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On the Third and Fourth Hankel Determinants for a Subclass of Analytic Functions

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Abstract

The objective of this paper is to investigate the third and fourth Hankel determinants for the class of functions with bounded turning associated with Bernoulli's lemniscate. The fourth Hankel determinants for 2-fold symmetric and 3-fold symmetric functions are also studied.

Keywords Bounded turning functions · Lemniscate of Bernoulli · Hankel determinant

Mathematics Subject Classification Primary 30C45; Secondary 30C80

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1 Introduction

Let $\mathcal{H}$ represent the family of analytic functions in the region $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. By the notation $\mathcal{A}$, we mean a set consisting of functions $f \in \mathcal{H}$ of the Taylor series form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

The symbol $\mathcal{S}$ denotes a family of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$. It is familiar that for the function $f \in \mathcal{S}$ of form (1.1), the coefficients of this function satisfy the sharp inequality $|a_n| \leq n$ for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. This outstanding result was proposed by Bieberbach [12] as a conjecture in 1916 and it remained a challenge for researchers for a long period of time. Finally, after almost 69 years, de-Branges [16] in 1985 proved this fundamental result. Many subfamilies of the set $\mathcal{S}$ of univalent functions were introduced with respect to geometric point of view of their image domains, such as the families $\mathcal{C}$, $\mathcal{S}^*$, $\mathcal{K}$ of convex, starlike and close-to-convex univalent functions, respectively, and these are defined as:

$$\mathcal{C} := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}) \right\},$$

$$\mathcal{S}^* := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}) \right\},$$

$$\mathcal{K} := \left\{ f : f \in \mathcal{S} \text{ and } \Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (g \in \mathcal{S}^*;\ z \in \mathbb{D}) \right\}.$$

In particular, let $\mathcal{R}$ denote the subclass of $\mathcal{K}$ with $g(z) = z$.

In 1996, Sokół and Stankiewicz [43] introduced a subfamily $\mathcal{SL}$ of the set $\mathcal{S}$, defined as:

$$\mathcal{SL} := \left\{ f : f \in \mathcal{S} \text{ and } \left| \left(\frac{zf'(z)}{f(z)}\right)^2 - 1 \right| < 1 \quad (z \in \mathbb{D}) \right\}.$$

The geometrical interpretation of $f \in \mathcal{SL}$ is that, for any $z \in \mathbb{D}$, the ratio $zf'(z)/f(z)$ lies in the region bounded by the right half side of the Bernoulli's lemniscate

$$\left| w^2 - 1 \right| = 1.$$

Equivalently, by using the familiar subordination, a function $f \in \mathcal{SL}$ satisfies the relationship

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} \quad (z \in \mathbb{D}).$$

This set was further studied by different researchers, see the work of Ali et al. [1], Kumar et al. [24], Omar and Halim [34], Raza and Malik [39] and Sokół [44]. We now define a subclass $\mathcal{KL}$ of univalent functions f of form (1.1) as follows:

$$\mathcal{KL} := \left\{ f : f \in \mathcal{S} \text{ and} \left| \left(\frac{zf'(z)}{g(z)} \right)^2 - 1 \right| < 1 \quad (g \in \mathcal{SL};\ z \in \mathbb{D}) \right\}.$$

Alternatively, a function $f \in \mathcal{KL}$ if and only if

$$\frac{zf'(z)}{g(z)} \prec \sqrt{1+z} \quad (g \in \mathcal{SL};\ z \in \mathbb{D}). \tag{1.2}$$

We note that if $g(z) = z$, then the family $\mathcal{KL}$ reduced to the class $\mathcal{RL}$ which is defined in terms of subordination as:

$$\mathcal{RL} := \left\{ f : f \in \mathcal{S} \text{ and } f'(z) \prec \sqrt{1+z} \quad (z \in \mathbb{D}) \right\}. \tag{1.3}$$

For the given parameters $q,\ n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ for a function $f \in \mathcal{S}$ of form (1.1) was defined by Pommerenke [36,37] (see also [3,4]) as:

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}. \tag{1.4}$$

The growth of $H_{q,n}(f)$ for different fixed integer q and n has been studied for different subfamilies of univalent functions. We include here a few of them. The sharp bounds of $|H_{2,2}(f)|$ for the subfamilies $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ of the set $\mathcal{S}$ were investigated by Janteng et al. [19,20]. They proved the bounds:

$$|H_{2,2}(f)| \leq \begin{cases} 1 & (f \in \mathcal{S}^*), \\ \frac{1}{8} & (f \in \mathcal{C}), \\ \frac{4}{9} & (f \in \mathcal{R}). \end{cases}$$

The problem of this determinant was studied by many researchers for different subfamilies of analytic and univalent functions, see [10,18,23,28,30,33,35,45].

The third-order Hankel determinant is given by:

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5, \tag{1.5}$$

the estimation of $|H_{3,1}(f)|$ is so hard as to find the value of $|H_{2,2}(f)|$. The first article on $H_{3,1}(f)$ shows up in 2010 by Babalola [8], in which he got the upper bound of $|H_{3,1}(f)|$ for the groups of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$. Later on, many researchers distributed their work concerning $\left|H_{3,1}(f)\right|$ for various subfamilies of analytic and univalent functions, see [2,3,6,11,14,39–42].

In 2017, Zaprawa [46] improved the consequences of Babalola [8] by proving

$$\left|H_{3,1}(f)\right| \leq \begin{cases} 1 \ \ (f \in \mathcal{S}^*), \\ \frac{49}{540} \ \ (f \in \mathcal{C}), \\ \frac{41}{60} \ \ (f \in \mathcal{R}). \end{cases}$$

and asserted that these inequalities are as yet not sharp. Further for the sharpness, he thought about the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ comprising of functions with m-fold symmetry and acquired the sharp bounds. Recently, Kowalczyk et al. [22] and Lecko et al. [27] get the sharp inequalities

$$\left|H_{3,1}(f)\right| \leq \tfrac{4}{135} \text{ and } \left|H_{3,1}(f)\right| \leq \tfrac{1}{9},$$

for the familiar sets $\mathcal{C}$ and $\mathcal{S}^*(1/2)$, respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates to the family of starlike functions of order $1/2$. Additionally, in 2018, the authors [26] obtained an improved bound $\left|H_{3,1}(f)\right| \leq 8/9$ for $f \in \mathcal{S}^*$, yet not the best possible. Moreover, in 2018, Arif et al. [5] studied the problem of fourth Hankel determinant for the class of bounded turning functions at the first time and successfully obtained the bound

$$\left|H_{4,1}(f)\right| \leq \frac{73757}{94500} \approx 0.7805.$$

Recently, this determinant was studied in [7] for a subclass of starlike function connected with Bernoulli's lemniscate.

In this paper, we make a contribution to the subject by deducing the third and fourth Hankel determinants for the class $\mathcal{RL}$ of bounded turning functions associated with Bernoulli's lemniscate.

2 A Set of Lemmas

To derive the bounds of Hankel determinants, we need the following results involving the class $\mathcal{P}$ of functions with positive real part.

Lemma 2.1 *If $p \in \mathcal{P}$ and of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}), \tag{2.1}$$

then, for $n,\ k \in \mathbb{N}$, the following sharp inequalities hold,

$$|c_{n+k} - \lambda c_n c_k| \leq 2 \quad (0 \leq \lambda \leq 1), \tag{2.2}$$

$$|c_n| \leq 2, \tag{2.3}$$

$$\left|c_1^3 - 2c_1c_2 + c_3\right| \leq 2, \tag{2.4}$$

and for a complex number μ,

$$\left|c_2 - \mu c_1^2\right| \leq 2\max\{1, |2\mu - 1|\}. \tag{2.5}$$

Inequalities (2.2), (2.3), (2.4) and (2.5) are given in [13,29,31] and [32], respectively.

Lemma 2.2 ([38]) *Let the parameters* $\delta,\ \lambda,\ \rho$ *and* σ *satisfy the conditions* $0 < \delta < 1$, $0 < \sigma < 1$, *and*

$$\begin{aligned} &8\sigma(1-\sigma)\left[(\delta\lambda - 2\rho)^2 + (\delta(\sigma+\delta) - \lambda)^2\right] \\ &\quad + \delta(1-\delta)(\lambda - 2\sigma\delta)^2 \leq 4\delta^2(1-\delta)^2\sigma(1-\sigma). \end{aligned} \tag{2.6}$$

If $p \in \mathcal{P}$, *then*

$$\left|\rho c_1^4 + \sigma c_2^2 + 2\delta c_1 c_3 - \frac{3}{2}\lambda c_1^2 c_2 - c_4\right| \leq 2.$$

Lemma 2.3 ([25,29]) *If* $h \in \mathcal{P}$ *and* $c_1 > 0$, *then*

$$c_2 = \frac{1}{2}[c_1^2 + (4 - c_1^2)x], \tag{2.7}$$

$$c_3 = \frac{1}{4}[c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y], \tag{2.8}$$

and

$$\begin{aligned} c_4 = \frac{1}{8}\big[&c_1^4 + 3c_1^2(4 - c_1^2)x + (4 - 3c_1^2)(4 - c_1^2)x^2 + c_1^2(4 - c_1^2)x^3 \\ &+ 4(4 - c_1^2)(1 - |x|^2)(c_1 y - c_1 x y - \overline{x} y^2) + 4(4 - c_1^2)(1 - |x|^2)(1 - |y|^2)z\big], \end{aligned} \tag{2.9}$$

hold for some $x,\ y,\ z \in \overline{\mathbb{D}} := \{z : |z| \leq 1\}$.

3 Bound of $\left|H_{3,1}(f)\right|$ for the Class $\mathcal{RL}$

In this section, we derive the bound of $\left|H_{3,1}(f)\right|$ for the class $\mathcal{RL}$.

