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Bohr-Type Inequalities with One Parameter for Bounded Analytic Functions of Schwarz Functions

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Abstract

In this article, some Bohr-type inequalities with one parameter or involving convex combination for bounded analytic functions of Schwarz functions are established. Some previous inequalities are generalized. All the results are sharp.

Keywords Bohr radius · Bohr-type inequality · Bounded analytic functions · Convex combination · Schwarz functions

Mathematics Subject Classification 30A10 · 30B10

1 Introduction

Let $\mathcal{B}$ denote the class of analytic functions f defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| \leq 1$ for $z \in \mathbb{D}$. The Bohr's theorem states that if $f \in \mathcal{B}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$\sum_{n=0}^{\infty} |a_n||z|^n \leq 1 \quad for \quad |z| = r \leq 1/3, \tag{1.1}$$

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the constant $1/3$ is sharp and the above inequality is called the classical Bohr inequality. Bohr originally established inequality (1.1) only for $r \leq 1/6$ in 1914 [16]. Later, the value $1/3$ was obtained independently by Riesz, Schur and Wiener. There are many articles that have shown the constant $1/3$ cannot be improved (see [29,31]).

For background information related to Bohr's phenomenon, we refer to the recent surveys by Ali et al. [8], Bénéteau et al. [10], Ismagilov et al. [21], Kayumova et al. [27] and the references therein. In particular, [11] includes the Bohr phenomenon on the subordination classes of concave univalent functions, [12] discusses some Bohr inequalities for logarithmic power series, and [13] initiates a study of the Bohr radius problem for derivatives of analytic functions. Some harmonic versions of Bohr's inequality were discussed in [18,24,26]. In recent years, many results related to Bohr's theorem are obtained in the setting of several complex variables. Boas and Khavinson [15] obtained some multidimensional generalizations of Bohr's theorem, and Aizenberg [5] extended it for further studies. For more information about Bohr inequality and related investigations, we refer to the recent articles [7,15,23].

It is worth pointing out that the Bohr radius has been discussed for certain power series in $\mathbb{D}$, as well as for analytic functions from $\mathbb{D}$ into other domains, such as convex domains [1,6], concave wedge domains [4], the punctured unit disk [3] and the exterior of the closed unit disk [2].

In order to state our main results, we recall the following several Bohr-type inequalities.

Theorem 1.1 [22] Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then

$$|f(z)| + \sum_{k=1}^{\infty} |a_k||z|^k \leq 1 \quad for \quad |z| = r \leq \sqrt{5} - 2$$

and the radius $\sqrt{5} - 2$ cannot be improved. Moreover,

$$|f(z)|^2 + \sum_{k=1}^{\infty} |a_k||z|^k \leq 1 \quad for \quad |z| = r \leq \frac{1}{3}$$

and the radius $\frac{1}{3}$ cannot be improved.

Theorem 1.2 [28] Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then

$$|f(z)| + \sum_{k=1}^{\infty} |a_{2k}||z|^{2k} \leq 1 \quad for \quad |z| = r \leq \sqrt{2} - 1$$

and the radius $\sqrt{2} - 1$ cannot be improved.

In [28], combining Theorem 2.5 and Remark 2.7, we get the following theorem.

Theorem 1.3 [28] Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. For $s \in \mathbb{N}$, then

$$|f(z)| + \sum_{k=1}^{\infty} |a_{sk}||z|^{sk} \le 1 \quad for \quad |z| = r \le R_s,$$

where R_s is positive root of the equation $\varphi_s(r) = 0$, $\varphi_s(r) = r^{s+1} + 3r^s + r - 1$. The radius R_s is the best possible.

Theorem 1.4 [28] Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then

$$|f(z)| + |f'(z)||z| + \sum_{k=2}^{\infty} |a_k||z|^k \le 1 \quad for \quad |z| = r \le \frac{\sqrt{17}-3}{4}$$

and the radius $\frac{\sqrt{17}-3}{4}$ cannot be improved.

In this paper, let $\mathcal{B}_m = \{\omega \in \mathcal{B} : \omega(0) = \cdots = \omega^{(m-1)}(0) = 0, \omega^{(m)}(0) \neq 0\}$ and $\mathcal{B}_n = \{\omega \in \mathcal{B} : \omega(0) = \cdots = \omega^{(n-1)}(0) = 0, \omega^{(n)}(0) \neq 0\}$ be the classes of Schwarz functions, where $m, n \in \mathbb{N} = \{1, 2, \cdots\}$. Our aims of this article are to generalize the above theorems and establish some new Bohr-type inequalities with one parameter or involving convex combination for bounded analytic functions of Schwarz functions.

2 Some Lemmas

In order to establish our main results, we need the following some lemmas which will play the key role in proving the main results of this paper.

Lemma 2.1 *(Schwarz-Pick lemma) Let $\phi(z)$ be analytic in the open unit disk $\mathbb{D}$ and $|\phi(z)| < 1$. Then*

$$\frac{|\phi(z_1) - \phi(z_2)|}{|1 - \overline{\phi(z_1)}\phi(z_2)|} \le \frac{|z_1 - z_2|}{|1 - \overline{z_1} z_2|} \quad for \quad z_1, z_2 \in \mathbb{D},$$

and equality holds for distinct $z_1, z_2 \in \mathbb{D}$ if and only if ϕ is a Möbius transformation. In particular,

$$|\phi'(z)| \le \frac{1 - |\phi(z)^2|}{1 - |z^2|} \quad for \quad z \in \mathbb{D},$$

and equality holds for some $z \in \mathbb{D}$ if and only if ϕ is a Möbius transformation.

