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On the σ-Nilpotent Norm and the σ-Nilpotent Hypernorm of a Finite Group

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Abstract

A subgroup $\mathcal{N}_\sigma(G)$ of a finite group G is the σ-nilpotent norm of G if $\mathcal{N}_\sigma(G)$ is the intersection of the normalizers of the σ-nilpotent residuals of all subgroups of G. Let $\mathcal{N}_\sigma^0(G) = 1$ and define $\mathcal{N}_\sigma^{i+1}(G)/\mathcal{N}_\sigma^i(G) = \mathcal{N}_\sigma(G/\mathcal{N}_\sigma^i(G))$ for $i = 0, 1, 2, \ldots$. By $\mathcal{N}_\sigma^\infty(G)$ denote the terminal term of the ascending series and say that $\mathcal{N}_\sigma^\infty(G)$ is the σ-nilpotent hypernorm of G. We study the influence of the σ-nilpotent norm and σ-nilpotent hypernorm of G on the structure of a finite group G. In particular, we proved that $G = \mathcal{N}_\sigma^\infty(G)$ if and only if $G^{\mathfrak{N}_\sigma}$ is nilpotent, where $G^{\mathfrak{N}_\sigma}$ is the σ-nilpotent residual of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. The notation and terminology used in this paper are standard as in [8,12].

Baer in [1] considered the intersection of normalizers of all subgroups of G, which is called the norm of G, and denoted by $\mathcal{N}(G)$. Much investigation has focused on using the concepts of the norm to determine the structure of groups (see, for example, [1–4,22]).

Recall that a class of groups $\mathfrak{F}$ is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. If $\mathfrak{F}$ is a non-empty formation, then the $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup N of G with $G/N \in \mathfrak{F}$. Li and Shen in [21] considered $D(G)$, where $D(G)$ is the intersection of the normalizers of derived subgroups of all subgroups of G. Shen et al. in [24] introduced $S(G)$, which is the intersection of the normalizers of nilpotent residuals of all subgroups of G. Recall that a group G is called p-decomposable if there exists a subgroup H of G such that $G = P \times H$ for the Sylow p-subgroup P of G. In 2020, Fu, Shen and Yan [9] introduced $\mathcal{N}^{\mathcal{D}_p}(G)$, which is the intersection of the normalizers of p-decomposable residuals of all subgroups of G.

In recent years, a new theory of σ-groups has been established by Skiba and Guo (See [13,14,25–27]).

In fact, following Shemetkov [23], $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [25,26], G is said to be: σ-primary if $|\sigma(G)| \leq 1$; σ-soluble if every chief factor of G is σ-primary; σ-nilpotent if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \cdots, G_n$. Clearly, a σ-nilpotent group is σ-soluble. We use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the class of all σ-soluble groups and the class of all σ-nilpotent groups, respectively. If $G \notin \mathfrak{N}_\sigma$ but every proper subgroup of G belongs to $\mathfrak{N}_\sigma$, then G is called an $\mathfrak{N}_\sigma$-critical or a minimal non-σ-nilpotent group.

Remark 1.1 When $\sigma = \sigma^1 = \{\{2\}, \{3\}, \cdots\}$ (we use here the notation in [26]), then σ-soluble groups and σ-nilpotent groups are just soluble groups and nilpotent groups respectively, and an $\mathfrak{N}_{\sigma^1}$-critical group G (that is, G is not nilpotent but every proper subgroup of G is nilpotent) is a

Schmidt group. Let $\sigma = \{\{p\}, p'\}$, then σ-soluble groups and σ-nilpotent groups are just p-soluble groups and p-decomposable groups, respectively. For the set $\pi = \{p_1, \ldots, p_n\}$ of primes, we deal with the partition $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$ [26]. Then G is: $\sigma^{1\pi}$-soluble if and only if G is π-soluble; $\sigma^{1\pi}$-nilpotent if and only if G is π-special [7], that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$.

This new theory of σ-groups is actually the development and popularization of the famous Sylow theorem, the Hall theorem of the soluble groups and the Chunihin theorem of π-soluble groups. A series of studies have been caused (See, for example, [13,15,18,25,26,30–32]).

In 2021, Hu et al. [18] introduced the notion of σ-nilpotent norm as follows:

Definition 1.2 ([18, Definition 1.2]) A subgroup $\mathcal{N}_\sigma(G)$ of G is called the σ-nilpotent norm of G if $\mathcal{N}_\sigma(G)$ is the intersection of the normalizers of the σ-nilpotent residuals of all subgroups of G, that is,

$$\mathcal{N}_\sigma(G) = \bigcap_{H \leq G} N_G(H^{\mathfrak{N}_\sigma}).$$

Now let $\mathcal{N}_\sigma^0(G) = 1$ and define $\mathcal{N}_\sigma^{i+1}(G)/\mathcal{N}_\sigma^i(G) = \mathcal{N}_\sigma(G/\mathcal{N}_\sigma^i(G))$ for $i = 0, 1, 2, \ldots$. Then, there exists a series of normal subgroups of G:

$$1 = \mathcal{N}_\sigma^0(G) \leq \mathcal{N}_\sigma^1(G) \leq \mathcal{N}_\sigma^2(G) \leq \cdots \leq \mathcal{N}_\sigma^n(G) = \mathcal{N}_\sigma^{n+1}(G) = \cdots.$$

Denote by $\mathcal{N}_\sigma^\infty(G)$ the terminal term of this ascending series and say that $\mathcal{N}_\sigma^\infty(G)$ is the σ-nilpotent hypernorm of G.

Remark 1.3 The σ-nilpotent norm $\mathcal{N}_\sigma(G)$ of G covers many possible definitions. For example, in one case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$, the σ-nilpotent norm $\mathcal{N}_\sigma(G)$ is $S(G)$, that is, the intersection of the normalizers of nilpotent residuals of all subgroups in G.

