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Isolated Subsemigroups of Order-Preserving and Decreasing Transformation Semigroups

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Abstract

In this paper, we classify all isolated, completely isolated, and (left/right) convex subsemigroups of $\mathcal{C}_n$, the semigroup of all order-preserving and decreasing transformations on $X_n = \{1, \ldots, n\}$ under its natural order. Moreover, we find the rank of each convex subsemigroup of $\mathcal{C}_n$.

Keywords Order-preserving/decreasing transformation · Rank · Nilpotent subsemigroups · Isolated subsemigroups

Mathematics Subject Classification 20M20

1 Introduction

For an arbitrary set X, the set $\mathcal{T}_X$ of all transformations of X, that is of all maps from X to itself, is a semigroup under composition, and called the full transformation semigroup on X. If $X = X_n = \{1, \ldots, n\}$ with its natural order, then $\mathcal{T}_X$ is denoted by $\mathcal{T}_n$. A transformation $\alpha \in \mathcal{T}_n$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and *decreasing* (*increasing*) if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. The subsemigroup of all order-preserving transformations in $\mathcal{T}_n$ are denoted by $\mathcal{O}_n$, and the subsemigroup of all order-decreasing (order-increasing) transformations in $\mathcal{T}_n$ is denoted by $\mathcal{D}_n$ ($\mathcal{D}_n^+$). The subsemigroup of all order-preserving and decreasing (increasing) transformations in $\mathcal{T}_n$ is denoted by $\mathcal{C}_n$ ($\mathcal{C}_n^+$), i.e., $\mathcal{C}_n = \mathcal{O}_n \cap \mathcal{D}_n$ ($\mathcal{C}_n^+ =$

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$\mathcal{O}_n \cap \mathcal{D}_n^+$). In [14, Corollary 2.7.], Umar proved that $\mathcal{D}_n$ and $\mathcal{D}_n^+$ are isomorphic, and it is also well-known that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are isomorphic semigroups (for example, see Remarks on [8, page 290]).

For any transformation $\alpha \in \mathcal{T}_n$, the *kernel*, the *image*, the *fix*, and the *shift* of α are defined, respectively, by

$$\begin{aligned}&\ker(\alpha) = \{(x, y) : x\alpha = y\alpha \text{ for all } x, y \in X_n\}, \quad \operatorname{im}(\alpha) = \{x\alpha : x \in X_n\},\\ &\operatorname{fix}(\alpha) = \{x \in X_n : x\alpha = x\}, \text{ and } \operatorname{shift}(\alpha) = \{x \in X_n : x\alpha \neq x\}.\end{aligned}$$

The set of all idempotent elements of S is denoted by $E(S)$, that is, $E(S) = \{e \in S \mid e^2 = e\}$. For any $\alpha \in \mathcal{T}_n$, it is clear that α is an idempotent if and only if $\operatorname{fix}(\alpha) = \operatorname{im}(\alpha)$. The set of all nilpotent elements of a semigroup S with zero 0 is denoted by $N(S)$, that is, $N(S) = \{a \in S \mid a^k = 0 \text{ for some } k \in \mathbb{N}\}$. It is a known fact that a finite semigroup S with zero is nilpotent, $S^m = \{0\}$ for a positive integer m, if and only if the unique idempotent of S is the zero element (see, for example [5, Proposition 8.1.2]). It is clear that

$$\varepsilon = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \varepsilon^+ = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \end{pmatrix}$$

are the zero elements of $\mathcal{C}_n$ and $\mathcal{C}_n^+$, respectively. As shown in [10, Lemma 1.4.], an element α of $\mathcal{C}_n$ is nilpotent if and only if $\operatorname{fix}(\alpha) = \{1\}$. As on [4, page 241], for any $\alpha \in \mathcal{C}_n$, we can use the following tabular form:

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix}, \tag{1}$$

where $\operatorname{im}(\alpha) = \{1 = a_1 < \cdots < a_r\}$, and $a_i\alpha^{-1} = A_i$ for each $1 \leq i \leq r$. Thus, $\{A_1, \ldots, A_r\}$ is an *ordered convex partition* of X_n, that is $x < y$ for all $x \in A_i$ and $y \in A_{i+1}$ where $1 \leq i \leq r-1$, and moreover, each A_i $(1 \leq i \leq r)$ is a convex subset of X_n provided that for all $x, y \in A_i$, $x \leq z \leq y$ implies $z \in A_i$ (for example, see [1,2,8]). As proved in [11, Theorem 2.1], $|\mathcal{C}_n| = |\mathcal{C}_n^+| = C_n$, n-th *Catalan number* (see, for example [6]). For this reason, $\mathcal{C}_n$ is also called the *Catalan monoid* on X_n under its natural order.

For a non-empty subset A of a semigroup S, the *subsemigroup generated by* A, the smallest subsemigroup of S containing A, is denoted by $\langle A \rangle$. If there exists a finite subset A of S such that $S = \langle A \rangle$, then S is called a *finitely generated semigroup*. The *rank* of a finitely generated semigroup S is defined by

$$\operatorname{rank}(S) = \min\{\,|A| : \langle A \rangle = S\,\},$$

where $|A|$ denotes the cardinality of A. An element s of a semigroup S is called *indecomposable*, if $s \neq xy$ for all $x, y \in S$, that is, if $s \in S \setminus S^2$. An element s of a semigroup S is called *irreducible* provided that the condition $s = xy$ for $x, y \in S$ implies $s = x$ or $s = y$. It is clear that every generating set of S must contain all indecomposable and irreducible elements of S. Thus, if A is a non-empty finite set which consists of irreducible elements and $S = \langle A \rangle$, then A is the minimum generating set of S, and so rank $(S) = |A|$.

