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On the Neighbor-Distinguishing Indices of Planar Graphs

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Abstract

Let G be a simple graph with no isolated edges. The neighbor-distinguishing edge coloring of G is a proper edge coloring of G such that any pair of adjacent vertices have different sets consisting of colors assigned on their incident edges. The neighbor-distinguishing index of G, denoted by $\chi'_a(G)$, is the minimum number of colors in such an edge coloring of G. In this paper, we show that if G is a connected planar graph with maximum degree $\Delta \geq 14$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$, and $\chi'_a(G) = \Delta + 1$ if and only if G contains a pair of adjacent vertices of maximum degree. This improves a result in [W. Wang, D. Huang, A characterization on the adjacent vertex distinguishing index of planar graphs with large maximum degree, SIAM J. Discrete Math. 29(2015), 2412–2431], which says that every connected planar graph G with $\Delta \geq 16$ has $\Delta \leq \chi'_a(G) \leq \Delta + 1$, and $\chi'_a(G) = \Delta + 1$ if and only if G contains a pair of adjacent vertices of maximum degree.

Keywords Neighbor-distinguishing edge coloring · Planar graph · Maximum degree · Discharging method

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1 Introduction

Graphs considered in this paper are finite and simple. A graph G is called *planar* if G can be drawn in the plane without any two of its edges crossing. A graph G that is already drawn in the plane in this manner is a *plane* graph. Let G be a plane graph with vertex set $V(G)$, edge set $E(G)$, face set $F(G)$, minimum degree $\delta(G)$, and maximum degree $\Delta(G)$ (for short, Δ). Let $|G| = |V(G)|$ and $\|G\| = |E(G)|$ denote the order and size of the graph G, respectively. For positive integers p, q with $p \leq q$, let $[p, q]$ denote the set of integers $\{p, p+1, \ldots, q\}$.

A *proper edge k-coloring* of a graph G is a mapping ϕ from $E(G)$ to $[1, k]$ such that any two adjacent edges are assigned distinct colors. The *chromatic index* $\chi'(G)$ of G is the smallest k such that G has a proper edge k-coloring. For a proper edge k-coloring ϕ of G and $v \in V(G)$, denote by $C_\phi(v) = \{\phi(uv)|uv \in E(G)\}$ the set of colors of edges incident with v. The proper edge k-coloring ϕ is called *neighbor-distinguishing* (or ϕ is an NDE-k-coloring) if $C_\phi(u) \neq C_\phi(v)$ for each edge $uv \in E(G)$. Two adjacent vertices u and v are said to be *conflict* if $C_\phi(u) = C_\phi(v)$. The *neighbor-distinguishing index* $\chi'_a(G)$ of G is the smallest k such that G has an NDE-k-coloring.

A graph G is called *normal* if it contains no isolated edges. If G has an NDE-coloring, then G is normal. It holds trivially that $\chi'_a(G) \geq \chi'(G) \geq \Delta$ for any graph G. Moreover, if G contains two adjacent vertices of maximum degree, then $\chi'_a(G) \geq \Delta + 1$. Zhang, Liu and Wang [11] introduced the NDE-coloring of graphs and proposed the following conjecture:

Conjecture 1 *If G is a connected graph with $|G| \geq 3$ and $G \neq C_5$, then $\chi'_a(G) \leq \Delta + 2$.*

Balister et al. [2] affirmed Conjecture 1 for bipartite graphs and subcubic graphs. Akbari et al. [1] proved that $\chi'_a(G) \leq 3\Delta$ for any graph G. This result is gradually improved to that $\chi'_a(G) \leq 2.5\Delta + 5$ in [10], to that $\chi'_a(G) \leq 2.5\Delta$ in [9], and to that $\chi'_a(G) \leq 2\Delta + 2$ in [7]. Using probabilistic method, Hatami [4] showed that every graph G with sufficiently large Δ has $\chi'_a(G) \leq \Delta + 300$. Recently, Joret and Lochet [6] improved this result by replacing 300 with 19. Suppose that G be a planar graph. Bonamy et al. [3] showed that if $\Delta \geq 12$, then $\chi'_a(G) \leq \Delta + 1$. Furthermore, in 2015, Wang and Huang [8] showed that (1) if $\Delta \geq 15$, then $\chi'_a(G) \leq \Delta + 1$; and (2) if $\Delta \geq 16$, then $\chi'_a(G) = \Delta + 1$ if and only if G contains a pair of adjacent vertices of maximum degree.

This paper focuses on improving the result of Wang and Huang [8]. Namely, we will show the following:

Theorem 1 *Let G be a planar graph with $\Delta \geq 14$. Then $\Delta \leq \chi'_a(G) \leq \Delta + 1$; and $\chi'_a(G) = \Delta + 1$ if and only if G contains a pair of adjacent vertices of maximum degree.*

2 Notation

Let G be a plane graph and $H \subseteq G$. If the boundary vertices of a face f of H are $u_1, u_2, \ldots, u_k$ in a cyclic ordering, then we write $f = [u_1 u_2 \cdots u_k]$. For $x \in V(H) \cup$

$F(H)$, let $d_H(x)$ denote the degree of x in H. A vertex of degree k (at least k, at most k, respectively) in H is called a *k-vertex* (*k^+-vertex*, *k^--vertex*, respectively). Similarly, we can define k-face, k^+-face and k^--face. For a vertex $v \in V(H)$, let $N_H(v)$ denote the set of neighbors of v in H. Let $N_k^H(v)$ denote the set of k-vertices adjacent to v in H, and set $d_k^H(v) = |N_k^H(v)|$. Set $N_{i_1,i_2,\ldots,i_k}^H(v) = N_{i_1}^H(v) \cup N_{i_2}^H(v) \cup \ldots \cup N_{i_k}^H(v)$. Similarly, we can define $N_{k^+}^H(v)$, $N_{k^-}^H(v)$, $d_{k^+}^H(v)$ and $d_{k^-}^H(v)$. For a face $f \in F(H)$, let $d_k^H(f)$ ($d_{k^+}^H(f)$, $d_{k^-}^H(f)$, respectively) denote the number of k-vertices (k^+-vertices, k^--vertices, respectively) incident with f in H. If no confusion is caused in the context, we omit the letter G in $d_G(v)$, $d_k^G(v)$, $d_{k^+}^G(v)$, $d_{k^-}^G(v)$, $d_k^G(f)$, $d_{k^+}^G(f)$ and $d_{k^-}^G(f)$.

A 3-cycle in the plane graph H is *special* if it has one 2-vertex, and *bad* if it has two 3-vertices. A 4-cycle in H is *special* if it has two 2-vertices. A 3-face in H is *special* or *bad* if its boundary forms a special or bad 3-cycle. A 4-face in H is *special* if its boundary forms a special 4-cycle. A vertex v of H is called *small* if $d_H(v) \in [1, 5]$.

Given a graph G, denote by $n_i(G)$ the number of i-vertices in $V(G)$ for $i \in [1, \Delta(G)]$. We say that G is *smaller* than a graph H if ($\|G\|$, $n_t(G)$, $n_{t-1}(G)$, …, $n_1(G)$) precedes ($\|H\|$, $n_t(H)$, $n_{t-1}(H)$, …, $n_1(H)$) regarding the standard lexicographic order, where $t = \max\{\Delta(G), \Delta(H)\}$. A graph is called *minimal* regarding a given property P if no smaller graph meets the property P.

Suppose that ϕ is an NDE-k-coloring of the graph G and $uv \in E(G)$. If $d_G(u) \neq d_G(v)$, then $C_\phi(u) \neq C_\phi(v)$. An edge uv is said to be *legally colored* if its color is different from that of its adjacent edges in $E(G)$ and no pair of new conflict vertices are produced.

3 Proof of Theorem 1

It suffices to show that $\chi'_a(G) = \Delta + 1$ if and only if G contains adjacent Δ-vertices. The necessity is clear by the foregoing discussion. To prove sufficiency, it is enough to show that if G is a planar graph with $\Delta \geq 14$ and without adjacent Δ-vertices, then $\chi'_a(G) \leq \Delta$. Assume that this is not true. Let G be a minimal counterexample of Theorem 1, that is, G has no NDE-Δ-coloring, while any other planar graph H smaller than G admits an NDE-Δ-coloring. Obviously, G is connected. Since G contains no adjacent Δ-vertices, it is easy to derive that no 1-vertex is adjacent to a Δ-vertex. Let $C = [1, \Delta]$ denote a set of Δ colors. Then $|C| \geq 14$.

3.1 Structural Properties of *G*

In all of the figures of this section, black bullets represent vertices which have no incident edges in $E(G)$ other than those shown, but white bullets represent vertices which might be adjacent to other vertices in $V(G)$ not in the configuration. A cut vertex v is called *nontrivial* if there exist at least two components of $G - v$ with order at least 2. We denote simply $\{1, 2, 3, 4, 5, 6, 14\}$ by $\{1 - 6, 14\}$; $\{1, 3, 4, 5, 6, 13, 14\}$ by $\{1, 3 - 6, 13, 14\}$; etc.

Claim 1 *G does not contain any nontrivial cut vertex.*

Claim 2 *There is no edge* $uv \in E(G)$ *with* $d(u) = 1$ *and* $d(v) \leq 7$.

Claim 3 *If* v *is a* k*-vertex with* $k \in [2, 5]$, *then* $d_{5^-}(v) = d_k(v) \leq 1$.

Claim 4 *There is no edge* $uv \in E(G)$ *with* $d(u) = d(v) = 2$.

Claim 1 is given in [8]. Claims 2–4 are established in [5].

Claim 5 *Let* $v \in V(G)$.

(1) *If* $d_1(v) \geq k \in [1, 4]$, *then* $d_s(v) = 0$ *for* $s \in [2, k+1]$.
(2) *If* $d(v) \leq \Delta - 1$, *then* $d_2(v) \leq 1$.
(3) v *is not incident with any special 4-cycle.*
(4) *If* v *is incident with a special 3-cycle, then* $d_2(v) = 1$.

Proof (1) Assume that $d_s(v) \geq 1$ for $s \in [2, k+1] \subseteq [2, 5]$. Let $v_1, v_2, \ldots, v_{k+1}$ be the neighbors of v such that $d(v_i) = 1$ for $i \in [1, k]$ and $d(v_{k+1}) = s$. Let $N(v_{k+1}) = \{u_1, u_2, \ldots, u_{s-1}, v\}$, as depicted in Fig. 1 a. By Claim 3, we consider two cases below:

*Case*1 $d_s(v_{k+1}) = 0$. We split v_{k+1} into one 1-vertex x_1 and one $(s-1)$-vertex x_2 such that x_1 is adjacent to v and x_2 is adjacent to $u_1, u_2, \ldots, u_{s-1}$, yielding a smaller graph H. Then $\|H\| = \|G\|$, $n_s(H) < n_s(G)$, $n_{s-1}(H) > n_{s-1}(G)$ and $n_1(H) > n_1(G)$. By the minimality of G, H has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. Note that $\varphi(vx_1) \notin [1, k]$. Sticking x_1 and x_2, we restore the graph G. If $\varphi(vx_1) \notin C_\varphi(x_2)$, then φ is an NDE-Δ-coloring of G. Otherwise, we exchange the colors of vx_1 and vv_j for $j \in [1, k] \backslash (C_\varphi(x_2) \backslash \{\varphi(vx_1)\})$ to obtain an NDE-Δ-coloring of G, which is a contradiction.