Theorem 3.1 *Let* $f \in \mathcal{RL}$. *Then*

$$|a_2| \leq \tfrac{1}{4},\ |a_3| \leq \tfrac{1}{6},\ |a_4| \leq \tfrac{1}{8},\ |a_5| \leq \tfrac{1}{10}, \tag{3.1}$$

and these inequalities are the best possible.

Proof If $f \in \mathcal{RL}$, then we can rewrite (1.3) in terms of Schwarz functions $w(z)$ as

$$f'(z) = \sqrt{1 + w(z)} \quad (z \in \mathbb{D}). \tag{3.2}$$

Also, if the function $p \in \mathcal{P}$, then

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \dots, \tag{3.3}$$

and this further gives

$$w(z) = \frac{p(z) - 1}{p(z) + 1}.$$

Putting the value of w in (3.2), we obtain

$$f'(z) = \sqrt{\frac{2p(z)}{1 + p(z)}}. \tag{3.4}$$

Using the series form (3.3) of p, we have

$$\begin{aligned}\sqrt{\frac{2p(z)}{1 + p(z)}} = 1 &+ \frac{1}{4}c_1 z + \frac{1}{4}\left(c_2 - \frac{5}{8}c_1^2\right) z^2 + \frac{1}{4}\left(c_3 - \frac{5}{4}c_1 c_2 + \frac{13}{32}c_1^3\right) z^3 \\ &+ \frac{1}{4}\left(c_4 - \frac{5}{8}c_2^2 - \frac{5}{4}c_1 c_3 + \frac{39}{32}c_1^2 c_2 - \frac{141}{512}c_1^4\right) z^4 + \cdots.\end{aligned} \tag{3.5}$$

Similarly, using (1.1), we know that

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we can easily obtain the following coefficients,

$$a_2 = \frac{1}{8}c_1, \tag{3.7}$$

$$a_3 = \frac{1}{12}\left(c_2 - \frac{5}{8}c_1^2\right), \tag{3.8}$$

$$a_4 = \frac{1}{16}\left(c_3 - \frac{5}{4}c_1 c_2 + \frac{13}{32}c_1^3\right), \tag{3.9}$$

and

$$a_5 = \frac{1}{20}\left(c_4 - \frac{5}{8}c_2^2 - \frac{5}{4}c_1 c_3 + \frac{39}{32}c_1^2 c_2 - \frac{141}{512}c_1^4\right). \tag{3.10}$$

By virtue of (2.2) and (2.3), we get the bounds

$$|a_2| \leq \tfrac{1}{4} \text{ and } |a_3| \leq \tfrac{1}{6}.$$

To prove the sharpness of the fourth coefficient, we consider

$$\begin{aligned}
|a_4| &= \frac{1}{16}\left|c_3 - \frac{5}{4}c_1c_2 + \frac{13}{32}c_1^3\right| \\
&= \frac{1}{16}\left|\frac{13}{32}\left(c_3 - 2c_1c_2 + c_1^3\right) + \frac{7}{16}\left(c_3 - c_1c_2\right) + \frac{5}{32}c_3\right| \\
&\leq \frac{1}{16}\left[\frac{13}{32}\left|\left(c_3 - 2c_1c_2 + c_1^3\right)\right| + \frac{7}{16}\left|c_3 - c_1c_2\right| + \frac{5}{32}\left|c_3\right|\right] \\
&\leq \frac{1}{8}\left(\frac{13}{32} + \frac{7}{16} + \frac{5}{32}\right) = \frac{1}{8},
\end{aligned}$$

where we have used (2.2) , (2.3) and (2.4) . For the proof of $|a_5| \leq \frac{1}{10}$, consider relationship (3.10) and compare it with (2.6), yields

$$\rho = \frac{141}{512},\ \delta = \frac{5}{8},\ \sigma = \frac{5}{8},\ \lambda = \frac{13}{16}.$$

These constants satisfy all the conditions of Lemma 2.2, and hence, the result follows.

To see the sharpness of the results, consider the function $f_n : \mathbb{D} \to \mathbb{C}$ defined by

$$f_n(z) = \int_0^z \sqrt{1+\zeta^n}\,d\zeta = z + \frac{1}{2(n+1)}z^{n+1} + \cdots \quad (n = 1, 2, 3, 4), \quad (3.11)$$

we know that $f_n \in \mathcal{RL}$, it follows that the inequalities in (3.1) are sharp by taking $n = 1, 2, 3, 4$. □

From (3.11), we conjecture the following result.

Conjecture 3.2 *Let $f \in \mathcal{RL}$. Then*

$$|a_n| \leq \frac{1}{2n} \quad (n \geq 2).$$

Theorem 3.3 *Let $f \in \mathcal{KL}$. Then for $n \geq 2$,*

$$|na_n - b_n|^2 \leq \sum_{k=1}^{n-1}\left(|k\gamma a_k - b_k|^2 - |ka_k - b_k|^2\right), \quad (3.12)$$

where b_n are the coefficients of $g \in \mathcal{SL}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \ (z \in \mathbb{D}), \quad (3.13)$$

and

$$\gamma = \sqrt{2} - 1. \tag{3.14}$$

Proof From (1.2), we have

$$\frac{zf'(z)}{g(z)} \prec \sqrt{1+z} \prec \frac{1+z}{1+\gamma z},$$

where $g \in \mathcal{SL}$. So we can rewrite

$$\frac{zf'(z)}{g(z)} = \frac{1+w(z)}{1+\gamma w(z)},$$

where $w(0) = 0$, $|w(z)| < 1$ and $w(z) = \sum_{n=1}^{\infty} c_n z^n$ for $z \in \mathbb{D}$. Thus, we know that

$$zf'(z) - g(z) = w(z)[g(z) - \gamma z f'(z)].$$

Now, using (1.1) and (3.13), we get

$$\sum_{k=1}^{\infty}(ka_k - b_k)z^k = w(z)\sum_{k=1}^{\infty}(b_k - k\gamma a_k)z^k \quad (a_1 = b_1 = 1).$$

Rewrite it as

$$\sum_{k=1}^{n}(ka_k - b_k)z^k + \sum_{k=n+1}^{\infty}(ka_k - b_k)z^k$$
$$= w(z)\left[\sum_{k=1}^{n-1}(b_k - k\gamma a_k)z^k + \sum_{k=n}^{\infty}(b_k - k\gamma a_k)z^k\right],$$

it follows that

$$\sum_{k=1}^{n}(ka_k - b_k)z^k + \sum_{k=n+1}^{\infty}(ka_k - b_k)z^k - w(z)\sum_{k=n}^{\infty}(b_k - k\gamma a_k)z^k$$
$$= w(z)\sum_{k=1}^{n-1}(b_k - k\gamma a_k)z^k.$$

By applying the same method as in Clunie and Keogh [15], we now write

$$\sum_{k=1}^{n}(ka_k - b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k = w(z)\sum_{k=1}^{n-1}(b_k - k\gamma a_k)z^k,$$

for some d_k $(n+1 \leq k < \infty)$, where d_k can be expressed in terms of the coefficients a_k, b_k and c_k as

$$d_k = (ka_k - b_k) - \sum_{j=1}^{k-n} (b_{k-j} - (k-j)\gamma a_{k-j})c_j.$$

This gives

$$\begin{aligned}\left|\sum_{k=1}^{n}(ka_k - b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k\right|^2 &= \left|w(z)\sum_{k=1}^{n-1}(b_k - k\gamma a_k)z^k\right|^2\\ &\leq \left|\sum_{k=1}^{n-1}(b_k - k\gamma a_k)z^k\right|^2.\end{aligned}$$

Now, we consider

$$\sum_{k=1}^{n}(ka_k - b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k = \sum_{k=1}^{\infty} e_k z^k,$$

which is an analytic function in $\mathbb{D}$. Parseval's theorem [17] gives

$$\int_0^{2\pi}\left|\sum_{k=1}^{\infty} e_k (re^{i\theta})^k\right|^2 d\theta = 2\pi \sum_{k=1}^{\infty} |e_k|^2 r^{2k}.$$

For any r $(0 < r < 1)$, by integrating the above relation with respect to θ from 0 to 2π, we obtain

$$\sum_{k=1}^{n} |ka_k - b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} |b_k - k\gamma a_k|^2 r^{2k}.$$

Therefore,

$$\sum_{k=1}^{n} |ka_k - b_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} |b_k - k\gamma a_k|^2 r^{2k}.$$

When $r \to 1$, we deduce that

$$\sum_{k=1}^{n} |ka_k - b_k|^2 \leq \sum_{k=1}^{n-1} |b_k - k\gamma a_k|^2,$$

which leads to the desired result. □

If we take $g(z) = z$, then $b_1 = 1$ and $b_k = 0$ for all $k = 2, 3, \ldots, n-1$. We easily get the following Corollary.

Corollary 3.4 *Let $f \in \mathcal{RL}$. Then for $n \geq 3$,*

$$n^2|a_n|^2 \leq (1-\gamma)^2 + \left(\gamma^2 - 1\right)\sum_{k=2}^{n-1} k^2|a_k|^2. \tag{3.15}$$

By setting $n = 6$ and $n = 7$ in (3.15), we obtain the following bounds of $|a_6|$ and $|a_7|$.

Corollary 3.5 *Let $f \in \mathcal{RL}$. Then*

$$|a_6| \leq \frac{2-\sqrt{2}}{6} \quad \textit{and} \quad |a_7| \leq \frac{2-\sqrt{2}}{7}. \tag{3.16}$$

Theorem 3.6 *Let $f \in \mathcal{RL}$. Then, for $\lambda \in \mathbb{R}$,*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{6}\max\left\{1, \ \frac{1}{8}\left|2+3\lambda\right|\right\},$$

and this inequality is sharp.