Lemma 2.2 ([19]) *Suppose $f(z)$ is analytic in the open unit disk $\mathbb{D}$ and $|f(z)| \le 1$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $|a_n| \le 1 - |a_0|^2$ for all $n \in \mathbb{N}$.*

Lemma 2.3 *For* $0 \le x \le x_0 \le 1$, *it holds that*

$$\Phi(x) := x + A(1 - x^2) \le \Phi(x_0) \quad whenever \quad 0 \le A \le 1/2, \tag{2.1}$$

and similarly,

$$\Psi(x) := x^2 + A(1 - x^2) \le \Psi(x_0) \quad whenever \quad 0 \le A \le 1. \tag{2.2}$$

The proof is simple, we omit it.

3 Main Results

In Theorem 3.1, we give a kind of convex combination form for refined classical Bohr inequality as follows.

Theorem 3.1 *Suppose that* $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a := |a_0|$ *and* $\omega \in \mathcal{B}_m$ *for* $m \in \mathbb{N}$. *Then for* $t \in [0, 1)$, *we have*

$$t|f(\omega(z))| + (1 - t)\sum_{k=0}^{\infty} |a_k||\omega(z)|^k \le 1 \tag{3.1}$$

for $|z| = r \le R_{t,m}$, *where*

$$R_{t,m} = \begin{cases} \sqrt[m]{\frac{1-2\sqrt{1-t}}{4t-3}} \quad for \quad t \in [0, \frac{3}{4}) \bigcup (\frac{3}{4}, 1), \\ \sqrt[m]{\frac{1}{2}} \quad for \quad t = \frac{3}{4}. \end{cases}$$

The radius $R_{t,m}$ *is the best possible.*

Proof According to the assumption, $f \in \mathcal{B}$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$, by the Schwarz lemma and the Schwarz-Pick lemma, respectively, we obtain

$$|\omega(z)| \le |z|^m, \quad |f(z)| \le \frac{|z| + a}{1 + a|z|}$$

for $z \in \mathbb{D}$. It follows that

$$|f(\omega(z)| \le \frac{|\omega(z)| + a}{1 + a|\omega(z)|} \le \frac{r^m + a}{1 + ar^m}, \quad |z| = r < 1. \tag{3.2}$$

Using inequality (3.2) and Lemma 2.2, we have

$$t|f(\omega(z))| + (1 - t)\sum_{k=0}^{\infty} |a_k||\omega(z)|^k$$

$$\le t\frac{r^m+a}{1+ar^m}+(1-t)a+(1-t)(1-a^2)\frac{r^m}{1-r^m}$$
$$:= A_m(a,r,t).$$

Now, we need to show that $A_m(a,r,t)\le 1$ holds for $r\le R_{t,m}$. It is equivalent to show $A(a,r,t)\le 0$, where

$$\begin{aligned}A(a,r,t)&=[A_m(a,r,t)-1](1+ar^m)(1-r^m)\\&=-r^{2m}(1-t)a^3+[-r^{2m}(1-t)]a^2\\&\quad+[(1-r^m)^2+(1-t)r^{2m}]a-r^{2m}t+2r^m-1.\end{aligned}$$

Obviously,

$$\begin{aligned}\frac{\partial A(a,r,t)}{\partial a}=&-3r^{2m}(1-t)a^2-2r^{2m}(1-t)a+(1-r^m)^2+(1-t)r^{2m}\\:=&B(a,r,t).\end{aligned}$$

Observe that $B(a,r,t)$ is a continuous and decreasing function of $a\in[0,1)$ for fixed $t\in[0,1)$ and $r\in(0,1)$. Then we have $B(a,r,t)\ge B(1,r,t)=(4t-3)r^{2m}-2r^m+1$. Next, we divide it into two cases to discuss.

Case 1. If $t\in[0,\frac{3}{4})\bigcup(\frac{3}{4},1)$, then we have $B(a,r,t)\ge B(1,r,t)\ge 0$ for $r\le\sqrt[m]{\frac{1-2\sqrt{1-t}}{4t-3}}$, where $\sqrt[m]{\frac{1-2\sqrt{1-t}}{4t-3}}$ is the unique root in $(0,1)$ of the equation $(4t-3)r^{2m}-2r^m+1=0$. It follows that the $A(a,r,t)$ is an increasing function of a for $a\in[0,1)$. Thus, $A(a,r,t)\le A(1,r,t)=0$ for $r\le\sqrt[m]{\frac{1-2\sqrt{1-t}}{4t-3}}$.

Case 2. If $t=\frac{3}{4}$, then we have $B(a,r,t)\ge B(1,r,t)=-2r^m+1\ge 0$ for $r\le\sqrt[m]{\frac{1}{2}}$. Thus, $A(a,r,t)\le A(1,r,t)=0$ for $r\le\sqrt[m]{\frac{1}{2}}$.