*Case*2 $d_s(v_{k+1}) = 1$, i.e., $d(u_{s-1}) = d(v_{k+1}) = s$. Then $s \in [3, 5]$ by Claim 4. Let $H = G - v_{k+1}u_{s-1}$. Then $\|H\| < \|G\|$, $n_s(H) < n_s(G)$ and $n_{s-1}(H) > n_{s-1}(G)$. By the minimality of G, H has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. Note that $\varphi(vv_{k+1}) \notin [1, k]$. If $C_\varphi(u_{s-1}) \neq C_\varphi(v_{k+1})$, then we color $v_{k+1}u_{s-1}$ properly. Otherwise, we exchange the colors of vv_t and vv_{k+1} for $t \in [1, k] \backslash (C_\varphi(v_{k+1}) \backslash \{\varphi(vv_{k+1})\})$ and color $v_{k+1}u_{s-1}$ properly. So we always get an NDE-Δ-coloring of G, which is a contradiction.

(2) Assume that $d_2(v) \geq 2$. Let v_1, v_2 be the neighbors of v with $d(v_1) = d(v_2) = 2$. We consider the following three cases.

*Case*1 Each v_i $(i = 1, 2)$ is not incident with a 3-face and v, v_1, v_2 are not incident with a 4-face (see Fig. 1b). Let $N(v_1) = \{v, w\}$ and $N(v_2) = \{v, u\}$. We get a smaller graph H by contracting vv_1 and vv_2. By the minimality of G, H has an NDE-Δ-coloring φ with $\varphi(vw) = 1$ and $\varphi(vu) = 2$. Reversely we subdivide vw, vu with v_1, v_2 respectively. By coloring vv_1 and v_2u with 2, vv_2 and v_1w with 1, we get an NDE-Δ-coloring of G, which is a contradiction.

*Case*2 v, v_1, v_2 are incident with a 4-face $[vv_1uv_2]$ (see Fig. 1c). We get a smaller graph H by splitting each v_i $(i = 1, 2)$ into two 1-vertices x_i and y_i such that x_i is adjacent to v and y_i is adjacent to u. By the minimality of G, H has an NDE-Δ-coloring φ. Next, we can stick x_i and y_i together for $i = 1, 2$, and exchange the colors of vx_1 and vx_2 if necessary. This gives an NDE-Δ-coloring of G, which is a contradiction.

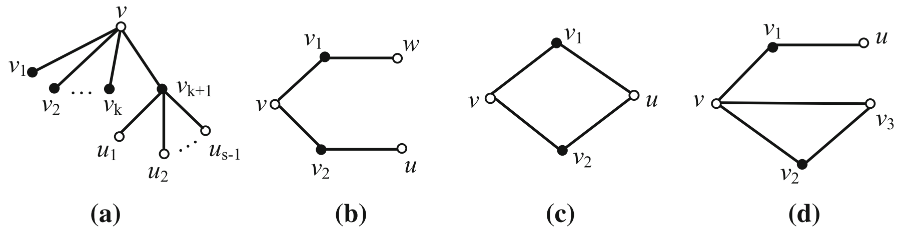

Fig. 1 Configurations used in the proof of Claim 5

*Case*3 Some v_i, say v_2, is incident with some 3-face $[vv_2v_3]$ (see Fig. 1d). Let $N(v_1) = \{v, u\}$. We get a smaller graph H by splitting v_1 into two 1-vertices x_1 and y_1 such that x_1 is adjacent to v and y_1 is adjacent to u. By the minimality of G, H has an NDE-Δ-coloring φ. Sticking x_1 and y_1, we restore the graph G. If $\varphi(vx_1) \neq \varphi(uy_1)$, then φ is an NDE-Δ-coloring of G. Otherwise, $\varphi(vx_1) = \varphi(uy_1)$. If $\varphi(v_2v_3) \neq \varphi(vx_1)$, then we exchange the colors of vx_1 and vv_2. If $\varphi(v_2v_3) = \varphi(vx_1)$, then we recolor v_2v_3 and vx_1 with $\varphi(vv_3)$, vv_3 with $\varphi(v_2v_3)$. So we always get an NDE-Δ-coloring of G, which is a contradiction.

In view of the proof of Cases 2 and 3 in the statement (2), we can prove the statements (3) and (4). □

Claim 6 *There is no edge $uv \in E(G)$ such that $6 \le d(v) \le 7$ and $d(u) \le 5$.*

Proof Assume that G contains an edge uv with $6 \le d(v) = k \le 7$ and $d(u) = s \le 5$. Let $N(v) = \{v_1, v_2, \ldots, v_{k-1}, u\}$ and $N(u) = \{u_1, u_2, \ldots, u_{s-1}, v\}$. By Claim 3, u has at most one conflict vertex, say u_1 (if it exists). Let $H = G - uv$, which has an NDE-Δ-coloring φ such that $\varphi(vv_i) = i$ for $i \in [1, k-1]$ and $\varphi(uu_j) = a_j$ for $j \in [1, s-1]$.

Without loss of generality, assume that $k = 7$ and $s = 5$ (for other cases, we have an easier proof). Let $a_j \in [1, 10]$ for $j \in [1, 4]$. Suppose that uv cannot be legally colored. That is, $C_\varphi(v_i) = \{1{-}6, i{+}10\}$ for $i \in [1, 4]$ or $C_\varphi(u_1) = \{a_1, a_2, a_3, a_4, 14\}$ and $C_\varphi(v_i) = \{1-6, i+10\}$ for $i \in [1, 3]$.

*Case*1 $C_\varphi(v_i) = \{1-6, i+10\}$ for $i \in [1, 4]$. For each $i \in [1, 4]$, we recolor vv_i with a color $b_i \in [7, 14] \backslash \{i+10\}$ such that v_i does not conflict with its neighbors, and color uv with a color in $[11, 14] \backslash \{b_i, i+10\}$.

If $C_\varphi(u_1) \notin \{\{a_1, a_2, a_3, a_4, t\} \mid t \in [11, 14]\}$, then we have at least $2 \times 4 = 8$ different ways to recolor or color some edges incident with v. So assume that $C_\varphi(u_1) = \{a_1, a_2, a_3, a_4, 11\}$. Notice that uv can be colored with a color in $[12, 14] \backslash \{b_i, i+10\}$. Then there are at least $1 \times 4 = 4$ different ways to recolor or color some edges incident with v. Since v has at most two conflict vertices other than v_1, v_2, v_3, v_4, φ can be extended to G, which is a contradiction.

*Case*2 $C_\varphi(u_1) = \{a_1, a_2, a_3, a_4, 14\}$ and $C_\varphi(v_i) = \{1-6, i+10\}$ for $i \in [1, 3]$. For each $i \in [1, 3]$, we recolor vv_i with a color $b_i \in [7, 14] \backslash \{i+10\}$ such that v_i does not conflict with its neighbors, and color uv with a color in $[11, 13] \backslash \{b_i, i+10\}$.

If there exists a color $b_i \notin [11, 13]$ for $i \in [1, 3]$, then we have at least four different ways to recolor or color some edges incident with v. So assume that $b_i \in [11, 13]$

for all $i \in [1, 3]$. We claim that there exists $b_s \neq b_t$ for $s, t \in [1, 3]$. Otherwise, $b_1 = b_2 = b_3 \in [11, 13]$, contradicting the choice of b_i. Say $b_1 \neq b_2$. Next, we recolor vv_1 with b_1, vv_2 with b_2, and color uv with $[11, 13]\backslash\{b_1, b_2\}$. Then we have at least $3+1 = 4$ different ways to recolor or color some edges incident with v. Since v has at most three conflict vertices other than v_1, v_2, v_3, we can extend φ to G, which is a contradiction. □

In the following discussion, if v is a l-vertex of G, then we denote by $v_1, v_2, \ldots, v_l$ the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_l)$.

Claim 7 *Let v be a k-vertex of G with $k = 8$ or 9.*

(1) *If $d_{4^-}(v) \geq 1$, then $d_{5^-}(v) \leq k - 7$.*
(2) *v is not in a bad 3-cycle.*

Proof (1) Assume that $d_{5^-}(v) \geq k - 6$. Let $d(v_1) = s \leq 4$, $d(v_p) \leq 5$ for $p \in [2, k-6]$, and $N(v_1) = \{u_1, u_2, \ldots, u_{s-1}, v\}$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, k]$. By Claim 3, we consider the following two cases.

*Case*1 $d_s(v_1) = 0$. Then v_1 has no conflict vertices. Without loss of generality, assume that $\varphi(v_1u_j) = a_j \in [1, k+j-1] \subseteq [1, k+2]$ for $j \in [1, s-1]$. First, we can color vv_1 with any color in $[k+3, 14]$. Second, for each $i \in [2, k-6]$, v_i has at most one conflict vertex by Claim 3. Recolor vv_i with a color $b_i \in [k, 14]\backslash C_\varphi(v_i)$ such that v_i does not conflict with its neighbors, and color vv_1 with a color in $[k+3, 14]\backslash\{b_i\}$. So there are at least $(14-k-2)+(14-k-3)\times(k-6-1) = -k^2 + 17k - 65 = 7$ different ways to recolor or color some edges incident with v. As v has at most $k - (k-6) = 6$ conflict vertices, φ can be extended to G, which is a contradiction.

*Case*2 $d_s(v_1) = 1$, i.e., $d(u_{s-1}) = d(v_1) = s$. By Claims 3 and 6, each vertex in $(N(v_1) \cup N(u_{s-1}))\backslash\{v_1, u_{s-1}\}$ is an 8^+-vertex. Recolor v_1u_{s-1} with a color in $[1, k-1]\backslash(C_\varphi(v_1) \cup C_\varphi(u_{s-1}))$. Let $a \in C_\varphi(u_{s-1})\backslash C_\varphi(v_1)$. Note that if vv_1 can be colored with some color in $[k, 14]\backslash\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{s-2})\}$, then u_{s-1} and v_1 do not conflict with each other. Without loss of generality, assume that $\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{s-2})\} \subseteq [1, k+2]$. Analogous to the analysis of Case 1, φ can be extended to G, which is a contradiction.

