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Ground State Solution of Critical *p*-Biharmonic Equation Involving Hardy Potential

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Abstract

In this paper, we consider the following critical p-biharmonic equation involving Hardy potential

$$\Delta_p^2 u - \Delta_q u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = |u|^{p_*-2}u, \quad x \in \mathbb{R}^N,$$

where $2 \le p < \frac{N}{2}$, $0 < \mu < \mu_{N,p} = \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $q = p^* = \frac{Np}{N-p}$, and $p_* = \frac{Np}{N-2p}$. The existence of ground state solution to above equation is established by using the Nehari manifold and some analysis techniques. Our result extends the existing results in the literature.

Keywords p-Biharmonic equation · Hardy potential · Nehari manifold · Critical exponent · Nonlinear elliptic equations

Mathematics Subject Classification 35J60 · 31A30 · 35J75

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1 Introduction

In this paper, we study the following critical p-biharmonic equation involving Hardy potential:

$$\Delta_p^2 u - \Delta_q u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = |u|^{p_*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $2 \le p < \frac{N}{2}, 0 < \mu < \mu_{N,p} = \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is p-biharmonic operator and $\Delta_q u = div(|\nabla u|^{q-2}\nabla u)$ is q-Laplace operator, $q = p^* = \frac{Np}{N-p}$, and $p_* = \frac{Np}{N-2p}$ denotes the critical Sobolev exponent.

In recent years, the nonlinear elliptic equations with singularities become an interesting topic. It arises from physical modeling, such as non-Newtonian fluid, viscous fluids, elastic mechanic, boundary layer, see, for instance, [1]. Recently, the existence and multiplicity of ground state solutions, positive solutions and sign-changing solutions of p-biharmonic equations with singular potential have been studied extensively. For more related works, we refer to [2–12] and the references therein. In particular, Dhifli–Alsaedi [3] studied the following p-biharmonic equation:

$$\begin{cases} \Delta_p^2 u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} - \Delta_p u = g(x)u^{m-1} + \lambda f(x)u^{q-1}, & x \in \mathbb{R}^N, \\ u(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $0 < m < 1 < p < q < p_*, N > 2p, \lambda > 0$. Under some appropriate conditions on functions f and g, the authors showed that Eq. (1.2) has at least two positive solutions by using the fibering maps and Nehari manifold.

Yang–Zhang–Liu [13] dealt with the following p-biharmonic equation:

$$\Delta_p^2 u - \mu \frac{|u|^{p-2}u}{|x|^{2p}} = a(x)|u|^{r-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $1 < p < \frac{N}{2}, p < r < p_*$. By applying the method of invariant sets of descending flow, the existence of sign-changing solutions of Eq. (1.3) was obtained. By using Nehari manifold, Su–Shi [14] investigated the existence of ground state solutions for the following equation:

$$\Delta^2 u - \mu \frac{u}{|x|^4} - \Delta_p u = |u|^{2_*-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $N \ge 5$, $p = 2^* = \frac{2N}{N-2}$, $2_* = \frac{2N}{N-4}$. Furthermore, Su–Liu–Feng [10] established the existence of ground state solutions for the thin film epitaxy equation via the generalized versions of Lions-type theorem.

Inspired by the above-mentioned works, **it is natural to ask a question whether Eq. (1.1) admits ground state solution?** As far as we know, there is no result about the ground state solution for p-biharmonic equations with Hardy potential in current literature. Therefore, in the present paper, we shall give a positive answer to the above question. Our main result is the following theorem.

Theorem 1.1 *Assume that* $2 \leq p < \frac{N}{2}$, $q = p^* = \frac{Np}{N-p}$ *and* $\mu \in (0, \mu_{N,p})$ *hold. Then, Eq.* (1.1) *has at least a ground state solution.*

2 Proof of Theorem 1.1

The space $W_0^{2,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$, where the norm is $\|u\|_0^p = \int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x$. According to [15], $\mu_{N,p}$ is the best constant in the following Rellich inequality:

$$\mu_{N,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \mathrm{d}x \leq \int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x, \quad \forall u \in W_0^{2,p}(\mathbb{R}^N). \tag{2.1}$$

For the above inequality, we refer to [16] for more details. We define

$$E = \left\{ u \in W_0^{2,p}(\mathbb{R}^N) \cap W_0^{1,p}(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} \left(|\Delta u|^p - \mu \frac{|u|^p}{|x|^{2p}} \right) \mathrm{d}x < \infty \right\}.$$

