently, many authors have studied this problem for some special Finsler metrics [5,8,12,17,22].

In this paper, we mainly study warped product metrics which have been introduced by Chen–Shen–Zhao using the concept of the warped product structure on an n-dimensional manifold $M := I \times \breve{M}$, where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric, [7]. The Finsler warped product metric F can be expressed in the following form:

$$F(u, v) = \breve{\alpha}(\breve{u}, \breve{v})\phi\left(u^1, \frac{v^1}{\breve{\alpha}(\breve{u}, \breve{v})}\right), \tag{1}$$

where $u = (u^1, \breve{u})$, $v = v^1 \frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$. According to [7, Lemma 3.1], this class of Finsler metrics includes spherically symmetric Finsler metrics.

Below are three important examples:

(1) Funk warped product Let $\mathbb{B}^n$ denote the unit open ball in $\mathbb{R}^n$. The metric

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n \cong \mathbb{R}^n,$$

is a Randers metric, called the Funk metric on $\mathbb{B}^n$. Its warped product form can be expressed in the following

$$F = \breve{\alpha}_+ \frac{\sqrt{s^2 + r^2(1-r^2)} + sr}{1-r^2},$$

where $r = u^1$, $s = \frac{v^1}{\breve{\alpha}_+}$ and $\breve{\alpha}_+$ is the standard Euclidean metric on the unit sphere S^{n-1}.

(2) Berwald warped product Define

$$F := \frac{\left[\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle\right]^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}},$$

where $y \in T_x\mathbb{B}^n \cong \mathbb{R}^n$, [24]. We call F the Berwald's metric. Its warped product form is

$$F = \breve{\alpha}_+ \frac{\left[\sqrt{s^2 + r^2(1-r^2)} + rs\right]^2}{(1-r^2)^2\sqrt{s^2 + r^2(1-r^2)}}.$$

(3) Bryant warped product Denote

$$\begin{aligned}
A &= \Big[|y|^2\cos\theta + (|x|^2|y|^2 - \langle x, y\rangle^2)\Big]^2 + \Big(|y|^2\sin\theta\Big)^2,\\
B &= |y|^2\cos\theta + (|x|^2|y|^2 - \langle x, y\rangle^2),\\
C &= \langle x, y\rangle\sin\theta,\\
D &= |x|^4 + 2|x|^2\cos\theta + 1.
\end{aligned}$$

For an angle θ with $0 \le \theta < \pi$, the Bryant's metric is defined by

$$F = \sqrt{\frac{\sqrt{A} + B}{2D} + \frac{C^2}{D^2}} + \frac{C}{D},$$

on the whole region $\mathbb{R}^n$. Putting

$$\begin{aligned}
\bar{A} &= \Big[(r^2 + s^2)\cos\theta + r^4\Big]^2 + \Big[(r^2 + s^2)\sin\theta\Big]^2,\\
\bar{B} &= (r^2 + s^2)\cos\theta + r^4,\\
\bar{C} &= rs\sin\theta,\\
\bar{D} &= r^4 + 2r^2\cos\theta + 1,
\end{aligned}$$

the warped product form is

$$F = \breve{\alpha}_+ \left(\sqrt{\frac{\sqrt{\bar{A}} + \bar{B}}{2\bar{D}} + \frac{\bar{C}^2}{\bar{D}^2}} + \frac{\bar{C}}{\bar{D}} \right).$$

In [7], B. Chen, Z. Shen, and L. Zhao have obtained the formula of the flag curvature and Ricci curvature of Finsler warped product metrics and characterized these metrics to be Einstein. H. Liu and X. Mo have gave a local characterization of the metrics with vanishing Douglas curvature [14]. Then, H. Liu, X. Mo, and H. Zhang have found equations that characterize the metrics of constant flag curvature and have constructed explicitly many new warped product Douglas metrics of constant Ricci [15]. In [10], M. Gabrani, B. Rezaei, and E.S. Sevim have obtained a differential equation which characterizes a Finsler warped product metric with isotropic E-curvature. Moreover, they have characterized the Landsberg Finsler warped product metrics [11]. For more progress on the warped and doubly warped product structure in Finsler geometry, see [2,13,18,19,25].