(2) Assume that v is in a bad 3-cycle. If $d(v) = 8$, then $d_{5^-}(v) \leq 1$ by (1). Otherwise, $d(v) = 9$. Let $d(v_1) = d(v_2) = 3$ and $v_1v_2 \in E(G)$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, 9]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$. Notice that if vv_1 or vv_2 can be recolored, then we can establish an NDE-Δ-coloring of G by coloring v_1v_2 properly. Then vv_1 or vv_2 can be recolored with any color in $[10, 14]$. So there are at least $2 \times 5 = 10$ different ways to recolor some edges incident with v. As v has at most seven conflict vertices, we can extend φ to G, which is a contradiction. □

Claim 8 *Let v be a 10-vertex of G.*

(1) *If $d_{4^-}(v) = m$ and $d_{k^-}(v) \geq 1$, then $d_{6^+}(v) \geq (5-k)m + 1$ for $k \in [1, 4]$.*

(2) *v is not in a bad 3-cycle.*

Proof (1) Assume that $d_{6^+}(v) \leq (5-k)m$. Let $d(v_1) = k \leq 4$, $d(v_p) \leq 4$ for $p \in [2, m]$, and $N(v_1) = \{u_1, u_2, \ldots, u_{k-1}, v\}$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, 10]$. By Claim 3, we consider the following two cases.

*Case*1 $d_k(v_1) = 0$. Then v_1 has no conflict vertices. Without loss of generality, assume that $\varphi(v_1u_j) = a_j \in [1, 9+j] \subseteq [1, 8+k]$ for $j \in [1, k-1]$. First, we can color vv_1 with any color in $[9+k, 14]$. Second, for each $i \in [2, m]$, v_i has at most one conflict vertex by Claim 3. Recolor vv_i with a color $b_i \in [10, 14]\backslash C_\varphi(v_i)$ such that v_i does not conflict with its neighbors, and color vv_1 with a color in $[9+k, 14]\backslash\{b_i\}$. So we have at least $(6-k) + (m-1)(5-k) = (5-k)m + 1$ different ways to recolor or color some edges incident with v. As v has at most $(5-k)m$ conflict vertices, φ can be extended to G, which is a contradiction.

*Case*2 $d_k(v_1) = 1$, i.e., $d(u_{k-1}) = d(v_1) = k$. By Claims 3 and 6, each vertex in $(N(v_1) \cup N(u_{k-1}))\backslash\{v_1, u_{k-1}\}$ is an 8^+-vertex. Recolor v_1u_{k-1} with a color in $[1, 9]\backslash(C_\varphi(v_1) \cup C_\varphi(u_{k-1}))$. Let $a \in C_\varphi(u_{k-1})\backslash C_\varphi(v_1)$. Note that if vv_1 can be colored with some color in $[10, 14]\backslash\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{k-2})\}$, then u_{k-1} and v_1 do not conflict with each other. Without loss of generality, assume that $\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{k-2})\} \subseteq [1, 8+k]$. Analogous to the analysis of Case 1, φ can be extended to G, which is a contradiction.

(2) Assume that v is in a bad 3-cycle. Let $d(v_1) = d(v_2) = 3$ and $v_1v_2 \in E(G)$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, 10]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$. Notice that if vv_1 or vv_2 can be recolored, then we can establish an NDE-Δ-coloring of G by coloring v_1v_2 properly. First, we can recolor vv_1 or vv_2 with any color in $[11, 14]$. Second, we recolor vv_1 with 11, vv_2 with 12. So we have at least $2 \times 4 + 1 = 9$ different ways to recolor some edges incident with v. While v has at most eight conflict vertices, we can extend φ to G, deriving a contradiction. □

Claim 9 *Let v be an 11-vertex of G.*

(1) *If $d_{4^-}(v) = m$ and $d_{k^-}(v) \geq 1$, then $d_{6^+}(v) \geq (4-k)m + 1$ for $k \in [1, 3]$.*
(2) *If v is in a bad 3-cycle, then $d_{6^+}(v) \geq 9$.*

Proof (1) Assume that $d_{6^+}(v) \leq (4-k)m$. Let $d(v_1) = k \leq 3$, $d(v_p) \leq 4$ for $p \in [2, m]$, and $N(v_1) = \{u_1, u_2, \ldots, u_{k-1}, v\}$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, 11]$. □

Remark 1 If $d(v_2) \leq 3$, or $d(v_2) = 4$ and $d_4(v_2) = 0$, then we can recolor vv_2 with a color $b_2 \in [11, 14]\backslash C_\varphi(v_2)$ such that v_2 does not conflict with its neighbors. If $d(v_2) = 4$ and $d_4(v_2) = 1$, say, $x_2 \in N(v_2)$ with $d(x_2) = d(v_2) = 4$, then we recolor v_2x_2 with a color in $[1, 10]\backslash(C_\varphi(v_2) \cup C_\varphi(x_2))$. Let $\alpha_2 \in C_\varphi(x_2)\backslash C_\varphi(v_2)$. Then we can recolor vv_2 with a color $b_2 \in [11, 14]\backslash((C_\varphi(v_2)\backslash\{\varphi(v_2x_2), \varphi(vv_2)\}) \cup \{\alpha_2\})$. Hence, there exists a color $b_2 \in [11, 14]$ such that if vv_2 is recolored with b_2, then v_2 does not conflict with its neighbors.

Now, by Claim 3, we have to consider the following two cases.

*Case*1 $d_k(v_1) = 0$. Then v_1 has no conflict vertices. Without loss of generality, assume that $\varphi(v_1u_j) = a_j \in [1, 10 + j] \subseteq [1, 9 + k]$ for $j \in [1, k - 1]$. First, we can color vv_1 with any color in $[10 + k, 14]$. Second, for each $i \in [2, m]$, v_i has at most one conflict vertex by Claim 3. As in Remark 1, we can recolor vv_i with a color $b_i \in [11, 14]$ such that v_i does not conflict with its neighbors, and color vv_1 with a color in $[10+k, 14]\backslash\{b_i\}$. So there are at least $(5-k)+(m-1)(4-k) = (4-k)m+1$ different ways to recolor or color some edges incident with v. As v has at most $(4-k)m$ conflict vertices, φ can be extended to G, which is a contradiction.

*Case*2 $d_k(v_1) = 1$, i.e., $d(u_{k-1}) = d(v_1) = k$. By Claims 3 and 6, each vertex in $(N(v_1) \cup N(u_{k-1}))\backslash\{v_1, u_{k-1}\}$ is an 8^+-vertex. Recolor v_1u_{k-1} with a color in $[1, 10]\backslash(C_\varphi(v_1) \cup C_\varphi(u_{k-1}))$. Let $a \in C_\varphi(u_{k-1})\backslash C_\varphi(v_1)$. Note that if vv_1 can be colored with some color in $[11, 14]\backslash\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{k-2})\}$, then u_{k-1} and v_1 do not conflict with each other. Without loss of generality, assume that $\{a, \varphi(v_1u_1), \ldots, \varphi(v_1u_{k-2})\} \subseteq [1, 9+k]$. Analogous to the analysis of Case 1, φ can be extended to G, which is a contradiction.

(2) Assume that $d_{6^+}(v) \le 8$. Let $d(v_1) = d(v_2) = 3$ and $v_1v_2 \in E(G)$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, 11]$. If $C_\varphi(v_1) \ne C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$. Notice that if vv_1 or vv_2 can be recolored, then we can establish an NDE-Δ-coloring of G by coloring v_1v_2 properly. First, we can recolor vv_1 or vv_2 with any color in $[12, 14]$. Second, we recolor vv_1 with 12, vv_2 with 13 or 14. Third, we recolor vv_1 with 13, vv_2 with 14. So we have at least $2 \times 3 + 2 + 1 = 9$ different ways to recolor some edges incident with v. As v has at most eight conflict vertices, we can extend φ to G, which is a contradiction. □

Claim 10 *Let v be a 12-vertex of G.*

(1) *If $d_1(v) \ge 1$ and $d_{3^-}(v) \ge 2$, then $d_{6^+}(v) \ge 2d_{4^-}(v) + 1$.*
(2) *If $d_{2^-}(v) \ge 1$, then $d_{6^+}(v) \ge d_{3^-}(v) + 1$.*
(3) *If v is in a bad 3-cycle, then $d_{6^+}(v) \ge 5$.*
(4) *If v is in a special 3-cycle and $d_{3^-}(v) \ge 2$, then $d_{6^+}(v) \ge 5$.*

Proof (1) Assume that $d_{6^+}(v) \le 2d_{4^-}(v)$. Let $d(v_1) = 1$ and $d(v_2) \le 3$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, 12]$.
First, we can color vv_1 with any color in $[12, 14]$. Second, v_2 has at most one conflict vertex by Claim 3. Analogous to the analysis of Remark 1 in Claim 9(1), we can recolor vv_2 with a color $b_2 \in [12, 14]$ such that v_2 does not conflict with its neighbors, and color vv_1 with any color in $[12, 14]\backslash\{b_2\}$. Third, for each v_i with $d(v_i) \le 4$ and $i \ge 3$, v_i has at most one conflict vertex by Claim 3. By Remark 1 in Claim 9(1), we can recolor vv_i with a color $b_i \in \{1, 12 - 14\}$ such that v_i does not conflict with its neighbors. If $b_i \in [12, 14]$, then we color vv_1 with any color in $[12, 14]\backslash\{b_i\}$. If $b_i = 1$, then we recolor vv_2 with $b_2 \in [12, 14]$ and color vv_1 with any color in $[12, 14]\backslash\{b_2\}$. So at least $3 + 2(d_{4^-}(v) - 1) = 2d_{4^-}(v) + 1$ different ways can be used to recolor or color some edges incident with v. As v has at most $2d_{4^-}(v)$ conflict vertices, φ can be extended to G, which is a contradiction.

(2) Assume that $d_{6^+}(v) \le d_{3^-}(v)$. Let $d(v_1) \le 2$ and x be the neighbor of v_1 other than v (if it exists). Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$

for $i \in [2, 12]$. Suppose that $\varphi(v_1x) \in [1, 12]$, if x exists.

First, v_1 has no conflict vertex by Claim 4. Then we can color vv_1 with any color in $\{13, 14\}$. Second, for each v_i with $d(v_i) \leq 3$ and $i \geq 2$, v_i has at most one conflict vertex by Claim 3. As for Remark 1 in Claim 9(1), we can recolor vv_i with a color $b_i \in [12, 14]$ such that v_i does not conflict with its neighbors, and color vv_1 with a color in $\{13, 14\}\backslash\{b_i\}$. So we have at least $2 + (d_{3^-}(v) - 1) = d_{3^-}(v) + 1$ different ways to recolor or color some edges incident with v. As v has at most $d_{3^-}(v)$ conflict vertices, φ can be extended to G, which is a contradiction.

(3) Assume that $d_{6^+}(v) \leq 4$. Let $d(v_1) = d(v_2) = 3$ and $v_1v_2 \in E(G)$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, 12]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$. Notice that if vv_1 or vv_2 can be recolored, then we can establish an NDE-Δ-coloring of G by coloring v_1v_2 properly. First, vv_1 or vv_2 can be recolored with any color in $\{13, 14\}$. Second, we recolor vv_1 with 13, vv_2 with 14. So we have at least $2 \times 2 + 1 = 5$ different ways to recolor some edges incident with v. While v has at most four conflict vertices, we can extend φ to G, deriving a contradiction.

(4) Assume that $d_{6^+}(v) \leq 4$. Let $d(v_1) = 2$, $d(v_2) \leq 3$ and vv_1v_xv is a special 3-cycle with $d(v_x) \geq 8$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, 12]$. We consider the following two cases.