A semigroup S is called a *band* if $S = E(S)$. If S is a commutative band, then S is called a *semilattice*. For any non-empty set X, let $\mathcal{SL}_X$ be the set of all subsets of X. With the multiplication in $\mathcal{SL}_X$ defined by $A \cdot B = A \cap B$, $\mathcal{SL}_X$ is a semilattice: it is called the *free semilattice* on X. Notice that $\mathcal{SL}_X$ can be defined to be the set of all subsets of X with the multiplication $A \cdot B = A \cup B$, which is more common definition of the free semilattice on X, but the first definition is more useful for this research. If we consider the map $\varphi : (\mathcal{SL}_X, \cap) \to (\mathcal{SL}_X, \cup)$ defined by $A\varphi = X \setminus A$, then we see that these two definitions of the free semilattice on X are equivalent up to the isomorphism. If X is a finite set with n elements, then we suppose that $X = X_n = \{1, \ldots, n\}$, and we denote the free semilattice on X_n by $\mathcal{SL}_n$ instead of $\mathcal{SL}_{X_n}$. (For other terms in semigroup theory, we refer to [7].)

A proper subsemigroup of a semigroup S is called *maximal* if it is not contained any other proper subsemigroup of S, and it is called the *maximum subsemigroup* if it is unique. A subsemigroup T of a semigroup S is called

- *isolated* provided that for all $x \in S$, the condition $x^n \in T$ for some $n \in \mathbb{N}$ implies $x \in T$;
- *completely isolated* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$ or $y \in T$;
- *right convex* provided that, for all $x, y \in S$, $xy \in T$ implies $y \in T$;
- *left convex* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$;
- *convex* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$ and $y \in T$.

It is clear from the definitions that every completely isolated subsemigroup is isolated that every left (right) convex subsemigroup is completely isolated, and that every convex subsemigroup is both left and right convex. Moreover, any left (right) convex subsemigroup of a monoid S must contain the identity of S. We refer the readers to [3–5] for details and more properties on these subsemigroups. The (completely) isolated and (left/right) convex subsemigroups of some special semigroups have been studied by several authors, (for examples, see [12,13]). In [12], all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{T}_n$ are classified. In [13], all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{IS}_n$, all injective partial transformations in X_n, are classified. As we have recently focused in $\mathcal{C}_n$ (see [9,15,16]), we are concerned with the classification of all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{C}_n$.

Given $\zeta \in E(\mathcal{C}_n)$, it is known that

$$\mathcal{C}_n(\zeta) = \{\alpha \in \mathcal{C}_n : \alpha^m = \zeta \text{ for some } m \in \mathbb{N}\}$$

is a subsemigroup of $\mathcal{C}_n$ with zero element ζ (see [16, Proposition 2]). In the second section, we firstly show that for any $\zeta \in E(\mathcal{C}_n)$, $\mathcal{C}_n(\zeta)$ is the unique isolated nilpotent subsemigroup of $\mathcal{C}_n$ with zero element ζ. Then, we show that a subsemigroup T of $\mathcal{C}_n$ is isolated if and only if $T = \bigcup_{\zeta \in E(T)} \mathcal{C}_n(\zeta)$ and $\mathrm{sl}\,(T) = \{\,\mathrm{fix}\,(\zeta) : \zeta \in E(T)\,\}$ is a subsemigroup of the free semilattice $\mathcal{SL}_n$. For any isolated subsemigroup T of $\mathcal{C}_n$, we prove that T is completely isolated if and only if $\mathrm{sl}\,(T)$ is a completely isolated subsemigroup of $\mathcal{SL}_n(1) = \{Y : 1 \in Y \subseteq X_n\}$, which is clearly isomorphic to $\mathcal{SL}_{n-1}$. In the third section, for a (completely isolated) subsemigroup T of $\mathcal{C}_n$, we prove that T is convex if and only if $T = \mathcal{C}_n[Y] = \{\alpha \in \mathcal{C}_n : Y \subseteq \mathrm{fix}\,(\alpha)\}$, where $Y = \bigcap_{\zeta \in E(T)} \mathrm{fix}\,(\zeta)$. Finally, we find the cardinality and the minimum generating set of $\mathcal{C}_n[Y]$, and so we conclude that $\mathrm{rank}\,(\mathcal{C}_n[Y]) = n - |Y| + 1$.

2 Completely Isolated Subsemigroups of $\mathcal{C}_n$

Let ζ be any idempotent element of a semigroup S and let

$$S(\zeta) = \{\alpha \in S : \alpha^m = \zeta \ \text{ for some } \ m \in \mathbb{N}\}.$$

The following lemma, which was proved in [5, Lemma 5.3.4], is very useful to classify all isolated subsemigroups of $\mathcal{C}_n$.

Lemma 2.1 *Let S be a semigroup, and let T be a subsemigroup of S.*

(i) If T is isolated, then $S(\zeta) \subseteq T$, for all $\zeta \in E(T)$.
(ii) If T is isolated, and if S is finite, then $T = \bigcup_{\zeta \in E(T)} S(\zeta)$. □

For any $\zeta \in E(\mathcal{C}_n)$, it follows from [16, Proposition 2] that

$$\mathcal{C}_n(\zeta) = \{\alpha \in \mathcal{C}_n : \alpha^m = \zeta \text{ for some } m \in \mathbb{N}\}$$

is a subsemigroup of $\mathcal{C}_n$ with zero element ζ. Furthermore, if T is a nilpotent subsemigroup of $\mathcal{C}_n$ with zero ζ, then it is clear that T is a subset of $\mathcal{C}_n(\zeta)$, and so $\mathcal{C}_n(\zeta)$ is the maximum nilpotent subsemigroup of $\mathcal{C}_n$ with zero element ζ. Notice that $\mathcal{C}_n(\varepsilon) = N(\mathcal{C}_n)$, where $\varepsilon = \begin{pmatrix} 1 & \cdots & n \\ 1 & \cdots & 1 \end{pmatrix}$.