Throughout this paper, our index conventions are as follows:

$$1 \leq A \leq B \leq \ldots \leq n, \qquad 2 \leq i \leq j \leq \ldots \leq n.$$

Theorem 1 *Let $F = \breve{\alpha}\phi(r,s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Then, F is of isotropic S-curvature if and only if it is of isotropic E-curvature.*

It is known that any spherically symmetric metric is a Finsler warped product metric, [7]. One of the examples of Finsler warped product metrics is given as follows, [16]:

$$F(u,v) = \breve{\alpha}_+ \left[\sqrt{(r^2+s^2)f(r) + k^2r^2f(r)^2s^2} + krf(r)s \right],$$

where $r = u^1$, $s = \frac{v^1}{\breve{\alpha}_+}$ and $\breve{\alpha}_+$ is the standard Euclidean metric on the unit sphere S^{n-1}. One can be easily seen that the metric has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, with $c = \frac{k}{4}\frac{2f(r)+rf'(r)}{[1+k^2r^2f(r)]f(r)}$, where k is a constant. Hence, the example satisfies Theorem 1.

Theorem 2 *Let $F = \breve{\alpha}\phi(r,s)$ be a Douglas warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. If F has isotropic S-curvature with respect to the volume form $\mathrm{d}V$, then either*

1. *F is a Randers warped product (Riemann warped product included), or*
2. *F is a Berwald warped product which can be formulated by*

$$F = \breve{\alpha}\Upsilon[s^2 e^{4(\int \xi(r)\mathrm{d}r)}], \tag{2}$$

where Υ is any differentiable function.

It is known that if F is Riemannian or Berwaldian, then F is a Douglas metric and has a vanishing S-curvature. Also, if the Randers warped product metrics satisfy (32) and (33), then it is obviously seen that it is a Douglas metric of isotropic S-curvature with respect to the volume form $\mathrm{d}V$.

Corollary 1 *Let $F = \breve{\alpha}\phi(r, s)$ be a Douglas Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. If F has vanishing S-curvature with respect to the volume form $\mathrm{d}V$, then F must be Berwaldian.*

2 Preliminaries

Let G^A be the geodesic coefficients of a Finsler metric F on an n-dimensional manifold M, which are defined by

$$G^A := \frac{1}{4}g^{AB}\{[F^2]_{u^C v^B}v^C - [F^2]_{u^B}\},$$

where $g_{AB}(u, v) = \left[\frac{1}{2}F^2\right]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$.

Lemma 1 *The spray coefficients G^A of a warped product metric $F = \breve{\alpha}\phi(r, s)$ are given by [7]*

$$G^1 = \Phi\breve{\alpha}^2, \qquad G^i = \breve{G}^i + \Psi\breve{\alpha}^2\breve{l}^i, \tag{3}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and

$$\begin{cases} \Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \\ \Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_s\omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \end{cases} \tag{4}$$

where $\omega = \phi^2$. Φ and Ψ can be rewritten as follows:

$$\Phi = s\Psi + A, \tag{5}$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A, \tag{6}$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{7}$$

The Berwald curvature $\mathbf{B} = B_{C\ DE}^{\ A}\mathrm{d}u^C \otimes \mathrm{d}u^D \otimes \mathrm{d}u^E \otimes \frac{\partial}{\partial u^A}$ of a Finsler metric F is defined by

$$B_{C\ DE}^{\ A} := \frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E}.$$

F is called a Berwald metric if $B = 0$. Furthermore, F is said to be of isotropic Berwald curvature if its Berwald curvature $B_{C}{}^{A}{}_{DE}$ satisfies

$$B_{C}{}^{A}{}_{DE} = \tau(u)(F_{v^C v^D}\delta^A_E + F_{v^C v^E}\delta^A_D + F_{v^D v^E}\delta^A_C + F_{v^C v^D v^E} v^A),$$

where $\tau(u)$ is a scalar function. H. Liu and X. Mo have characterized the Finsler warped product metrics to be Berwaldian. They have obtained the following lemma:

Lemma 2 [14] *Let $F = \breve{\alpha}\phi(r,s)$ be a warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then, F is a Berwald metric if and only if*

$$\Phi = a(r)s^2 + b(r), \quad \Psi = c(r)s, \tag{8}$$

where $a = a(r)$, $b = b(r)$ and $c = c(r)$ are differentiable functions and Φ and Ψ are defined in (4).

Substituting (5) into (8), we obtain

$$A = m(r)s^2 + b(r), \quad \Psi = c(r)s,$$

where $m(r) = a(r) - c(r)$.