Proof From (3.7) and (3.8), we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{12}\left|c_2 - \frac{10+3\lambda}{16}c_1^2\right| \leq \frac{1}{6}\max\left\{1, \ \left|\frac{2+3\lambda}{8}\right|\right\},$$

where we have used inequality (2.5) . This result is sharp for the functions

$$f_1(z) = \int_0^z \sqrt{1+\zeta}\,d\zeta = z + \frac{1}{4}z^2 - \frac{1}{24}z^3 + \cdots, \tag{3.17}$$

and

$$f_2(z) = \int_0^z \sqrt{1+\zeta^2}\,d\zeta = z + \frac{1}{6}z^3 - \frac{1}{40}z^5 + \cdots. \tag{3.18}$$

□

For $\lambda = 1$, we obtain the following Corollary.

Corollary 3.7 *Let $f \in \mathcal{RL}$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}, \tag{3.19}$$

and this inequality is sharp for the function f_2 given by (3.18).

Theorem 3.8 *Let $f \in \mathcal{RL}$. Then*

$$|a_2a_3 - a_4| \leq \frac{1}{8}, \tag{3.20}$$

and this bound is the best possible.

Proof From (3.7), (3.8), (3.9), we know that

$$\begin{aligned}
|a_2a_3 - a_4| &= \frac{1}{16}\left|\frac{49}{96}c_1^3 - \frac{17}{12}c_1c_2 + c_3\right| \\
&= \frac{1}{16}\left|\frac{49}{96}\left(c_3 - 2c_1c_2 + c_1^3\right) + \frac{19}{48}\left(c_3 - c_1c_2\right) + \frac{3}{32}c_3\right| \\
&\leq \frac{1}{16}\left\{\frac{49}{96}\left|\left(c_3 - 2c_1c_2 + c_1^3\right)\right| + \frac{19}{48}\left|(c_3 - c_1c_2)\right| + \frac{3}{32}\left|c_3\right|\right\} \\
&\leq \frac{1}{16}\left(\frac{49}{48} + \frac{19}{24} + \frac{3}{16}\right) = \frac{1}{8},
\end{aligned}$$

where we have used triangle inequality along with Lemma 2.1. This result is sharp for the function

$$f_3(z) = \int_0^z \sqrt{1+\zeta^3}\,d\zeta = z + \frac{1}{8}z^4 - \frac{1}{56}z^7 + \cdots. \tag{3.21}$$

□

The result given below has been proved in [28], but we still present it for the reader.

Theorem 3.9 *Let $f \in \mathcal{RL}$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}. \tag{3.22}$$

This inequality is sharp.

Motivated by the papers [9,21,22,27], we now determine the sharp bound of third-order Hankel determinant for the family $\mathcal{RL}$ of bounded turning functions connected with Bernoulli's lemniscate.

Theorem 3.10 *Let $f \in \mathcal{RL}$. Then*

$$\left|H_{3,1}(f)\right| \leq \frac{1}{64}. \tag{3.23}$$

This result is sharp.

Proof From (1.5), we have

$$H_{3,1}(f) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \tag{3.24}$$

Putting the values of a_j $(j = 2, 3, 4, 5)$ in (3.7)–(3.10) to (3.24), we get

$$\begin{aligned} H_{3,1}(f) = \frac{3479}{35389440}\bigg(& c_1^6 - \frac{15504}{3479}c_1^4c_2 + \frac{8640}{3479}c_1^3c_3 + \frac{19392}{3479}c_1^2c_2^2 - \frac{119808}{3479}c_1^2c_4 \\ & + \frac{207360}{3479}c_1c_2c_3 - \frac{112640}{3479}c_2^3 + \frac{147456}{3479}c_2c_4 - \frac{138240}{3479}c_3^2\bigg). \end{aligned}$$

By using (2.7)–(2.9) and letting $c_1 = c \in [0, 2]$, $t = 4 - c^2$, also, by straightforward algebraic computations, we have

$$\begin{aligned} H_{3,1}(f) = \frac{3479}{35389440}\bigg[& \frac{25}{497}c^6 + \frac{36864}{3479}t^2x^3 - \frac{14080}{3479}t^3x^3 \\ & - \frac{23040}{3479}c^2tx^2 - \frac{5760}{3479}c^4tx^3 + \frac{6480}{3479}c^4tx^2 \\ & - \frac{120}{497}c^4tx + \frac{576}{3479}c^2t^2x^4 - \frac{19008}{3479}c^2t^2x^3 \\ & + \frac{7536}{3479}c^2t^2x^2 - \frac{34560}{3479}t^2\left(1 - |x|^2\right)^2 y^2 \\ & - \frac{1440}{3479}c^3t\left(1 - |x|^2\right) y \\ & + \frac{23040}{3479}c^3tx\left(1 - |x|^2\right) y + \frac{23040}{3479}c^2t\overline{x}\left(1 - |x|^2\right) y^2 \\ & - \frac{23040}{3479}c^2t\left(1 - |x|^2\right)\left(1 - |y|^2\right) z \\ & - \frac{2304}{3479}ct^2x^2\left(1 - |x|^2\right) y - \frac{36864}{3479}t^2x\overline{x}\left(1 - |x|^2\right) y^2 \\ & + \frac{19584}{3479}ct^2x\left(1 - |x|^2\right) y + \frac{36864}{3479}t^2x\left(1 - |x|^2\right)\left(1 - |y|^2\right) z\bigg]. \end{aligned}$$

Therefore,

$$H_{3,1}(f) = \frac{3479}{35389440}\left[v_1(c, x) + v_2(c, x)y + v_3(c, x)y^2 + \Psi(c, x, y)z\right],$$

where $x,\ y,\ z \in \overline{\mathbb{D}}$, and

$$\begin{aligned} v_1(c,x) &= \frac{25}{497}c^6 + \left(4-c^2\right)\Bigg[\left(4-c^2\right)\left(-\frac{19456}{3479}x^3 - \frac{704}{497}c^2x^3 \right. \\ &\quad \left. + \frac{576}{3479}c^2x^4 + \frac{7536}{3479}c^2x^2\right) \\ &\quad - \frac{23040}{3479}c^2x^2 - \frac{5760}{3479}c^4x^3 + \frac{6480}{3479}c^4x^2 - \frac{120}{497}c^4x\Bigg], \\ v_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right) \\ &\quad \left[\left(4-c^2\right)\left(\frac{19584}{3479}cx - \frac{2304}{3479}cx^2\right) + \frac{23040}{3479}c^3x - \frac{1440}{3479}c^3\right], \\ v_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-\frac{2304}{3479}x^2 - \frac{34560}{3479}\right) + \frac{23040}{3479}c^2\overline{x}\right], \end{aligned}$$

and

$$\Psi(c,x,y) = \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|y|^2\right)\left[-\frac{23040}{3479}c^2 + \frac{36864}{3479}x\left(4-c^2\right)\right].$$

Now, by using $|x| = x$, $|y| = y$ and utilizing the fact $|z| \le 1$, we get

$$\begin{aligned} \left|H_{3,1}(f)\right| &\le \frac{3479}{35389440}\left(|v_1(c,x)| + |v_2(c,x)|\, y + |v_3(c,x)|\, y^2 + |\Psi(c,x,y)|\right) \\ &= \frac{3479}{35389440}G(c,x,y), \end{aligned} \tag{3.25}$$

where

$$G(c,x,y) = g_1(c,x) + g_2(c,x)\,y + g_3(c,x)\,y^2 + g_4(c,x)\left(1-y^2\right), \tag{3.26}$$

with

$$\begin{aligned} g_1(c,x) &= \frac{25}{497}c^6 + \left(4-c^2\right)\Bigg[\left(4-c^2\right)\left(\frac{19456}{3479}x^3 + \frac{704}{497}c^2x^3 \right. \\ &\quad \left. + \frac{576}{3479}c^2x^4 + \frac{7536}{3479}c^2x^2\right) \\ &\quad + \frac{23040}{3479}c^2x^2 + \frac{5760}{3479}c^4x^3 + \frac{6480}{3479}c^4x^2 + \frac{120}{497}c^4x\Bigg], \end{aligned}$$

$$g_2(c,x) = \left(4-c^2\right)\left(1-|x|^2\right)$$
$$\left[\left(4-c^2\right)\left(\frac{19584}{3479}cx+\frac{2304}{3479}cx^2\right)+\frac{23040}{3479}c^3x+\frac{1440}{3479}c^3\right],$$
$$g_3(c,x) = \left(4-c^2\right)\left(1-|x|^2\right)$$
$$\left[\left(4-c^2\right)\left(\frac{2304}{3479}x^2+\frac{34560}{3479}\right)+\frac{23040}{3479}c^2x\right],$$

and

$$g_4(c,x) = \left(4-c^2\right)\left(1-|x|^2\right)\left[\frac{23040}{3479}c^2+\frac{36864}{3479}x\left(4-c^2\right)\right].$$

Let the closed cuboid be $\Lambda : [0,2]\times[0,1]\times[0,1]$. We have to obtain the points of maxima inside Λ, inside the six faces and on the twelve edges in order to maximize G.

(I) Let $(c,x,y)\in(0,2)\times(0,1)\times(0,1)$. Now, to find points of maxima inside Λ, we take partial derivative of (3.26) with respect to y and get

$$\frac{\partial G}{\partial y} = \frac{288}{3479}(4-p^2)(1-x^2)\{16y(x-1)[(x-15)(4-c^2)+10c^2]$$
$$+c[4x(4-c^2)(17+2x)+5c^2(1+16x)]\}.$$

For $\frac{\partial G}{\partial y}=0$, yields

$$y=\frac{c[4x(4-c^2)(17+2x)+5c^2(1+16x)]}{16(x-1)[(4-c^2)(15-x)-10c^2]} := y_0.$$

If y_0 is a critical point inside Λ, then $y_0\in(0,1)$, which is possible only if

$$5c^3(1+16x)+4cx(4-c^2)(17+2x)$$
$$+16(1-x)(4-c^2)(15-x) < 160c^2(1-x), \tag{3.27}$$

and

$$c^2 > \frac{4(15-x)}{25-x}. \tag{3.28}$$

Now, we have to obtain the solutions which satisfy both inequalities (3.27) and (3.28) for the existence of the critical points. Let

$$g(x) := \frac{4(15-x)}{25-x}.$$

Since $g'(x) < 0$ for $(0, 1)$, $g(x)$ is decreasing in $(0, 1)$. Hence $c^2 > 7/3$ and a simple exercise shows that (3.27) does not hold in this case for all values of $x \in (0, 1)$ and there is no critical point of G in $(0, 2) \times (0, 1) \times (0, 1)$.