Case 1 $\varphi(v_1v_x) \notin [12, 14]$. First, we can color vv_1 with any color in $[12, 14]$. Second, v_2 has at most one conflict vertex by Claim 3. As for Remark 1 in Claim 9(1), we can recolor vv_2 with a color $b_2 \in [12, 14]$ such that v_2 does not conflict with its neighbors, and color vv_1 with a color in $[12, 14]\backslash\{b_2\}$. So we have at least $3 + 2 = 5$ different ways to recolor or color some edges incident with v. As v has at most four conflict vertices, we can extend φ to G, which is a contradiction.

Case 2 $\varphi(v_1v_x) \in [12, 14]$, say $\varphi(v_1v_x) = 12$. First, vv_1 can be colored with any color in $\{13, 14\}$. Second, v_2 has at most one conflict vertex by Claim 3. As for Remark 1 in Claim 9(1), we can recolor vv_2 with a color $b_2 \in [12, 14]$ such that v_2 does not conflict with its neighbors, and color vv_1 with a color in $\{13, 14\}\backslash\{b_2\}$. Third, we exchange the colors of v_1v_x and vv_x, and color vv_1 with any color in $\{13, 14\}$. Hence, we have at least $2 + 1 + 2 = 5$ different ways to recolor or color some edges incident with v. Since v has at most four conflict vertices, we can extend φ to G, which is a contradiction. □

Claim 11 *Let v be a k-vertex of G with $k \in [13, \Delta - 1]$ and $d_1(v) \geq 1$. If $d_{2^-}(v) \geq s$ for $s \in [2, 4]$, then $d_{6^+}(v) \geq d_{(s+1)^-}(v) + 1$.*

Proof Assume that $d_{6^+}(v) \leq d_{(s+1)^-}(v)$. Let $d(v_1) = 1$ and $d(v_i) \leq 2$ for $i \in [2, s]$. Then $G - vv_1$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i - 1$ for $i \in [2, k]$. If $d(v_i) = 2$ for $i \in [2, s]$, then $d_2(v_i) = 0$ by Claim 4.

First, we can color vv_1 with any color in $\{k, k+1\}$. Second, for each i with $i \in [2, s]$, we recolor vv_i with a color $b_i \in \{k, k+1\}\backslash C_\varphi(v_i)$ and color vv_1 with $\{k, k+1\}\backslash\{b_i\}$. Third, for each v_i with $d(v_i) \leq s + 1 \leq 5$ and $i \geq s + 1$, Analogous to the analysis of Remark 1 in Claim 9(1), we can recolor vv_i with a color $c_i \in ([1, s - 1] \cup \{k, k + 1\})\backslash C_\varphi(v_i)$ such that v_i does not conflict with its neighbors. If $c_i \in [1, s - 1]$, say $c_i = j$, then we recolor vv_{j+1} with a color $b_j \in \{k, k + 1\}\backslash C_\varphi(v_{j+1})$, and color vv_1

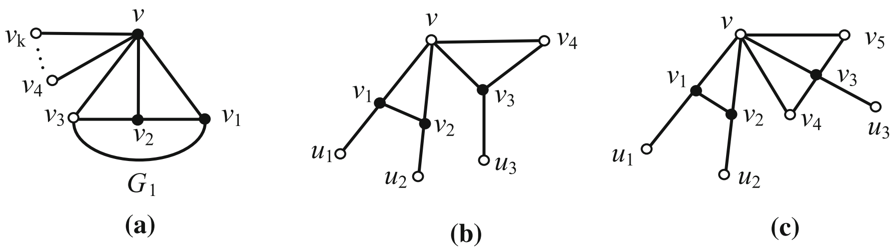

Fig. 2 Configurations used in Claim 12(2)–(4)

with $\{k, k+1\}\backslash\{b_j\}$. If $c_i \in \{k, k+1\}$, then we color vv_1 with $\{k, k+1\}\backslash\{c_i\}$. So there are at least $2 + (s-1) + (d_{(s+1)^-}(v) - s) = d_{(s+1)^-}(v) + 1$ different ways to recolor or color some edges incident with v. As v has at most $d_{(s+1)^-}(v)$ conflict vertices, φ can be extended to G, which is a contradiction. □

Claim 12 *Let v be a k-vertex of G with $k \geq 8$.*

(1) *v is in at most one bad 3-cycle. Further, if v is in a bad 3-cycle, then $d_{2^-}(v) = 0$.*
(2) *If v is in G_1 of Fig. 2a, then $d_{3^-}(v) = 2$.*
(3) *G does not contain a configuration Fig. 2b.*
(4) *G does not contain a configuration Fig. 2c.*

Proof (1) First, we show that v is in at most one bad 3-cycle. Assume that v is in two bad 3-cycles. Let $d(v_i) = 3$ for $i \in [1, 4]$ and $v_1v_2, v_3v_4 \in E(G)$. We denote by u_i the third neighbor of v_i for $i \in [1, 4]$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$, i.e., $\varphi(v_1u_1) = 2$ and $\varphi(v_2u_2) = 1$. Note that $\varphi(v_3u_3) \neq 4$ or $\varphi(v_4u_4) \neq 3$, say $\varphi(v_3u_3) \neq 4$. Remove the color of v_3v_4. Recolor vv_3 with a color in $\{1, 2\}\backslash\{\varphi(v_3u_3)\}$, say 1, vv_1 with 3. Then we color v_1v_2, v_3v_4 properly to get an NDE-Δ-coloring of G, which is a contradiction.

Next, we show that if v is in a bad 3-cycle, then $d_{2^-}(v) = 0$. Assume that $d_{2^-}(v) \geq 1$. Let $d(v_1) = d(v_2) = 3$, $d(v_3) \leq 2$ and $v_1v_2 \in E(G)$. We denote by u_i the third neighbor of v_i for $i = 1, 2$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$, i.e., $\varphi(v_1u_1) = 2$ and $\varphi(v_2u_2) = 1$. Recolor vv_3 with a color in $\{1, 2\}\backslash C_\varphi(v_3)$, say 1, and vv_1 with 3. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G, which is a contradiction.

(2) Assume that $d_{3^-}(v) \geq 3$, say $d(v_4) = 3$, as shown in Fig. 2a. We use the same symbols as in Fig. 2a. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$, i.e., $\varphi(v_1v_3) = 2$ and $\varphi(v_2v_3) = 1$. As for Remark 1 in Claim 9(1), we recolor vv_4 with a color $\alpha \in [1, 3]$ such that v_4 does not conflict with its neighbors. If $\alpha \in [1, 2]$, say $\alpha = 1$, then we recolor vv_1 with 4. If $\alpha = 3$, then we recolor vv_1 with 4, vv_3 with 1, v_2v_3

with 3. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G, which is a contradiction.

(3) Assume that the configuration Fig. 2b exists in G. It follows from Claims 3, 6 and 12(1) that $d(v_4) \geq 8$. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$, i.e., $\varphi(v_1u_1) = 2$ and $\varphi(v_2u_2) = 1$. By Claim 3, we consider the following two cases.
Case 1 $d_3(v_3) = 0$. If $|\{1, 2\} \cap C_\varphi(v_3)| \leq 1$, say $1 \notin C_\varphi(v_3)$, then we recolor vv_3 with 1, vv_1 with 3. Otherwise, $\{1, 2\} \subseteq C_\varphi(v_3)$, say $\varphi(v_3v_4) = 1$ and $\varphi(v_3u_3) = 2$. We recolor vv_1 and v_3v_4 with 4, vv_4 with 1. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G, which is a contradiction.
Case 2 $d_3(v_3) = 1$, i.e., $d(u_3) = d(v_3) = 3$. By Claims 2, 3 and 6, each vertex in $N(u_3)\backslash\{v_3\}$ is an 8^+-vertex. Remove the color of v_3u_3. Let $\alpha \in C_\varphi(u_3)\backslash C_\varphi(v_3)$. If $|\{1, 2\} \cap \{\varphi(v_3v_4), \alpha\}| \leq 1$, say $1 \notin \{\varphi(v_3v_4), \alpha\}$, then we recolor vv_3 with 1, vv_1 with 3. Otherwise, $\{\varphi(v_3v_4), \alpha\} = \{1, 2\}$, say $\varphi(v_3v_4) = 1$ and $\alpha = 2$. We recolor vv_1 and v_3v_4 with 4, vv_4 with 1. Then we color v_1v_2, v_3u_3 properly to get an NDE-Δ-coloring of G, which is a contradiction.

(4) Assume that the configuration Fig. 2c exists in G. Then $G - v_1v_2$ has an NDE-Δ-coloring φ with $\varphi(vv_i) = i$ for $i \in [1, k]$. If $C_\varphi(v_1) \neq C_\varphi(v_2)$, then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Suppose that $C_\varphi(v_1) = C_\varphi(v_2) = \{1, 2\}$, i.e., $\varphi(v_1u_1) = 2$ and $\varphi(v_2u_2) = 1$. We deal with the following three cases by symmetry.

Case 1 $d_4(v_3) = 0$. By Claims 2, 3 and 6, $d(v_4), d(v_5), d(u_3) \geq 8$. If $|\{1, 2\} \cap C_\varphi(v_3)| \leq 1$, say $1 \notin C_\varphi(v_3)$, then we recolor vv_3 with 1, vv_1 with 3. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G. Otherwise, $\{1, 2\} \subseteq C_\varphi(v_3)$. By symmetry, we consider the following two possibilities.

- $\varphi(v_3v_4) = 1$ and $\varphi(v_3v_5) = 2$. Let $\gamma \in \{4, 5\}\backslash(\varphi(v_3u_3))$, say $\gamma = 4$. We exchange the colors of vv_4 and v_3v_4, and recolor vv_1 with 4. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G, which is a contradiction.
- $\varphi(v_3v_4) = 1$ and $\varphi(v_3u_3) = 2$. If $\varphi(v_3v_5) \neq 4$, then we exchange the colors of vv_4 and v_3v_4, and recolor vv_1 with 4. If $\varphi(v_3v_5) = 4$, then we exchange the colors of vv_4 and v_3v_4, the colors of vv_5 and v_3v_5, and recolor vv_1 with 5. Then we color v_1v_2 properly to get an NDE-Δ-coloring of G, which is a contradiction.

Case 2 $d_4(v_3) = 1$ and $d(u_3) = 4$. By Claims 2, 3 and 6, each vertex in $(N(v_3) \cup N(u_3))\backslash\{v_3, u_3\}$ is an 8^+-vertex. Remove the color of v_3u_3. Let $\alpha \in C_\varphi(u_3)\backslash C_\varphi(v_3)$. If $|\{1, 2\} \cap \{\varphi(v_3v_4), \varphi(v_3v_5), \alpha\}| \leq 1$, say $1 \notin \{\varphi(v_3v_4), \varphi(v_3v_5), \alpha\}$, then we recolor vv_3 with 1, vv_1 with 3. Then we color v_1v_2, v_3u_3 properly to get an NDE-Δ-coloring of G. Otherwise, $\{1, 2\} \subseteq \{\varphi(v_3v_4), \varphi(v_3v_5), \alpha\}$. By symmetry, we consider the following two possibilities.