For $\mu \in (0, \mu_{N,p})$, E is equipped with the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} \left(|\Delta u|^p - \mu \frac{|u|^p}{|x|^{2p}} \right) \mathrm{d}x \right]^{\frac{1}{p}}.$$

Furthermore, we denote the best Sobolev's constant by

$$S_\mu := \inf_{u \in E \backslash \{0\}} \left\{ \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x\right)^{\frac{p}{p_*}}} \right\}. \tag{2.2}$$

The energy functional $I(u) : E \to \mathbb{R}$ associated with Eq. (1.1) can be given by

$$I(u) = \frac{1}{p}\|u\|^p + \frac{1}{p^*} \int_{\mathbb{R}^N} |\nabla u|^{p^*} \mathrm{d}x - \frac{1}{p_*} \int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x,$$

and the Nehari manifold of E is defined by

$$\aleph = \{u \in E \backslash \{0\} | \langle I'(u), u \rangle = 0\}.$$

Denote

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad \bar{c} = \inf_{u \in \aleph} I(u), \quad \bar{\bar{c}} = \inf_{u \in E \backslash \{0\}} \sup_{t \geq 0} I(tu),$$

where $\Gamma = \{\gamma \in C([0, 1], E) | \gamma(0) = 0, \ I(\gamma(1)) < 0\}$.

Lemma 2.1 *Assume that the assumptions of Theorem 1.1 hold. Then, the following conclusions are true.*
(i) *For every $u \in E \setminus \{0\}$, there exists only one $t_u > 0$ such that $t_u u \in \aleph$ and $I(t_u u) = \max_{t>0} I(tu)$.*
(ii) $c = \bar{c} = \bar{\bar{c}}$.
(iii) *There exists a $(PS)_c$ sequence $\{u_n\} \subset \aleph$ of I with $c > 0$.*

Proof (i) For any $u \in E \setminus \{0\}$ and $t \in (0, +\infty)$, we set

$$g(t) := I(tu) = \frac{t^p}{p}\|u\|^p + \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x - \frac{t^{p_*}}{p_*}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x.$$

Then, we have

$$g'(t) = t^{p-1}\|u\|^p + t^{p^*-1}\int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x - t^{p_*-1}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x.$$

By $p < p^* < p_*$, we know that $g'(\cdot) > 0$ for $t > 0$ enough small, and $g'(\cdot) < 0$ for t enough large. Then, there exists $t_u > 0$ such that $g'(t_u) = 0$.

To prove the uniqueness of t_u, let us assume that $0 < \bar{t} < \bar{\bar{t}}$ satisfy $g'(\bar{t}) = g'(\bar{\bar{t}}) = 0$. Then,

$$0 = \frac{g'(\bar{t})}{\bar{t}^{p^*-1}} = \bar{t}^{p-p^*}\|u\|^p + \int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x - \bar{t}^{p_*-p^*}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x$$

and

$$0 = \frac{g'(\bar{\bar{t}})}{\bar{\bar{t}}^{p^*-1}} = \bar{\bar{t}}^{p-p^*}\|u\|^p + \int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x - \bar{\bar{t}}^{p_*-p^*}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x.$$

According to $0 = \frac{g'(\bar{t})}{\bar{t}^{p^*-1}} = \frac{g'(\bar{\bar{t}})}{\bar{\bar{t}}^{p^*-1}}$, we obtain

$$(\bar{t}^{p-p^*} - \bar{\bar{t}}^{p-p^*})\|u\|^p = (\bar{t}^{p_*-p^*} - \bar{\bar{t}}^{p_*-p^*})\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x.$$

On the one hand, since $p - p^* < 0$ and $0 < \bar{t} < \bar{\bar{t}}$, then $\bar{t}^{p-p^*} - \bar{\bar{t}}^{p-p^*} > 0$. On the other hand, by $p_* - p^* > 0$ and $0 < \bar{t} < \bar{\bar{t}}$, we have $\bar{t}^{p_*-p^*} - \bar{\bar{t}}^{p_*-p^*} < 0$. This is a contradiction. Hence, for any $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \aleph$. So $g(t)$ admits a unique critical point $t_u \in (0, +\infty)$ such that $g(t)$ attains its maximum at t_u.