In the case when $\sigma = \{\{p\}, p'\}$, the σ-nilpotent norm $\mathcal{N}_\sigma(G)$ is $N^{\mathcal{D}_p}(G)$, that is, the intersection of the normalizers of p-decomposable residuals of all subgroups of G.

In the other case when $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$, the σ-nilpotent norm is the π-special norm, that is, the intersection of the normalizers of π-special residuals of all subgroups of G.

In [18], the authors studied the relationship of the σ-nilpotent length with the σ-nilpotent norm of G, and get some important results. In this paper, we continue to study the influence of the σ-nilpotent norm and σ-nilpotent hypernorm of G on the structure of G.

For any σ-nilpotent group G, it is easy to see that $\mathcal{N}_\sigma(G) = \mathcal{N}_\sigma^\infty(G) = G$. If $\sigma = \{\{2\}, \{2\}'\}$ and $G = S_4$, the symmetry group of degree four. Then $\mathcal{N}_\sigma(G) = \mathcal{N}_\sigma^\infty(G) = 1$.

Motivated by the above observations, the following question naturally arise:

Question 1.4 *What is the structure of G under the condition that* $\mathcal{N}_\sigma^\infty(G) = G$?

We use $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ to denote the class of all finite group G with $G^{\mathfrak{N}_\sigma}$ nilpotent.

In this paper, we give the affirmative answer to this above problem and get the following theorem:

Theorem 1.5 *Let G be a finite group. Then, the following statements are equivalent:*

(i) $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$;
(ii) $G/\mathcal{N}_\sigma^\infty(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$;
(iii) $G = \mathcal{N}_\sigma^\infty(G)$;
(iv) $\mathcal{N}_\sigma(G/N) > 1$ *for any proper normal subgroup* N *of* G.

For a formation $\mathfrak{X}$, G is called an $\mathfrak{X}$-group if $G \in \mathfrak{X}$.

Remark 1.6 It follows from the Theorem 1.5 that G is an $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-group if and only if $\mathcal{N}_\sigma^\infty(G) = G$.

In [24], Shen, Shi and Qian introduced the definition of S-groups:

Definition 1.7 A group G is called an S-group if $G = S(G)$, that is, the nilpotent residuals of all subgroups of G are normal.

The authors in [24] discussed some characters and the structure of S-groups. Recently, the S-groups are also studied by Guo and Gong, see [10,16].

Here, we generalize the definition of S-groups and give the following definition:

Definition 1.8 A group G is called an N_σ-group if $G = \mathcal{N}_\sigma(G)$, that is, the σ-nilpotent residuals of all subgroups of G are normal.

In this paper, we would also study the properties and structure of N_σ-groups.

The paper is organized as follows. In Sect. 2, we prove some basic properties of the subgroups $\mathcal{N}_\sigma(G)$ and $\mathcal{N}_\sigma^\infty(G)$. In Sect.. 3, we give the structure of G under the condition that $\mathcal{N}_\sigma^\infty(G) = G$ and prove Theorem 1.5. In Sect. 4, we study the properties and structure of N_σ-groups.

2 Preliminaries

In this section, we give some basic properties of the subgroups $\mathcal{N}_\sigma(G)$ and $\mathcal{N}_\sigma^\infty(G)$.

Lemma 2.1 *(*See [18, Proposition 2.5] and [6, Lemma 2.4]*)*

(1) *If* $M \le G$*, then* $M \cap \mathcal{N}_\sigma(G) \le \mathcal{N}_\sigma(M)$ *and so* $M \cap \mathcal{N}_\sigma^\infty(G) \le \mathcal{N}_\sigma^\infty(M)$.
(2) *If* $N \trianglelefteq G$*, then* $\mathcal{N}_\sigma(G)N/N \le \mathcal{N}_\sigma(G/N)$ *and so* $\mathcal{N}_\sigma^\infty(G)N/N \le \mathcal{N}_\sigma^\infty(G/N)$.
(3) *If* $N \trianglelefteq G$ *and* $N \le \mathcal{N}_\sigma^\infty(G)$*, then* $\mathcal{N}_\sigma^\infty(G/N) = \mathcal{N}_\sigma^\infty(G)/N$.

Lemma 2.2 (1) *The class* $\mathfrak{N}_\sigma$ *of all* σ*-nilpotent groups is closed under taking direct products, homomorphic images and subgroups. Moreover, if* H *is a normal subgroup of* G *and* $H/H \cap \Phi(G)$ *is* σ*-nilpotent, then* H *is* σ*-nilpotent. [25, Lemma 2.5]*

(2) *The class* $\mathfrak{S}_\sigma$ *of all* σ*-soluble groups is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the* σ*-soluble group by a* σ*-soluble group is a* σ*-soluble group as well. [28, Lemma 2.1]*

Remark 2.3 From the above Lemma 2.2(1), we know that $\mathfrak{N}_\sigma$ is a subgroup closed saturated formation.

Recall that $\Phi(G)$ is the intersection of all maximal subgroups of G and $F(G)$ is the Fitting subgroup of G, that is the maximal normal nilpotent subgroup of G.

Lemma 2.4 (See [23, Ch. V, Theorem 26.1]) *If G is a Schmidt group, then $G = P \rtimes Q$, where $P = G^{\mathfrak{N}} = G'$ is a Sylow p-subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q-subgroup of G with $\langle x^q \rangle \leq Z(G) \cap \Phi(G)$. Hence $Q^G = G$.*

Lemma 2.5 (See [5, Theorem 1.2]) *If G is an $\mathfrak{N}_\sigma$-critical group, then G is a Schmidt group.*

Proposition 2.6 *If $G = \mathcal{N}_\sigma(G)$, then $G^{\mathfrak{N}_\sigma}$ is nilpotent.*

Proof Suppose it is false and let G be a counterexample of minimal order. Then:

(1) For every proper subgroup H of G, $H^{\mathfrak{N}_\sigma}$ is nilpotent and so $H^{\mathfrak{N}_\sigma} \leq F(G)$.
For every proper subgroup H of G, $H = H \cap \mathcal{N}_\sigma(G) \leq \mathcal{N}_\sigma(H)$ by the hypothesis and Lemma 2.1(1) and so $H = \mathcal{N}_\sigma(H)$. Hence, H satisfies the hypothesis of the proposition. The choice of G implies that $H^{\mathfrak{N}_\sigma}$ is nilpotent. From the definition of $\mathcal{N}_\sigma(G)$, $H^{\mathfrak{N}_\sigma}$ is normal in $G = \mathcal{N}_\sigma(G)$. Hence $H^{\mathfrak{N}_\sigma} \leq F(G)$. Therefore Claim (1) holds.