- $\varphi(v_3v_4) = 1$ and $\varphi(v_3v_5) = 2$. Let $\gamma \in \{4, 5\}\backslash\{\alpha\}$, say $\gamma = 4$. We exchange the colors of vv_4 and v_3v_4, and recolor vv_1 with 4. Then we color v_1v_2, v_3u_3 properly to get an NDE-Δ-coloring of G, which is a contradiction.
- $\varphi(v_3v_4) = 1$ and $\alpha = 2$. If $\varphi(v_3v_5) \neq 4$, then we exchange the colors of vv_4 and v_3v_4, and recolor vv_1 with 4. If $\varphi(v_3v_5) = 4$, then we exchange the colors of vv_4

Table 1 The relation between $d_G(v)$ and $d_H(v)$

$d_G(v)$	$2 \le d_G(v) \le 7$	8	9	10	11	12	13	$\cdots$
$d_H(v)$	$= d_G(v)$	≥ 7	≥ 7	≥ 9	≥ 9	≥ 9	≥ 7	$\cdots$

and v_3v_4, the colors of vv_5 and v_3v_5, and recolor vv_1 with 5. Then we color v_1v_2, v_3u_3 properly to get an NDE-Δ-coloring of G, which is a contradiction.

Case 3 $d_4(v_3) = 1$ and $d(v_5) = 4$. By Claims 2, 3 and 6, each vertex in $(N(v_3) \cup N(v_5)) \setminus \{v_3, v_5\}$ is an 8^+-vertex. Remove the color of v_3v_5. Let $\beta \in C_\varphi(v_5) \setminus C_\varphi(v_3)$. If $|\{1, 2\} \cap \{\varphi(v_3v_4), \varphi(v_3u_3), \beta\}| \le 1$, say $1 \notin \{\varphi(v_3v_4), \varphi(v_3u_3), \beta\}$, then we recolor vv_3 with 1, vv_1 with 3. Then we color v_1v_2, v_3v_5 properly to get an NDE-Δ-coloring of G. Otherwise, $\{1, 2\} \subseteq \{\varphi(v_3v_4), \varphi(v_3u_3), \beta\}$. By symmetry, we consider the following two possibilities.

- $\varphi(v_3v_4) = 1$ and $\varphi(v_3u_3) = 2$. We exchange the colors of vv_4 and v_3v_4, and recolor vv_1 with 4. Then we color v_1v_2, v_3v_5 properly to get an NDE-Δ-coloring of G, which is a contradiction.
- $1 \in \{\varphi(v_3v_4), \varphi(v_3u_3)\}$ and $\beta = 2$. If $5 \notin C_\varphi(v_3)$ or $C_\varphi(v_5) \setminus (\varphi(v_3v_5)) \ne \{1, 2, 5\}$, then we recolor vv_3 with 2, vv_2 with 3. If $5 \in C_\varphi(v_3)$ and $C_\varphi(v_5) \setminus (\varphi(v_3v_5)) = \{1, 2, 5\}$, then we recolor vv_5 with 3, vv_3 with 2, vv_2 with 5. Then we color v_1v_2, v_3v_5 properly to get an NDE-Δ-coloring of G, which is a contradiction.

□

3.2 Discharging Analysis on *H*

Let H be the graph obtained from G by removing all 1-vertices. Note that $d_H(v) = d_G(v) - d_1^G(v)$. By Claims 2, 7–11, we give the relationship between $d_G(v)$ and $d_H(v)$ in Table 1. It follows that $\delta(H) \ge 2$. By Claim 1, H is 2-connected. Some structural properties of H are collected as follows.

Claim 13 *Let v be a vertex of H. Then the following statements hold.*

(0) *If $2 \le d_H(v) \le 6$, then $d_G(v) = d_H(v)$.*
(1) *If $d_H(v) = 2$, then $d_{5^-}^H(v) = 0$.*
(2) *If $d_H(v) = k$ for $k \in [3, 5]$, then $d_{5^-}^H(v) = d_k^H(v) \le 1$.*
(3) *If $d_H(v) = 6, 7$, then $d_{5^-}^H(v) = 0$.*
(4) *Let $d_H(v) = 8$. If $d_{4^-}^H(v) \ge 1$, then $d_{5^-}^H(v) \le 1$.*
(5) *Let $d_H(v) = 9$. Then v is not in a bad 3-cycle. If $d_{4^-}^H(v) \ge 1$, then $d_{5^-}^H(v) \le 2$.*
(6) *Let $d_H(v) = 10$. Then v is not in a bad 3-cycle. If $d_{3^-}^H(v) \ge 1$, then $d_{6^+}^H(v) \ge 2d_{4^-}^H(v) + 1$. If $d_{4^-}^H(v) \ge 1$, then $d_{6^+}^H(v) \ge d_{4^-}^H(v) + 1$.*
(7) *Let $d_H(v) = 11$. If $d_{3^-}^H(v) \ge 1$, then $d_{6^+}^H(v) \ge d_{4^-}^H(v) + 1$. If v is in a bad 3-cycle, then $d_{6^+}^H(v) \ge 9$.*
(8) *Let $d_H(v) = 12$. If $d_2^H(v) \ge 1$, then $d_{6^+}^H(v) \ge d_{3^-}^H(v) + 1$. If v is in a bad 3-cycle, then $d_{6^+}^H(v) \ge 5$. If v is in a special 3-cycle and $d_{3^-}^H(v) \ge 2$, then $d_{6^+}^H(v) \ge 5$.*

Proof (0) holds clearly by Table 1. Both (1) and (2) hold by Claims 3, 4, 13(0).

(3) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) > d_H(v)$, by Table 1, then $d_G(v) = 8$, 9 or $d_G(v) \geq 13$. If $d_G(v) = 8$, then $d_1^G(v) = 1$. By Claim 7(1), $d_{5^-}^G(v) = 1$, further to say that $d_{5^-}^H(v) = d_{5^-}^G(v) - d_1^G(v) = 0$. If $d_G(v) = 9$, then $d_1^G(v) = 2$. By Claim 7(1), $d_{5^-}^G(v) \leq 2$, further to say that $d_{5^-}^H(v) = d_{5^-}^G(v) - d_1^G(v) \leq 2 - 2 = 0$. If $d_G(v) \geq 13$, then $d_1^G(v) \geq 6$. By Claim 5(1), $d_s^G(v) = 0$ for $s \in [2, 5]$, further to say that $d_{5^-}^H(v) = 0$.

(4) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) > d_H(v)$, by Table 1, then $d_G(v) = 9$ or $d_G(v) \geq 13$. If $d_G(v) = 9$, then $d_1^G(v) = 1$. By Claim 7(1), $d_{5^-}^G(v) \leq 2$, further to say that $d_{5^-}^H(v) = d_{5^-}^G(v) - d_1^G(v) \leq 1$. If $d_G(v) \geq 13$, then $d_1^G(v) \geq 5$. By Claim 5(1), $d_s^G(v) = 0$ for $s \in [2, 5]$, further to say that $d_{5^-}^H(v) = 0$.

(5) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) = 10$, then $d_1^G(v) = 1$. By Claim 8(1), $d_{6^+}^H(v) = d_{6^+}^G(v) \geq 4d_{4^-}^G(v) + 1 = 4d_{4^-}^H(v) + 5$, further to say that $d_{4^-}^H(v) = 0$. If $d_G(v) = 11$, then $d_1^G(v) = 2$. By Claim 9(1), $d_{6^+}^H(v) = d_{6^+}^G(v) \geq 3d_{4^-}^G(v) + 1 = 3d_{4^-}^H(v) + 7$, further to say that $d_{4^-}^H(v) = 0$. If $d_G(v) \geq 12$, then $d_1^G(v) \geq 3$. By Claim 5(1), $d_s^G(v) = 0$ for $s \in [2, 4]$, further to say that $d_{4^-}^H(v) = 0$.

(6) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) = 11$, then $d_1^G(v) = 1$. By Claim 9(1), $d_{6^+}^H(v) = d_{6^+}^G(v) \geq 3d_{4^-}^G(v) + 1 = 3d_{4^-}^H(v) + 4$, further to say that $d_{4^-}^H(v) \leq 1$. If $d_G(v) = 12$, then $d_1^G(v) = 2$. By Claim 10(1), $d_{6^+}^H(v) = d_{6^+}^G(v) \geq 2d_{4^-}^G(v) + 1 = 2d_{4^-}^H(v) + 5$, further to say that $d_{4^-}^H(v) \leq 1$. If $d_G(v) \geq 13$, then $d_1^G(v) \geq 3$. By Claim 5(1), $d_s^G(v) = 0$ for $s \in [2, 4]$, further to say that $d_{4^-}^H(v) = 0$.

(7) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) = 12$, then $d_1^G(v) = 1$. By Claim 12(1), v is not in a bad 3-cycle. If $d_{3^-}^H(v) \geq 1$, implying $d_{3^-}^G(v) \geq 2$, by Claim 10(1), then $d_{6^+}^H(v) = d_{6^+}^G(v) \geq 2d_{4^-}^G(v) + 1 \geq 2d_{4^-}^H(v) + 3$. If $d_G(v) \geq 13$, then $d_1^G(v) \geq 2$. By Claim 5(1), $d_s^G(v) = 0$ for $s \in [2, 3]$, further to say that $d_{3^-}^H(v) = 0$.

(8) It holds trivially if $d_G(v) = d_H(v)$. If $d_G(v) \geq 13$, then $d_1^G(v) \geq 1$. By Claim 12(1), v is not in a bad 3-cycle. By Claim 5(1), $d_2^G(v) = 0$, further to say that $d_2^H(v) = 0$. □

Using Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we can deduce

$$\sum_{v \in V(H)} (d_H(v) - 6) + \sum_{f \in F(H)} (2d_H(f) - 6) = -12.$$

We first define the weight function w on H by $w(x) = d_H(x) - 6$ if $x \in V(H)$ and $w(x) = 2d_H(x) - 6$ if $x \in F(H)$. Next, we design some discharging rules and redistribute weights accordingly. Once the discharging is finished, the resultant weight function w' is produced, so that the sum of all weights is kept fixed when the discharging is in process. However, we can show that $w'(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This leads to the following contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -12 < 0,$$

and hence demonstrates the nonexistence of such a counterexample.

The discharging rules are defined as follows:

(R1) Every 4-face f gives 2 to its incident 2-vertex if $d_2^H(f) = 1$; otherwise, it gives 1 to each incident small vertex.

(R2) Every 5^+-face f gives 2 to each incident 2-vertices, and $\frac{\omega(f)-2d_2^H(f)}{d_3^H(f)+d_4^H(f)+d_5^H(f)}$ to each incident small vertex of degree at least 3.
For a small vertex u, let $\beta(u)$ denote the total sum of charges transferred into u after (R1)-(R2) were carried out.