(ii) First, we prove $I(u) \geq I(tu)$ for $t \geq 0$. Let $u \in \aleph$. Then, we have

$$\langle I'(u), u\rangle = 0 \Leftrightarrow \int_{\mathbb{R}^N} |\nabla u|^{p^*} dx = -\|u\|^p + \int_{\mathbb{R}^N} |u|^{p_*} dx.$$

It follows that

$$I(u) - I(tu) = \frac{1}{p}(1 - t^p)\|u\|^p + \frac{1}{p^*}(1 - t^{p^*}) \int_{\mathbb{R}^N} |\nabla u|^{p^*} \mathrm{d}x - \frac{1}{p_*}(1 - t^{p_*}) \int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x$$
$$= \left(\frac{1 - t^p}{p} + \frac{t^{p^*} - 1}{p^*} \right) \|u\|^p + \left(\frac{1 - t^{p^*}}{p^*} + \frac{t^{p_*} - 1}{p_*} \right) \int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x. \tag{2.3}$$

Since $2 \le p < p^* < p_*$, it is easy to see that

$$\frac{1 - t^p}{p} + \frac{t^{p^*} - 1}{p^*} \ge 0, \quad \text{and} \quad \frac{1 - t^{p^*}}{p^*} + \frac{t^{p_*} - 1}{p_*} \ge 0.$$

Thus, we have $I(u) \ge I(tu)$ for $t \ge 0$. From (i), it is obvious that $c = \bar{c}$.

Next, we prove $\bar{c} = \bar{\bar{c}}$. By the definition of c, we can select a sequence $\{u_n\} \subset E$ such that

$$c \le \max_{t \ge 0} I(tu_n) \le c + \frac{1}{n}, \quad \forall n \in \mathbb{N}^*. \tag{2.4}$$

For any $u \in E \backslash \{0\}$ and $t > 0$ large enough, we have $g(t) = I(tu) < 0$ and then there exists $t_n = t(u_n) > 0$ and $s_n > t_n$ such that

$$I(t_n u_n) = \max_{t \ge 0} I(tu_n), \quad I(s_n u_n) < 0, \quad \forall n \in \mathbb{N}^*. \tag{2.5}$$

Let $\gamma_n(\bar{t}) = \bar{t} s_n u_n$, $\bar{t} \in [0, 1]$, then $\gamma_n \in \Gamma$. It follows from (2.4) and (2.5) that

$$\sup_{\bar{t} \in [0,1]} I(\gamma_n(\bar{t})) = \max_{t \ge 0} I(tu_n) < c + \frac{1}{n}, \quad \forall n \in \mathbb{N}^*,$$

which indicates $\bar{\bar{c}} < c$. Obviously, E can be separated into two parts by the manifold $\aleph$ as follows:

$$E^+ = \{u \in E | \langle I'(u), u \rangle > 0\} \cup \{0\} \text{ and } E^- = \{u \in E | \langle I'(u), u \rangle < 0\}.$$

By $p < p^* < p_*$, it easy to obtain that $I(u) \ge \frac{1}{p^*}\langle I'(u), u \rangle$ for any $u \in E$. It follows that $I(u) \ge 0$ for all $u \in E^+$ and indicates that E^+ conclude a little ball which is around the origin. Thus, every $\gamma \in \Gamma$ has to cross $\aleph$ for $\gamma(0) \in E^+$ and $\gamma(1) \in E^-$. So $\bar{c} \le \bar{\bar{c}}$. Combining with $\bar{\bar{c}} \le c$ and $c = \bar{c}$, we have $c = \bar{c} = \bar{\bar{c}}$.

(iii) Set

$$\Phi(u) = \langle I'(u), u \rangle = \|u\|^p + \int_{\mathbb{R}^N} |\nabla u|^{p^*} \mathrm{d}x - \int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x,$$

then

$$\langle \Phi'(u), u \rangle = p\|u\|^p + p^* \int_{\mathbb{R}^N} |\nabla u|^{p^*} \mathrm{d}x - p_* \int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x.$$