Next we show the radius $R_{t,m}$ is sharp. For $a\in[0,1)$, let

$$\omega(z)=z^m,\quad f(z)=\frac{a+z}{1+az}=a+(1-a^2)\sum_{k=1}^{\infty}(-a)^{k-1}z^k,\quad z\in\mathbb{D}. \tag{3.3}$$

Taking $z=r$, then the left side of inequality (3.1) reduces to

$$t|f(r^m)|+(1-t)\sum_{k=0}^{\infty}|a_k|r^{mk}=t\frac{a+r^m}{1+ar^m}+(1-t)a+(1-t)(1-a^2)\frac{r^m}{1-ar^m}. \tag{3.4}$$

Now we just need to show that if $r>R_{t,m}$, then there exists an a, such that the right side of (3.4) is greater than 1. That is

$$(1-a)[2r^{2m}(1-t)a^2-((2t-1)r^{2m}-r^m)a+r^m-1]>0. \tag{3.5}$$

Let

$$C(a,r,t)=2r^{2m}(1-t)a^2-((2t-1)r^{2m}-r^m)a+r^m-1.$$

Then we have

$$\begin{aligned}\frac{\partial C(a,r,t)}{\partial a}=&4r^{2m}(1-t)a-(2t-1)r^{2m}+r^m\\:=&D(a,r,t).\end{aligned}$$

Observe that $D(a,r,t)$ is a continuous and increasing function of $a\in[0,1)$ for each fixed $t\in[0,1)$ and $r\in(0,1)$. Then we have

$$D(a,r,t)\geq D(0,r,t)=(1-2t)r^{2m}+r^m\geq 0$$

for any $t\in[0,1)$ and $r\in(0,1)$. It means that

$$C(a,r,t)\leq C(1,r,t)=(3-4t)r^{2m}+2r^m-1=-B(1,r,t).$$

Furthermore, the monotonicity of $B(1,r,t)$ leads that if $r>R_{t,m}$, then $B(1,r,t)<0$. Namely, if $r>R_{t,m}$, then $C(1,r,t)>0$. Hence, by the continuity of $C(a,r,t)$, we have

$$\lim_{a\to 1^-}C(a,r,t)=C(1,r,t)>0.$$

Therefore, if $r>R_{t,m}$, then there exists an $a\in[0,1)$, such that inequality (3.5) holds. This proves the sharpness and proof of Theorem 3.1 is complete. □

Remark 3.1 1. If $\omega(z)=z$, then Theorem 3.1 reduces to Theorem 3.1 of [32].
2. If $\omega(z)=z$, $t=0$, then Theorem 3.1 reduces to the classical Bohr inequality.

Theorem 3.2 *Suppose that $f\in\mathcal{B}$, $f(z)=\sum_{k=0}^{\infty}a_kz^k$, $a:=|a_0|$ and $\omega_m\in\mathcal{B}_m$, $\omega_n\in\mathcal{B}_n$ for $m,n\in\mathbb{N}$. Then for $\lambda\in(0,\infty)$, we have*

$$|f(\omega_m(z))|+\lambda\sum_{k=1}^{\infty}|a_k||\omega_n(z)|^k\leq 1 \tag{3.6}$$

for $|z|=r\leq R_{\lambda,m,n}$, where $R_{\lambda,m,n}$ is the unique root in $(0,1)$ of the equation

$$(2\lambda-1)r^{m+n}+2\lambda r^n+r^m+r^n-1=0$$

and the radius $R_{\lambda,m,n}$ is the best possible. Moreover,

$$|f(\omega_m(z))|^2+\lambda\sum_{k=1}^{\infty}|a_k||\omega_n(z)|^k\leq 1 \tag{3.7}$$

for $|z| = r \leq R_{2,\lambda,m,n}$, *where* $R_{2,\lambda,m,n}$ *is the unique root in* $(0, 1)$ *of the equation*

$$(\lambda - 1)r^{2m+n} + 2\lambda r^{m+n} + (\lambda + 1)r^n + r^{2m} - 1 = 0$$

and the radius $R_{2,\lambda,m,n}$ *is the best possible.*

Proof Firstly, we consider the first part. By the Schwarz lemma and the Schwarz-Pick lemma, respectively, we obtain

$$|\omega_m(z)| \leq |z|^m, \quad |\omega_n(z)| \leq |z|^n, \quad and \quad |f(\omega_m(z))| \leq \frac{r^m + a}{1 + ar^m}, \tag{3.8}$$

for $z \in \mathbb{D}$. Then by Lemma 2.2, we obtain

$$\begin{aligned} |f(\omega_m(z))| + \lambda \sum_{k=1}^{\infty} |a_k||\omega_n(z)|^k \leq & \frac{a + r^m}{1 + ar^m} + \lambda(1 - a^2)\frac{r^n}{1 - r^n} \\ := & A_{m,n}(a, r, \lambda). \end{aligned}$$

We just need to show that $A_{m,n}(a, r, \lambda) \leq 1$ holds for $r \leq R_{\lambda,m,n}$. That is to prove $A(a, r, \lambda) \leq 0$, where

$$\begin{aligned} A(a, r, \lambda) = & \ (a + r^m)(1 - r^n) + \lambda(1 - a^2)r^n(1 + ar^m) - (1 + ar^m)(1 - r^n) \\ = & \ (1 - a)[r^{m+n}\lambda a^2 + r^n\lambda(1 + r^m)a + r^n\lambda - r^{m+n} + r^m + r^n - 1] \\ \leq & \ (1 - a)[r^{m+n}\lambda + r^n\lambda(1 + r^m) + r^n\lambda - r^{m+n} + r^m + r^n - 1] \\ = & \ (1 - a)[(2\lambda - 1)r^{m+n} + 2\lambda r^n + r^m + r^n - 1]. \end{aligned}$$

Obviously, it is enough to show that $(2\lambda - 1)r^{m+n} + 2\lambda r^n + r^m + r^n - 1 \leq 0$ holds for $r \leq R_{\lambda,m,n}$. Let $g(r) = (2\lambda - 1)r^{m+n} + 2\lambda r^n + r^m + r^n - 1$, then we have

$$\begin{aligned} g'(r) = & \ (m + n)(2\lambda - 1)r^{m+n-1} + 2n\lambda r^{n-1} + mr^{m-1} + nr^{n-1} \\ = & \ 2(m + n)\lambda r^{m+n-1} + mr^{m-1}(1 - r^n) + nr^{n-1}(1 - r^m) + 2n\lambda r^{n-1} \\ > & \ 0. \end{aligned}$$

We conclude that $g(r)$ is an increasing function of $r \in (0, 1)$ for fixed $\lambda \in (0, \infty)$. Meanwhile, we observe that $g(0) = -1 < 0$ and $g(1) = 4\lambda > 0$. Then there is a unique root $R_{\lambda,m,n} \in (0, 1)$ such that $g(r) = 0$. Hence, $g(r) \leq 0$ holds for $r \leq R_{\lambda,m,n}$.