(2) $\Phi(G) = 1$
If $\Phi(G) \neq 1$, then $G/\Phi(G) = \mathcal{N}_\sigma(G)\Phi(G)/\Phi(G) \leq \mathcal{N}_\sigma(G/\Phi(G))$ by Lemma 2.1(2). So $G/\Phi(G)$ satisfies the hypothesis of the proposition. The choice of G implies that $(G/\Phi(G))^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}\Phi(G)/\Phi(G) \cong G^{\mathfrak{N}_\sigma}/G^{\mathfrak{N}_\sigma} \cap \Phi(G)$ is nilpotent. Then $G^{\mathfrak{N}_\sigma}$ is nilpotent, a contradiction. Therefore Claim (2) holds.

(3) $F(G) > 1$.
Assume that $F(G) = 1$. Then $H^{\mathfrak{N}_\sigma} = 1$ for every proper subgroup H of G by Claim (1). This shows that every proper subgroup of G is σ-nilpotent. Therefore G is either a σ-nilpotent group or an $\mathfrak{N}_\sigma$-critical group. If G is a σ-nilpotent group, then $G^{\mathfrak{N}_\sigma} = 1$, a contradiction. Assume that G is an $\mathfrak{N}_\sigma$-critical group. Then by Lemma 2.5, G is a Schmidt group. It follows from Lemma 2.4 that $G^{\mathfrak{N}} = G^{\mathfrak{N}_\sigma} \leq G' < G$. Hence $G^{\mathfrak{N}_\sigma}$ is nilpotent. This contradiction shows that Claim (3) holds.

(4) The finial contradiction.

By Claim (3), there exists a minimal normal subgroup N of G such that $N \leq F(G)$. Then N is an elementary abelian group. By Claim (2), there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Since $G/NM^{\mathfrak{N}_\sigma} = MNM^{\mathfrak{N}_\sigma}/NM^{\mathfrak{N}_\sigma} \simeq M/M^{\mathfrak{N}_\sigma} \in \mathfrak{N}_\sigma$, $G^{\mathfrak{N}_\sigma} \leq NM^{\mathfrak{N}_\sigma}$. By Claim (1), $M^{\mathfrak{N}_\sigma} \leq F(G)$. Hence $G^{\mathfrak{N}_\sigma} \leq F(G)$. The final contradiction completes the proof. □

Proposition 2.7 *For any group G, the subgroup $\mathcal{N}_\sigma^\infty(G)$ of G is σ-soluble.*

Proof In view of Lemma 2.2(2), we only need to show that $\mathcal{N}_\sigma(G)$ is σ-soluble. Write $X = \mathcal{N}_\sigma(G)$. Let M be a maximal subgroup of X. If $M^{\mathfrak{N}_\sigma} > 1$, then $M^{\mathfrak{N}_\sigma}$ is normal in X. By Lemmas 2.1(1) and 2.1(2) and induction, $X/M^{\mathfrak{N}_\sigma}$ and $M^{\mathfrak{N}_\sigma}$ are σ-soluble. Therefore X is σ-soluble by Lemma 2.2(2). If $M^{\mathfrak{N}_\sigma} = 1$ for all maximal subgroups of X, then X is either a σ-nilpotent group or an $\mathfrak{N}_\sigma$-critical group. Therefore by Lemmas 2.4 and 2.5, X is σ-soluble. The proof is completed. □

Theorem 2.8 *Suppose that G has no σ-primary Schmit subgroups. Then $G^{\mathfrak{N}_\sigma} \cap \mathcal{N}_\sigma(G) = 1$ if and only if $Z(G^{\mathfrak{N}_\sigma}) = 1$.*

Proof Assume that $G^{\mathfrak{N}_\sigma} \cap \mathcal{N}_\sigma(G) = 1$. By [9, Lemma 2.1] and Lemma 2.2, we have $C_G(G^{\mathfrak{N}_\sigma}) \le \mathcal{N}_\sigma(G)$. It follows that $Z(G^{\mathfrak{N}_\sigma}) = C_G(G^{\mathfrak{N}_\sigma}) \cap G^{\mathfrak{N}_\sigma} \le \mathcal{N}_\sigma(G) \cap G^{\mathfrak{N}_\sigma} = 1$. Hence, the necessity holds.

Next we prove the sufficiency, that is, if $Z(G^{\mathfrak{N}_\sigma}) = 1$, then $G^{\mathfrak{N}_\sigma} \cap \mathcal{N}_\sigma(G) = 1$.

Suppose that $Z(G^{\mathfrak{N}_\sigma}) = 1$ and $G^{\mathfrak{N}_\sigma} \cap \mathcal{N}_\sigma(G) > 1$. Hence there exists a minimal normal subgroup N of G such that $N \le G^{\mathfrak{N}_\sigma} \cap \mathcal{N}_\sigma(G)$. Since $\mathcal{N}_\sigma(G)$ is σ-soluble by Proposition 2.7, N is a σ_i-group for some i. Set $C = C_G(N)$.