(II) To find points of maxima inside the six faces of Λ. We deal with each face individually.

When $c = 0$, $G(c, x, y)$ reduces to

$$\begin{aligned} h_1(x, y) &:= G(0, x, y) \\ &= \frac{4096\left[9(1-x^2)(y^2(x-1)(x-15)+16x)+76x^3\right]}{3479} \\ &\quad (x,\ y \in (0,\ 1)). \end{aligned} \tag{3.29}$$

h_1 has no optimal point in $(0, 1) \times (0, 1)$ since

$$\frac{\partial h_1}{\partial y} = \frac{73728\left[y(1-x^2)(x-1)(x-15)\right]}{3479} \neq 0 \quad (x,\ y \in (0, 1)). \tag{3.30}$$

When $c = 2$, $G(c, x, y)$ reduces to

$$G(2, x, y) = \frac{1600}{497} \quad (x,\ y \in (0, 1)). \tag{3.31}$$

When $x = 0$, $G(c, x, y)$ reduces to $G(c, 0, y)$, given by

$$\begin{aligned} h_2(c, y) := &\left[\frac{34560}{3479}(4-c^2)^2 - \frac{23040}{3479}(4-c^2)c^2\right]y^2 + \frac{1440}{3479}(4-c^2)c^3 y \\ &+ \frac{25}{497}c^6 + \frac{23040}{3479}(4-c^2)c^2, \end{aligned} \tag{3.32}$$

where $c \in (0, 2)$ and $y \in (0, 1)$. We solve $\frac{\partial h_2}{\partial y} = 0$ and $\frac{\partial h_2}{\partial c} = 0$ to find the points of maxima. By solving $\frac{\partial h_2}{\partial y} = 0$, we obtain

$$y = -\frac{c^3}{16(12-5c^2)} =: y_1. \tag{3.33}$$

For the given range of y, y_1 should belong to $(0, 1)$, which is possible only if $c > c_0$, $c_0 \approx 1.549193338$. $\frac{\partial h_2}{\partial c} = 0$ implies that

$$(-24576 + 7680c^2)y^2 + (576c - 240c^3)y + 35c^4 - 3072c^2 + 6144 = 0. \tag{3.34}$$

By substituting (3.33) into (3.34), we get

$$-40368c^6 + 415c^8 + 263424c^4 - 589824c^2 + 442368 = 0. \tag{3.35}$$

A calculation given in the solution of (3.35) in $(0, 2)$ is c approximately equal to 1.420061367. Thus, h_2 has no optimal point in $(0, 2) \times (0, 1)$.

When $x = 1$, $G(c, x, y)$ reduces to

$$h_3(c, y) := G(c, 1, y) = \frac{311296 + 145152c^2 - 55584c^4 + 135c^6}{3479} \quad (c \in (0, 2)). \quad (3.36)$$

By solving $\frac{\partial h_3}{\partial c} = 0$, we obtain that the critical points are $c =: c_0 = 0$ and

$$c := c_1 = \frac{4}{15}\sqrt{1930 - 5\sqrt{146161}} \approx 1.145412714.$$

Here, c_0 is the minimum point of h_3. Thus, h_3 achieves its maximum

$$\frac{-457485033472 + 1197350912\sqrt{146161}}{2348325} \approx 116.804$$

at c_1.

When $y = 0$, $G(c, x, y)$ reduces to

$$h_4(c, x) := G(c, x, 0) = \frac{1}{3479}\left(\begin{array}{l} 175c^6 + (4 - c^2)\big[(4 - c^2)(7536c^2x^2 + 576x^4c^2 \\ -17408x^3 + 4928x^3c^2 + 36864x) + 840c^4x + 5760c^4x^3 \\ +6480c^4x^2 + 23040c^2\big] \end{array}\right).$$

A calculation shows that there does not exist any solution for the system of equations $\frac{\partial h_4}{\partial x} = 0$ and $\frac{\partial h_4}{\partial c} = 0$ in $(0, 2) \times (0, 1)$.

When $y = 1$, $G(c, x, y)$ reduces to

$$G(c, x, 1) = \frac{1}{3479}\left(\begin{array}{l} 175c^6 + (4 - c^2)[(4 - c^2)(576x^4c^2 + 2304cx^2 + 19584cx \\ -2304x^4c + 19456x^3 + 7536c^2x^2 - 19584x^3c + 34560 \\ -32256x^2 + 4928x^3c^2 - 2304x^4) + 5760c^4x^3 - 1440x^2c^3 \\ -23040x^3c^3 + 840c^4x + 23040c^2x + 23040c^2x^2 + 6480c^4x^2 \\ +23040c^3x + 1440c^3 - 23040x^3c^2] \end{array}\right) =: h_5(c, x).$$

A calculation shows that there does not exist any solution for the system of equations $\frac{\partial h_5}{\partial x} = 0$ and $\frac{\partial h_5}{\partial c} = 0$ in $(0, 2) \times (0, 1)$.

(III) Now, we are going to find the maxima of $G(c, x, y)$ on the edges of Λ. By putting $y = 0$ in (3.32), we have

$$G(c, 0, 0) =: m_1(c) = \frac{175c^6 + 92160c^2 - 23040c^4}{3479}.$$

We note that $m_1'(c) = 0$ for $c =: \lambda_0 = 0$ and

$$c =: \lambda_1 = (4/35)\sqrt{3360 - 70\sqrt{2094}} \approx 1.431005319 \in [0, 2],$$

where λ_0 is minimum point and maximum point of m_1 is achieved at λ_1 . We can see that

$$G(c, 0, 0) \leq (-6256852992 + 137232384\sqrt{2094})/852355 \approx 26.90720149 \quad (c \in [0, 2]).$$

Solving equation (3.32) at $y = 1$, we get

$$G(c, 0, 1) = m_2(c) := \frac{175c^6 + 5760c^3 - 1440c^5 + 552960 - 276480c^2 + 34560c^4}{3479}.$$

Since $m_2'(c) < 0$ for $[0, 2]$, we know that $m_2(c)$ is decreasing in $[0, 2]$ and the maximum is achieved at $c = 0$. Thus,

$$G(c, 0, 1) \leq \frac{552960}{3479} \approx 158.9422248 \quad (c \in [0, 2]).$$

By taking $c = 0$ in (3.32), we get

$$G(0, 0, y) = \frac{552960y^2}{3479}.$$

A simple calculation gives

$$G(0, 0, y) \leq \frac{552960}{3479} \approx 158.9422248 \quad (y \in [0, 1]).$$

As we see that Eq. (3.36) is independent of y, we have

$$G(c, 1, 1) = G(c, 1, 0) = m_3(c) := \frac{311296 + 145152c^2 - 55584c^4 + 135c^6}{3479}.$$

Now, $m_3'(c) = 0$ for $c =: \lambda_0 = 0$ and

$$c =: \lambda_1 = \frac{4}{15}\sqrt{1930 - 5\sqrt{146161}} \approx 1.145412714 \in [0, 2],$$

where λ_0 is minimum point, and maximum point of $m_3(c)$ is achieved at λ_1. We conclude that

$$G(c, 1, 1) = G(c, 1, 0)$$

$$\leq \frac{-457485033472 + 1197350912\sqrt{146161}}{2348325}$$
$$\approx 116.804 \quad (c \in [0, 2]).$$

By setting $c = 0$ in Eq. (3.36), we get

$$G(0, 1, y) = \frac{311296}{3479} \approx 89.4785858.$$

As (3.31) is independent of x and y, thus, we have

$$\begin{aligned} G(2, 1, y) &= G(2, 0, y) \\ &= G(2, x, 0) = G(2, x, 1) \\ &= \frac{1600}{497} \approx 3.219315895 \quad (x,\ y \in [0, 1]). \end{aligned}$$

By taking $y = 0$ in (3.29), we get

$$G(0, x, 0) = m_4(x) := \frac{-278528x^3 + 589824x}{3479}.$$

Now, $m_4'(x) = 0$ for

$$x =: x_0 = \frac{2\sqrt{51}}{17} \approx 0.8401680504 \in [0, 1],$$

we know that $m_4(x)$ is an increasing function for $x \leq x_0$ and decreasing for $x_0 \leq x$. Hence, $m_4(x)$ has its maximum at $x = x_0$, and we conclude that

$$G(0, x, 0) \leq \frac{786432\sqrt{51}}{59143} \approx 94.96048292 \quad (x \in [0, 1]).$$

By putting $y = 1$ in (3.29), we get

$$G(0, x, 1) =: m_5(x) = \frac{-36864x^4 + 311296x^3 - 516096x^2 + 552960}{3479}.$$

By observing that $m_5'(x) < 0$ for $[0, 1]$, $m_5(x)$ is decreasing in $[0, 1]$ and hence achieves its maxima at $x = 0$. Thus,

$$G(0, x, 1) \leq \frac{552960}{3479} \approx 158.9422248 \quad (x \in [0, 1]).$$

From above cases, we conclude that

$$G(c, x, y) \leq \frac{552960}{3479}$$

on $[0, 2] \times [0, 1] \times [0, 1]$. It follows from (3.25) that

$$H_{3,1}(f) \leq \frac{3479}{35389440} G(c, x, y) \leq \frac{1}{64} \approx 0.01562.$$

If $f \in \mathcal{RL}$, then sharp bound for this Hankel determinant is determined by

$$H_{3,1}(f) = \frac{1}{64} \approx 0.01562,$$

with an extremal function

$$f_3(z) = \int_0^z \sqrt{1 + \zeta^3} d\zeta = z + \frac{1}{8} z^4 - \frac{1}{56} z^7 + \cdots . \tag{3.37}$$

□

4 Bound of $\left|H_{4,1}(f)\right|$ for the Class $\mathcal{RL}$

In this section, we investigate the bound of $\left|H_{4,1}(f)\right|$ for the class $\mathcal{RL}$.