To show that the radius $R_{\lambda,m,n}$ is the best possible. For $a \in [0, 1)$, let

$$\begin{aligned} & \omega_m(z) = z^m, \quad \omega_n(z) = z^n \quad and \\ & f(z) = \frac{a + z}{1 + az} = a + (1 - a^2)\sum_{k=1}^{\infty}(-a)^{k-1}z^k, \quad z \in \mathbb{D}. \end{aligned} \tag{3.9}$$

Taking $z = r$, then the left side of inequality (3.6) reduces to

$$|f(r^m)| + \lambda \sum_{k=1}^{\infty} |a_k| r^{nk} = \frac{a + r^m}{1 + ar^m} + \lambda(1 - a^2)\frac{r^n}{1 - ar^n}. \tag{3.10}$$

Now we just need to show that if $r > R_{\lambda,m,n}$, then there exists an $a \in [0, 1)$, such that the right side of (3.10) is greater than 1. That is

$$(1 - a)(r^{m+n}\lambda a^2 + r^n[\lambda(1 + r^m) + (1 - r^m)]a + r^n\lambda + r^m - 1) > 0. \tag{3.11}$$

Let

$$B(a, r, \lambda) = r^{m+n}\lambda a^2 + r^n[\lambda(1 + r^m) + (1 - r^m)]a + r^n\lambda + r^m - 1.$$

Obviously, $B(a, r, \lambda)$ is a continuous and increasing function of $a \in [0, 1)$ for each fixed $\lambda \in (0, \infty)$ and $r \in (0, 1)$. Then $B(a, r, \lambda) \le B(1, r, \lambda) = (2\lambda - 1)r^{m+n} + (2\lambda + 1)r^n + r^m - 1 = g(r)$ for $\lambda \in (0, \infty)$ and $r \in (0, 1)$. Meanwhile, the monotonicity of $g(r)$ leads to that if $r > R_{\lambda,m,n}$, then $B(1, r, \lambda) > 0$. Hence, by the continuity of $B(a, r, \lambda)$, if $r > R_{\lambda,m,n}$, we have

$$\lim_{a \to 1^-} B(a, r, \lambda) = B(1, r, \lambda) > 0.$$

Therefore, if $r > R_{\lambda,m,n}$, then there exists an a, such that inequality (3.11) holds.

Next, we prove the second part. As in the previous case, by (3.8) and Lemma 2.2, it follows easily that

$$|f(\omega_m(z))|^2 + \lambda \sum_{k=1}^{\infty} |a_k||\omega_n(z)|^k \le \left(\frac{a + r^m}{1 + ar^m}\right)^2 + \lambda(1 - a^2)\frac{r^n}{1 - r^n}. \tag{3.12}$$

We know above inequality (3.12) is smaller than or equal to 1 for $r \le R_{2,\lambda,m,n}$ provided $A_2(a, r, \lambda) \le 0$, where

$$\begin{aligned} A_2(a, r, \lambda) =&(1 - a^2)[r^{2m+n}\lambda a^2 + 2r^{m+n}\lambda a - r^{2m+n} + r^n\lambda + r^{2m} + r^n - 1] \\ \le&(1 - a^2)[(\lambda - 1)r^{2m+n} + 2\lambda r^{m+n} + (\lambda + 1)r^n + r^{2m} - 1]. \end{aligned}$$

It is sufficient for us to prove $(\lambda - 1)r^{2m+n} + 2\lambda r^{m+n} + (\lambda + 1)r^n + r^{2m} - 1 \le 0$ holds for $r \le R_{2,\lambda,m,n}$. Let $k(r) = (\lambda - 1)r^{2m+n} + 2\lambda r^{m+n} + (\lambda + 1)r^n + r^{2m} - 1$, then we obtain

$$\begin{aligned} k'(r) =&(2m + n)(\lambda - 1)r^{2m+n-1} + 2(m + n)\lambda r^{m+n-1} \\ &+ n(\lambda + 1)r^{n-1} + 2mr^{2m-1} \\ =&(2m + n)\lambda r^{2m+n-1} + nr^{n-1}(1 - r^{2m}) \\ &+ 2mr^{2m-1}(1 - r^n) + 2(m + n)\lambda r^{m+n-1} \end{aligned}$$

$$+n\lambda r^{n-1} > 0.$$

Obviously, $k(r)$ is an increasing function of $r \in (0, 1)$ for fixed $\lambda \in (0, \infty)$. And we also have $k(0) = -1 < 0$ and $k(1) = 4\lambda > 0$. Then there is a unique root $R_{2,\lambda,m,n} \in (0, 1)$ such that $k(r) = 0$. Hence, $k(r) \leq 0$ holds for $r \leq R_{2,\lambda,m,n}$.

The sharpness part follows similarly. Thus the proof of Theorem 3.2 is complete. □

In Theorem 3.2, setting $\omega_m(z) = \omega_n(z) = \omega(z)$, then we have the following corollary.