(R3) Let v be an 8^+-vertex adjacent to a small k-vertex u. If $d_k^H(u) = 1$, then v gives $\max\{\frac{6-d_H(u)-\beta(u)}{d_H(u)-1}, 0\}$ to u; if $d_k^H(u) = 0$, then v gives $\max\{\frac{6-d_H(u)-\beta(u)}{d_H(u)}, 0\}$ to u.

For $x, y \in V(H) \cup F(H)$, let $\tau(x \to y)$ denote the charge transferred from x to y according to the above rules.

Observation 1 (Wang and Huang [8]) Let f be a face of H and v be a small vertex incident with f.

(1) Every face f is incident with at most $\lfloor\frac{2d_H(f)}{3}\rfloor$ small vertices.
(2) Let $d_H(f) = 5$. If $d_2^H(f) \geq 1$, then $\tau(f \to v) \geq 1$. Otherwise, $\tau(f \to v) \geq \frac{4}{3}$.
(3) Let $d_H(f) = 6$. If $d_2^H(f) \geq 1$, then $\tau(f \to v) \geq 2$. Otherwise, $\tau(f \to v) \geq \frac{3}{2}$.
(4) Let $d_H(f) \geq 7$. Then $\tau(f \to v) \geq 2$.

Observation 2 *Let v be an 8^+-vertex of H, u be a small vertex adjacent to v, and G_1 be the configuration of Fig. 2a.*

(1) Let $d_H(u) = 2$. If v is in a special 3-face, then $\tau(v \to u) \leq 1$. Otherwise, $\tau(v \to u) = 0$.
(2) Let $d_H(u) = 3$. If v is in G_1, then $\tau(v \to u) \leq \frac{3}{2}$. Otherwise, $\tau(v \to u) \leq 1$.
(3) If $d_H(u) = 4$, then $\tau(v \to u) \leq \frac{2}{3}$.
(4) If $d_H(u) = 5$, then $\tau(v \to u) \leq \frac{1}{4}$.
(5) If v is in a bad 3-face and u is not in this bad 3-face, then $\tau(v \to u) \leq \frac{1}{2}$.
(6) If v is in G_1 and u is not in G_1, then $\tau(v \to u) \leq \frac{1}{3}$.

Proof Both (3) and (4) hold clearly by Observation 1 and (R3).

(1) If v is in a special 3-face, then $d_2^H(v) = 1$ by Claim 5(4). Then the faces incident with u are this special 3-face and a 4^+-face. By (R1) and (R2), $\beta(u) \geq 2$. Hence, $\tau(v \to u) \leq \frac{6-2-2}{2} = 1$. Otherwise, the faces incident with u are all 4^+-faces. By (R1) and (R2), $\beta(u) \geq 4$. Hence, $\tau(v \to u) \leq \frac{6-2-4}{2} = 0$.
(2) If $d_3^H(u) = 0$, then $\tau(v \to u) = \frac{6-3-\beta(u)}{3} \leq 1$. Suppose that $d_3^H(u) = 1$. Let v, w_1, w_2 be the neighbors of u in the clockwise order with $d_H(w_1) = 3$. Let f_{uvw_1} be the face with uv and uw_1 as boundary edges. Similarly, we can define $f_{w_1uw_2}$ and f_{vuw_2}. If f_{uvw_1} or $f_{w_1uw_2}$ is a 4^+-face, by (R1) and Observation 1, then $\beta(u) \geq 1$. Hence, $\tau(v \to u) \leq \frac{6-3-1}{3-1} = 1$. Suppose that $d_H(f_{uvw_1}) = d_H(f_{w_1uw_2}) = 3$. Then v is in a bad 3-face vuw_1v. By Claim 12(1), $d_2^H(v) = 0$. If f_{vuw_2} is a 4-face with $d_2^H(f_{vuw_2}) = 0$, or a 5^+-face, by (R1) and Observation 1, then $\beta(u) \geq 1$.

Hence, $\tau(v \to u) \leq \frac{6-3-1}{3-1} = 1$. Suppose that $d_H(f_{vuw_2}) = 3$. Then u is incident with three 3-faces, i.e., v is in G_1, and $\tau(v \to u) = \frac{6-3-0}{3-1} = \frac{3}{2}$.

(5) If v is in a bad 3-face, then $d_2^H(v) = 0$ by Claim 12(1). Then $d_H(u) = 3, 4, 5$. If $d_H(u) = 3$, then vu is not incident with a 3-face by Claim 12(3). By (R1) and Observation 1, $\beta(u) \geq 2$. Hence, $\tau(v \to u) \leq \frac{6-3-2}{3-1} = \frac{1}{2}$. If $d_H(u) = 4$, then one of the faces incident with vu is at least a 4^+-face by Claim 12(4). By (R1) and Observation 1, $\beta(u) \geq 1$. Hence, $\tau(v \to u) \leq \frac{6-4-1}{4-1} = \frac{1}{3} < \frac{1}{2}$. If $d_H(u) = 5$, then $\tau(v \to u) \leq \frac{1}{4} < \frac{1}{2}$ by Observation 2(4).

(6) If v is in G_1, then $d_{3^-}^H(v) = 2$ by Claim 12(2). Then $d_H(u) = 4, 5$. If $d_H(u) = 4$, then one of the faces incident with vu is at least a 4^+-face by Claim 12(4). By (R1) and Observation 1, $\beta(u) \geq 1$. Hence, $\tau(v \to u) \leq \frac{6-4-1}{4-1} = \frac{1}{3}$. If $d_H(u) = 5$, then $\tau(v \to u) \leq \frac{1}{4} < \frac{1}{3}$ by Observation 2(4). □

Observation 3 *Suppose that $v \in V(H)$ is an 8^+-vertex which is not in any bad 3-face. Let $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+s}, v_{i+s+1}$ be $s+2$ consecutive neighbors of v. If $d_H(v_{i+j}) \in [3, 5]$ for $j \in [1, s]$ and $s \geq 2$, then $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq \frac{s}{3} + 1$. In particular,*

(1) *If neither v_{i+1} nor v_{i+s} is in a bad 3-face, then $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq \frac{s}{3} + \frac{2}{3}$.*

(2) *If exactly one of v_{i+1} and v_{i+s} is in a bad 3-face, then $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq \frac{s}{3} + \frac{5}{6}$.*

(3) *If $s = 2$ and $d_H(v_i), d_H(v_{i+3}) \notin [3, 5]$, then $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{3}{2}$.*

Proof Let f_t be the face with vv_t and vv_{t+1} as boundary edges for $t \in [i, i+s]$.

Let $j \in [i+2, i+s-1]$ and $d_H(v_j) = k$. If $d_k^H(v_j) = 0$, then f_{j-1} and f_j incident with vv_j are either 4-faces without 2-vertices or 5^+-faces. Hence, $\tau(v \to v_j) \leq \frac{6-k-2}{k} \leq \frac{1}{3}$ by (R1) and Observation 1. If $k \geq 4$ and $d_k^H(v_j) = 1$, then $d_{5^-}^H(v_j) = d_k^H(v_j) = 1$ by Claim 13(2). Then one of f_{j-1} and f_j is at least a 4^+-face. Hence, $\tau(v \to v_j) \leq \frac{6-k-1}{k-1} \leq \frac{1}{3}$ by (R1) and Observation 1. If $k = 3$ and $d_3^H(v_j) = 1$, then one of f_{j-1} and f_j is a 4^+-face and the other is a 5^+-face. Hence, $\tau(v \to v_j) \leq \frac{6-3-1-\frac{4}{3}}{3-1} = \frac{1}{3}$ by (R1) and Observation 1.

Let $j \in \{i+1, i+s\}$, say $j = i+1$. If $d_H(v_j) \geq 4$, then $\tau(v \to v_j) \leq \frac{2}{3}$ by Observation 2(3)–(4). If $d_H(v_j) = 3$ and $d_3^H(v_j) = 0$, then $d_{5^-}^H(v_j) = d_3^H(v_j) = 0$ by Claim 13(2). Hence, $d_H(f_j) \geq 4$ and $\tau(v \to v_j) \leq \frac{6-3-1}{3} = \frac{2}{3}$ by (R1) and Observation 1. Suppose that $d_H(v_j) = 3$ and $d_3^H(v_j) = 1$. Let v, w_1, w_2 be the neighbors of v_j in the clockwise order. If $d_H(w_1) = 3$, then $d_H(f_{j-1}) \geq 4$. By Claim 13(1)–(3), $d_H(w_2) \geq 8$ and $d_H(f_j) \geq 4$. Hence, $\tau(v \to v_j) \leq \frac{6-3-2}{3-1} = \frac{1}{2}$ by (R1) and Observation 1. If $d_H(w_2) = 3$, then $d_{5^-}^H(w_2) = d_3^H(w_2) = 1$ by Claim 13(2). Then f_j is either a 5-face without 2-vertices or a 6^+-face. Hence, $\tau(v \to v_j) \leq \frac{6-3-\frac{4}{3}}{3-1} = \frac{5}{6}$ by (R1) and Observation 1. Specially, suppose that v_j is not in a bad 3-face. For $l \in \{1, 2\}$ with $d_H(w_l) = 3$, the faces incident with $v_j w_l$ are all 4^+-faces. Hence, $\tau(v \to v_j) \leq \frac{6-3-2}{3-1} = \frac{1}{2}$ by (R1) and Observation 1.

Consequently, $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq 2 \cdot \frac{5}{6} + \frac{1}{3}(s-2) = \frac{s}{3} + 1$. In particular,

(1) If neither v_{i+1} nor v_{i+s} is in a bad 3-face, then $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq 2 \cdot \frac{2}{3} + \frac{1}{3}(s-2) = \frac{s}{3} + \frac{2}{3}$.

(2) If exactly one of v_{i+1} and v_{i+s} is in a bad 3-face, then $\sum_{j=1}^{s} \tau(v \to v_{i+j}) \leq \frac{2}{3} + \frac{5}{6} + \frac{1}{3}(s-2) = \frac{s}{3} + \frac{5}{6}$.