The E-curvature $\mathbf{E} = E_{AB}\mathrm{d}u^A \otimes \mathrm{d}u^B$ of F is defined by

$$E_{AB} := \frac{1}{2}\frac{\partial^2}{\partial v^A \partial v^B}\left(\frac{\partial G^C}{\partial v^C}\right). \tag{9}$$

Moreover, F is said to have isotropic E-curvature if there is a scalar function $\kappa = \kappa(u)$ on M such that

$$\mathbf{E} = \frac{1}{2}(n+1)\kappa F^{-1} h, \tag{10}$$

where h is a family of bilinear forms $h_v = h_{AB}\mathrm{d}u^A \otimes \mathrm{d}u^B$, which are defined by $h_{AB} := F F_{v^A v^B}$.

The well-known non-Riemannian quantity, S-curvature, is given by

$$S(u, v) := \frac{d}{\mathrm{d}t}[\tau(c(t), \dot{c}(t))] \,|_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = u$ and $\dot{c}(0) = v$, [20]. According to the given definition, S-curvature measures the rate of change of the distortion on $(T_u M, F_u)$ in the direction $v \in T_u M$. For Berwald metrics, the S-curvature is zero, [21]. In local coordinates, the S-curvature is defined by

$$\mathbf{S} = \frac{\partial G^C}{\partial v^C} - v^C \frac{\partial}{\partial u^C}\left[\ln \sigma_{BH}\right], \tag{11}$$

where $\mathrm{d}V_F = \sigma_F(u)\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^n$ is the Busemann–Hausdorff volume form. A Finsler metric is of isotropic S-curvature given by

$$\mathbf{S} = (n+1)cF, \tag{12}$$

where $c = c(u)$ is a scalar function on M.

Moreover,

$$\mathbf{D} = D^A_{BCD}\mathrm{d}x^B \otimes \mathrm{d}x^C \otimes \mathrm{d}x^D$$

is a tensor on $TM \setminus \{0\}$ which is called the Douglas tensor, where

$$D^A_{BCD} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^D}\left(G^A - \frac{1}{n+1}\frac{\partial G^C}{\partial v^C}v^A\right). \tag{13}$$

A Finsler metric F is called Douglas metric if $\mathbf{D} = 0$. For a Berwald metrics, the spray coefficients G^A are quadratic in y. It follows that $\mathbf{D} = 0$, (13). The Berwald metrics are Douglas metric. H. Liu and X. Mo have proved that a warped product Finsler metric $F = \breve{\alpha}\phi(r,s)$ is of Douglas type if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r),$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions, [14].

3 *S*-Curvature of Finsler Warped Product Metrics

In this section, we study the S-curvature and isotropic S-curvature of the warped product metrics:

The concept of the S-curvature in Finsler manifold can be determined by the volume form $\mathrm{d}V = \sigma(u)\mathrm{d}u$. Hence, in a local coordinate (u^A, v^A), the Busemann–Hausdorff volume form $\mathrm{d}V_{BH} = \sigma_{BH}(u)\mathrm{d}u$ given as follows:

$$\sigma_{BH}(u) = \frac{Vol(\mathbb{B}^n(1))}{Vol\{(v^A) \in \mathbb{R}^n : F\left(u, v^A \frac{\partial}{\partial u^A}\right) < 1\}},$$

where Vol denotes the Euclidean volume and $\mathbb{B}^n(1)$ is a unit ball in $\mathbb{R}^n$.

For a warped product metric, we have the following lemma:

Lemma 3 *Let* $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ *be a warped product metric on an* n*-dimensional manifold* $M := I \times \breve{M}$. *Then, the Busemann–Hausdorff volume form* $\mathrm{d}V_{BH}$ *on* M *is obtained as follows:*

$$\mathrm{d}V_{BH} = h(r)\mathrm{d}V_\alpha, \tag{14}$$

where

$$h(r) = \frac{\int_0^{\pi} \sin^{n-2}(t)\mathrm{d}t}{r^{n-1}\int_{-\infty}^{\infty} \frac{1}{\phi^n(r,s)}\mathrm{d}s} \tag{15}$$

and

$$\mathrm{d}V_\alpha = r^{n-1}\sigma_{\breve{\alpha}}\mathrm{d}v^1\mathrm{d}\breve{v}^2\ldots\mathrm{d}\breve{v}^n, \quad \sigma_{\breve{\alpha}} = \sqrt{\det(\breve{a}_{ij})}. \tag{16}$$

Proof Note that

$$\int_{-\infty}^{\infty} \frac{1}{(1+u^2)^{\frac{n}{2}}}\mathrm{d}u = \int_0^{\pi} \sin^{n-2}(t)\mathrm{d}t,$$

where $u = \tan t$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{(r^2+s^2)^{\frac{n}{2}}}\mathrm{d}s = \frac{1}{r^{n-1}}\int_0^{\pi} \sin^{n-2}(t)\mathrm{d}t.$$