Corollary 3.1 *Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then for $\lambda \in (0, \infty)$, we have*

$$|f(\omega(z))| + \lambda \sum_{k=1}^{\infty} |a_k||\omega(z)|^k \leq 1$$

for $|z| = r \leq R_{\lambda,m}$, where

$$R_{\lambda,m} = \begin{cases} \sqrt[m]{\frac{\sqrt{\lambda^2+4\lambda}-(\lambda+1)}{2\lambda-1}} & for \quad \lambda \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, \infty), \\ \sqrt[m]{\frac{1}{3}} \quad for \quad \lambda = \frac{1}{2}. \end{cases}$$

The radius $R_{\lambda,m}$ is the best possible. Moreover,

$$|f(\omega(z))|^2 + \lambda \sum_{k=1}^{\infty} |a_k||\omega(z)|^k \leq 1$$

for $|z| = r \leq R_{2,\lambda,m}$, where

$$R_{2,\lambda,m} = \begin{cases} \sqrt[m]{\frac{\lambda+2-\sqrt{\lambda^2+8\lambda}}{2(1-\lambda)}} & for \quad \lambda \in (0, 1) \bigcup (1, \infty), \\ \sqrt[m]{\frac{1}{3}} \quad for \quad \lambda = 1. \end{cases}$$

The radius $R_{2,\lambda,m}$ is the best possible.

Remark 3.2 1. If $\omega(z) = z$, then Corollary 3.1 reduces to Theorem 3.3 of [32].
2. If $\omega(z) = z$, $\lambda = 1$, then Corollary 3.1 reduces to Theorem 1.1.

Theorem 3.3 *Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a := |a_0|$ and $\omega_m(z) \in \mathcal{B}_m$, $\omega_n(z) \in \mathcal{B}_n$ for $m, n \in \mathbb{N}$. Then for $\lambda \in (0, \infty)$ and $s \in \mathbb{N}$, we have*

$$|f(\omega_m(z))| + \lambda \sum_{k=1}^{\infty} |a_{sk}||\omega_n(z)|^{sk} \leq 1 \tag{3.13}$$

for $|z| = r \leq R_{\lambda,m,n,s}$*, where* $R_{\lambda,m,n,s}$ *is unique root in* $(0, 1)$ *of equation*

$$(2\lambda - 1)r^{ns+m} + (2\lambda + 1)r^{ns} + r^m - 1 = 0.$$

The radius $R_{\lambda,m,n,s}$ *is the best possible.*

Proof Inequality (3.8) and Lemma 2.2 lead to that

$$|f(\omega_m(z))| + \lambda \sum_{k=1}^{\infty} |a_{sk}||\omega_n(z)|^{sk} \leq \frac{a + r^m}{1 + ar^m} + \lambda(1 - a^2)\frac{r^{ns}}{1 - r^{ns}} := A_{m,n,s}(a, r, \lambda). \tag{3.14}$$

We know (3.14) is smaller than or equal to 1 provided $A'_{m,n,s}(a, r, \lambda) \leq 0$, where

$$\begin{aligned} &A'_{m,n,s}(a, r, \lambda) \\ &= (a + r^m)(1 - r^{ns}) + \lambda(1 - a^2)r^{ns}(1 + ar^m) - (1 + ar^m)(1 - r^{ns}) \\ &= (1 - a)[r^{ns+m}\lambda a^2 + r^{ns}\lambda(1 + r^m)a + r^{ns}\lambda - r^{ns+m} + r^{ns} + r^m - 1] \\ &\leq (1 - a)[r^{ns+m}\lambda + r^{ns}\lambda(1 + r^m) + r^{ns}\lambda - r^{ns+m} + r^{ns} + r^m - 1] \\ &= (1 - a)[(2\lambda - 1)r^{ns+m} + (2\lambda + 1)r^{ns} + r^m - 1]. \end{aligned}$$

Let $l(r) = (2\lambda - 1)r^{ns+m} + (2\lambda + 1)r^{ns} + r^m - 1 \leq 0$. Now, $A'_{m,n,s}(a, r, \lambda) \leq 0$ if $l(r) \leq 0$, which holds for $r \leq R_{\lambda,m,n,s}$. When $\lambda \in (0, \infty)$, we have

$$\begin{aligned} l'(r) &= (2\lambda - 1)(ns + m)r^{ns+m-1} + (2\lambda + 1)(ns)r^{ns-1} + mr^{m-1} \\ &= 2\lambda(ns + m)r^{ns+m-1} + nsr^{ns-1}(1 - r^m) + mr^{m-1}(1 - r^{ns}) + 2\lambda nsr^{ns-1} \\ &> 0. \end{aligned}$$

We claim that for any $\lambda \in (0, \infty)$, $l(r)$ is a monotonically increasing function of $r \in (0, 1)$. Meanwhile, we have $l(0)l(1) < 0$. Thus, there is a unique root $R_{\lambda,m,n,s} \in (0, 1)$ such that $l(r) = 0$. Hence, $l(r) \leq 0$ holds for $r \leq R_{\lambda,m,n,s}$.