The following proposition from [16] plays a major role throughout this paper.

Proposition 2.2 *Let ζ be any idempotent, and let α be any element of $\mathcal{C}_n$. Then, $\alpha \in \mathcal{C}_n(\zeta)$ if and only if* $\mathrm{fix}\,(\alpha) = \mathrm{fix}\,(\zeta)$. □

It is known that for any $\alpha, \beta \in \mathcal{D}_n$, $\mathrm{fix}\,(\alpha\beta) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\beta)$ (for example, see [10, Lemma 1.1]), and so we are able state the following technical proposition, which also plays an important role throughout in our paper.

Proposition 2.3 *(i) For any $\alpha, \beta \in \mathcal{C}_n$,* $\mathrm{fix}\,(\alpha\beta) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\beta)$.
(ii) For any $\alpha \in \mathcal{C}_n$ and for any $k \in \mathbb{N}$, $\mathrm{fix}\,(\alpha^k) = \mathrm{fix}\,(\alpha)$.

Proof (i) Since $\mathcal{C}_n$ is a subsemigroup of $\mathcal{D}_n$, it is an immediate consequence of Lemma 1.1 from [10].

(ii) The result follows from (i) by induction on k. □

Notice that Proposition 2.3 is not valid for all transformations in $\mathcal{T}_n$. Next, we have the following result:

Theorem 2.4 *For any $\zeta \in E(\mathcal{C}_n)$, $\mathcal{C}_n(\zeta)$ is the unique isolated nilpotent subsemigroup of $\mathcal{C}_n$ with zero element ζ.*

Proof We have just noticed that $\mathcal{C}_n(\zeta)$ is the maximal nilpotent subsemigroup of $\mathcal{C}_n$ with the zero element. Let $\text{fix}\,(\zeta) = \{1 = a_1 < a_2 < \cdots < a_p\}$. Given $\alpha \in \mathcal{C}_n$ and $k \in \mathbb{N}$, suppose that $\alpha^k \in \mathcal{C}_n(\zeta)$. Then, it follows from Propositions 2.2 and 2.3 (ii) that $\text{fix}\,(\alpha) = \text{fix}\,(\alpha^k) = \{1 = a_1 < a_2 < \cdots < a_p\}$, and so $\alpha \in \mathcal{C}_n(\zeta)$. Therefore, $\mathcal{C}_n(\zeta)$ is isolated.

Let T be any isolated nilpotent subsemigroup of $\mathcal{C}_n$ with zero element ζ. Since T is nilpotent, $E(T) = \{\zeta\}$, and so it follows from Lemma 2.1 (ii) that $T = \mathcal{C}_n(\zeta)$, as required. □

The following example shows that the union of isolated subsemigroups does not need to be even a semigroup.

Example 2.5 Let $\zeta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$ and $\zeta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix}$, which are two idempotents of $\mathcal{C}_4$. Then, $\mathcal{C}_4(\zeta_1)$ and $\mathcal{C}_4(\zeta_2)$ are two isolated subsemigoups of $\mathcal{C}_4$, but $\zeta_1\zeta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ is not in $\mathcal{C}_4(\zeta_1) \cup \mathcal{C}_4(\zeta_2)$. □

To complete the classification of isolated subsemigroups of $\mathcal{C}_n$, we state and prove the following lemma:

Lemma 2.6 *Let T be an isolated subsemigroup of $\mathcal{C}_n$.*

(i) For any idempotent element ζ of T, $\mathcal{C}_n(\zeta)$ is a subsemigroup of T.
(ii) If ζ_1 and ζ_2 are two idempotents elements of T, then there exists an idempotent element ζ_3 of T such that $\text{fix}\,(\zeta_1) \cap \text{fix}\,(\zeta_2) = \text{fix}\,(\zeta_3)$.

Proof (i) It is an immediate consequence of the concerning definitions.

(ii) For any $\zeta_1, \zeta_2 \in E(T)$, let $\zeta_1\zeta_2 = \alpha$. Since T is a finite subsemigroup, there exists a positive integer k such that α^k is an idempotent element of T, that is there exist a positive integer k and an idempotent $\zeta_3 \in E(T)$ such that $\alpha^k = \zeta_3$. Therefore, it follows from Proposition 2.3 that

$$\text{fix}\,(\zeta_1) \cap \text{fix}\,(\zeta_2) = \text{fix}\,(\zeta_1\zeta_2) = \text{fix}\,(\alpha) = \text{fix}\,(\alpha^k) = \text{fix}\,(\zeta_3),$$

as wanted. □

Notice that for any non-empty subset Y of X_n, there are many idempotents in $\mathcal{T}_n$ whose fix set is Y. Notice also that, for every α in $\mathcal{C}_n$, we have $1 \in \text{fix}\,(\alpha)$. However, for $Y \subseteq X_n$ which contains 1, there is a unique idempotent ζ in $\mathcal{C}_n$ such that $\text{fix}\,(\zeta) = Y$. Namely, if $Y = \{1 = a_1 < a_2 < \cdots < a_r\} \subseteq X_n$, then $\zeta = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ 1 & a_2 & \cdots & a_r \end{pmatrix}$ where