Theorem 4.1 *Let $f \in \mathcal{RL}$. Then*

$$\left|a_5 - a_3^2\right| \leq \frac{1}{10}. \tag{4.1}$$

The bound is sharp.

Proof From (3.8) and (3.10), we obtain

$$\left|a_5 - a_3^2\right| = \frac{1}{20} \left| \frac{1519}{4608} c_1^4 - \frac{401}{288} c_1^2 c_2 + \frac{5}{4} c_3 c_1 + \frac{55}{72} c_2^2 - c_4 \right|. \tag{4.2}$$

Comparing the right side of (4.2) with

$$\left| \rho c_1^4 + 2\delta c_1 c_3 + \sigma c_2^2 - \frac{3}{2} \lambda c_1^2 c_2 - c_4 \right|,$$

we get $\rho = \frac{1519}{4608}$, $\delta = \frac{5}{8}$, $\sigma = \frac{55}{72}$ and $\lambda = \frac{401}{432}$. It follows that

$$8\sigma(1-\sigma)\left[(\delta\lambda - 2\rho)^2 + (\delta(\sigma+\delta) - \lambda)^2\right] + \delta(1-\delta)(\lambda - 2\sigma\delta)^2 = 0.014429,$$

and

$$4\delta^2(1-\delta)^2\sigma(1-\sigma) = 0.03963.$$

By Lemma 2.2, we deduce that

$$\left|a_5 - a_3^2\right| \le \frac{1}{10}.$$

For the sharpness, we consider the function $f_4 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_4(z) = \int_0^z \sqrt{1+\zeta^4}\, d\zeta = z + \frac{1}{10} z^5 - \frac{1}{72} z^9 + \cdots .$$

□

Theorem 4.2 *Let $f \in \mathcal{RL}$. Then*

$$|a_2 a_5 - a_3 a_4| \le 0.02645871784. \tag{4.3}$$

Proof From the coefficient bounds given in (3.7), (3.8), (3.9) and (3.10), along with $c_1 = c$, we have

$$|a_2 a_5 - a_3 a_4| = \frac{1}{122880}\left|-49c^5 + 176c^3 c_2 - 560c^2 c_3 + 320cc_2^2 + 768cc_4 - 640c_2 c_3\right|.$$

Let $t = 4 - c_1^2$ and using Lemma 2.3, we obtain

$$\begin{aligned}
&|a_2 a_5 - a_3 a_4| \\
&= \frac{1}{122880}\Big\{-5c^5 + 96c^3 x^3 t + 80cx^3 t^2 - 384ct\bar{x}\left(1-|x|^2\right) y^2 - 384c^2 tx\left(1-|x|^2\right) y \\
&\quad - 68x^2 tc^3 - 160t^2\left(1-|x|^2\right) xy - 80x^2 t^2 c + 384ct\left(1-|x|^2\right)\left(1-|y|^2\right) z \\
&\quad -56c^2 t\left(1-|x|^2\right) y + 16txc^3 + 384tx^2 c\Big\}.
\end{aligned}$$

Since $t = 4 - c^2$, we get

$$|a_2 a_5 - a_3 a_4| = \frac{1}{122880}\left(f_1(c,x) + f_2(c,x)\, y + f_3(c,x)\, y^2 + f_4(c,x,y)\, z\right),$$

where

$$\begin{aligned}
f_1(c,x) &= -5c^5 + \left(4-c^2\right)\Big[\left(4-c^2\right)\left(80x^3 c - 80x^2 c\right) \\
&\qquad +96x^3 c^3 - 68x^2 c^3 + 16xc^3 + 384x^2 c\Big], \\
f_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[-160\left(4-c^2\right) x - 384xc^2 - 56c^2\right], \\
f_3(c,x) &= -384c\left(4-c^2\right)\left(1-|x|^2\right)\bar{x},
\end{aligned}$$

and

$$f_4(c,x,y) = 384c\left(1-|x|^2\right)\left(1-|y|^2\right).$$

Now, by using $|x| = x$, $|y| = y$ and taking $|z| \leq 1$, we obtain

$$\begin{aligned}|a_2a_5 - a_3a_4| &\leq \frac{1}{122880}\left(|f_1(c,x)| + |f_2(c,x)|\,y + |f_3(c,x)|\,y^2 + |f_4(c,x,y)|\right)\\ &\leq Q(c,x,y),\end{aligned} \tag{4.4}$$

where

$$Q(c,x,y) := \frac{1}{122880}\left[q_1(c,x) + q_2(c,x)y + q_3(c,x)y^2 + q_4(c,x)(1-y^2)\right] \tag{4.5}$$

with

$$\begin{aligned}q_1(c,x) &:= 5c^5 + \left(4-c^2\right)\Big[\left(4-c^2\right)\left(80x^3c + 80x^2c\right)\\ &\qquad +96x^3c^3 + 68x^2c^3 + 16xc^3 + 384x^2c\Big],\\ q_2(c,x) &:= \left(4-c^2\right)\left(1-x^2\right)\left[160\left(4-c^2\right)x + 384xc^2 + 56c^2\right],\\ q_3(c,x) &:= 384c\left(4-c^2\right)\left(1-x^2\right)x,\end{aligned}$$

and

$$q_4(c,x) := 384c\left(1-x^2\right).$$

Let the closed cuboid be $\Lambda : [0,2]\times[0,1]\times[0,1]$. We have to obtain the points of maxima inside Λ, inside the six faces and on the twelve edges in order to maximize Q.

(I) Let $(c,x,y) \in (0,2)\times(0,1)\times(0,1)$. Now, to find points of maxima inside Λ, we take partial derivative of (4.5) with respect to y and get

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \frac{1}{15360}(1-x^2)\Big\{[96c(4-c^2)x - 96c]y + 20(4-c^2)^2x\\ &\qquad +48(4-c^2)c^2x + 7(4-c^2)c^2\Big\}.\end{aligned}$$

For $\frac{\partial Q}{\partial y} = 0$, yields

$$y = \frac{20(4-c^2)^2x + 48(4-c^2)c^2x + 7(4-c^2)c^2}{96c[1-(4-c^2)x]} := y_0.$$

If y_0 is a critical point inside Λ, then $y_0 \in (0, 1)$, which is possible only if

$$20(4-c^2)^2x + 48(4-c^2)c^2x + 7(4-c^2)c^2 < 96c[1-(4-c^2)x], \tag{4.6}$$

and

$$c^2 > \frac{4x-1}{x} =: g_1(x). \tag{4.7}$$

We have to obtain the solutions which satisfy both inequalities (4.6) and (4.7) for the existence of the critical points. A simple exercise shows that (4.6) does not hold in this case for all values of $x \in (0, 1)$, and there is no critical point of Q in $(0, 2) \times (0, 1) \times (0, 1)$.

(II) To find points of maxima inside the six faces of Λ. We deal with each case individually.

When $c = 0$, $Q(c, x, y)$ reduces to

$$t_1(x, y) := Q(0, x, y) = \frac{x(1-x^2)y}{48} \quad (x,\ y \in (0, 1)). \tag{4.8}$$

t_1 has no critical point in $(0, 1) \times (0, 1)$ since

$$\frac{\partial t_1}{\partial y} = \frac{x(1-x^2)}{48} \neq 0 \quad (x,\ y \in (0, 1)). \tag{4.9}$$

When $c = 2$, $Q(c, x, y)$ reduces to

$$t_2(x, y) := Q(2, x, y) = \frac{24(1-x^2)(1-y^2)+5}{3840} \quad (x,\ y \in (0, 1)). \tag{4.10}$$

t_2 has no critical point in $(0, 1) \times (0, 1)$ since

$$\frac{\partial t_2}{\partial y} = -\frac{(1-x^2)y}{80} \neq 0 \quad (x,\ y \in (0, 1)). \tag{4.11}$$

When $x = 0$, $Q(c, x, y)$ reduces to

$$t_3(c, y) := Q(c, 0, y) = \frac{-384cy^2 + 56(4-c^2)c^2y + 5c^5 + 384c}{122880}, \tag{4.12}$$

where $c \in (0, 2)$ and $y \in (0, 1)$. A calculation shows that there does not exist any solution for the system of equations $\frac{\partial t_3}{\partial y} = 0$ and $\frac{\partial t_3}{\partial c} = 0$ in $(0, 2) \times (0, 1)$.