Now, we show that the radius $R_{\lambda,m,n,s}$ is the best possible, we still consider the function $\omega_m(z)$, $\omega_n(z)$, $f(z)$ as in (3.9). Taking $z = r$, then the left side of inequality (3.13) reduces to

$$|f(r^m)| + \lambda \sum_{k=1}^{\infty} |a_{sk}|r^{nsk} = \frac{a + r^m}{1 + ar^m} + \lambda(1 - a^2)\frac{a^{s-1}r^{ns}}{1 - a^s r^{ns}}. \tag{3.15}$$

Now to show that if $r > R_{\lambda,m,n,s}$, then there exists an $a \in [0, 1)$, such that the right side of (3.15) is greater than 1. That is

$$(1 - a)(r^{ns+m}\lambda a^{s+1} + r^{ns}[1 - r^m + (r^m + 1)\lambda]a^s + \lambda r^{ns}a^{s-1} + r^m - 1) > 0. \tag{3.16}$$

Let

$$B_{m,n,s}(a,r,\lambda) = r^{ns+m}\lambda a^{s+1} + r^{ns}[1-r^m+(r^m+1)\lambda]a^s + \lambda r^{ns}a^{s-1} + r^m - 1.$$

Obviously, $B_{m,n,s}(a,r,\lambda)$ is a continuous and increasing function of $a \in [0,1)$ for each fixed $\lambda \in (0,\infty)$ and $r \in (0,1)$. Then

$$B_{m,n,s}(a,r,\lambda) \le B_{m,n,s}(1,r,\lambda) = (2\lambda-1)r^{ns+m} + (2\lambda+1)r^{ns} + r^m - 1 = l(r)$$

holds for $\lambda \in (0,\infty)$ and $r \in (0,1)$. Furthermore, according to the monotonicity of $l(r)$, we have if $r > R_{\lambda,m,n,s}$, then $B_{m,n,s}(1,r,\lambda) > 0$. Hence, by the continuity of $B_{m,n,s}(a,r,\lambda)$, if $r > R_{\lambda,m,n,s}$, we have

$$\lim_{a\to 1^-} B_{m,n,s}(a,r,\lambda) = B_{m,n,s}(1,r,\lambda) > 0.$$

Therefore, if $r > R_{\lambda,m,n,s}$, then there exists an a, such that inequality (3.16) holds. □

Remark 3.3 If $\lambda = 1$, then Theorem 3.3 reduces to Theorem 3.3 of [20].

In Theorem 3.3, setting $\omega_m(z) = \omega_n(z) = \omega(z)$; $\omega_m(z) = \omega_n(z) = \omega(z)$ and $s = 2$, then we have Corollaries 3.2, 3.3, respectively.

Corollary 3.2 *Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then for $\lambda \in (0,\infty)$ and $s \in \mathbb{N}$, we have*

$$|f(\omega(z))| + \lambda \sum_{k=1}^{\infty} |a_{sk}||\omega(z)|^{sk} \le 1$$

for $|z| = r \le R_{\lambda,m,s}$, where $R_{\lambda,m,s}$ is unique root in $(0,1)$ of equation

$$(2\lambda-1)r^{ms+m} + (2\lambda+1)r^{ms} + r^m - 1 = 0.$$

The radius $R_{\lambda,m,s}$ is the best possible.

Remark 3.4 1. If $\omega(z) = z$, then Corollary 3.2 reduces to Theorem 3.2 of [32].
2. If $\omega(z) = z$ and $\lambda = 1$, then Corollary 3.2 reduces to Theorem 1.3.

Corollary 3.3 *Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then for $\lambda \in (0,\infty)$, we have*

$$|f(\omega(z))| + \lambda \sum_{k=1}^{\infty} |a_{2k}||\omega(z)|^{2k} \le 1$$

for $|z| = r \le R_{\lambda,m,2}$, where

$$R_{\lambda,m,2} = \begin{cases} \sqrt[m]{\frac{\sqrt{2\lambda}-1}{2\lambda-1}} & for \quad \lambda \in [0,\frac{1}{2}) \bigcup (\frac{1}{2},\infty), \\ \sqrt[m]{\frac{1}{2}} & for \quad \lambda = \frac{1}{2}. \end{cases}$$

The radius $R_{\lambda,m,2}$ *is the best possible.*

Remark 3.5 If $\omega(z) = z$ and $\lambda = 1$, then Corollary 3.3 reduces to Theorem 1.2.

Theorem 3.4 *Suppose that* $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a := |a_0|$ *and* $\omega \in \mathcal{B}_m$ *for* $m \in \mathbb{N}$. *Then for* $\lambda \in (0, \infty)$, *we have*

$$|f(\omega(z))| + |f'(\omega(z))||\omega(z)| + \lambda \sum_{k=2}^{\infty} |a_k||\omega(z)|^k \leq 1 \tag{3.17}$$

for $|z| = r \leq R_\lambda$, *where*

$$R_\lambda = \begin{cases} r_\lambda, & for \quad \lambda \in (\frac{1}{2}, \infty) \\ r^*, & for \quad \lambda \in (0, \frac{1}{2}] \end{cases}$$

is the best possible, and the radii r_λ *and* r^* *are the unique roots in* $(0, \sqrt[m]{\sqrt{2}-1})$ *of the equations*

$$2\lambda r^{4m} + (4\lambda - 1)r^{3m} + (2\lambda - 1)r^{2m} + 3r^m - 1 = 0$$

and

$$r^{4m} + r^{3m} + 3r^m - 1 = 0,$$

respectively.