$A_i = \{a_i, a_i + 1, \ldots, a_{i+1} - 1\}$, with $1 \leq i \leq r - 1$ and $A_r = \{a_r, a_r + 1, \ldots, n\}$. Moreover, it is clear that the subset of X_n

$$\mathrm{sl}\,(\mathcal{C}_n) = \{\mathrm{fix}\,(\zeta) : \zeta \in E(\mathcal{C}_n)\} = \{Y \subseteq X_n : 1 \in Y\}$$

is a subsemigroup of $\mathcal{SL}_n$. Furthermore, for $n \geq 2$, $\mathrm{sl}\,(\mathcal{C}_n)$ and $\mathcal{SL}_{n-1}$ are isomorphic semigroups. Now, we state and prove one of our main results: it presents a complete description of all isolated proper subsemigroups of $\mathcal{C}_n$.

Theorem 2.7 *Let T be a non-empty subset of $\mathcal{C}_n$. If $E(T)$ is not empty, then T is an isolated subsemigroup of $\mathcal{C}_n$ if and only if the following two conditions are satisfied:*

(a) $T = \bigcup\limits_{\zeta \in E(T)} \mathcal{C}_n(\zeta)$ *and*
(b) $\mathrm{sl}\,(T) = \{\,\mathrm{fix}\,(\zeta) \;:\; \zeta \in E(T)\,\}$ *is a subsemigroup of the free semilattice $\mathcal{SL}_n$.*

Proof ($\Rightarrow$) Since $\mathcal{C}_n$ is finite, it follows from Lemma 2.1 (ii) that $T = \bigcup\limits_{\zeta \in E(T)} \mathcal{C}_n(\zeta)$. For any $\zeta_1, \zeta_2 \in E(T)$, it follows from Lemma 2.6 (ii) that there exists an idempotent element ζ_3 of T such that $\mathrm{fix}\,(\zeta_1) \cap \mathrm{fix}\,(\zeta_2) = \mathrm{fix}\,(\zeta_3)$, and so $\mathrm{sl}\,(T)$ is a subsemigroup of $\mathcal{SL}_n$.

($\Leftarrow$) For a non-empty subset T of $\mathcal{C}_n$, suppose that $E(T)$ is not empty, $T = \bigcup\limits_{\zeta \in E(T)} \mathcal{C}_n(\zeta)$, and that $\mathrm{sl}\,(T) = \{\mathrm{fix}\,(\zeta) : \zeta \in E(T)\}$ is a subsemigroup of $\mathcal{SL}_n$. For $\alpha, \beta \in T$, there exist two idempotents $\zeta_1, \zeta_2 \in E(T)$ such that $\alpha \in \mathcal{C}_n(\zeta_1)$ and $\beta \in \mathcal{C}_n(\zeta_2)$. From Proposition 2.2, we have $\mathrm{fix}\,(\alpha) = \mathrm{fix}\,(\zeta_1)$ and $\mathrm{fix}\,(\beta) = \mathrm{fix}\,(\zeta_2)$. Then, it follows from Proposition 2.3 (i) that

$$\mathrm{fix}\,(\alpha\beta) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\beta) = \mathrm{fix}\,(\zeta_1) \cap \mathrm{fix}\,(\zeta_2).$$

Let ζ_3 be the unique idempotent in $\mathcal{C}_n$ with $\mathrm{fix}\,(\zeta_3) = \mathrm{fix}\,(\zeta_1) \cap \mathrm{fix}\,(\zeta_2)$. It follows from (b) and (a) that $\mathrm{fix}\,(\zeta_3) \in \mathrm{sl}\,(T)$ and $\mathcal{C}_n(\zeta_3) \subseteq T$. Since $\mathrm{fix}\,(\alpha\beta) = \mathrm{fix}\,(\zeta_3)$, it follows Proposition 2.2 that $\alpha\beta \in \mathcal{C}_n(\zeta_3)$. Therefore, T is a subsemigroup, and from Theorem 2.4, is clearly isolated, as required. □

For a subsemigroup T of $\mathcal{C}_n$, notice that if $T = \{1_n\}$, where 1_n denotes the identity element of $\mathcal{C}_n$, then, since 1_n is irreducible, $T = \{1_n\}$ is isolated. If $1_n \in T \neq \{1_n\}$, then it is clear that T is isolated exactly when $T \setminus \{1_n\}$ is an isolated subsemigroup of $\mathcal{C}_n$.

Now, we focus on completely isolated subsemigroups of $\mathcal{C}_n$. The following example shows that for any non-identity $\zeta \in E(\mathcal{C}_n)$, $\mathcal{C}_n(\zeta)$ is not a completely isolated, and so is not, in general, (right/left) convex subsemigroup of $\mathcal{C}_n$.

Example 2.8 Let $\zeta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 3 \end{pmatrix}$, which is an idempotent element of $\mathcal{C}_5$. If we consider two transformations, namely $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 4 & 5 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 \end{pmatrix}$, then we have $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 4 \end{pmatrix}$. It follows from Proposition 2.2 that $\alpha\beta \in \mathcal{C}_5(\zeta)$.

However, neither α nor β are in $\mathcal{C}_5(\zeta)$. Therefore, $\mathcal{C}_5(\zeta)$ is not completely isolated, and so not (left/right) convex. □

The following lemma is given in [5, Exercise 5.3.1] which is an immediate consequence of the considered definitions.