When $x = 1$, $Q(c, x, y)$ reduces to

$$t_4(c, y) := Q(c, 1, y) = \frac{4096c - 944c^3 - 15c^5}{122880} \quad (c \in (0, 2)). \tag{4.13}$$

Solving $\frac{\partial t_4}{\partial c} = 0$, we obtain optimal point at

$$c =: c_0 = \frac{2}{15}\sqrt{-1062 + 6\sqrt{36129}} \approx 1.181018790.$$

Thus, t_4 achieves its maximum at c_0 that is

$$\left(\frac{5081}{450000} - \frac{59\sqrt{36129}}{1350000}\right) \cdot \sqrt{-1062 + 6\sqrt{36129}} \approx 0.02643184534.$$

When $y = 0$, $Q(c, x, y)$ reduces to

$$\begin{aligned} t_5(c, x) &:= Q(c, x, 0) \\ &= \frac{1}{122880}\left(\begin{array}{l} 5c^5 - 384cx^2 + 384c + (4 - c^2)\big[(4 - c^2)(80x^3c + 80cx^2) \\ +96x^3c^3 + 384cx^2 + 68x^2c^3 + 16xc^3\big] \end{array}\right). \end{aligned}$$

A calculation shows that there exists a unique solution $(c, x) \approx (1.894283613, 0.3232411338)$ for the system of equations $\frac{\partial t_5}{\partial x} = 0$ and $\frac{\partial t_5}{\partial c} = 0$ in $(0, 2) \times (0, 1)$. We conclude that

$$t_5(c, x) := Q(c, x, 0) \approx 0.006930539679.$$

When $y = 1$, $Q(c, x, y)$ reduces to

$$\begin{aligned} Q(c, x, 1) &= \frac{1}{122880}\left(\begin{array}{l} 5c^5 + (4 - c^2)[(4 - c^2)(-160x^3 + 160x + 80cx^3 + 80cx^2) \\ +384cx - 384cx^3 + 384cx^2 + 68x^2c^3 + 16xc^3 + 96x^3c^3 \\ -384x^3c^2 + 384c^2x + 56c^2 - 56x^2c^2] \end{array}\right) \\ &=: t_6(c, x). \end{aligned}$$

A calculation shows that there exists a unique solution

$$(c, x) \approx (1.176425264, 0.9758744867)$$

for the system of equations $\frac{\partial t_6}{\partial x} = 0$ and $\frac{\partial t_6}{\partial c} = 0$ in $(0, 2) \times (0, 1)$. We deduce that

$$t_6(c, x) := Q(c, x, 1) \approx 0.02645871784.$$

(III) Now, we are going to find the maxima of $Q(c, x, y)$ on the edges of Λ.
By putting $y = 0$ in (4.8), we get $Q(0, x, 0) = 0$, where $x \in [0, 1]$.
By setting $y = 1$ in (4.8), we get

$$Q(0, x, 1) = n_5(x) := \frac{x - x^3}{48}.$$

By noting that $n_5'(x) = 0$ for $x =: x_0 = 1/\sqrt{3}$ in $[0, 1]$. Since $n_5(x)$ is increasing for $x \le x_0$ and decreasing for $x_0 \le x$, it achieves maxima at x_0. Hence

$$Q(0, x, 1) \le \frac{\sqrt{3}}{216} \approx 0.008018753741 \quad (x \in [0, 1]).$$

By taking $y = 0$ in (4.10), we get

$$Q(2, x, 0) = n_4(x) := \frac{29 - 24x^2}{3840}.$$

Since $n_4'(x) < 0$ for $x \in [0, 1]$, we know that $n_4(x)$ is decreasing in $[0, 1]$. Thus, it achieves maxima at $x = 0$. Hence

$$Q(2, x, 0) \le \frac{29}{3840} \approx 0.007552083 \quad (x \in [0, 1]).$$

By putting $y = 1$ in (4.10), we get

$$Q(2, x, 1) \le \frac{1}{768} \approx 0.001302083 \quad (x \in [0, 1]).$$

By setting $y = 0$ in (4.12), we have

$$Q(c, 0, 0) =: n_1(c) = \frac{5c^5 + 384c}{122880}.$$

Since $n_1'(c) > 0$ for $c \in [0, 2]$, we see that $n_1(c)$ is increasing in $[0, 2]$ and hence attains its maximum value at $c = 2$. Thus,

$$Q(c, 0, 0) \le \frac{29}{3840} \approx 0.007552083 \quad (c \in [0, 2]).$$

Solving Eq. (4.12) at $y = 1$, we get

$$Q(c, 0, 1) = n_2(c) := \frac{5c^5 - 56c^4 + 224c^2}{122880}.$$

It is easy to verify that the function $n_2'(c) = 0$ for $c =: c_0 = 0$ and

$$c =: c_1 \approx 1.555704593$$

in the interval $[0, 2]$. We observe that c_0 is the point of minima and the maximum value of $n_2(c)$ is approximately equal to 0.002113230011, which is attained at c_1, thus,

$$Q(c, 0, 1) \le 0.002113230011 \quad (c \in [0, 2]).$$

By putting $c = 0$ in (4.12), we get

$$Q(0, 0, y) \leq 0 \quad (y \in [0, 1]).$$

By setting $c = 2$ in (4.12), we obtain

$$Q(2, 0, y) = \frac{29 - 24y^2}{3840}.$$

A simple calculation gives

$$Q(2, 0, y) \leq \frac{29}{3840} \approx 0.007552083 \quad (y \in [0, 1]).$$

As we see that Eq. (4.13) is independent of y, so we have

$$Q(c, 1, 1) = Q(c, 1, 0) = n_3(c) := \frac{4096c - 944c^3 - 15c^5}{122880}.$$

Now, $n_3'(c) = 0$ for

$$c =: c_0 = \frac{2}{15}\sqrt{-1062 + 6\sqrt{36129}} \approx 1.18101879 \in [0, 2]$$

and $n_3(c)$ achieves its maximum at c_0. We conclude that

$$\begin{aligned} Q(c, 1, 1) = Q(c, 1, 0) &\leq \left(\frac{5081}{450000} - \frac{59}{1350000}\sqrt{36129}\right)\sqrt{-1062 + 6\sqrt{36129}} \\ &\approx 0.02643184534 \quad (c \in [0, 2]). \end{aligned}$$

By putting $c = 0$ in Eq. (4.13), we get

$$Q(0, 1, y) \leq 0 \quad (y \in [0, 1]).$$

By setting $c = 2$ in Eq. (4.13), we have

$$Q(2, 1, y) \leq \frac{1}{768} \approx 0.001302083 \quad (y \in [0, 1]).$$

Therefore, by virtue of all the above cases, we deduce that inequality (4.3) holds. □

Theorem 4.3 *Let $f \in \mathcal{RL}$. Then*

$$\left|a_3 a_5 - a_4^2\right| \leq \frac{1}{64}. \tag{4.14}$$

The bound is sharp.

Proof From (3.8), (3.9) and (3.10), along with $c_1 = c$, we have

$$\left|a_3a_5 - a_4^2\right| = \frac{1}{3932160}\Big|285c^6 - 1392c^4c_2 + 320c^3c_3 + 2368c^2c_2^2 - 10240c^2c_4 + 17920cc_2c_3 - 10240c_2^3 + 16384c_2c_4 - 15360c_3^2\Big|.$$

Let $t = 4 - c_1^2$ and using Lemma 2.3, we obtain

$$\begin{aligned}
&\left|a_3a_5 - a_4^2\right| \\
&= \frac{1}{3932160}\Big\{5c^6 - 56c^4xt + 368c^4x^2t + 464c^2x^2t^2 - 1024c^2tx^2 - 256c^4tx^3 \\
&- 1472x^3t^2c^2 + 64t^2x^4c^2 - 3840t^2\left(1-|x|^2\right)^2 y^2 + 1024c^2t\bar{x}\left(1-|x|^2\right)y^2 \\
&+ 1024c^3t\left(1-|x|^2\right)xy - 1024c^2t\left(1-|x|^2\right)\left(1-|y|^2\right)z + 896cxt^2\left(1-|x|^2\right)y \\
&- 256ct^2x^2\left(1-|x|^2\right)y + 4096xt^2\left(1-|x|^2\right)\left(1-|y|^2\right)z - 224c^3t\left(1-|x|^2\right)y \\
&- 1280x^3t^3 + 4096x^3t^2 - 4096xt^2\bar{x}\left(1-|x|^2\right)y^2\Big\}.
\end{aligned}$$

Since $t = 4 - c^2$, we know that

$$\left|a_3a_5 - a_4^2\right| = \frac{1}{3932160}\left(u_1(c,x) + u_2(c,x)\,y + u_3(c,x)\,y^2 + u_4(c,x,y)\,z\right),$$

where

$$\begin{aligned}
u_1(c,x) &= 5c^6 + (4-c^2)^2(64x^4c^2 - 192x^3c^2 + 464x^2c^2 - 1024x^3) \\
&\quad + (4-c^2)(-1024x^2c^2 + 368c^4x^2 - 56c^4x - 256c^4x^3), \\
u_2(c,x) &= -32c(4-c^2)(1-|x|^2)[c^2(-32x+7) + (4-c^2)x(8x-28)], \\
u_3(c,x) &= -256(4-c^2)(1-|x|^2)[(4-c^2)(15+x^2) - 4c^2\bar{x}],
\end{aligned}$$

and

$$u_4(c,x,y) = 1024(4-c^2)(1-|x|^2)(1-|y|^2)[4x(4-c^2) - c^2].$$

Now, by using $|x| = x$, $|y| = y$ and taking $|z| \le 1$, we obtain

$$\begin{aligned}
\left|a_3a_5 - a_4^2\right| &\le \frac{1}{3932160}\Big(|u_1(c,x)| + |u_2(c,x)|\,y + |u_3(c,x)|\,y^2 + |u_4(c,x,y)|\Big) \\
&\le S(c,x,y),
\end{aligned}$$

where

$$S(c,x,y) = \frac{1}{3932160}\left[s_1(c,x) + s_2(c,x)\,y + s_3(c,x)\,y^2 + s_4(c,x)\left(1-y^2\right)\right]$$

(4.15)

with

$$s_1(c,x) = 5c^6 + (4-c^2)^2(64x^4c^2 + 192x^3c^2 + 464x^2c^2 + 1024x^3) \\ + (4-c^2)(1024x^2c^2 + 368c^4x^2 + 56c^4x + 256c^4x^3),$$
$$s_2(c,x) = 32c(4-c^2)(1-x^2)[c^2(32x+7) + (4-c^2)x(8x+28)],$$
$$s_3(c,x) = 256(4-c^2)(1-x^2)[(4-c^2)(15+x^2) + 4c^2x],$$

and

$$s_4(c,x) = 1024(4-c^2)(1-x^2)[4x(4-c^2) + c^2].$$

Let the closed cuboid be $\Lambda : [0,2] \times [0,1] \times [0,1]$. We have to obtain the points of maxima inside Λ, inside the six faces and on the twelve edges in order to maximize Q.