Proof By inequality (3.2), Schwarz-Pick lemma, Lemma 2.2 and Lemma 2.3 (2.1), respectively. Then we have

$$\begin{aligned} &|f(\omega(z))| + |f'(\omega(z))||\omega(z)| + \lambda \sum_{k=2}^{\infty} |a_k||\omega(z)|^k \\ &\leq \frac{a + r^m}{1 + ar^m} + \left[1 - \left(\frac{a + r^m}{1 + ar^m}\right)^2\right] \frac{r^m}{1 - r^{2m}} + \lambda(1 - a^2)\frac{r^{2m}}{1 - r^m} \\ &= \frac{a + r^m}{1 + ar^m} + \frac{(1 - a^2)r^m}{(1 + ar^m)^2} + \lambda(1 - a^2)\frac{r^{2m}}{1 - r^m} \\ &= 1 + \frac{(1 - a)\Phi(a, r, \lambda)}{(1 + ar^m)^2(1 - r^m)}, \end{aligned}$$

for $r \leq \sqrt[m]{\sqrt{2}-1}$, where

$$\begin{aligned} \Phi(a, r, \lambda) = {} & r^{4m}\lambda a^3 + (2r^{3m}\lambda + r^{4m}\lambda)a^2 + (r^{2m}\lambda + 2r^{3m}\lambda + r^{2m} - r^{3m})a \\ & +3r^m + r^{2m}\lambda - 2r^{2m} - 1 \end{aligned}$$

and $\sqrt[m]{\sqrt{2}-1}$ is the unique root in $(0,1)$ of the equation $r^{2m}+2r^m-1=0$. Observe that $\Phi(a,r,\lambda)$ is a monotonically increasing function of $a\in[0,1)$ for each fixed $\lambda\in[0,\infty)$ and $r\in(0,1)$. Then we have

$$\Phi(a,r,\lambda)\leq\Phi(1,r,\lambda)=2\lambda r^{4m}+(4\lambda-1)r^{3m}+(2\lambda-1)r^{2m}+3r^m-1.$$

Now, we need to show that $\Phi(1,r,\lambda)\leq 0$ holds for $r\leq R_\lambda$.

Case 1. If $\lambda\in(\frac{1}{2},\infty)$, $\Phi(1,r,\lambda)$ is a continuous and increasing function of $r\in(0,1)$ and

$$\Phi(1,0,\lambda)\Phi(1,\sqrt[m]{\sqrt{2}-1},\lambda)<0.$$

Thus r_λ is unique root in $(0,\sqrt[m]{\sqrt{2}-1})$ of $\Phi(1,r,\lambda)$ and $\Phi(1,r,\lambda)\leq 0$ for $r\leq r_\lambda$.

Case 2. If $\lambda\in[0,\frac{1}{2}]$, then we have $\Phi(1,r,\lambda)\leq r^{4m}+r^{3m}+3r^m-1$. Let $j(r)=r^{4m}+r^{3m}+3r^m-1$. One can verify that $j(r)$ is a continuous and increasing function of $r\in(0,1)$ and $j(0)j(\sqrt[m]{\sqrt{2}-1})<0$. Thus r^* is unique root in $(0,\sqrt[m]{\sqrt{2}-1})$ of $j(r)$ and $j(r)\leq 0$ for $r\leq r^*$. Then $\Phi(1,r,\lambda)\leq j(r)\leq 0$ for $r\leq r^*$. Hence, inequality (3.17) holds for $r\leq R_\lambda$.

To show the radius R_λ is sharp, we consider the function $\omega(z)$ and $f(z)$ is same as (3.3). Taking $z=r$, then the left side of inequality (3.17) gives

$$|f(r^m)|+|f'(r^m)||r^m|+\lambda\sum_{k=2}^{\infty}|a_k||r|^{mk}=\frac{a+r^m}{1+ar^m}+\frac{(1-a^2)r^m}{(1+ar^m)^2}+\lambda(1-a^2)\frac{ar^{2m}}{1-ar^m}.$$

Next, we show that if $r>R_\lambda$, then there exists an $a\in[0,1)$, such that the right side of above equality is greater than 1. It is equivalent to show that

$$\begin{aligned}&r^{4m}\lambda a^4+(2r^{3m}\lambda+r^{4m}\lambda)a^3+(r^{2m}\lambda+2r^{3m}\lambda-r^{3m})a^2\\&+(r^m+r^{2m}\lambda-r^{2m})a+2r^m-1>0.\end{aligned}\tag{3.18}$$

Let

$$\begin{aligned}P(a,r,\lambda)=&\,r^{4m}\lambda a^4+(2r^{3m}\lambda+r^{4m}\lambda)a^3+(r^{2m}\lambda+2r^{3m}\lambda-r^{3m})a^2\\&+(r^m+r^{2m}\lambda-r^{2m})a+2r^m-1.\end{aligned}$$

Next, we divide it into two cases to show that there exists an $a\in[0,1)$, such that (3.18) holds for $r>R_\lambda$.

Case 1. If $\lambda\in(\frac{1}{2},\infty)$, one can verify that the function $P(a,r,\lambda)$ is an increasing function with respect to $a\in[0,1)$ for each fixed $\lambda\in(\frac{1}{2},\infty)$ and $r\in(0,1)$. Thus $P(a,r,\lambda)\leq P(1,r,\lambda)=2\lambda r^{4m}+(4\lambda-1)r^{3m}+(2\lambda-1)r^{2m}+3r^m-1=\Phi(1,r,\lambda)$.

According to the monotonicity of $\Phi(1, r, \lambda)$, if $r > r_\lambda$, then $P(1, r, \lambda) = \Phi(1, r, \lambda) > 0$. In the same way, if $r > r_\lambda$, we have

$$\lim_{a\to 1^-} P(a, r, \lambda) = P(1, r, \lambda) > 0.$$

Hence, if $r > r_\lambda$, then there exists an a, such that inequality (3.18) holds.