According to (i), we have $\aleph \neq \emptyset$ and $\inf_{u\in\aleph} I(u) = \bar{c} = c$. Applying Ekeland's variational principle, there exists $\{u_n\} \subset \aleph$ and $\lambda_n \in \mathbb{R}$ such that $I(u_n) \to \bar{c}$ and $I'(u_n) - \lambda_n \Phi'(u_n) \to 0$, as $n \to \infty$. Then, we get

$$I(u_n) = I(u_n) - \frac{1}{p^*}\langle I'(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u\|^p.$$

Therefore, $\{u_n\}$ is bounded in E. In view of

$$|\langle I'(u_n), u_n\rangle - \langle \lambda_n \Phi'(u_n), u_n\rangle| \leq \|I'(u_n) - \lambda_n \Phi'(u_n)\|\|u_n\|,$$

then we have

$$|\langle I'(u_n), u_n\rangle - \langle \lambda_n \Phi'(u_n), u_n\rangle| \to 0,$$

as $n \to \infty$. Since $|\langle I'(u_n), u_n\rangle| = 0$ and $\langle \Phi'(u_n), u_n\rangle \neq 0$, we can easily get $\lambda_n \to 0$. Applying Hölder's and Sobolev's inequalities, we know

$$\begin{aligned}
\|\Phi'(u_n)\| &= \sup_{\|\varphi\|=1} |\langle \Phi'(u_n), \varphi\rangle| \\
&= \sup_{\|\varphi\|=1} \left| p\int_{R^N} |\Delta u|^{p-2}\Delta u \Delta\varphi - p\mu \frac{|u|^{p-2}u\varphi}{|x|^{2p}}\mathrm{d}x \right. \\
&\quad \left. + p^*\int_{R^N} |\nabla u|^{p^*-2}\nabla u \nabla\varphi \mathrm{d}x - p_*\int_{R^N} |u|^{p_*-2}u\varphi \mathrm{d}x \right| \\
&\leq \sup_{\|\varphi\|=1} \left\{ \left| p\left(\int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\Delta\varphi|^p \mathrm{d}x\right)^{\frac{1}{p}} \right| \right. \\
&\quad + \left| p\mu\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} \frac{|\varphi|^p}{|x|^{2p}}\mathrm{d}x\right)^{\frac{1}{p}} \right| \\
&\quad + \left| p^*\left(\int_{\mathbb{R}^N} |\nabla u|^{p^*} \mathrm{d}x\right)^{\frac{p^*-1}{p^*}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \right| \\
&\quad \left. + \left| p_*\left(\int_{\mathbb{R}^N} |u|^{p_*} \mathrm{d}x\right)^{\frac{p_*-1}{p_*}} \left(\int_{\mathbb{R}^N} |\varphi|^{p_*} \mathrm{d}x\right)^{\frac{1}{p_*}} \right| \right\} \\
&\leq C.
\end{aligned}$$

Therefore, we have

$$\|I'(u_n)\| \leq \|I'(u_n) - \lambda_n \Phi'(u_n)\| + |\lambda_n|\|\Phi'(u_n)\| = o(1).$$

It shows that $I'(u_n) \to 0$ and then $\{u_n\} \subset \aleph$ is the $(PS)_c$ sequence of I. Next, we prove that $c > 0$. For any $u \in \aleph$, it follows that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p^*}\langle I'(u), u\rangle \\ &= \frac{1}{p}\|u\|^p + \frac{1}{p^*}\int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x \\ &\quad - \frac{1}{p^*}\|u\|^p - \frac{1}{p^*}\int_{\mathbb{R}^N} |\nabla u|^{p^*}\mathrm{d}x + \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u\|^p + \left(\frac{1}{p^*} - \frac{1}{p_*}\right)\int_{\mathbb{R}^N} |u|^{p_*}\mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u\|^p, \end{aligned}$$

which implies our desired results. The proof is completed. □

Lemma 2.2 *Assume that the assumptions described in Theorem 1.1 hold. Let $\{u_n\} \subset \aleph$ be a $(PS)_c$ sequence of I with $c > 0$. Then, there exists $C_1 > 0$ such that $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_*}\mathrm{d}x = C_1$.*

Proof It follows from the proof of Lemma 2.1-(iii) that $\{u_n\}$ is uniformly bounded in E. We divide the following proof into two steps.

Step 1. There is a constant $C > 0$ independent of n such that $0 \leq Q_n = \int_{\mathbb{R}^N} |u_n|^{p_*}\mathrm{d}x \leq C$, which means that $\{Q_n\}$ is a bounded sequence in $\mathbb{R}$. By Bolzano–Weierstrass theorem, we know that there is an accumulation point Q_0.