(I) Let $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. Now, to find points of maxima inside Λ, we take partial derivative of (4.15) with respect to y and get

$$\frac{\partial S}{\partial y} = \frac{1}{122880}(4-c^2)(1-x^2)\big\{16y(x-1)[(4-c^2)(x-15) + 4c^2] \\ + c[4x(4-c^2)(7+2x) + c^2(7+32x)]\big\}.$$

For $\frac{\partial S}{\partial y} = 0$, yields

$$y = \frac{c[4x(4-c^2)(7+2x) + c^2(7+32x)]}{16(x-1)[(4-c^2)(15-x) - 4c^2} := y_0.$$

If y_0 is a critical point inside Λ, then $y_0 \in (0,1)$, which is possible only if

$$c^3(7+32x) + 4cx(4-c^2)(7+2x) + 16(1-x)(4-c^2)(15-x) < 64c^2(1-x), \tag{4.16}$$

and

$$c^2 > \frac{4(15-x)}{19-x}. \tag{4.17}$$

Now, we have to obtain the solutions which satisfy both inequalities (4.16) and (4.17) for the existence of the critical points.

Let

$$g_2(x) := \frac{4(15-x)}{19-x}.$$

Since $g_2'(x) < 0$ for $(0, 1)$, $g_2(x)$ is decreasing in $(0, 1)$. Hence $c^2 > 28/9$, a simple exercise shows that (4.16) does not hold in this case for all values of $x \in (0, 1)$, and there is no critical point of S in $(0, 2) \times (0, 1) \times (0, 1)$.

(II) Now, to find points of maxima inside the six faces of Λ. We deal with each case individually.

When $c = 0$, $S(c, x, y)$ reduces to

$$\hbar_1(x, y) := S(0, x, y) = \frac{(1 - x^2)(y^2(x - 1)(x - 15) + 16x) + 4x^3}{960} \quad (x,\ y \in (0, 1)). \tag{4.18}$$

$\hbar_1$ has no optimal point in $(0, 1) \times (0, 1)$ since

$$\frac{\partial \hbar_1}{\partial y} = \frac{y(1 - x^2)(x - 1)(x - 15)}{480} \neq 0 \quad (x,\ y \in (0, 1)). \tag{4.19}$$

When $c = 2$, $S(c, x, y)$ reduces to

$$S(2, x, y) = \frac{1}{12288} \quad (x,\ y \in (0, 1)). \tag{4.20}$$

When $x = 0$, $S(c, x, y)$ reduces to $S(c, 0, y)$, given by

$$\hbar_2(c, y) := \frac{256y^2(240 - 136c^2 + 19c^4) + 224c^3 y(4 - c^2) + 5c^6 + 4096c^2 - 1024c^4}{3932160}, \tag{4.21}$$

where $c \in (0, 2)$ and $y \in (0, 1)$. We solve $\frac{\partial \hbar_2}{\partial y} = 0$ and $\frac{\partial \hbar_2}{\partial c} = 0$ to find the points of maxima. By solving $\frac{\partial \hbar_2}{\partial y} = 0$, we obtain

$$y = -\frac{7c^3}{16(60 - 19c^2)} =: y_1. \tag{4.22}$$

For the given range of y, y_1 should belong to $(0, 1)$, which is possible only if $c > c_0$, $c_0 \approx 1.777046633$. A calculation shows that $\frac{\partial \hbar_2}{\partial c} = 0$, which implies that

$$(-34816 + 9728c^2)y^2 + (1344c - 560c^3)y + 15c^4 + 4096 - 2048c^2 = 0. \tag{4.23}$$

By substituting (4.22) into (4.23), we get

$$-377160c^6 + 1311c^8 + 3083408c^4 - 8355840c^2 + 7372800 = 0. \tag{4.24}$$

A calculation gives the solution of (4.24) in $(0, 2)$ that is $c \approx 1.413134655$. Thus $\hbar_2$ has no optimal point in $(0, 2) \times (0, 1)$.

When $x = 1$, $S(c, x, y)$ reduces to

$$\hbar_3(c, y) := S(c, 1, y) = \frac{45c^6 - 3040c^4 + 7424c^2 + 16384}{3932160} \quad (c \in (0, 2)). \tag{4.25}$$

By solving $\frac{\partial \hbar_3}{\partial c} = 0$, we obtain the critical point are $c =: c_0 = 0$ and

$$c := c_1 = \frac{4}{45}\sqrt{2850 - 15\sqrt{32185}} \approx 1.120751931,$$

where c_0 is minimum point and maximum point of $\hbar_3$ is achieved at c_1 that is

$$-\frac{45071}{209952} + \frac{6437\sqrt{32185}}{5248800} \approx 0.00534108401.$$

When $y = 0$, $S(c, x, y)$ reduces to

$$\begin{aligned}\hbar_4(c, x) &:= S(c, x, 0)\\ &= \frac{1}{3932160}\\ &\quad \begin{pmatrix} 5c^6 + (4 - c^2)[(4 - c^2)(64x^4c^2 + 464x^2c^2 - 3072x^3 + 4096x \\ +192x^3c^2) + 368c^4x^2 + 56c^4x + 256c^4x^3 + 1024c^2] \end{pmatrix}.\end{aligned}$$

A calculation shows that there does not exist any solution for the system of equations $\frac{\partial \hbar_4}{\partial x} = 0$ and $\frac{\partial \hbar_4}{\partial c} = 0$ in $(0, 2) \times (0, 1)$.

When $y = 1$, $S(c, x, y)$ reduces to

$$\begin{aligned}S(c, x, 1) &= \frac{1}{3932160}\\ &\quad \begin{pmatrix} 5c^6 + (4 - c^2)\big[(4 - c^2)(896cx + 256cx^2 + 192x^3c^2 - 896cx^3 \\ +1024x^3 - 256cx^4 + 64x^4c^2 + 3840 + 464x^2c^2 - 256x^4 - 3584x^2) \\ +368c^4x^2 - 224c^3x^2 + 256c^4x^3 - 1024c^3x^3 + 1024c^3x + 56c^4x \\ +224c^3 + 1024x^2c^2 - 1024x^3c^2 + 1024c^2x\big] \end{pmatrix}\\ &=: \hbar_5(c, x).\end{aligned}$$

A calculation shows that there does not exist any solution for the system of equations $\frac{\partial \hbar_5}{\partial x} = 0$ and $\frac{\partial \hbar_5}{\partial c} = 0$ in $(0, 2) \times (0, 1)$.

(III) We are going to find the maxima of $S(c, x, y)$ on the edges of Λ. By taking $y = 0$ in (4.21), we have

$$S(c, 0, 0) =: l_1(c) = \frac{5c^6 + 4096c^2 - 1024c^4}{3932160}.$$

Now, $l_1'(c) = 0$ for $c =: \lambda_0 = 0$ and

$$c =: \lambda_1 = \frac{8}{15}\sqrt{240 - 15\sqrt{241}} \approx 1.424846645 \in [0, 2],$$

where λ_0 is minimum point and maximum point of l_1 is λ_1, we see that

$$S(c,0,0) \leq \frac{-7472+482\sqrt{241}}{10125} \approx 0.0010520697 \quad (c \in [0,2]).$$

By solving Eq. (4.21) at $y=1$, we get

$$S(c,0,1) = l_2(c) := \frac{5c^6-224c^5+3840c^4+896c^3-30720c^2+61440}{3932160}.$$

Since $l_2'(c) < 0$ for $[0,2]$, we know that $l_2(c)$ is decreasing in $[0,2]$ and hence maxima is achieved at $c=0$. Thus,

$$S(c,0,1) \leq \frac{1}{64} \approx 0.015625 \quad (c \in [0,2]).$$

By setting $c=0$ in (4.21), we get $S(0,0,y) = y^2/64$. A simple calculation gives

$$S(0,0,y) \leq \frac{1}{64} \approx 0.015625 \quad (y \in [0,1]).$$

As we see that Eq. (4.25) is independent of y, we have

$$S(c,1,1) = S(c,1,0) = l_3(c) := \frac{45c^6-3040c^4+7424c^2+16384}{3932160}.$$

Now, $l_3'(c) = 0$ for $c =: c_0 = 0$ and

$$c =: c_1 = \frac{4}{45}\sqrt{2850-15\sqrt{32185}} \approx 1.120751931,$$

where c_0 is minimum point and maximum point of l_3 is achieved at c_1. We conclude that

$$S(c,1,1) = S(c,1,0) \leq -\frac{45071}{209952} + \frac{6437\sqrt{32185}}{5248800} \approx 0.00534108401 \quad (c \in [0,2]).$$

By taking $c=0$ in Eq. (4.25), we get

$$S(0,1,y) = \frac{1}{240} \approx 0.004166666667 \quad (y \in [0,1]).$$

As (4.20) is independent of c, x and y, thus we have

$$\begin{aligned} S(2,1,y) &= S(2,0,y) = S(2,x,0) = S(2,x,1) \\ &= \frac{1}{12288} \approx 0.00008138 \quad (x,\ y \in [0,1]). \end{aligned}$$

By putting $y=0$ in (4.18), we get

$$S(0,x,0)=l_4(x):=\frac{-3x^3+4x}{240}.$$

Since $l_4'(x)=0$ for $x=: x_0=2/3$ in $[0,1]$, we know that $l_4(x)$ is increasing for $x\le x_0$ and $l_4(x)$ is decreasing for $x_0\le x$. It achieves maximum at x_0. Hence

$$S(0,x,0)\le\frac{1}{135}\approx 0.007407407 \quad (x\in[0,1]).$$

By setting $y=1$ in (4.18), we get

$$S(0,x,1)=: l_5(x)=\frac{-x^4+4x^3-14x^2+15}{960}.$$

Since $l_5'(x)<0$ for $[0,1]$, we see that $l_5(x)$ is decreasing in $[0,1]$ and achieves its maxima at $x=0$. Thus,

$$S(0,x,1)\le\frac{1}{64}\approx 0.015625 \quad (x\in[0,1]).$$

By means of all the above cases, we deduce that inequality (4.14) holds. The result is sharp for the function $f_3(z)$ given by (3.37). □