We show that the factor G/C is a σ-nilpotent group. Suppose it is false. Then, there exists a minimal non-σ-nilpotent subgroup K/C of G/C, that is, K/C is not σ-nilpotent but all of whose proper subgroups are σ-nilpotent. Choose a subgroup L of K such that $K = CL$ and $C \cap L \le \Phi(L)$. Then by Remark 2.3, $L/\Phi(L)$ is a minimal non-σ-nilpotent subgroup. By Lemmas 2.4 and 2.5, $L/\Phi(L) = (Q/\Phi(L))(P/\Phi(L))$, where $Q/\Phi(L)$ is a normal Sylow q-subgroup of $L/\Phi(L)$ and $P/\Phi(L)$ is a cyclic Sylow p-subgroup of $L/\Phi(L)$. Since $L/\Phi(L)$ is not σ-nilpotent, clearly p and q belong to different partitions, that is, $p \in \sigma_i$ and $q \in \sigma_j$, where $\sigma_i \cap \sigma_j = \emptyset$. By Lemma 2.4, we also know that $(L/\Phi(L))^{\mathfrak{N}_\sigma} = L^{\mathfrak{N}_\sigma}\Phi(L)/\Phi(L)$ is a q-subgroup, so $L^{\mathfrak{N}_\sigma}$ is nilpotent. Take the Sylow q-subgroup Q_0 of $L^{\mathfrak{N}_\sigma}$. Then Q_0 is characterize in $L^{\mathfrak{N}_\sigma}$. By definition of $\mathcal{N}_\sigma(G)$, $\mathcal{N}_\sigma(G)$ normalizers $L^{\mathfrak{N}_\sigma}$. Hence $\mathcal{N}_\sigma(G)$ normalizers Q_0. If $N \cap Q_0 = 1$, then $Q_0 \le C \cap L \le \Phi(L)$. It follows that $(L/\Phi(L))^{\mathfrak{N}_\sigma} = 1$. Hence $L/\Phi(L)$ is σ-nilpotent, a contradiction. Assume $N \cap Q_0 \ne 1$. It follows from N is a minimal normal subgroup of G and $N \cap Q_0$ is a normal q-subgroup of N that N is an elementary abelian q-subgroup of G. Now we show that G/C is p-nilpotent for every prime $p \ne q$. Assume that it is false, then there exists a minimal non-p-nilpotent subgroup H/C of G/C. Choose a subgroup M of H such that $H = CM$ and $C \cap M \le \Phi(M)$. Then $M/\Phi(M)$ is a minimal-non-p-nilpotent subgroup. In the light of a theorem of Ito [19, Chapert IV, Theorem 5.4], $M/\Phi(M) = (P^*/\Phi(M))(R/\Phi(M))$ is a Schmit group of order $p^m r^n$, where $P^*/\Phi(M)$ is a normal Sylow p-subgroup of $M/\Phi(M)$ and $R/\Phi(M)$ is a cyclic Sylow r-subgroup of $M/\Phi(M)$, $r \ne p$ is a prime, $m, n \ge 1$. It is clear that p and r belong to different partitions. In fact, if $p, r \in \sigma_j$ for some j. Then $M/\Phi(M)$ is a σ-primary Schmit group. Since $M/\Phi(M)$ is a σ_j-group, M is a σ_j-group. Clearly M is not nilpotent, so there exists a Schmit subgroup S of M. Hence, S is a σ-primary Schmit group of G, which contradicts the hypothesis. Therefore, p and r belong to different partitions. Then $(M/\Phi(M))^{\mathfrak{N}_\sigma} = (M/\Phi(M))^{\mathfrak{N}} = P^*/\Phi(M)$. Now as $M^{\mathfrak{N}_\sigma}\Phi(M)/\Phi(M) = (M/\Phi(M))^{\mathfrak{N}_\sigma} = (M/\Phi(M))^{\mathfrak{N}} = M^{\mathfrak{N}}\Phi(M)/\Phi(M)$ is a p-subgroup, $M^{\mathfrak{N}_\sigma}$ is nilpotent. Take the Sylow p-subgroup P_0 of $M^{\mathfrak{N}_\sigma}$. Then P_0 is characterize in $M^{\mathfrak{N}_\sigma}$. By definition of $\mathcal{N}_\sigma(G)$, N normalizes $M^{\mathfrak{N}_\sigma}$, and so N normalizes P_0. Since $N \cap P_0 = 1$, we have that $P_0 \le C \cap M \le \Phi(M)$. It follows that $(M/\Phi(M))^{\mathfrak{N}} = 1$. Hence $M/\Phi(M)$ is nilpotent, a contradiction. Therefore G/C is p-nilpotent for every prime $p \ne q$. We now show that G/C is q-closed. Let $H(p)/C$ be the normal Hall p'-subgroup of G/C for every prime p with $p \ne q$. Write $T/C = \bigcap_{p \ne q} H(p)/C$. It is clear that T/C is a normal Sylow q-subgroup of G/C. Therefore G/C is q-closed. It follows that G/T is a q'-group. Then by [23, Chapter III, Lemma 11.2], there exists a q'-subgroup W of G such that $G = TW$. Since

$[N, W^{\mathfrak{N}_\sigma}] \subseteq N \cap W^{\mathfrak{N}_\sigma} = 1$, $W^{\mathfrak{N}_\sigma} \leq C$. It follows that G/T is σ-nilpotent. Thus $G^{\mathfrak{N}_\sigma} \leq T$. Take Q_1 is a Sylow q-subgroup of T containing N. Then $N \cap Z(Q_1) \neq 1$. Hence, there exists an element x such that $1 \neq x \in N \cap Z(Q_1)$. Now $C \leq C_G(x)$ and $Q_1 \leq C_G(x)$. Since $T = CQ_1$, we have that $T \leq C_G(x)$. Hence $G^{\mathfrak{N}_\sigma} \leq T \leq C_G(x)$ and $x \in C_G(G^{\mathfrak{N}_\sigma}) \cap N \leq C_G(G^{\mathfrak{N}_\sigma}) \cap G^{\mathfrak{N}_\sigma} = Z(G^{\mathfrak{N}_\sigma})$, contrary to $Z(G^{\mathfrak{N}_\sigma}) = 1$. This contradiction shows that G/C is a σ-nilpotent group.