Let

$$\alpha = \sqrt{(v^1)^2 + r^2\breve{\alpha}^2}, \tag{17}$$

where $\breve{\alpha} = \sqrt{\sum_{i=2}^n(\breve{v}^i)^2}$. Here, we have considered the special coordinate system at $(r,\breve{u}) \in M$ such that the Riemannian metric α is expressed in the form (17), $v = v^1\frac{\partial}{\partial r} + \sum_{i=2}^n \breve{v}^i\frac{\partial}{\partial \breve{u}^i}$, and

$$\begin{aligned} \alpha &= \breve{\alpha}\sqrt{\left(\frac{v^1}{\breve{\alpha}}\right)^2 + r^2} \\ &= \breve{\alpha}\sqrt{s^2+r^2}, \end{aligned} \tag{18}$$

where $s = \frac{v^1}{\breve{\alpha}}$. Consider a warped product metric $F = \breve{\alpha}\phi(r,s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$. The Busemann–Hausdorff volume form $\mathrm{d}V_{BH} = \sigma(r,\breve{u})\,\mathrm{d}v^1\,\mathrm{d}\breve{v}^2\ldots\mathrm{d}\breve{v}^n$ is given by

$$\sigma(r,\breve{u}) = \frac{Vol(\mathbb{B}^n(1))}{Vol(\Omega)},$$

where

$$\Omega = \{(v^1,\breve{v}^2,\ldots,\breve{v}^n) \in \mathbb{R}^n \mid \breve{\alpha}\phi\left(\frac{v^1}{\breve{\alpha}},r\right) < 1\}$$

and $Vol(\Omega) = \int_\Omega dv^1 d\breve{v}^2 \cdots d\breve{v}^n$. Let $\Omega^{'} := \{(s, \breve{v}^2, \ldots, \breve{v}^n) \in \mathbb{R}^n \mid \breve{\alpha}\phi(s,r) < 1\}$. Define

$$\begin{aligned} f : \Omega^{'} &\longrightarrow \Omega \\ (s, \breve{v}^i) &\longrightarrow (v^1, \breve{v}^i), \end{aligned}$$

by

$$\begin{cases} v^1 = s\breve{\alpha}, \\ \breve{v}^i = \breve{v}^i. \end{cases}$$

Then,

$$\begin{aligned} Vol(\Omega) &= \int_\Omega dv^1 d\breve{v}^2 \cdots d\breve{v}^n \\ &= \int_{\Omega'} \left| \frac{\partial f}{\partial(s, \breve{v}^i)} \right| ds\, d\breve{v}^2 \cdots d\breve{v}^n \\ &= \int_{\Omega'} \breve{\alpha}\, ds\, d\breve{v}^2 \cdots d\breve{v}^n. \end{aligned}$$

Note that

$$\frac{\partial f}{\partial(s, \breve{v}^i)} = \begin{pmatrix} \breve{\alpha} & s\frac{v1}{\breve{\alpha}} \\ 0 & \delta^i_j \end{pmatrix}.$$

Thus,

$$\left| \frac{\partial f}{\partial(s, \breve{v}^i)} \right| = \breve{\alpha}.$$

$\Omega^{'}$ is a family of disks. For each $-\infty < s < \infty$, $\breve{v}^i$ is controlled by $\breve{\alpha} < \frac{1}{\phi(r,s)}$.

Let $D_s := \left\{(\breve{v}^2, \ldots \breve{v}^n) \in \mathbb{R}^{n-1} | \breve{\alpha} < \frac{1}{\phi(r,s)}\right\}$. Then,

$$\begin{aligned} Vol(\Omega) &= \int_{\Omega'} \breve{\alpha}\, ds\, d\breve{v}^2 \cdots d\breve{v}^n \\ &= \int_{-\infty}^{\infty} \left[\int_{D_s} \breve{\alpha}\, d\breve{v}^2 \cdots d\breve{v}^n \right] ds \\ &= \int_{-\infty}^{\infty} \int_0^R [\int_{S^{n-2}(t)} t\, dA] dt\, ds \quad (where\ \ R = \frac{1}{\phi(r,s)}) \\ &= \int_{-\infty}^{\infty} \int_0^R t^{n-1} Vol(S^{n-2}(1)) dt\, dA \\ &= \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{R^n}{n} Vol(S^{n-2}(1))\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \frac{Vol(S^{n-2}(1))}{n\,\phi^n(r,s)}\mathrm{d}s \\
&= \frac{Vol(S^{n-2}(1))}{n} \int_{-\infty}^{\infty} \frac{1}{\phi^n(r,s)}\mathrm{d}s.
\end{aligned}
$$