Let us define $H \subset [0, C] \subset \mathbb{R}$ be the set of all accumulation points of $\{Q_n\}$. By $Q_0 \in H$, so $H \neq \emptyset$. It follows from the definition of the superior limit and H that $\limsup_{n\to\infty} Q_n = \sup H$. Using $H \subset [0, C]$ and the supremum and infimum principle, we can get the existence of $\sup H$. Then, there is $C_1 \in [0, C]$ such that $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_*}\mathrm{d}x = C_1$.

Step 2. We prove that $C_1 > 0$. By contradiction, we assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_*}\mathrm{d}x = 0. \tag{2.6}$$

From the Gagliardo–Nirenberg inequality, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p^*}\mathrm{d}x \leq C\left(\lim_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|^{p}\mathrm{d}x\right)^{\frac{p^*}{2p}} \left(\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_*}\mathrm{d}x\right)^{\frac{2p-p^*}{2p}} = 0. \tag{2.7}$$

Combining (2.6), (2.7) and $\{u_n\}$ is a $(PS)_c$ sequence of I, we get

$$c + o(1) = \frac{1}{p}\|u_n\|^p$$

and $\|u_n\|^p = o(1)$, which indicates $c = 0$. This contradicts $c > 0$. Hence, we get $\limsup\limits_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p_*}\mathrm{d}x = C_1 > 0$. The proof is completed. □

Lemma 2.3 *Assume that the assumptions described in Theorem 1.1 hold. Let* $\{u_n\} \subset E$ *be a* $(PS)_c$ *sequence of* I *at* $c > 0$*, and* $u_n \rightharpoonup 0$ *weakly in* E*. Then, there exists* $\varepsilon > 0$ *satisfying that*

$$either \quad \lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = 0 \quad or \quad \limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x \geq \varepsilon,$$

where $B_1(0)$ *denotes a sphere with a center at 0 and radius of 1.*

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence of I at $c > 0$. For any $\varphi \in E$, we have

$$\begin{aligned} \langle I'(u_n), \varphi\rangle = &\int_{\mathbb{R}^N} |\Delta u_n|^{p-2}\Delta u_n \Delta\varphi \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|u_n|^{p-2}u_n\varphi}{|x|^{2p}}\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} |\nabla u_n|^{p^*-2}\nabla u_n \nabla\varphi \mathrm{d}x - \int_{\mathbb{R}^N} |u_n|^{p_*-2}u_n\varphi \mathrm{d}x. \end{aligned} \tag{2.8}$$

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be a cutoff function satisfying $supp(\psi) = \overline{B_2(0)}$ and $\psi = 1$ in $B_1(0)$. The embedding

$$E \hookrightarrow L^r(\overline{B_2(0)})$$

is compact for all $r \in [2, p_*)$.

Step 1. According to Rellich's compactness theorem, Sobolev's inequality and Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} |\Delta u_n|^{p-2}\Delta u_n \Delta(\psi^p u_n)\mathrm{d}x = \int_{\mathbb{R}^N} |\Delta(\psi u_n)|^p \mathrm{d}x + o(1) \tag{2.9}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p^*-2}\nabla u_n \nabla(\psi^p u_n)\mathrm{d}x = \int_{\mathbb{R}^N} \psi^p |\nabla u_n|^{p^*}\mathrm{d}x + o(1). \tag{2.10}$$

According to Hölder's inequality, one gets

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^{p_*}\psi^p \mathrm{d}x &= \int_{\mathbb{R}^N} (\psi u_n)^p |u_n|^{p_*-p}\mathrm{d}x \leq \int_{B_2(0)} |\psi u_n|^p |u_n|^{p_*-p}\mathrm{d}x \\ &\leq \left(\int_{B_2(0)} |\psi u_n|^{p_*}\mathrm{d}x\right)^{\frac{p}{p_*}} \left(\int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x\right)^{\frac{p_*-p}{p_*}} \\ &\leq \|\psi u_n\|^p_{L^{p_*}(\mathbb{R}^N)} \left(\int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x\right)^{\frac{p_*-p}{p_*}} \\ &\leq \frac{1}{S_\mu}\|\psi u_n\|^p \|u_n\|^{p_*-p}_{L^{p_*}(B_2(0))}. \end{aligned} \tag{2.11}$$