Now G/C is σ-nilpotent, that is, $(G/C)^{\mathfrak{N}_\sigma} = 1$. Hence, $G^{\mathfrak{N}_\sigma} \leq C = C_G(N)$. Consequently $N \leq C_G(G^{\mathfrak{N}_\sigma}) \cap G^{\mathfrak{N}_\sigma} = Z(G^{\mathfrak{N}_\sigma})$, contrary to $Z(G^{\mathfrak{N}_\sigma}) = 1$. The proof is completed. □

Remark 2.9 We do not know whether the result of Theorem 2.8 is still true if we remove the hypothesis that G has no σ-primary Schmit subgroups. Note also that when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \cdots\}$, then clearly G has no σ-primary Schmit subgroups and so the hypothesis of Theorem 2.8 natural holds. Hence our Theorem 2.8 covers the Theorem 2.5 in [24].

3 The Proof of Theorem 1.5

Let $\mathfrak{M}$ and $\mathfrak{H}$ be non-empty formations, then the *Gaschütz product* $\mathfrak{M} \circ \mathfrak{H}$ of $\mathfrak{M}$ and $\mathfrak{H}$ is defined as follows: $G \in \mathfrak{M} \circ \mathfrak{H}$ if and only if $G^{\mathfrak{H}} \in \mathfrak{M}$. In particular, $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma} = \mathfrak{N} \circ \mathfrak{N}_\sigma$ is the class of finite groups G with $G^{\mathfrak{N}_\sigma}$ nilpotent.

In order to prove Theorem 1.5, we first need to prove the following two propositions.

Proposition 3.1 *(1) $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ is a formation;*

(2) $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ is subgroup closed, that is , if $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ and $H \leq G$, then $H \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$.

(3) $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ if and only if $G/\Phi(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$.

Proof (1) It follows that [8, IV, Theorem 1.8(a)].

(2) It is well known that the class $\mathfrak{N}$ of all nilpotent groups is subgroup closed. By Lemma 2.2, we also know that $\mathfrak{N}_\sigma$ is subgroup closed. Now we only need to prove the more general result: If two formations $\mathfrak{M}$ and $\mathfrak{H}$ are subgroup closed, then $\mathfrak{M} \circ \mathfrak{H}$ is subgroup closed. In fact, if $G \in \mathfrak{M} \circ \mathfrak{H}$ and $H \leq G$, then $G^{\mathfrak{H}} \in \mathfrak{M}$. Since $\mathfrak{H}$ is subgroup closed and $G/G^{\mathfrak{H}} \in \mathfrak{H}$, we have that $HG^{\mathfrak{H}}/G^{\mathfrak{H}} \in \mathfrak{H}$. By the isomorphism $HG^{\mathfrak{H}}/G^{\mathfrak{H}} \simeq H/H \cap G^{\mathfrak{H}}$, we see that $H/H \cap G^{\mathfrak{H}} \in \mathfrak{H}$, and so $H^{\mathfrak{H}} \leq G^{\mathfrak{H}} \in \mathfrak{M}$. But as $\mathfrak{M}$ is subgroup closed, we obtain that $H^{\mathfrak{H}} \in \mathfrak{M}$. Hence $H \in \mathfrak{M} \circ \mathfrak{H}$.

(3) It follows from [11, Theorems 3.1.11 and 3.1.20].

□

Remark 3.2 The Proposition 3.1 shows that $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ is a subgroup closed saturated formation.

Proposition 3.3 *The following statements are equivalent:*

(1) $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$

(2) $G/\mathcal{N}_\sigma(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$

Proof (1) $\Longrightarrow$ (2):

It follows from Proposition 3.1(1).

(2) $\Longrightarrow$ (1):

Assume that this is false and let G be a counterexample of minimal order. We proceed via the following steps.

(1) For any proper subgroup L of G, $L^{\mathfrak{N}_\sigma}$ is nilpotent.
Since $G/\mathcal{N}_\sigma(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, we have that $L\mathcal{N}_\sigma(G)/\mathcal{N}_\sigma(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ by Proposition 3.1(2). It follows that $L\mathcal{N}_\sigma(G)/\mathcal{N}_\sigma(G) \cong L/(\mathcal{N}_\sigma(G) \cap L)$ that $L/(\mathcal{N}_\sigma(G) \cap L) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$. By Lemma 2.1(1), $\mathcal{N}_\sigma(G) \cap L \leq \mathcal{N}_\sigma(L)$. So $L/\mathcal{N}_\sigma(L) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$. Hence $L \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ by the minimal choice of G.

(2) For any nontrivial normal subgroup K of G, $(G/K)^{\mathfrak{N}_\sigma}$ is nilpotent.
Since $G/\mathcal{N}_\sigma(G)K \cong (G/\mathcal{N}_\sigma(G))/(\mathcal{N}_\sigma(G)K/\mathcal{N}_\sigma(G))$ and $G/\mathcal{N}_\sigma(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, we have that $G/\mathcal{N}_\sigma(G)K \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$. By Lemma 2.1(2), $\mathcal{N}_\sigma(G)K/K \leq \mathcal{N}_\sigma(G/K)$. This shows that G/K satisfies the hypothesis of the proposition. Therefore $(G/K)^{\mathfrak{N}_\sigma}$ is nilpotent by the choice of G.

(3) $\Phi(G) = 1$.
If $\Phi(G) \neq 1$, then $(G/\Phi(G))^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}\Phi(G)/\Phi(G)$ is nilpotent by Claim (2). So $G^{\mathfrak{N}_\sigma}$ is nilpotent, that is, $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, a contradiction. Therefore $\Phi(G) = 1$.