By (18), it follows that

$$
\begin{aligned}
Vol(\Omega) &= \frac{Vol(S^{n-2}(1))}{n} \int_{-\infty}^{\infty} \frac{1}{(s^2+r^2)^{\frac{n}{2}}}\mathrm{d}s \\
&= \frac{Vol(S^{n-2}(1))}{n} \frac{1}{r^{n-1}} \int_0^{\pi} \sin^{n-2}(t)\mathrm{d}t.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathrm{d}V_\alpha &= \frac{Vol(\mathbb{B}^n(1))}{Vol(\Omega)}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n \\
&= r^{n-1}\frac{Vol(\mathbb{B}^n(1))n}{Vol(S^{n-2}(1))\int_0^{\pi}\sin^{n-2}(t)\mathrm{d}t}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n.
\end{aligned}
$$

Note that

$$
\begin{aligned}
Vol(\mathbb{B}^n(1)) &= \int_0^1 Vol(S^{n-1}(t))\mathrm{d}t \\
&= \frac{1}{n}Vol(S^{n-1}(1)) \\
&= \frac{1}{n}\int_0^{\pi}\sin^{n-2}(t)\mathrm{d}t.Vol(S^{n-2}(1)).
\end{aligned}
$$

Then, $\mathrm{d}V_\alpha = r^{n-1}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n$. Go back to $F=\breve{\alpha}\phi(r,s)$, we obtain

$$
\begin{aligned}
\mathrm{d}V_{BH} &= \frac{Vol(\mathbb{B}^n(1))}{Vol(\Omega)}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n \\
&= \frac{Vol(\mathbb{B}^n(1))n}{Vol(S^{n-2}(1))}.\frac{1}{\int_{-\infty}^{\infty}\frac{1}{\phi^n(r,s)}\mathrm{d}s}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n \\
&= \frac{\int_0^{\pi}\sin^{n-2}(t)\mathrm{d}t}{\int_{-\infty}^{\infty}\frac{1}{\phi^n(r,s)}\mathrm{d}s}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n \\
&= \frac{\int_0^{\pi}\sin^{n-2}(t)\mathrm{d}t}{r^{n-1}\int_{-\infty}^{\infty}\frac{1}{\phi^n(r,s)}\mathrm{d}s}\mathrm{d}V_\alpha \\
&= h(r)\mathrm{d}V_\alpha.
\end{aligned}
$$

In the above formula, $\mathrm{d}V_\alpha$ is given in a special coordinate system. Thus for a general base, we have

$$\mathrm{d}V_\alpha = r^{n-1}\sigma_{\breve{\alpha}}\mathrm{d}v^1\mathrm{d}\breve{v}^2\cdots\mathrm{d}\breve{v}^n.$$

□

Now, we prove the following propositions:

Proposition 1 *Let $F = \breve{\alpha}\phi(r, s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Then, the S-curvature of F with respect to the volume form $\mathrm{d}V$ is given by*

$$\mathbf{S} = \breve{\alpha}\left[(n+1)\Psi + A_s + sg(r)\right], \tag{19}$$

where $g(r) := -\frac{f'(r)}{f(r)}$ and Ψ and A are defined by (6) and (7), respectively.

Proof By (3), we have

$$\frac{\partial G^1}{\partial v^1} = \Phi_s\breve{\alpha}, \qquad \frac{\partial G^m}{\partial v^m} = \frac{\partial \breve{G}^m}{\partial v^m} + (n\Psi - s\Psi_s)\breve{\alpha}. \tag{20}$$

By Lemma 3, $\mathrm{d}V = h(r)\sigma_\alpha\mathrm{d}u = r^{n-1}h(r)\sigma_{\breve{\alpha}}\mathrm{d}u = f(r)\sigma_{\breve{\alpha}}\mathrm{d}u$. Thus we have

$$\begin{aligned} v^A\frac{\partial}{\partial u^A}\left[\ln\sigma(u)\right] &= v^1\frac{\partial}{\partial r}\left[\ln f(r) + \ln\sigma_{\breve{\alpha}}\right] + v^m\frac{\partial}{\partial u^m}\left[\ln f(r) + \ln\sigma_{\breve{\alpha}}\right] \\ &= s\breve{\alpha}\frac{f'(r)}{f(r)} + v^m\frac{\partial}{\partial u^m}(\ln\sigma_{\breve{\alpha}}). \end{aligned} \tag{21}$$