(3) If neither v_{i+1} nor v_{i+2} is in a bad 3-face, then $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{4}{3} < \frac{3}{2}$ by (1). If exactly one of v_{i+1} and v_{i+2} is in a bad 3-face, then $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{3}{2}$ by (2). Suppose that v_{i+1} and v_{i+2} are all incident with bad 3-faces. Then $d_H(v_{i+1}) = d_H(v_{i+2}) = 3$ and $d_3^H(v_{i+1}) = d_3^H(v_{i+2}) = 1$. Let v, x_1, x_2 be the neighbors of v_{i+1} in the clockwise order. Let v, y_1, y_2 be the neighbors of v_{i+2} in the clockwise order. If $d_H(x_1) = 3$, then f_i is either a 4-face without 2-vertices or a 5^+-face. By Claim 13(1)-(3), $d_H(x_2) \geq 8$ and $d_H(f_{i+1}) \geq 4$. Hence, $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{6-3-2}{3-1} + \frac{6-3-1}{3-1} = \frac{3}{2}$ by (R1) and Observation 1. If $d_H(y_2) = 3$, then $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{3}{2}$ by symmetry. If $d_H(x_2) = d_H(y_1) = 3$, then f_{i+1} is either a 6-face without 2-vertices or a 7^+-face. Hence, $\tau(v \to v_{i+1}) + \tau(v \to v_{i+2}) \leq \frac{6-3-\frac{3}{2}}{3-1} + \frac{6-3-\frac{3}{2}}{3-1} = \frac{3}{2}$ by (R1) and Observation 1. □

Let $f \in F(H)$. If $d_H(f) = 3$, then $\omega'(f) = \omega(f) = 2 \times 3 - 6 = 0$. If $d_H(f) = 4$, then by (R1), f gives at most 2 to all its incident small vertices. Hence, $\omega'(f) \geq \omega(f) - 2 = 2 \times 4 - 6 - 2 = 0$. If $d_H(f) \geq 5$, then by (R2),

$$\begin{aligned}\omega'(f) \geq \omega(f) - 2d_2^H(f) - \frac{\omega(f) - 2d_2^H(f)}{d_3^H(f) + d_4^H(f) + d_5^H(f)}(d_3^H(f) \\ + d_4^H(f) + d_5^H(f)) = 0.\end{aligned}$$

Let $v \in V(H)$ be a k-vertex. Then $k \geq 2$. Let $v_0, v_1, \ldots, v_{k-1}$ denote the neighbors of v in the clockwise order. For $0 \leq i \leq k-1$, we use f_i to denote the incident face of v in H with vv_i and vv_{i+1} as boundary edges, where all indices are taken as modulo k. If $6 \leq k \leq 7$, then $\omega'(v) = \omega(v) = k - 6 \geq 0$ by (R3) and Claim 13(3).

Case 1 $2 \leq k \leq 5$. If $d_k^H(v) = 1$, then $d_{7^-}^H(v) = 1$ by Claim 13(1)-(3). Hence, by (R1)-(R3), $\omega'(v) \geq k - 6 + \beta(v) + (k-1)\frac{6-k-\beta(v)}{k-1} = 0$. If $d_k^H(v) = 0$, then $d_{7^-}^H(v) = 0$ by Claim 13(1)-(3). Hence, by (R1)-(R3), $\omega'(v) \geq k - 6 + \beta(v) + k \cdot \frac{6-k-\beta(v)}{k} = 0$.

Case 2 $k = 8$. Then $\omega(v) = 2$. If $d_{4^-}^H(v) \geq 1$, then $d_{5^-}^H(v) \leq 1$ by Claim 13(4). Then v is not in a bad 3-face. Hence, $\omega'(v) \geq 2 - d_{5^-}^H(v) \geq 2 - 1 = 1$ by Observation 2(1)-(3). If $d_{4^-}^H(v) = 0$, then $\omega'(v) \geq 2 - \frac{1}{4}d_5^H(v) \geq 2 - \frac{1}{4} \times 8 = 0$ by Observation 2(4).

Case 3 $k = 9$. Then $\omega(v) = 3$ and v is not in a bad 3-face by Claim 13(5). If $d_{4^-}^H(v) \geq 1$, then $d_{5^-}^H(v) \leq 2$ by Claim 13(5). Hence, $\omega'(v) \geq 3 - d_{5^-}^H(v) \geq 3 - 1 \times 2 =$

1 by Observation 2(1)-(4). If $d^H_{4^-}(v) = 0$, then $\omega'(v) \ge 3 - \frac{1}{4}d^H_5(v) \ge 3 - \frac{1}{4} \times 9 = \frac{3}{4}$ by Observation 2(4).

Case 4 $k = 10$. Then $\omega(v) = 4$ and v is not in a bad 3-face by Claim 13(6). If $d^H_{3^-}(v) \ge 1$, then $d^H_{6^+}(v) \ge 2d^H_{4^-}(v)+1$ by Claim 13(6), which implies that $d^H_{4^-}(v) \le 3$. Hence, $\omega'(v) \ge 4 - d^H_{4^-}(v) - \frac{1}{4}(10 - d^H_{4^-}(v) - d^H_{6^+}(v)) \ge \frac{7}{4} - \frac{1}{4}d^H_{4^-}(v) \ge 1$ by Observation 2(1)-(4). If $d^H_{3^-}(v) = 0$ and $d^H_4(v) \ge 1$, then $d^H_{6^+}(v) \ge d^H_4(v) + 1$ by Claim 13(6), which implies that $d^H_4(v) \le 4$. Hence, $\omega'(v) \ge 4 - \frac{2}{3}d^H_4(v) - \frac{1}{4}(10 - d^H_4(v) - d^H_{6^+}(v)) \ge \frac{7}{4} - \frac{1}{6}d^H_4(v) \ge \frac{13}{12}$ by Observation 2(3)-(4). If $d^H_{4^-}(v) = 0$, then $\omega'(v) \ge 4 - \frac{1}{4}d^H_5(v) \ge 4 - \frac{1}{4} \times 10 = \frac{3}{2}$ by Observation 2(4).

Case 5 $k = 11$. Then $\omega(v) = 5$. If v is in a bad 3-face, then $d^H_{6^+}(v) \ge 9$ by Claim 13(7). Hence, $\omega'(v) \ge 5 - \frac{3}{2} \times 2 = 2$ by Observation 2. Otherwise, v is not in a bad 3-face.

Case 5.1 $d^H_{3^-}(v) \ge 1$. By Claim 13(7), $d^H_{6^+}(v) \ge d^H_{4^-}(v) + 1$, which implies that $d^H_{4^-}(v) \le 5$. Hence, $\omega'(v) \ge 5 - d^H_{4^-}(v) - \frac{1}{4}(11 - d^H_{4^-}(v) - d^H_{6^+}(v)) \ge \frac{5}{2} - \frac{1}{2}d^H_{4^-}(v) \ge 0$ by Observation 2(1)-(4).

Case 5.2 $d^H_{3^-}(v) = 0$. Let $p = d^H_{6^+}(v)$, $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ be 6^+-vertices with $0 = i_1 < i_2 < \cdots < i_p$, and $n_j = |\{l | i_j + 1 \le l \le i_{j+1} - 1, 4 \le d_H(v_l) \le 5\}|$ for $j \in [1, p]$. Then $i_{p+1} - 1 = 10$ and $\sum_{j=1}^{p} n_j = 11 - p$. If $p \ge 4$, then $\omega'(v) \ge 5 - \frac{2}{3}(11 - p) = \frac{2}{3}p - \frac{7}{3} \ge \frac{1}{3}$ by Observation 2(3)–(4). Assume that $p \le 3$. Note that every vertex in $N_H(v) \backslash \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ is not in a bad 3-face. By Observations 2 and 3, $\sum_{l=i_j+1}^{i_{j+1}-1} \tau(v \to v_l) \le \frac{n_j}{3} + \frac{2}{3}$ for $j \in [1, p]$. Hence, $\omega'(v) \ge 5 - \sum_{j=1}^{p}(\frac{n_j}{3} + \frac{2}{3}) = \frac{4-p}{3} \ge \frac{1}{3}$.

Case 6 $k = 12$. Then $\omega(v) = 6$ and $d^H_2(v) \le 1$ by Claim 5(2). If v is in a bad 3-face, then $d^H_{6^+}(v) \ge 5$ by Claim 13(8). Hence, $\omega'(v) \ge 6 - \frac{3}{2} \times 2 - \frac{1}{2}(12 - 2 - d^H_{6^+}(v)) = \frac{1}{2}d^H_{6^+}(v) - 2 \ge \frac{1}{2}$ by Observation 2. So assume that v is not in a bad 3-face.

Case 6.1 $d^H_2(v) = 1$. By Claim 13(8), $d^H_{6^+}(v) \ge d^H_{3^-}(v) + 1$. Then $\omega'(v) \ge 6 - d^H_{3^-}(v) - \frac{2}{3}(12 - d^H_{3^-}(v) - d^H_{6^+}(v)) = \frac{2}{3}d^H_{6^+}(v) - \frac{1}{3}d^H_{3^-}(v) - 2$ by Observation 2. If $d^H_{6^+}(v) \ge 5$, then $\omega'(v) \ge \frac{1}{3}d^H_{6^+}(v) - \frac{5}{3} \ge 0$. If $d^H_{3^-}(v) \ge 4$, then $\omega'(v) \ge \frac{1}{3}d^H_{3^-}(v) - \frac{4}{3} \ge 0$. Assume that $d^H_{3^-}(v) \le 3$ and $d^H_{6^+}(v) \le 4$. Let $p = d^H_{6^+}(v)$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ be 6^+-vertices with $0 = i_1 < i_2 < \cdots < i_p$.

Suppose that v is in a special 3-face. If $2 \le d^H_{3^-}(v) \le 3$, then $d^H_{6^+}(v) \ge 5$ by Claim 13(8). Otherwise, $d^H_{3^-}(v) = 1$. Let v_{i_p+1} be a 2-vertex, $n_j = |\{l | i_j + 1 \le l \le i_{j+1} - 1, 4 \le d_H(v_l) \le 5\}|$ for $j \in [1, p-1]$, and $n_p = |\{l | i_p + 2 \le l \le i_{p+1} - 1, 4 \le d_H(v_l) \le 5\}|$. Then $i_{p+1} - 1 = 11$ and $\sum_{j=1}^{p} n_j = 11 - p$. By Observations 2 and 3, $\sum_{l=i_j+1}^{i_{j+1}-1} \tau(v \to v_l) \le \frac{n_j}{3} + \frac{2}{3}$ for $j \in [1, p-1]$, and $\sum_{l=i_p+2}^{i_{p+1}-1} \tau(v \to v_l) \le \frac{n_p}{3} + \frac{2}{3}$. Hence, $\omega'(v) \ge 6 - 1 - \sum_{j=1}^{p}(\frac{n_j}{3} + \frac{2}{3}) = \frac{4-p}{3} \ge 0$.