(4) The minimal normal subgroup of G is unique.
Assume N is a minimal normal subgroup of G. By Claim (2), $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$ is nilpotent. If $N \nsubseteq G^{\mathfrak{N}_\sigma}$, then $N \cap G^{\mathfrak{N}_\sigma} = 1$. From the isomorphism $G^{\mathfrak{N}_\sigma}N/N \simeq G^{\mathfrak{N}_\sigma}$, we have $G^{\mathfrak{N}_\sigma}$ is nilpotent, a contradiction. Hence, any minimal normal subgroup of G is contained in $G^{\mathfrak{N}_\sigma}$. If there exist two different minimal normal subgroups of G, says N_1, N_2. Then $N_1 \cap N_2 = 1$ and $G^{\mathfrak{N}_\sigma}/N_1$, $G^{\mathfrak{N}_\sigma}/N_2$ are both nilpotent. So we have $G^{\mathfrak{N}_\sigma}$ is nilpotent, a contradiction. Thus the minimal normal subgroup of G is unique.

(5) The finial contradiction.

Let N be the unique minimal normal subgroup of G. Clearly, $\mathcal{N}_\sigma(G) > 1$. Then, $N \leq \mathcal{N}_\sigma(G)$. If N is solvable, then N is an elementary abelian group. Since $\Phi(G) = 1$ by Claim (3), there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Then $G^{\mathfrak{N}_\sigma} \leq NM^{\mathfrak{N}_\sigma}$. By Claim (1), $M^{\mathfrak{N}_\sigma}$ is nilpotent. Now as $N \leq \mathcal{N}_\sigma(G)$ and $\mathcal{N}_\sigma(G)$ normalizes $M^{\mathfrak{N}_\sigma}$, we have that $M^{\mathfrak{N}_\sigma}$ is normal in G. The minimality of N implies that $N \cap M^{\mathfrak{N}_\sigma} = 1$ and so $NM^{\mathfrak{N}_\sigma} = N \times M^{\mathfrak{N}_\sigma}$. Since N is an elementary abelian group and $M^{\mathfrak{N}_\sigma}$ is nilpotent, we conclude that $G^{\mathfrak{N}_\sigma}$ is nilpotent. The contradiction shows that N is not soluble, and so N is a direct product of copies of a nonabelian simple group. Since N is the unique minimal normal subgroup of G, we have $C_G(N) = 1$. Let H be any proper subgroup of G. By Claim (1), $H^{\mathfrak{N}_\sigma}$ is nilpotent. Therefore $H^{\mathfrak{N}_\sigma} \cap N = 1$. Since $N \leq \mathcal{N}_\sigma(G)$ and $\mathcal{N}_\sigma(G)$ normalizes $H^{\mathfrak{N}_\sigma}$, N normalizes $H^{\mathfrak{N}_\sigma}$. Hence $H^{\mathfrak{N}_\sigma} \leq C_G(N) = 1$ and so $H^{\mathfrak{N}_\sigma} = 1$ for any proper subgroup H of G. It follows that G is either a σ-nilpotent group or an $\mathfrak{N}_\sigma$-critical group. If G is a σ-nilpotent group, then $G^{\mathfrak{N}_\sigma} = 1$, and so $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, a contradiction. If G is an $\mathfrak{N}_\sigma$-critical group, then $G^{\mathfrak{N}_\sigma} \leq G^{\mathfrak{N}}$ is nilpotent by Lemmas 2.5 and 2.4, a contradiction also. The proposition is proved. □

The Proof of Theorem 1.5.

Proof $(i) \Longrightarrow (ii)$: It follows from Proposition 3.1(1).

$(ii) \Longrightarrow (iii)$: Firstly, we prove a fact that if $X > 1$ is an $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-group, then $\mathcal{N}_\sigma(X) > 1$. In fact, if $X \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, then $X^{\mathfrak{N}_\sigma}$ is nilpotent, and so $C_X(X^{\mathfrak{N}_\sigma}) > 1$. But $C_X(X^{\mathfrak{N}_\sigma}) \leq \mathcal{N}_\sigma(X)$ by Remark 2.3 and [9, Lemma 2.1], so we have $\mathcal{N}_\sigma(X) > 1$. Using the fact and the noting that $\mathcal{N}_\sigma(G/\mathcal{N}_\sigma^\infty(G)) = 1$, we get $G = \mathcal{N}_\sigma^\infty(G)$.

$(iii) \Longrightarrow (i)$: By Lemma 2.1(3), $\mathcal{N}_\sigma^\infty(G/\mathcal{N}_\sigma(G)) = \mathcal{N}_\sigma^\infty(G)/\mathcal{N}_\sigma(G)$. Then by induction, $G/\mathcal{N}_\sigma(G) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$. It follows from Proposition 3.3 that $G \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$.

$(i) \Longrightarrow (iv)$: Using the fact which is proved in $(ii) \Longrightarrow (iii)$, it is obvious.

$(iv) \Longrightarrow (iii)$: By definition, $\mathcal{N}_\sigma^{i+1}(G)/\mathcal{N}_\sigma^i(G) = \mathcal{N}_\sigma(G/\mathcal{N}_\sigma^i(G))$ for $i = 0, 1, 2, \ldots$. By the hypothesis, $\mathcal{N}_\sigma(G/\mathcal{N}_\sigma^i(G)) > 1$ if $\mathcal{N}_\sigma^i(G) < G$, so $\mathcal{N}_\sigma^{i+1}(G) > \mathcal{N}_\sigma^i(G)$ for any $\mathcal{N}_\sigma^i(G) < G$. Hence, the terminal term $\mathcal{N}_\sigma^\infty(G)$ of the ascending series must be G. □

Corollary 3.4 *If H is an $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-subgroup of G, then $\mathcal{N}_\sigma^\infty(G)H$ is an $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-group. Consequently, $\mathcal{N}_\sigma^\infty(G)$ is contained in every maximal $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-subgroup of G.*

Proof Let $M = \mathcal{N}_\sigma^\infty(G)H$. By Lemma 2.1(1), we have $\mathcal{N}_\sigma^\infty(G) \leq \mathcal{N}_\sigma^\infty(M)$. Hence $M/\mathcal{N}_\sigma^\infty(M) \leq \mathcal{N}_\sigma^\infty(M)H/\mathcal{N}_\sigma^\infty(M) \cong H/(H \cap \mathcal{N}_\sigma^\infty(M))$. Since $H \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$, $M/\mathcal{N}_\sigma^\infty(M) \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$ by Proposition 3.1. It follows from Theorem 1.5 that $M \in \mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$. In particular, we take H is any maximal $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-subgroup of G. Then $M = \mathcal{N}_\sigma^\infty(G)H = H$, and so $\mathcal{N}_\sigma^\infty(G) \leq H$. Therefore $\mathcal{N}_\sigma^\infty(G)$ is contained in every maximal $\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}$-subgroup of G. □

Following [12], we use $Int_{\mathfrak{X}}(G)$ to denotes the intersection of all $\mathfrak{X}$-maximal subgroups of G, where $\mathfrak{X}$ is a class of finite groups.