Plugging (20) and (21) into (11) yields

$$\mathbf{S} = \breve{\alpha}\left[\Phi_s + (n\Psi - s\Psi_s) - s\frac{f'(r)}{f(r)}\right]. \tag{22}$$

By (5), it is easy to see that (22) is equivalent to (19). □

Proposition 2 *Let $F = \breve{\alpha}\phi(r, s)$, $r = u^1$, $s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Then, F is of isotropic S-curvature if and only if*

$$(n+1)\Psi + A_s + sg(r) = (n+1)c\phi, \tag{23}$$

where $c = c(u)$ and $g(r) := -\frac{f'(r)}{f(r)}$.

Proof By (12) and (19), we complete the proof. □

By using (6) and (7), one can see that (23) is equivalent to the following equation

$$\begin{aligned} &\phi(\phi - s\phi_s)_r\phi_{sss} + \left[(n+1)\phi_s(\phi - s\phi_s)_r + s\phi\phi_{rss}\right]\phi_{ss} \\ &+ \left[(n+1)(s\phi_r - 2c\phi^2) + 2sg(r)\phi\right]\phi_{ss}^2 = 0. \end{aligned}$$

4 *E*-Curvature of Warped Product Metrics

In this section, we characterize warped product metrics with isotropic E-curvature. Throughout this section and the next section, we always assume that the dimension is greater than two.

The following identities are obvious for a warped product metric $F = \breve{\alpha}\phi(r, s)$:

$$\breve{\alpha}_{v^1} = 0, \quad s_{v^1} = \tfrac{1}{\breve{\alpha}}, \quad s_{v^j} = -\tfrac{s\breve{l}_j}{\breve{\alpha}}, \quad \breve{\alpha}^2_{v^j} = 2\breve{\alpha}\breve{l}_j, \quad \breve{l}_m\breve{l}^m = 1, \tag{24}$$

where $\breve{l}_j := \breve{\alpha}_{v^j}$.

The E-curvature of a warped product Finsler metric is computed in [10], and it is given at below:

$$\begin{aligned} E =& E_{AB}\mathrm{d}u^A \otimes \mathrm{d}u^B \\ =& E_{11}\mathrm{d}u^1 \otimes \mathrm{d}u^1 + E_{1j}\mathrm{d}u^1 \otimes \mathrm{d}u^j + E_{i1}\mathrm{d}u^i \otimes \mathrm{d}u^1 + E_{ij}\mathrm{d}u^i \otimes \mathrm{d}u^j, \end{aligned} \tag{25}$$

where

$$\begin{aligned} E_{11} &= \frac{1}{2}\left[\frac{\partial}{\partial v^1 \partial v^1}\left(\frac{\partial G^1}{\partial v^1}\right) + \frac{\partial}{\partial v^1 \partial v^1}\left(\frac{\partial G^m}{\partial v^m}\right)\right] \\ &= \frac{1}{2\breve{\alpha}}\left[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\right], \\ E_{1j} &= \frac{1}{2}\left[\frac{\partial}{\partial v^1 \partial v^j}\left(\frac{\partial G^1}{\partial v^1}\right) + \frac{\partial}{\partial v^1 \partial v^j}\left(\frac{\partial G^m}{\partial v^m}\right)\right] \\ &= \frac{-s}{2\breve{\alpha}}\left[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\right]\breve{l}_j, \\ E_{i1} &= \frac{1}{2}\left[\frac{\partial}{\partial v^i \partial v^1}\left(\frac{\partial G^1}{\partial v^1}\right) + \frac{\partial}{\partial v^i \partial v^1}\left(\frac{\partial G^m}{\partial v^m}\right)\right] \\ &= \frac{-s}{2\breve{\alpha}}\left[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\right]\breve{l}_i, \\ E_{ij} &= \frac{1}{2}\left[\frac{\partial}{\partial v^i \partial v^j}\left(\frac{\partial G^1}{\partial v^1}\right) + \frac{\partial}{\partial v^i \partial v^j}\left(\frac{\partial G^m}{\partial v^m}\right)\right] \\ &= \frac{1}{2\breve{\alpha}}\Big\{s^2\left[(n-2)\Psi_{ss} - s\Psi_{sss} + \Phi_{sss}\right]\breve{l}_i\breve{l}_j \\ &\quad + \Big[n(\Psi - s\Psi_s) + s^2\Psi_{ss} + \Phi_s - s\Phi_{ss}\Big]\breve{h}_{ij}\Big\}, \end{aligned}$$

where $\breve{h}_{ij} := \breve{\alpha}(\breve{l}_i)_{v^j}$.