We choose $\varphi = \psi^p u_n$ in (2.8), and there holds

$$\langle I'(u_n), \psi^p u_n\rangle = \int_{\mathbb{R}^N} |\Delta u_n|^{p-2}\Delta u_n \Delta(\psi^p u_n)\mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|u_n|^{p-2}u_n(\psi^p u_n)}{|x|^{2p}}\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^{p^*-2}\nabla u_n \nabla(\psi^p u_n)\mathrm{d}x - \int_{\mathbb{R}^N} |u_n|^{p_*-2}u_n(\psi^p u_n)\mathrm{d}x.$$

Applying (2.9)–(2.11) and

$$\|\psi u_n\| = \left(\int_{\mathbb{R}^N} |\Delta(\psi u_n)|^p \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|\psi u_n|^p}{|x|^{2p}}\mathrm{d}x\right)^{\frac{1}{p}},$$

one has

$$\begin{aligned}\frac{1}{S_\mu}\|\psi u_n\|^p \|u_n\|^{p_*-p}_{L^{p_*}(B_2(0))} &\geq C\|\psi u_n\|^p + \int_{\mathbb{R}^N} \psi^p |\nabla u_n|^{p^*}\mathrm{d}x + o(1)\\ &\geq C\|\psi u_n\|^p + o(1).\end{aligned} \tag{2.12}$$

Step 2. In this step, we split our following proof into two aspects: **(I)** $\limsup_{n\to\infty} \|\psi u_n\| > 0$ and **(II)** $\lim_{n\to\infty} \|\psi u_n\| = 0$.

Case (I). According to $\limsup_{n\to\infty} \|\psi u_n\| > 0$ and (2.12), we get

$$\limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x \geq \left(C_2 S_\mu\right)^{\frac{p_*}{p_*-p}} > 0. \tag{2.13}$$

Similar to **Step 1** of Lemma 2.2, there exists $0 \leq C_3 < \infty$ such that

$$C_3 = \limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x.$$

In view of (2.13), we get $C_3 > 0$. Set $D_1 := \limsup_{n\to\infty} \int_{B_2(0)\setminus\overline{B_1(0)}} |u_n|^{p_*}\mathrm{d}x$, we have

$$\limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x \leq D_1 + \limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x. \tag{2.14}$$

According to the range of D_1, there are three subcases.

Case (I-1). If $D_1 = 0$, by (2.14), then

$$\limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = \limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x = C_3 > 0. \tag{2.15}$$

Case (I-2). If $D_1 \in (0, C_3)$, then there exists $C_4 = C_3 - D_1 > 0$ satisfying

$$\limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x \geq C_4 > 0. \tag{2.16}$$

Case (I-3). If $D_1 = C_3 = \limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x$. Then, we have the following two subsubcases: **(1)** $\lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x$ exists, and **(2)** $\lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x$ does not exist. If **(1)** happens, then (2.14) turns into

$$\limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x = D_1 + \limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x.$$

Substituting $D_1 = C_3 = \limsup_{n\to\infty} \int_{B_2(0)} |u_n|^{p_*}\mathrm{d}x$ into above equality, we can see that

$$\limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = \lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = 0. \tag{2.17}$$

If **(2)** happens, it follows that $\limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x > \liminf_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x \geq 0$, which indicates that there exists $C_5 > 0$ such that

$$\limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = C_5 > 0. \tag{2.18}$$

Case (II). From $\lim_{n\to\infty} \|\psi u_n\| = 0$ and Sobolev's inequality, we get

$$0 = \lim_{n\to\infty} \frac{\|\psi u_n\|^p}{S_\mu} \geq \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\psi u_n|^{p_*}\mathrm{d}x\right)^{\frac{p}{p_*}} \geq \lim_{n\to\infty} \left(\int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x\right)^{\frac{p}{p_*}},$$

which indicates

$$\lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = 0. \tag{2.19}$$

In conclusion, setting $\varepsilon = \min\{C_3, C_4, C_5\}$ and combining (2.15)–(2.19), we can deduce

$$\text{either} \quad \lim_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x = 0 \quad \text{or} \quad \limsup_{n\to\infty} \int_{B_1(0)} |u_n|^{p_*}\mathrm{d}x \geq \varepsilon,$$

The proof is completed. □

In order to obtain our main result, we also give the following general version Brezis–Lieb lemma.