Remark 3.5 The above Corollary 3.4 shows that $\mathcal{N}_\sigma^\infty(G) \leq Int_{\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}}(G)$. But we do not know whether $Int_{\mathcal{F}_{\mathfrak{N}\mathfrak{N}_\sigma}}(G) \leq \mathcal{N}_\sigma^\infty(G)$ is true.

4 The Properties and Structures of N_σ-groups

In this section, we discuss the class of N_σ-groups.

Recall that a group G is an N_σ-group if $G = \mathcal{N}_\sigma(G)$, that is, the σ-nilpotent residual of every subgroup of G is normal in G; a group G is an S-group if $G = S(G)$, that is, the nilpotent residual of every subgroup of G is normal in G. Firstly we discuss the relationship between N_σ-groups and S-groups.

The following example shows that N_σ-groups and S-groups are different.

Example 4.1 Let $G = A_5$ be the alternating group with degree 5 and $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{2, 3, 5\}$ and $\sigma_2 = \{2, 3, 5\}'$. Clearly, G is a σ-nilpotent, so every subgroup of G is σ-nilpotent by Lemma 2.2(1). Hence G is an N_σ-group. However, G is not an S-group. In fact, ${A_4}^{\mathfrak{N}} \neq 1$ is not normal in G.

Now we give the following proposition to show N_σ-groups and S-groups are same under certain conditions.

Recall (see [25,26,29]) that a set $\mathcal{H}$ of subgroups of G is said to be a complete Hall σ-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some

σ_i and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. G is said to be: σ-full if G possesses a complete Hall σ-set; a σ-full group of Sylow type if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma(G)$.

Proposition 4.2 *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{W_1, \ldots, W_t\}$ be a complete Hall σ-set of G such that W_i is a nilpotent σ_i-subgroup for all $i = 1, \ldots, t$. Then G is an N_σ-group if and only if G is an S-group.*

Proof Let H be any subgroup of G. It is clear that $H^{\mathfrak{N}_\sigma} \leq H^{\mathfrak{N}}$. Now we show $H^{\mathfrak{N}} \leq H^{\mathfrak{N}_\sigma}$. Since $H/H^{\mathfrak{N}_\sigma} \in \mathfrak{N}_\sigma$, we have that $H/H^{\mathfrak{N}_\sigma} = H_1/H^{\mathfrak{N}_\sigma} \times H_2/H^{\mathfrak{N}_\sigma} \times \ldots H_t/H^{\mathfrak{N}_\sigma}$, where $H_i/H^{\mathfrak{N}_\sigma}$ is a Hall σ_i-subgroup of $H/H^{\mathfrak{N}_\sigma}$ for all $i = 1, 2, \ldots, t$. By the hypothesis, we see that $H_i/H^{\mathfrak{N}_\sigma}$ is nilpotent. Hence $H/H^{\mathfrak{N}_\sigma}$ is nilpotent, that is, $H/H^{\mathfrak{N}_\sigma} \in \mathfrak{N}$. Therefore $H^{\mathfrak{N}} \leq H^{\mathfrak{N}_\sigma}$. It follows that $H^{\mathfrak{N}} = H^{\mathfrak{N}_\sigma}$. If G is an N_σ-group, then $H^{\mathfrak{N}} = H^{\mathfrak{N}_\sigma}$ is normal in G for every subgroup H of G. Hence G is an S-group. If G is an S-group, then $H^{\mathfrak{N}_\sigma} = H^{\mathfrak{N}}$ is normal in G for every subgroup H of G. It follows that G is an N_σ-group. Therefore, the proof is completed. □

Next we give some characters of N_σ-groups and some judging theorems for a group to be an N_σ-group.

Note that for any normal subgroup N of G, $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$ (see [11, Lemma 2.1.3]). Hence the following facts are clear from Definition 1.8, Lemma 2.2(1), Lemmas 2.4 and 2.5:

Proposition 4.3 (i) *The subgroups of an N_σ-group are N_σ-groups;*

(ii) *The quotient groups of an N_σ-group are N_σ-groups;*

(iii) *If G is a σ-nilpotent group or a minimal non-σ-nilpotent group, then G is an N_σ-group.*

Lemma 4.4 *G is a σ-nilpotent group if and only if there is a normal nilpotent subgroup N of G such that G/N' is a σ-nilpotent group.*

Proof The necessity follows from Lemma 2.2(1). We now show the sufficiency. Since N is nilpotent, then $N' \leq \Phi(G)$. Then as G/N' is a σ-nilpotent group, we have that $G/\Phi(G)$ is σ-nilpotent by Lemma 2.2(1). Hence G is a σ-nilpotent group by Remark 2.3. The lemma is proved. □

Remark 4.5 Through the proof process of the above Lemma 4.4, we can see that the Lemma 4.4 is also valid when the class of σ-nilpotent groups is replaced by any saturated formation. Hence Lemma 4.4 covers a famous result of Hall (see [17, page 2]).