Case 2. If $\lambda \in [0, \frac{1}{2}]$, then

$$\begin{aligned} P(a, r, \lambda) &\le P(a, r, \frac{1}{2}) \\ &= \frac{1}{2} r^{4m} a^4 + \left(\frac{1}{2} r^{4m} + r^{3m}\right) a^3 \\ &\quad + \frac{1}{2} r^{2m} a^2 + \left(-\frac{1}{2} r^{2m} + r^m\right) a + 2r^m - 1 \\ &\le r^{4m} + r^{3m} + 3r^m - 1 = j(r). \end{aligned}$$

According to the monotonicity of $j(r)$, if $r > r^*$, then $j(r) > 0$. It means that

$$\lim_{a\to 1^-} P(a, r, \lambda) = j(r) > 0.$$

Therefore, if $r > r^*$, then there exists an a, such that (3.18) holds. We complete the proof of theorem. □

Remark 3.6 1. If $\lambda = 1$, then Theorem 3.4 reduces to Corollary 4.5 of [20].
2. If $\omega(z) = z$, then Theorem 3.4 reduces to Theorem 3.4 of [32].
3. If $\omega(z) = z$ and $\lambda = 1$ in Theorem 3.4, then it reduces to Theorem 1.4.

Theorem 3.5 *Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a := |a_0|$ and $\omega \in \mathcal{B}_m$ for $m \in \mathbb{N}$. Then for $\lambda \in (0, \infty)$, we have*

$$|f(\omega(z))|^2 + |f'(\omega(z))||\omega(z)| + \lambda \sum_{k=2}^{\infty} |a_k||\omega(z)|^k \le 1 \tag{3.19}$$

for $|z| = r \le R_{2,\lambda}$, where

$$R_{2,\lambda} = \begin{cases} r_{2,\lambda}, & for \quad \lambda \in (1, \infty) \\ r_2^*, & for \quad \lambda \in (0, 1] \end{cases}$$

is the best possible, and the radii $r_{2,\lambda}$ and r_2^* are the unique roots in $(0, \sqrt[m]{\frac{\sqrt{5}-1}{2}})$ of the equations

$$\lambda r^{4m} + (2\lambda - 1) r^{3m} + \lambda r^{2m} + 2r^m - 1 = 0$$

and

$$r^{4m} + r^{3m} + r^{2m} + 2r^m - 1 = 0,$$

respectively.

Proof By inequality (3.2), Schwarz-Pick lemma, Lemma 2.2 and Lemma 2.3 (2.2), respectively. Then

$$\begin{aligned}
&|f(\omega(z))|^2 + |f'(\omega(z))||\omega(z)| + \lambda \sum_{k=2}^{\infty} |a_k||\omega(z)|^k \\
&\le \left(\frac{a + r^m}{1 + ar^m}\right)^2 + \left[1 - \left(\frac{a + r^m}{1 + ar^m}\right)^2\right] \frac{r^m}{1 - r^{2m}} + \lambda(1 - a^2)\frac{r^{2m}}{1 - r^m} \\
&= \left(\frac{a + r^m}{1 + ar^m}\right)^2 + \frac{(1 - a^2)r^m}{(1 + ar^m)^2} + \lambda(1 - a^2)\frac{r^{2m}}{1 - r^m} \\
&= 1 + \frac{(1 - a^2)\Psi(a, r, \lambda)}{(1 + ar^m)^2(1 - r^m)},
\end{aligned}$$

for $r \le \sqrt[m]{\frac{\sqrt{5}-1}{2}}$, where

$$\Psi(a, r, \lambda) = r^{4m}\lambda a^2 + 2r^{3m}\lambda a - r^{3m} + r^{2m}\lambda + 2r^m - 1$$

and $\sqrt[m]{\frac{\sqrt{5}-1}{2}}$ is the unique root in $(0, 1)$ of the equation $r^{2m} + r^m - 1 = 0$. Observe that $\Psi(a, r, \lambda)$ is a monotonically increasing function of $a \in [0, 1)$ for each fixed $\lambda \in [0, \infty)$ and $r \in (0, 1)$. Then we have

$$\Psi(a, r, \lambda) \le \Psi(1, r, \lambda) = r^{4m}\lambda + (2\lambda - 1)r^{3m} + r^{2m}\lambda + 2r^m - 1.$$

Next, we show that $\Psi(1, r, \lambda) \le 0$ holds for $r \le R_{2,\lambda}$.
Case 1. If $\lambda \in (1, \infty)$, $\Psi(1, r, \lambda)$ is a continuous and increasing function of $r \in (0, 1)$ and

$$\Psi(1, 0, \lambda) = -1 < 0, \quad \Psi\left(1, \sqrt[m]{\frac{\sqrt{5} - 1}{2}}, \lambda\right) = \lambda > 0.$$

Thus $r_{2,\lambda}$ is unique root in $(0, \sqrt[m]{\frac{\sqrt{5}-1}{2}})$ of $\Psi(1, r, \lambda)$ and $\Psi(1, r, \lambda) \le 0$ for $r \le r_{2,\lambda}$.
Case 2. If $\lambda \in [0, 1]$, then we have $\Psi(1, r, \lambda) \le r^{4m} + r^{3m} + r^{2m} + 2r^m - 1$. Let $s(r) = r^{4m} + r^{3m} + r^{2m} + 2r^m - 1$. One can verify that $s(r)$ is a continuous and

increasing function of $r \in (0, 1)$ and

$$s(0) = -1, \quad s\left(\sqrt[m]{\frac{\sqrt{5}-1}{2}}\right) = 1 > 0.$$

Thus r_2^* is unique root in $(0, \sqrt[m]{\frac{\sqrt{5}-1}{2}})$ of $s(r)$ and $s(r) \leq 0$ for $r \leq r_2^*$. Then $\Psi(1, r, \lambda) \leq s(r) \leq 0$ for $r \leq r_2^*$. Hence, inequality (3.19) holds for $r \leq R_{2,\lambda}$.

The sharpness part is similar to Theorem 3.4, and we omit it. Thus the proof of Theorem 3.5 is complete. □

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