Suppose that v is not in a special 3-face. Let $v_{i_{p+1}}$ be a 2-vertex and $n_j = |\{l | i_j + 1 \leq l \leq i_{j+1} - 1, 3 \leq d_H(v_l) \leq 5\}|$ for $j \in [1, p+1]$. Then $i_{p+2} - 1 = 11$ and $\sum_{j=1}^{p+1} n_j = 11 - p$. If $d_{3^-}^H(v) = 1$, then each vertex in $N_H(v) \backslash \{v_{i_1}, v_{i_2}, \ldots, v_{i_{p+1}}\}$ is a 4-vertex or a 5-vertex. By Observations 2 and 3, $\omega'(v) \geq 6 - \sum_{j=1}^{p+1} (\frac{n_j}{3} + \frac{2}{3}) = \frac{5-p}{3} \geq \frac{1}{3}$. If $d_{3^-}^H(v) = 2$, then there is $s \in [1, p+1]$ such that a 3-vertex is in $\{v_{i_s+1}, \ldots, v_{i_{s+1}-1}\}$. By Observations 2 and 3, $\omega'(v) \geq 6 - (\frac{n_s}{3} + \frac{5}{6}) - \sum_{j \in [1,p+1]}^{j \neq s} (\frac{n_j}{3} + \frac{2}{3}) = \frac{9-2p}{6} \geq \frac{1}{6}$. Suppose that $d_{3^-}^H(v) = 3$. If there is $s \in [1, p+1]$ such that two 3-vertices are in $\{v_{i_s+1}, \ldots, v_{i_{s+1}-1}\}$, by Observations 2 and 3, then $\omega'(v) \geq 6 - (\frac{n_s}{3} + 1) - \sum_{j \in [1,p+1]}^{j \neq s} (\frac{n_j}{3} + \frac{2}{3}) = \frac{4-p}{3} \geq 0$. Otherwise, there are $s, t \in [1, p+1]$ such that two 3-vertices are in $\{v_{i_s+1}, \ldots, v_{i_{s+1}-1}\}$ and $\{v_{i_t+1}, \ldots, v_{i_{t+1}-1}\}$, respectively. By Observations 2 and 3, $\omega'(v) \geq 6 - (\frac{n_s}{3} + \frac{5}{6}) - (\frac{n_t}{3} + \frac{5}{6}) - \sum_{j \in [1,p+1]}^{j \neq s,t} (\frac{n_j}{3} + \frac{2}{3}) = \frac{4-p}{3} \geq 0$.

Case 6.2 $d_2^H(v) = 0$. Let $p = d_{6^+}^H(v)$, $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ be 6^+-vertices with $0 = i_1 < i_2 < \cdots < i_p$, and $n_j = |\{l | i_j + 1 \leq l \leq i_{j+1} - 1, 3 \leq d_H(v_l) \leq 5\}|$ for $j \in [1, p]$. Then $i_{p+1} - 1 = 11$ and $\sum_{j=1}^{p} n_j = 12 - p$. If $p \geq 6$, then $\omega'(v) \geq 6 - (12 - p) = p - 6 \geq 0$ by Observation 2. If $p \leq 3$, then $\omega'(v) \geq 6 - \sum_{j=1}^{p} (\frac{n_j}{3} + 1) = \frac{6-2p}{3} \geq 0$ by Observations 2 and 3.

Suppose that $p = 4$. If there are five consecutive vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+5} \in N_{3,4,5}^H(v)$, then $\omega'(v) \geq 6 - (\frac{5}{3} + 1) - (12 - 4 - 5) = \frac{1}{3}$ by Observations 2 and 3. If there are $s, t \in [1, 4]$ with $n_s = 4$ and $n_t = 2$, then $\omega'(v) \geq 6 - (\frac{4}{3} + 1) - \frac{3}{2} - (12 - 4 - 6) = \frac{1}{6}$ by Observations 2 and 3. If there are $s, t \in [1, 4]$ with $n_s = n_t = 3$, then $\omega'(v) \geq 6 - 2 \times (\frac{3}{3} + 1) - (12 - 4 - 6) = 0$ by Observations 2 and 3. If there are $s, t, r \in [1, 4]$ with $n_s = 3$ and $n_t = n_r = 2$, then $\omega'(v) \geq 6 - (\frac{3}{3} + 1) - 2 \times \frac{3}{2} - (12 - 4 - 7) = 0$ by Observations 2 and 3. Otherwise, $n_1 = n_2 = n_3 = n_4 = 2$. By Observations 2 and 3, $\omega'(v) \geq 6 - 4 \times \frac{3}{2} = 0$.

Suppose that $p = 5$. If there are three consecutive vertices $v_{i+1}, v_{i+2}, v_{i+3} \in N_{3,4,5}^H(v)$, Observations 2 and 3 implies that $\omega'(v) \geq 6 - (\frac{3}{3} + 1) - (12 - 5 - 3) = 0$. Otherwise, there are $s, t \in [1, 5]$ with $n_s = n_t = 2$. By Observations 2 and 3, $\omega'(v) \geq 6 - 2 \times \frac{3}{2} - (12 - 5 - 4) = 0$.

Case 7 $k \geq 13$. If v is in G_1, then $\omega'(v) \geq k - 6 - 2 \times \frac{3}{2} - \frac{1}{3}(k - 2 - d_{6^+}^H(v)) = \frac{1}{3}(2k - 25 + d_{6^+}^H(v)) \geq 0$ by Observation 2. So assume that v is not in G_1.

Suppose that v is in a bad 3-face. By Claim 12(1), $d_2^H(v) = 0$ and v is in at most one bad 3-face. If $d_{6^+}^H(v) \geq 1$, then $\omega'(v) \geq k - 6 - 2 \times 1 - \frac{1}{2}(k - 2 - d_{6^+}^H(v)) = \frac{1}{2}(k - 14 + d_{6^+}^H(v)) \geq 0$ by Observation 2. Otherwise, $d_{6^+}^H(v) = 0$. Assume that $d_H(v_j) = s \in [3, 5]$ for $j \in [2, k-1]$. If $d_s^H(v_j) = 0$, then f_{j-1} and f_j are 4-

faces without 2-vertices or 5^+-faces. Hence, $\tau(v \to v_j) \le \frac{6-s-2}{s} \le \frac{1}{3}$ by (R1) and Observation 1. If $d_s^H(v_j) = 1$ and $s \ge 4$, then $d_{5^-}^H(v_j) = d_s^H(v_j) = 1$ by Claim 13(2). Then at least one of f_{j-1} and f_j is a 4^+-face. Hence, $\tau(v \to v_j) \le \frac{6-s-1}{s-1} \le \frac{1}{3}$ by (R1) and Observation 1. If $s = 3$ and $d_3^H(v_j) = 1$, then one of f_{j-1} and f_j is a 4^+-face and the other is a 5^+-face. Hence, $\tau(v \to v_j) \le \frac{6-3-1-\frac{4}{3}}{3-1} = \frac{1}{3}$ by (R1) and Observation 1. Therefore, $\sum_{j=2}^{k-1} \tau(v \to v_j) \le \frac{k-2}{3}$, and $\omega'(v) \ge k-6-2\times 1-\frac{k-2}{3} = \frac{1}{3}(2k-22) \ge \frac{4}{3}$. So assume that v is not in a bad 3-face.

Case 7.1 v is in a special 3-face. By Claim 5(4), $d_2^H(v) = 1$. Let $p = d_{6^+}^H(v)$, $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ be 6^+-vertices with $0 = i_1 < i_2 < \cdots < i_p$, and v_{i_p+1} be a 2-vertex. Let $n_j = |\{l | i_j + 1 \le l \le i_{j+1} - 1, 3 \le d_H(v_l) \le 5\}|$ for $j \in [1, p-1]$, and $n_p = |\{l | i_p + 2 \le l \le i_{p+1} - 1, 3 \le d_H(v_l) \le 5\}|$. Then $i_{p+1} - 1 = k - 1$ and $\sum_{j=1}^{p} n_j = k - p - 1$. By Observations 2 and 3, $\sum_{l=i_j+1}^{i_{j+1}-1} \tau(v \to v_l) \le \frac{n_j}{3} + 1$ for $j \in [1, p-1]$, and $\sum_{l=i_p+2}^{i_{p+1}-1} \tau(v \to v_l) \le \frac{n_p}{3} + 1$. If $p \le 3$, then $\omega'(v) \ge k - 6 - 1 - \sum_{j=1}^{p} (\frac{n_j}{3} + 1) = \frac{2k-2p-20}{3} \ge 0$. If $p \ge 6$, then $\omega'(v) \ge k-6-1-(k-p-1) = p-6 \ge 0$ by Observation 2.

Suppose that $p = 4$. If there are five consecutive vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+5} \in N_{3,4,5}^H(v)$, then $\omega'(v) \ge k - 6 - 1 - (\frac{5}{3} + 1) - (k - 1 - 4 - 5) = \frac{1}{3}$ by Observations 2 and 3. If there are $s, t \in [1, 4]$ with $n_s = 4$ and $n_t = 2$, then $\omega'(v) \ge k - 6 - 1 - (\frac{4}{3} + 1) - \frac{3}{2} - (k - 1 - 4 - 6) = \frac{1}{6}$ by Observations 2 and 3. If there are $s, t \in [1, 4]$ with $n_s = n_t = 3$, then $\omega'(v) \ge k - 6 - 1 - 2 \times (\frac{3}{3} + 1) - (k - 1 - 4 - 6) = 0$ by Observations 2 and 3. If there are $s, t, r \in [1, 4]$ with $n_s = 3$ and $n_t = n_r = 2$, then $\omega'(v) \ge k - 6 - 1 - (\frac{3}{3} + 1) - 2 \times \frac{3}{2} - (k - 1 - 4 - 7) = 0$ by Observations 2 and 3. Otherwise, $n_1 = n_2 = n_3 = n_4 = 2$. Then $k = 13$. By Observations 2 and 3, $\omega'(v) \ge 13 - 6 - 1 - 4 \times \frac{3}{2} = 0$.

Suppose that $p = 5$. If there are three consecutive vertices $v_{i+1}, v_{i+2}, v_{i+3} \in N_{3,4,5}^H(v)$, then $\omega'(v) \ge k - 6 - 1 - (\frac{3}{3} + 1) - (k - 1 - 5 - 3) = 0$ by Observations 2 and 3. Otherwise, there are $s, t \in [1, 5]$ with $n_s = n_t = 2$. By Observations 2 and 3, $\omega'(v) \ge k - 6 - 1 - 2 \times \frac{3}{2} - (k - 1 - 5 - 4) = 0$.

Case 7.2 v is not in a special 3-face. Let $q = d_2^H(v) + d_{6^+}^H(v)$, $v_{i_1}, v_{i_2}, \ldots, v_{i_q}$ be 6^+-vertices or 2-vertices with $0 = i_1 < i_2 < \cdots < i_q$, and $n_j = |\{l | i_j + 1 \le l \le i_{j+1} - 1, 3 \le d_H(v_l) \le 5\}|$ for $j \in [1, q]$. Then $i_{q+1} - 1 = k - 1$ and $\sum_{j=1}^{q} n_j = k - q$. By Observations 2 and 3, $\sum_{l=i_j+1}^{i_{j+1}-1} \tau(v \to v_l) \le \frac{n_j}{3} + 1$ for $j \in [1, q]$. If $q \le 4$, then $\omega'(v) \ge k - 6 - \sum_{j=1}^{q} (\frac{n_j}{3} + 1) = \frac{2k-2q-18}{3} \ge 0$. If $q \ge 6$, then $\omega'(v) \ge k - 6 - (k - q) = q - 6 \ge 0$ by Observation 2.

Suppose that $q = 5$. If there are three consecutive vertices $v_{i+1}, v_{i+2}, v_{i+3} \in N_{3,4,5}^H(v)$, then $\omega'(v) \ge k - 6 - (\frac{3}{3} + 1) - (k - 5 - 3) = 0$ by Observations 2 and

3. Otherwise, there are $s, t \in [1, 5]$ with $n_s = n_t = 2$. By Observations 2 and 3, $\omega'(v) \geq k - 6 - 2 \times \frac{3}{2} - (k - 5 - 4) = 0$. □

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