We have characterized the warped product Finsler metrics of isotropic E-curvature by the following proposition, [10]:

Proposition 3 *The warped product metric $F = \breve{\alpha}\phi(r, s)$ is of isotropic E-curvature if and only if*

$$(n+1)(\Psi - s\Psi_s) + A_s - sA_{ss} = (n+1)\kappa(\phi - s\phi_s), \tag{26}$$

where Ψ and A are defined by (6) and (7), respectively, and $\kappa = \kappa(u)$ is a scalar function on M.

5 Proof of Main Theorems

Proof of Theorem 1 Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Suppose that F is of isotropic S-curvature. That is, (23) holds. Differentiating (23) with respect to the variable s, we have

$$(n+1)\Psi_s + A_{ss} + g(r) = (n+1)c\phi_s. \tag{27}$$

(23)- $s\times$ (27) yields (26). Hence, F is of isotropic E-curvature.

Conversely, suppose that F is of isotropic E-curvature. Then, (26) holds. Let $\Psi = s\bar{\Psi}$, $A_s = s\bar{A}_s$, and $\phi = s\bar{\phi}$. Then, we have

$$\Psi - s\Psi_s = -s^2\bar{\Psi}_s, \quad A_s - sA_{ss} = -s^2\bar{A}_{ss}, \tag{28}$$

and

$$\phi - s\phi_s = -s^2\bar{\phi}_s. \tag{29}$$

Plugging (28) and (29) into (26), we obtain (for $s \neq 0$)

$$(n+1)\bar{\Psi}_s + \bar{A}_{ss} - (n+1)\kappa\bar{\phi}_s = 0. \tag{30}$$

Integrating (30) with respect to s yields

$$(n+1)\bar{\Psi} + \bar{A}_s - (n+1)\kappa\bar{\phi} + \gamma(r) = 0,$$

where $\gamma(r)$ is an integration constant. Thus,

$$(n+1)\Psi + A_s - (n+1)\kappa\phi + s\gamma(r) = 0.$$

Take $\kappa(u) = c(u)$, $\gamma(r) = g(r)$, we obtain (23). Therefore, F is of isotropic S-curvature. □

To prove Theorem 2, we first prove the following proposition:

Proposition 4 *Let $F = \breve{\alpha}\phi(r,s)$ be a Finsler warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. If F is a Douglas metric and has isotropic E-curvature, then either*

1. *F is a Randers warped product (Riemann warped product included), or*

2. *F is a Berwald warped product which can be formulated by*

$$F = \breve{\alpha}\Upsilon[s^2 e^{4(\int \xi(r)\mathrm{d}r)}], \tag{31}$$

where Υ is any differentiable function.

To prove Proposition 4, we need the following lemma:

Lemma 4 [11] *Let $F = \breve{\alpha}\phi(r, s)$ be a Douglas warped product metric, where $r = u^1$ and $s = \frac{v^1}{\breve{\alpha}}$. Then, F has isotropic E-curvature if and only if*

$$\Psi = \kappa(u)\phi + s\mathrm{d}(r), \tag{32}$$

$$A = \xi(r)s^2 + \eta(r), \tag{33}$$

where Ψ and A are defined by (6) and (7), respectively.

Proof Let $F = \breve{\alpha}\phi(r, s)$ be a Douglas Finsler warped product metric with isotropic E-curvature. By [14, Lemma 3.3] and (5), F has vanishing Douglas curvature if and only if

$$A = \xi(r)s^2 + \eta(r). \tag{34}$$

By Proposition 3, F is of isotropic E-curvature if and only if

$$(n+1)(\Psi - s\Psi_s) + A_s - sA_{ss} = (n+1)\kappa(\phi - s\phi_s). \tag{35}$$

Plugging (34) into (35), we get

$$\Psi - s\Psi_s = \kappa(\phi - s\phi_s). \tag{36}$$

Thus, there exists a C^∞ function $\mathrm{d} = \mathrm{d}(r)$ such that

$$\Psi = \kappa\phi + s\mathrm{d}(r). \tag{37}$$

Conversely, suppose that (34) and (37) hold. By (34) and (37), (35) holds. Hence, we obtain that F is a Douglas metric with isotropic E-curvature. □