Lemma 2.9 *A proper subsemigroup $T \subset S$ is completely isolated if and only if its complement $\overline{T} = S \setminus T$ is a subsemigroup. In particular, if T is completely isolated, then $\overline{T}$ is completely isolated as well.* □

Since the permutation group $\mathcal{S}_n$ and the singular transformation semigroup $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ are both subsemigroups of $\mathcal{T}_n$, it follows from Lemma 2.9 that both $\mathcal{S}_n$ and Sing_n are completely isolated subsemigroups of $\mathcal{T}_n$. Moreover, since $\mathcal{C}_n \setminus \{1_n\}$ and $\{1_n\}$ are both subsemigroups of $\mathcal{C}_n$, similarly, both $\mathcal{C}_n \setminus \{1_n\}$ and $\{1_n\}$ are completely isolated. It is also clear that a subsemigroup $T \neq \{1_n\}$ of $\mathcal{C}_n$ with 1_n is completely isolated if and only if $T \setminus \{1_n\}$ is completely isolated. Furthermore, if T is a completely isolated subsemigroup of $\mathcal{C}_n$, then since every completely isolated subsemigroup is isolated, it follows from Theorem 2.7 (b) that $\mathrm{sl}\,(T) = \{\mathrm{fix}\,(\zeta) : \zeta \in E(T)\}$ is a subsemigroup of $\mathcal{SL}_n$. For $n \geq 2$, we define the set

$$\mathrm{csl}\,(T) = \{\, Y : 1 \in Y \subseteq X_n,\ \ Y \notin \mathrm{sl}\,(T)\,\},$$

so that $\mathrm{sl}\,(T) \cap \mathrm{csl}\,(T) = \emptyset$ and $\mathrm{sl}\,(T) \cup \mathrm{csl}\,(T) = \mathcal{SL}_n(1)$, where

$$\mathcal{SL}_n(1) = \{\, Y : 1 \in Y \subseteq X_n \,\}.$$

It is clear that $\mathcal{SL}_n(1)$ is a subsemigroup of $\mathcal{SL}_n$ and isomorphic to $\mathcal{SL}_{n-1}$. Since completely isolated subsemigroups are isolated, we consider only isolated subsemigroups in the following theorem:

Theorem 2.10 *For any isolated subsemigroup T of $\mathcal{C}_n$, T is completely isolated if and only if* $\mathrm{sl}\,(T)$ *is a completely isolated subsemigroup of* $\mathcal{SL}_n(1)$.

Proof First of all, since $1 \in \mathrm{fix}\,(\alpha)$, for every $\alpha \in T$, it follows from Theorem 2.7 (b) that $\mathrm{sl}\,(T)$ is a subsemigroup of $\mathcal{SL}_n(1)$.

($\Rightarrow$) Suppose that T is completely isolated, and that, for $Y_1, Y_2 \in \mathcal{SL}_n(1)$, $Y_1 \cap Y_2 \in \mathrm{sl}\,(T)$. Let ζ_1, ζ_2 and ζ_3 be the idempotent elements in $\mathcal{C}_n$ such that $Y_1 = \mathrm{fix}\,(\zeta_1)$, $Y_2 = \mathrm{fix}\,(\zeta_2)$ and $Y_1 \cap Y_2 = \mathrm{fix}\,(\zeta_3)$, respectively. Since $Y_1 \cap Y_2 = \mathrm{fix}\,(\zeta_3) \in \mathrm{sl}\,(T)$, it follows from the uniqueness of ζ_3 that $\zeta_3 \in T$, and so from Lemma 2.6 (i), $\mathcal{C}_n(\zeta_3)$ is a subsemigroup of T. From Proposition 2.3 (i), since we have

$$\mathrm{fix}\,(\zeta_1\zeta_2) = \mathrm{fix}\,(\zeta_1) \cap \mathrm{fix}\,(\zeta_2) = Y_1 \cap Y_2 = \mathrm{fix}\,(\zeta_3),$$

it follows from Proposition 2.2 that $\zeta_1\zeta_2 \in \mathcal{C}_n(\zeta_3)$, and so $\zeta_1\zeta_2 \in T$. Since T is completely isolated, $\zeta_1 \in T$ or $\zeta_2 \in T$, and so $\zeta_1 \in E(T)$ or $\zeta_2 \in E(T)$. Thus, $Y_1 \in \mathrm{sl}\,(T)$ or $Y_2 \in \mathrm{sl}\,(T)$, as required.

($\Leftarrow$) Suppose that $\mathrm{sl}\,(T)$ is completely isolated, and that, for $\alpha, \beta \in \mathcal{C}_n$, $\alpha\beta \in T$. As above, let ζ_1, ζ_2 and ζ_3 be the elements in $\mathcal{C}_n$ such that $\mathrm{fix}\,(\alpha) = \mathrm{fix}\,(\zeta_1)$,

$\text{fix}(\beta) = \text{fix}(\zeta_2)$ and $\text{fix}(\alpha\beta) = \text{fix}(\zeta_3)$, respectively. From Proposition 2.2, we have $\alpha \in \mathcal{C}_n(\zeta_1)$, $\beta \in \mathcal{C}_n(\zeta_2)$ and $\alpha\beta \in \mathcal{C}_n(\zeta_3)$, and so since $\alpha\beta \in T$, it follows from Proposition 2.3 (ii) that $\zeta_3 \in T$. From Proposition 2.3 (i), since we have

$$\text{fix}(\zeta_3) = \text{fix}(\alpha\beta) = \text{fix}(\alpha) \cap \text{fix}(\beta) = \text{fix}(\zeta_1) \cap \text{fix}(\zeta_2) \in \text{sl}(T),$$

and since $\text{sl}(T)$ is completely isolated, it follows that $\text{fix}(\zeta_1)$ or $\text{fix}(\zeta_2)$ is in $\text{sl}(T)$, and so ζ_1 or ζ_2 is in T. Then, since $\alpha \in \mathcal{C}_n(\zeta_1)$ and $\beta \in \mathcal{C}_n(\zeta_2)$, it follows from Lemma 2.6 (i) that α or β is an element of T, as required. □