From Lemmas 4.4, 2.2 and Proposition 2.6, we directly obtain the following:

Theorem 4.6 *G is a σ-nilpotent group if and only if G is an N_σ-group and $G/(G^{\mathfrak{N}_\sigma})'$ is a σ-nilpotent group.*

Following Definition 3.10 in [20], let $G_{n+1} = [G_n, G]$, if a group G satisfies $G_{c+1} = 1$, then we say that G is *nilpotent* of class $\leq c$; the least such number c is called the *nilpotency class* of G.

Theorem 4.7 *If $G/(G^{\mathfrak{N}_\sigma})'$ is an $\mathfrak{N}_\sigma$-critical group and $G^{\mathfrak{N}_\sigma}$ is of nilpotency class 2, then G is an N_σ-group.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. Then there exists $M < G$ such that $M^{\mathfrak{N}_\sigma}$ is not normal in G. Consider $N_G(M^{\mathfrak{N}_\sigma})$. As $G^{\mathfrak{N}_\sigma}$ is nilpotent of class 2, $(G^{\mathfrak{N}_\sigma})' \leq Z(G^{\mathfrak{N}_\sigma})$. Since $M/M \cap G^{\mathfrak{N}_\sigma} \simeq MG^{\mathfrak{N}_\sigma}/G^{\mathfrak{N}_\sigma} \leq G/G^{\mathfrak{N}_\sigma} \in \mathfrak{N}_\sigma$ and $\mathfrak{N}_\sigma$ is subgroup closed by Lemma 2.2(1), $M^{\mathfrak{N}_\sigma} \leq G^{\mathfrak{N}_\sigma}$. Then $G > N_G(M^{\mathfrak{N}_\sigma}) \geq M(G^{\mathfrak{N}_\sigma})'$. Since $G/(G^{\mathfrak{N}_\sigma})'$ is an $\mathfrak{N}_\sigma$-critical group, $M(G^{\mathfrak{N}_\sigma})'/(G^{\mathfrak{N}_\sigma})'$ is σ-nilpotent. It follows from $M(G^{\mathfrak{N}_\sigma})'/(G^{\mathfrak{N}_\sigma})' \simeq M/M \cap (G^{\mathfrak{N}_\sigma})' \in \mathfrak{N}_\sigma$ that $M^{\mathfrak{N}_\sigma} \leq M \cap (G^{\mathfrak{N}_\sigma})' \leq Z(G^{\mathfrak{N}_\sigma})$. Hence $G^{\mathfrak{N}_\sigma} \leq N_G(M^{\mathfrak{N}_\sigma})$, and so $MG^{\mathfrak{N}_\sigma} \leq N_G(M^{\mathfrak{N}_\sigma}) < G$. Then $MG^{\mathfrak{N}_\sigma}/(G^{\mathfrak{N}_\sigma})'$ is σ-nilpotent as $G/(G^{\mathfrak{N}_\sigma})'$ is an $\mathfrak{N}_\sigma$-critical group. Hence by Lemma 4.4, $MG^{\mathfrak{N}_\sigma}$ is σ-nilpotent. It follows from Lemma 2.2(1) that M is σ-nilpotent and so $M^{\mathfrak{N}_\sigma} = 1$ is normal in G. The contradiction completes the proof. □

By using the same method of the Proof of Theorem 4.7, we can also get the following general result.

Theorem 4.8 *If $G/(G^{\mathfrak{N}_\sigma})_n$ is an $\mathfrak{N}_\sigma$-critical group and $G^{\mathfrak{N}_\sigma}$ is of nilpotency class n, where n is an integer and $n \geq 2$, then G is an N_σ-group.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. Then there is $M < G$ such that $M^{\mathfrak{N}_\sigma}$ is not normal in G. Consider $N_G(M^{\mathfrak{N}_\sigma})$. If $G^{\mathfrak{N}_\sigma}$ is nilpotent of class n, then $(G^{\mathfrak{N}_\sigma})_n \leq Z(G^{\mathfrak{N}_\sigma})$. Clearly $M^{\mathfrak{N}_\sigma} \leq G^{\mathfrak{N}_\sigma}$, so $G > N_G(M^{\mathfrak{N}_\sigma}) \geq M(G^{\mathfrak{N}_\sigma})_n$. Since $G/(G^{\mathfrak{N}_\sigma})_n$ is an $\mathfrak{N}_\sigma$-critical group, $M(G^{\mathfrak{N}_\sigma})_n/(G^{\mathfrak{N}_\sigma})_n$ is σ-nilpotent. It follows from $M(G^{\mathfrak{N}_\sigma})_n/(G^{\mathfrak{N}_\sigma})_n \simeq M/M \cap (G^{\mathfrak{N}_\sigma})_n \in \mathfrak{N}_\sigma$ that $M^{\mathfrak{N}_\sigma} \leq M \cap (G^{\mathfrak{N}_\sigma})_n \leq Z(G^{\mathfrak{N}_\sigma})$. Hence $G^{\mathfrak{N}_\sigma} \leq N_G(M^{\mathfrak{N}_\sigma})$, and so $MG^{\mathfrak{N}_\sigma} \leq N_G(M^{\mathfrak{N}_\sigma}) < G$. Therefore $MG^{\mathfrak{N}_\sigma}/(G^{\mathfrak{N}_\sigma})_n$ is σ-nilpotent as $G/(G^{\mathfrak{N}_\sigma})_n$ is an $\mathfrak{N}_\sigma$-critical group. Since $(G^{\mathfrak{N}_\sigma})_n \leq (G^{\mathfrak{N}_\sigma})'$, $MG^{\mathfrak{N}_\sigma}/(G^{\mathfrak{N}_\sigma})'$ is σ-nilpotent by Lemma2.2. Hence by Lemma 4.4, $MG^{\mathfrak{N}_\sigma}$ is σ-nilpotent. Consequently M is σ-nilpotent, and so $M^{\mathfrak{N}_\sigma} = 1$ is normal in G. The contradiction completes the proof. □

Remark 4.9 It is clear that Theorem 4.5 and 4.6 in [24] are corollaries of our Theorem 4.8.

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