Proof of Proposition 4 Suppose that (32) and (33) hold. Using (6) and (32), it yields

$$\frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A = \kappa\phi + s\mathrm{d}(r).$$

Plugging (33) into the above equation, it yields

$$2[\xi(r)s^2 + \eta(r)]\phi_s - s\phi_r + 2s\mathrm{d}(r)\phi + 2\kappa\phi^2 = 0. \tag{38}$$

Differentiating (38) with respect to the variable s, we get

$$\begin{aligned}&2[\xi(r)s^2+\eta(r)]\phi_{ss}-s\phi_{rs}-\phi_r+2\mathrm{d}(r)\phi+2s[2\xi(r)+\mathrm{d}(r)]\phi_s\\&\quad+4\kappa\phi\phi_s=0.\end{aligned} \tag{39}$$

On the other hand, by (7) and (33), we have

$$\xi(r)s^2+\eta(r)=\frac{s\phi_{rs}-\phi_r}{2\,\phi_{ss}}. \tag{40}$$

From (40), it follows that

$$2[\xi(r)s^2+\eta(r)]\phi_{ss}-s\phi_{rs}+\phi_r=0. \tag{41}$$

By (39)–(41), we have

$$s[2\xi(r)+\mathrm{d}(r)]\phi_s-\phi_r+\mathrm{d}(r)\phi+2\kappa\phi\phi_s=0. \tag{42}$$

By (42)$\times s$–(38), it follows that

$$[\mathrm{d}(r)s^2+2\kappa s\phi-2\eta(r)]\phi_s=\mathrm{d}(r)s\phi+2\kappa\phi^2. \tag{43}$$

Hence, (43) can be written as follows:

$$\left[(ds^2-2\eta)\left(\frac{1}{\phi^2}\right)\right]_s+\left[4\kappa s\left(\frac{1}{\phi}\right)\right]_s=0. \tag{44}$$

(1) If $ds^2-2\eta\neq 0$, then the solution of (44) is given by [9, Theorem 4.2]

$$\phi=\frac{2\kappa s+\sqrt{(4\kappa^2+\sigma\mathrm{d}(r))s^2-\sigma\eta(r)}}{\sigma}.$$

It follows that

$$F=\frac{2\kappa v^1+\sqrt{(4\kappa^2+\sigma\mathrm{d}(r))(v^1)^2-\sigma\eta(r)\breve{\alpha}^2}}{\sigma}. \tag{45}$$

We define the metric $\alpha=\sqrt{(4\kappa^2+\sigma\mathrm{d}(r))(v^1)^2-\sigma\eta(r)\breve{\alpha}^2}/\sigma$ and 1-form $\beta=\frac{2\kappa v^1}{\sigma}$ on $M:=I\times\breve{M}$. Therefore, F is a Randers warped product metric. In this case, when $\kappa=0$, then the metric F in (45) becomes a Riemannian warped product metric.

(2) If $ds^2-2\eta=0$ and $\kappa\neq 0$, then integrating the equation (44) concludes that

$$4\kappa s\left(\frac{1}{\phi}\right)+t(r)=0,$$

where $t(r)$ is a positive smooth function. We omit this, since the corresponding warped product metric is a singular Kropina metric.

(3) If $ds^2 - 2\eta = 0$ and $\kappa = 0$, note that $\phi > 0$ and $s \neq 0$. By (43), we have

$$\mathrm{d}(r) = 0, \quad \mathrm{d}(r)s^2 - 2\eta(r) = 0. \tag{46}$$

By (46), it follows that

$$\mathrm{d}(r) = 0, \quad \eta(r) = 0. \tag{47}$$

Plugging (47) into (42), it yields

$$2s\xi(r)\phi_s - \phi_r = 0. \tag{48}$$

In this case, we only solve (48). The characteristic equation of (48) is

$$\frac{\mathrm{d}r}{-1} = \frac{\mathrm{d}s}{2s\xi(r)},$$

which is equivalent to

$$\frac{\mathrm{d}s}{\mathrm{d}r} = -2s\xi(r).$$

Hence, the solution of (48) is

$$\phi = \Upsilon[s^2 e^{4(\int \xi(r)\mathrm{d}r)}],$$

where $\Upsilon(.)$ is a differentiable function [23, Lemma 4.1]. □

Proof of Theorem 2 Theorem 1 and Proposition 4 yield the proof of Theorem 2. □

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