For $n \geq 2$, let Y be a subset of X_n containing 1. Then, we consider the sets:

$$\mathcal{C}_n[Y] = \{\alpha \in \mathcal{C}_n : Y \subseteq \text{fix}(\alpha)\} \quad \text{and}$$
$$\overline{\mathcal{C}}_n[Y] = \{\alpha \in \mathcal{C}_n : Y \setminus \text{fix}(\alpha) \neq \emptyset\}.$$

First recall that if $\alpha \in \mathcal{C}_n$, then $1 \in \text{fix}(\alpha)$. Then, it is clear that $\mathcal{C}_n[\{1\}] = \mathcal{C}_n$ is isolated, and that $\overline{\mathcal{C}}_n[\{1\}] = \emptyset$. Moreover, $\mathcal{C}_n[X_n] = \{1_n\}$ and $\overline{\mathcal{C}}_n[X_n] = \mathcal{C}_n \setminus \{1_n\}$ are both completely isolated. Now, we state and prove a less trivial result:

Lemma 2.11 *For every proper subset $Y \neq \{1\}$ of X_n containing 1, $\mathcal{C}_n[Y]$ and $\overline{\mathcal{C}}_n[Y]$ are both completely isolated subsemigroups of $\mathcal{C}_n$.*

Proof Given $\alpha, \beta \in \mathcal{C}_n$, it follows from Proposition 2.3 (i) that for every $i \in Y$, $i \in \text{fix}(\alpha\beta)$ if and only if $i \in \text{fix}(\alpha)$ and $i \in \text{fix}(\beta)$, and so $\mathcal{C}_n[Y]$ is a subsemigroup of $\mathcal{C}_n$. Similarly, since

$$Y \setminus \text{fix}(\alpha\beta) = Y \setminus (\text{fix}(\alpha) \cap \text{fix}(\beta)) = (Y \setminus \text{fix}(\alpha)) \cup (Y \setminus \text{fix}(\beta)),$$

it follows that $\overline{\mathcal{C}}_n[Y]$ is a subsemigroup of $\mathcal{C}_n$. Now, the result follows from Lemma 2.9. □

In the above proof, it is also shown that $\overline{\mathcal{C}}_n[Y]$ is an ideal of $\mathcal{C}_n$.

Proposition 2.12 *If T is a completely isolated subsemigroup of $\mathcal{C}_n$, then T is a subsemigroup of $\mathcal{C}_n[Y]$ where $Y = \bigcap_{\zeta \in E(T)} \text{fix}(\zeta)$.*

Proof For $\alpha \in T$, it follows from Theorem 2.7 (a) and Proposition 2.2 that there exists $\zeta \in E(T)$ such that $\alpha \in \mathcal{C}_n(\zeta)$ and $\text{fix}(\alpha) = \text{fix}(\zeta)$. Thus, $Y \subseteq \text{fix}(\alpha)$, and so $\alpha \in \mathcal{C}_n[Y]$. □

In general, it is not the case that T is $\mathcal{C}_n[Y]$ where $Y = \bigcap_{\zeta \in E(T)} \text{fix}(\zeta)$. To demonstrate this fact, we give the following counter-example:

Example 2.13 Let

$$T = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix} \right\}.$$

It is easy to check that T is a subsemigroup of $\mathcal{C}_4$, and moreover, $\mathrm{sl}\,(T) = \{\{1,2\},\{1,2,3\}\}$ and $\mathrm{csl}\,(T) = \{\{1\},\{1,3\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\}\}$ which are both subsemigroups of $\mathcal{SL}_4(1)$. Therefore, it follows from Lemma 2.9 and Theorem 2.10 that T is completely isolated. However, $\begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,2\,4 \end{pmatrix} \in \mathcal{C}_4[\{1,2\}]$, and so $\mathcal{C}_4[\{1,2\}] \neq T$. □

In the next section, we characterize all situations where the example does not occur.

3 Convex Subsemigroups of $\mathcal{C}_n$

Next, we focus on (left/right) convex subsemigroups of $\mathcal{C}_n$. Since every (left/right) convex subsemigroup is completely isolated, in this section, we only consider completely isolated subsemigroups of $\mathcal{C}_n$.

Lemma 3.1 *For any completely isolated subsemigroup T of $\mathcal{C}_n$, T is left (right) convex if and only if* $\mathrm{sl}\,(T)$ *is a left (right) convex subsemigroup of* $\mathcal{SL}_n(1)$.

Proof The proofs are similar to the proof of Theorem 2.10. □

Since $\mathcal{SL}_n(1)$ is a commutative semigroup, a subsemigroup of $\mathcal{SL}_n(1)$ is left convex if and only if it is right convex, and so from Lemma 3.1, a subsemigroup of $\mathcal{C}_n$ is left if and only if it is right convex. Therefore, it is enough to classify only convex subsemigroups of $\mathcal{C}_n$. Moreover, if T is a convex subsemigroup of $\mathcal{C}_n$, then since for any $a \in T$, $1_n a = a 1_n = a \in T$, $1_n \in T$. From now on, we consider only completely isolated subsemigroups of $\mathcal{C}_n$ with identity.