Lemma 2.4 (Brezis–Lieb Lemma, [17]) *Let Ω be an open subset of $\mathbb{R}^N$, and let $\{u_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$. If*

(1) $\{u_n\}$ is bounded in $L^p(\Omega)$;

(2) $u_n \to u_0$ almost everywhere in Ω,

then

$$\lim_{n\to\infty} (\|u_n\|_{L^p}^p - \|u_n - u_0\|_{L^p}^p) = \|u_0\|_{L^p}^p. \tag{2.20}$$

The proof of Theorem 1.1 In view of Lemma 2.2, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_*}\mathrm{d}x = C_1 > 0. \tag{2.21}$$

Set $\delta = \min\{C_1, \frac{\varepsilon}{2}\}$, where $\varepsilon > 0$ is taken in Lemma 2.3. By (2.21), there exists a sequence $\{r_n\} \subset \mathbb{R}^+$ such that for any $\delta' \in (0, \delta)$, one has $\limsup_{n\to\infty}\int_{B_{r_n}(0)}|u_n|^{p_*}\mathrm{d}x = \delta'$. Define $\overline{u_n} := r_n^{\frac{N-2p}{p}} u_n(r_n x)$. Then, $\overline{u_n} \in E$ and

$$I(\overline{u_n}) \to c, \quad I'(\overline{u_n}) \to 0 \text{ and } \int_{B_1(0)}|\overline{u_n}|^{p_*}\mathrm{d}x = \delta', \tag{2.22}$$

as $n \to \infty$. By (2.22), it is easy to check $\{\overline{u_n}\}$ is bounded in E. Without loss of generality, we suppose there exists $\overline{u} \in E$ such that $u_n \rightharpoonup u$ in E.

We next prove $\overline{u} \not\equiv 0$. Argue by contradiction. Let $u \equiv 0$. It follows from Lemma 2.3 that we have

$$\text{either } \lim_{n\to\infty}\int_{B_1(0)}|u_n|^{p_*}\mathrm{d}x = 0 \text{ or } \limsup_{n\to\infty}\int_{B_1(0)}|u_n|^{p_*}\mathrm{d}x \ge \varepsilon,$$

which contradicts to (2.22) since $0 < \delta' < \delta = \min\{C_1, \frac{\varepsilon}{2}\}$.

Now, we prove $\overline{u_n} \to \overline{u}$ strongly in E. It follows from $\lim_{n\to\infty}\langle I'(\overline{u_n}), \varphi\rangle = o(1)$, $\overline{u_n} \rightharpoonup \overline{u}$ in E and Lemma 2.1 that

$$\langle I'(\overline{u}), \varphi\rangle = 0 \text{ and } \overline{u} \in \aleph.$$

Set

$$F(u) := \left(\frac{1}{p} - \frac{1}{p_*}\right)\|u_n\|^2 + \left(\frac{1}{p^*} - \frac{1}{p_*}\right)\int_{R^N}|\nabla u_n|^{p^*}\mathrm{d}x.$$

According to Lemma 2.4, Fatou lemma and $\overline{c} = c$, we have

$$\begin{aligned} c + o(1) = \lim_{n\to\infty} I(\overline{u_n}) - \lim_{n\to\infty}\tfrac{1}{p_*}\langle I'(\overline{u_n}), \overline{u_n}\rangle &= \lim_{n\to\infty} F(\overline{u_n}) \\ \ge F(\overline{u}) = I(\overline{u}) \ge \overline{c} = c. \end{aligned} \tag{2.23}$$

Thus, the above inequalities will be equalities. Applying $\lim_{n\to\infty} F(\overline{u_n}) = F(\overline{u})$ and Lemma 2.4 again, we get $\lim_{n\to\infty} F(\overline{u_n}) - \lim_{n\to\infty} F(\overline{u_n} - \overline{u}) = F(\overline{u}) + o(1)$. So $\lim_{n\to\infty} F(\overline{u_n} - \overline{u}) = 0$, which implies $\overline{u_n} \to \overline{u}$ strongly in E. Applying (2.23) again, we have $I'(\overline{u}) = c$. Thus, $\overline{u}$ is a ground state solution of Eq. (1.1). □

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