Theorem 4.4 *Let $f\in\mathcal{RL}$. Then*

$$|H_{4,1}(f)|\le 7.9206\times 10^{-3}. \tag{4.26}$$

Proof It is easy to see that

$$H_4(1)=a_7H_{3,1}(f)-a_6\Delta_1+a_5\Delta_2-a_4\Delta_3, \tag{4.27}$$

where $\Delta_1,\ \Delta_2,\ \Delta_3$ are determinants of order 3 and given by

$$\Delta_1=a_6\left(a_3-a_2^2\right)+a_5\left(a_2a_3-a_4\right)+a_4\left(a_2a_4-a_3^2\right), \tag{4.28}$$

$$\Delta_2=a_6\left(a_4-a_2a_3\right)-a_5\left(a_5-a_3^2\right)+a_4\left(a_2a_5-a_3a_4\right), \tag{4.29}$$

and

$$\Delta_3=a_6\left(a_2a_4-a_3^2\right)-a_5\left(a_2a_5-a_3a_4\right)+a_4\left(a_3a_5-a_4^2\right). \tag{4.30}$$

By applying triangle inequality on (4.28) , (4.29) and (4.30) , we get

$$|\Delta_1|\le|a_6|\left|a_3-a_2^2\right|+|a_5|\left|a_2a_3-a_4\right|+|a_4|\left|a_2a_4-a_3^2\right|, \tag{4.31}$$

$$|\Delta_2| \leq |a_6|\,|a_4 - a_2a_3| + |a_5|\left|a_5 - a_3^2\right| + |a_4|\,|a_2a_5 - a_3a_4|\,, \tag{4.32}$$

and

$$|\Delta_3| \leq |a_6|\left|a_2a_4 - a_3^2\right| + |a_5|\,|a_2a_5 - a_3a_4| + |a_4|\left|a_3a_5 - a_4^2\right|. \tag{4.33}$$

From Theorem 3.1, along with (3.16) , (3.20), (3.22), (4.1), (4.3) and (4.14), we obtain

$$|\Delta_1| \leq 3.2244 \times 10^{-2},$$
$$|\Delta_2| \leq 2.5511 \times 10^{-2},$$

and

$$|\Delta_3| \leq 7.3110 \times 10^{-3}.$$

Clearly, it follows from (4.27) that

$$\left|H_{4,1}(f)\right| \leq |a_7|\left|H_{3,1}(f)\right| + |a_6|\,|\Delta_1| + |a_5|\,|\Delta_2| + |a_4|\,|\Delta_3|\,. \tag{4.34}$$

By putting the values of $\left|H_{3,1}(f)\right|$, $|\Delta_1|$, $|\Delta_2|$ and $|\Delta_3|$ into (4.34), as well as with the help of Theorem 3.1, it follows that

$$\begin{aligned}\left|H_{4,1}(f)\right| \leq{}& \frac{2-\sqrt{2}}{7} \times \frac{1}{64} + \frac{2-\sqrt{2}}{6} \times \left(3.2244 \times 10^{-2}\right) \\ &+ \frac{1}{10} \times \left(2.5511 \times 10^{-2}\right) + \frac{1}{8} \times \left(7.3110 \times 10^{-3}\right) \approx 7.9206 \times 10^{-3}.\end{aligned}$$

□

5 Bounds of $\left|H_{4,1}(f)\right|$ for the Classes $\mathcal{RL}^{(2)}$ and $\mathcal{RL}^{(3)}$

For given $m \in \mathbb{N}$, a domain Λ is said to be m-fold symmetric if Λ is taken on itself by a rotation of Λ around the origin by an angle $2\pi/m$. It is easy to observe that the analytic function f is m-fold symmetric in $\mathbb{D}$, if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z) \quad (z \in \mathbb{D})\,.$$

By $\mathcal{S}^{(m)}$, we specify the collection of all univalent m-fold functions having the Taylor series expansion

$$f(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1} \quad (z \in \mathbb{D})\,. \tag{5.1}$$

The subclass $\mathcal{RL}^{(m)}$ of $\mathcal{S}^{(m)}$ is the set of m-fold symmetric functions with bounded turning subordinated with $\sqrt{1+z}$. More precisely,

$$\mathcal{RL}^{(m)} = \left\{ f \in \mathcal{S}^{(m)} : f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}} \text{ with } p \in \mathcal{P}^{(m)} \quad (z \in \mathbb{D}) \right\}, \quad (5.2)$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{n=1}^{\infty} c_{mn} z^{mn} \quad (z \in \mathbb{D}) \right\}. \quad (5.3)$$

Theorem 5.1 *If* $f \in \mathcal{RL}^{(2)}$, *then*

$$\left| H_{4,1}(f) \right| \leq \frac{1}{600}.$$

Proof Since $f \in \mathcal{RL}^{(2)}$, there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f \in \mathcal{RL}^{(2)}$, using the series form (5.1) and (5.3) with $m = 2$, we can write

$$f'(z) = 1 + 3a_3 z^2 + 5a_5 z^4 + 7a_7 z^6 + \cdots, \quad (5.4)$$

and

$$\sqrt{\frac{2p(z)}{p(z)+1}} = 1 + \frac{1}{4}c_2 z^2 + \left(\frac{1}{4}c_4 - \frac{5}{32}c_2^2 \right) z^4 + \left(\frac{1}{4}c_6 - \frac{5}{16}c_2 c_4 + \frac{13}{128}c_2^3 \right) z^6 + \cdots. \quad (5.5)$$

Comparing (5.4) with (5.5), we obtain

$$a_3 = \frac{1}{12}c_2,$$
$$a_5 = \frac{1}{5}\left(\frac{1}{4}c_4 - \frac{5}{32}c_2^2 \right),$$

and

$$a_7 = \frac{1}{7}\left(\frac{1}{4}c_6 - \frac{5}{16}c_2 c_4 + \frac{13}{128}c_2^3 \right),$$

while $a_2 = a_4 = a_6 = 0$, so it is clear that for $f \in \mathcal{RL}^{(2)}$,

$$H_{4,1}(f) := \left(a_5 - a_3^2\right)\left(a_3 a_7 - a_5^2\right).$$

Therefore, we get

$$a_5 - a_3^2 = \frac{1}{5}\left(\frac{1}{4}c_4 - \frac{5}{32}c_2^2\right) - \frac{1}{144}c_2^2 = \frac{1}{20}c_4 - \frac{11}{288}c_2^2 = \frac{1}{20}\left(c_4 - \frac{55}{72}c_2^2\right).$$

By using (2.2) of Lemma 2.1, we have

$$\left|a_5 - a_3^2\right| \le \frac{1}{10}. \tag{5.6}$$

Furthermore, we see that

$$\begin{aligned} a_3 a_7 - a_5^2 &= \frac{1}{2100}\left(-\frac{5}{4}c_2^2 c_4 + \frac{125}{256}c_2^4 - \frac{21}{4}c_4^2 + \frac{25}{4}c_2 c_6\right) \\ &= \frac{1}{2100}\left[-\frac{5}{4}c_2^2\left(c_4 - \frac{25}{64}c_2^2\right) - \frac{25}{4}\left(c_8 - c_2 c_6\right) + \frac{25}{4}\left(c_8 - \frac{21}{25}c_4^2\right)\right], \end{aligned}$$

and then with the help of triangle inequality, (2.2) and (2.3), which yields

$$\left|a_3 a_7 - a_5^2\right| \le \frac{1}{60}. \tag{5.7}$$

From (5.6) and (5.7), we conclude that

$$\left|H_{4,1}(f)\right| \le \frac{1}{600}.$$

□

Theorem 5.2 *If $f \in \mathcal{RL}^{(3)}$, then*

$$\left|H_{4,1}(f)\right| \le \frac{1}{896}.$$

Proof Let $f \in \mathcal{RL}^{(3)}$. Then there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$f'(z) = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f \in \mathcal{RL}^{(3)}$, by using the series form (5.1) and (5.3) when $m = 3$, we can write

$$f'(z) = 1 + 4a_4 z^3 + 7a_7 z^6 + \cdots, \tag{5.8}$$

and

$$\sqrt{\frac{2p(z)}{p(z)+1}} = 1 + \frac{1}{4}c_3 z^3 + \left(\frac{1}{4}c_6 - \frac{5}{32}c_3^2\right) z^6 + \cdots . \tag{5.9}$$

By comparing (5.8) and (5.9), we get

$$a_4 = \frac{1}{16}c_3, \tag{5.10}$$

and

$$a_7 = \frac{1}{7}\left(\frac{1}{4}c_6 - \frac{5}{32}c_3^2\right), \tag{5.11}$$

while $a_2 = a_3 = a_5 = a_6 = 0$. So it is clear that for $f \in \mathcal{RL}^{(3)}$, we have

$$H_{4,1}(f) := -a_4^2\left(a_7 - a_4^2\right).$$

From (5.10) and (5.11), we obtain

$$a_7 - a_4^2 = \frac{1}{28}\left(c_6 - \frac{47}{64}c_3^2\right).$$

By the virtue of triangle inequality and (2.2), it follows that

$$\left|a_7 - a_4^2\right| \le \frac{1}{14}. \tag{5.12}$$

Also, we observe that

$$|a_4| \le \frac{1}{8}, \tag{5.13}$$

therefore, in view of (5.12) and (5.13), we deduce that

$$\left|H_{4,1}(f)\right| \le \frac{1}{896}.$$

□

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