Theorem 3.2 *Let T be a (completely isolated) subsemigroup of $\mathcal{C}_n$ with identity. Then, T is convex if and only if $T = \mathcal{C}_n[Y]$ where $Y = \bigcap_{\zeta \in E(T)} \mathrm{fix}\,(\zeta)$.*

Proof Let $E(T) = \{\zeta_1, \ldots, \zeta_r\}$, $Y = \bigcap_{i=1}^{r} \mathrm{fix}\,(\zeta_i)$, and let ζ_Y be the unique idempotent with $\mathrm{fix}\,(\zeta_Y) = Y$. Since $\zeta_1 \cdots \zeta_r \in T$, it follows from Proposition 2.3 (i) and (ii), and Theorem 2.7 (a) that $\mathrm{fix}\,(\zeta_1 \cdots \zeta_r) = Y$, and that $\zeta_Y \in T$. Moreover, since every completely isolated subsemigroup is isolated, it follows from Lemma 2.6 (i) that $\mathcal{C}_n(\zeta_Y)$ is a subsemigroup of T.

(⇒) Suppose that T is a convex subsemigroup of $\mathcal{C}_n$. Given $\alpha \in \mathcal{C}_n[Y]$, since $\mathrm{fix}\,(\alpha\zeta_Y) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\zeta_Y) = \mathrm{fix}\,(\zeta_Y) = Y$, it follows from Proposition 2.2 and Theorem 2.7 (a) that $\alpha\zeta_Y \in \mathcal{C}_n(\zeta_Y)$, and so $\alpha\zeta_Y \in T$. From convexity of T, we have $\alpha \in T$, and so $\mathcal{C}_n[Y] \subseteq T$. It follows from Proposition 2.12 that $T = \mathcal{C}_n[Y]$.

(⇐) First of all, from Lemma 2.11, $T = \mathcal{C}_n[Y]$ is completely isolated. For $\alpha, \beta \in \mathcal{C}_n$, suppose that $\alpha\beta \in \mathcal{C}_n[Y]$. Since $Y \subseteq \mathrm{fix}\,(\alpha\beta)$, it follows from Proposition 2.3 (i) that $Y \subseteq \mathrm{fix}\,(\alpha)$ and $Y \subseteq \mathrm{fix}\,(\beta)$, and so both α and β are elements of $\mathcal{C}_n[Y]$, as required. □

For every proper subset $Y \neq \{1\}$ of X_n containing 1, we know from Lemma 2.11 that both $\mathcal{C}_n[Y]$ and $\overline{\mathcal{C}}_n[Y]$ are completely isolated. From Theorem 3.2, we know also that $\mathcal{C}_n[Y]$ is convex. However, $\overline{\mathcal{C}}_n[Y]$ is not convex in general. To illustrate this fact, we give the following counter example:

Example 3.3 Let $n = 4$ and $Y = \{1, 2\}$. Then,

$$\mathcal{SL}_4(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\},$$

and so $\alpha \in \overline{\mathcal{C}}_4[\{1, 2\}]$ if and only if $\text{fix}\,(\alpha) \in \{\{1\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}\}$. If we consider two elements $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 4 \end{pmatrix}$, then we see that $\beta\gamma = \gamma\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}$, which is an element of $\overline{\mathcal{C}}_4[\{1, 2\}]$. However, β is not an element of $\overline{\mathcal{C}}_4[\{1, 2\}]$, and so $\overline{\mathcal{C}}_4[\{1, 2\}]$ is not convex. □

Moreover, from Lemma 2.11, Theorems 2.7 and 3.2 , we have the following immediate result:

Corollary 3.4 *A subsemigroup T of $\mathcal{SL}_n$ is convex if and only if*

$$T = \{Y \in \mathcal{SL}_n : Y_T \subseteq Y\}$$

where $Y_T = \bigcap_{Y \in T} Y$, that is, Y_T is the zero of T. □

For every proper subset $Y \neq \{1\}$ of X_n containing 1, suppose that $E(\mathcal{C}_n[Y]) = \{\zeta_1, \zeta_2, \ldots, \zeta_r\}$. Then, it is clear from Lemma 2.1 (ii) that $\mathcal{C}_n[Y]$ is the union of its disjoint subsemigroups $\mathcal{C}_n(\zeta_1), \ldots, \mathcal{C}_n(\zeta_r)$. For each $1 \leq i \leq r$, let $\text{fix}\,(\zeta_i) = \{1 = a_{i,1} < a_{i,2} < \cdots < a_{i,p_i}\}$. Then, for each $1 \leq i \leq r$, it follows from [16, Theorem 3] that $|\mathcal{C}_n(\zeta_i)| = \prod_{j=1}^{p_i} C_{k_j-1}$ (recall that C_n is n-th Catalan number), and so we have

$$|\mathcal{C}_n[Y]| = \sum_{i=1}^{r} \left(\prod_{j=1}^{p_i} C_{k_j-1} \right)$$

where $k_j = a_{i,j+1} - a_{i,j}$ for $1 \leq j \leq p_i - 1$ and $k_{p_i} = n - a_{i,p_j} + 1$.

For all $1 \leq i \leq n-1$, we consider the transformations

$$\xi_i = \begin{pmatrix} 1 & \cdots & i & i+1 & i+2 & \cdots & n \\ 1 & \cdots & i & i & i+2 & \cdots & n \end{pmatrix},$$

which are idempotent elements of $\mathcal{C}_n$. It is known that $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\}$ is the minimum generating set for $\mathcal{C}_n \setminus \{1_n\}$, and so $\text{rank}\,(\mathcal{C}_n \setminus \{1_n\}) = n - 1$ for $n \geq 2$ (see [5, Theorem 14.4.5]).

Lemma 3.5 *For every proper subset $Y \neq \{1\}$ of X_n containing 1, $\mathcal{C}_n[Y]$ is generated by the set $\{\xi_i : i + 1 \in X_n \setminus Y\} \cup \{1_n\}$.*

Proof If $|Y| = n-1$, then it is clear that $\mathcal{C}_n[Y] = \{\xi_i\} \cup \{1_n\}$ for some $1 \le i \le n-1$, and so the result is clear. Suppose that $2 \le |Y| \le n-2$ (and that $n \ge 4$). Let $A = \{\xi_i : i+1 \in X_n \setminus Y\}$. Then, we show that A is a generating set for $\mathcal{C}_n[Y]$ by induction on the cardinality of shift. First of all, if $\alpha \in \mathcal{C}_n[Y] \setminus \{1_n\}$, then $1 \le |\text{shift}\,(\alpha)| \le n - |Y|$, and moreover, if $|\text{shift}\,(\alpha)| = 1$, then it is clear that $\alpha \in A$. Now, we suppose that $|\text{shift}\,(\alpha)| \ge 2$. Let i be the maximum element of shift (α), and let $i\alpha = j$. Hence, we have the inequalities $1 \le j \le i-1$, and $(j+1)\alpha \le j$. Then, consider the transformation α_i defined by

$$x\alpha_i = \begin{cases} x\alpha & \text{if } x \le i-1 \\ x & \text{otherwise} \end{cases}.$$

Then, it is clear that $\alpha_i \in \mathcal{C}_n[Y]$ and $|\text{shift}\,(\alpha_i)| = |\text{shift}\,(\alpha)| - 1$. Moreover, since $j+1, \ldots, i-1, i \in X_n \setminus Y$, we have $\xi_{i-1}, \xi_{i-2}, \ldots, \xi_j \in A$ and

$$\alpha = \alpha_i \xi_{i-1} \xi_{i-2} \cdots \xi_j,$$

as required. □

From the above proof, if shift $(\alpha) = \{x_1 < x_2 < \cdots < x_r\}$ and if $k = \sum_{i=1}^{r} (x_i - x_i\alpha)$, then we also have $\alpha \in A^k$, that is, α can be written as a product of k many elements of A.

Example 3.6 If $\alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,1\,1\,2\,5\,5\,5\,8\,8 \end{pmatrix} \in \mathcal{C}_9[\{1, 5\}]$, then we have

$$\begin{aligned} \alpha &= \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,1\,1\,2\,5\,5\,5\,8\,9 \end{pmatrix} \xi_8 \\ &= \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,1\,1\,2\,5\,5\,7\,8\,9 \end{pmatrix} \xi_6\xi_5\xi_8 \\ &\;\vdots \\ &= \xi_1\xi_2\xi_1\xi_3\xi_2\xi_5\xi_6\xi_5\xi_8, \end{aligned}$$

and $(2-2\alpha) + (3-3\alpha) + (4-4\alpha) + (6-6\alpha) + (7-7\alpha) + (9-9\alpha) = 9$, as claimed. □

Theorem 3.7 *For any subset Y of X_n containing* 1*, the set of idempotents* $\{\xi_i : i+1 \in X_n \setminus Y\} \cup \{1_n\}$ *is the minimum generating set of* $\mathcal{C}_n[Y]$*, and so if* $|Y| = m$*, then* rank $(\mathcal{C}_n[Y]) = n - m + 1$.

Proof If $Y = \{1\}$, then $\mathcal{C}_n[\{1\}] = \mathcal{C}_n$, as indicated above, $\{\xi_1, \xi_2, \ldots, \xi_{n-1}\} \cup \{1_n\}$ is the minimum generating set of $\mathcal{C}_n[\{1\}]$. If $Y = X_n$, then $\mathcal{C}_n[X_n] = \{1_n\}$, and so $\{1_n\}$ is the minimum generating set of $\mathcal{C}_n[X_n]$. Since $\xi_1, \xi_2, \ldots, \xi_{n-1}$ and 1_n are all irreducible elements of $\mathcal{C}_n$, the result follows Lemma 3.5. □

For every subset Y of X_n, since the convex subsemigroup

$$\mathcal{SL}_n[Y] = \{Z : Y \subseteq Z \subseteq X_n\}$$

of $\mathcal{SL}_n$ plays a crucial role to classify convex subsemigroups of $\mathcal{C}_n$, we should give the next result:

Theorem 3.8 *For any subset Y of X_n, the set $\{X_n \setminus \{i\} : i \in X_n \setminus Y\} \cup \{X_n\}$ is the minimum generating set of $\mathcal{SL}_n[Y]$, and so if $|Y| = m$, then* $\operatorname{rank}(\mathcal{SL}_n[Y]) = n - m + 1$.

Proof If $Y = X_n$, then $\mathcal{SL}_n[X_n] = \{X_n\}$, and so $\{X_n\}$ is the minimum generating set of $\mathcal{SL}_n[X_n]$. If $Y \subseteq Z \subsetneqq X_n$, then one can easily prove that

$$Z = \bigcap_{i \in X_n \setminus Z} X_n \setminus \{i\}.$$

Therefore, the set $\{X_n \setminus \{i\} : i \in X_n \setminus Y\} \cup \{X_n\}$, which contains irreducible elements, is the minimum generating set of $\mathcal{SL}_n[Y]$. □

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