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Almost Factorizable Glrac Semigroups

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Abstract

Glrac semigroups are common generalizations of left GC-lpp semigroups and left inverse semigroups. And, such a semigroup is a left restriction semigroup if and only if the projection set is a semilattice. So, glrac semigroup is also a generalization of left restriction semigroup. Permissible subsets of a glrac semigroup are introduced in this paper. In terms of permissible subsets, we define (uniquely) factorizable glrac semigroups and (uniquely) almost factorizable glrac semigroups. Many characterizations of (uniquely) factorizable glrac semigroups and (uniquely) almost factorizable glrac semigroups are obtained. As their applications, we establish the structures of uniquely factorizable left GC-lpp semigroups (left inverse semigroups, inverse semigroups, ample semigroups, left restriction semigroup, restriction semigroups) and uniquely almost factorizable left GC-lpp semigroups (left inverse semigroups, inverse semigroups, ample semigroups, left restriction semigroup, restriction semigroups). Our results enrich and extend the related results of almost factorizable restriction semigroups.

Keywords Glrac semigroup · (Almost) factorizable semigroup · Left GC-lpp semigroup · Reduced semigroup

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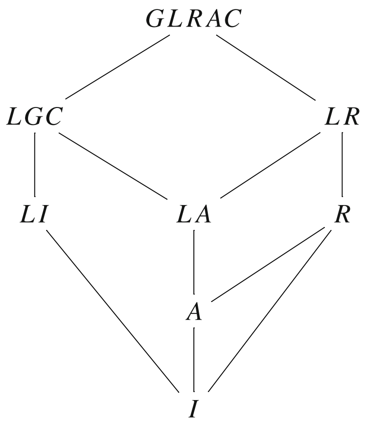
1 Introduction

An inverse semigroup is defined to be a regular semigroup whose set of idempotents forms a commutative subsemigroup; that is, a semilattice under its multiplication. Because inverse semigroups play an important role in the theory of semigroups and are extensively used in many branches of mathematics (see [28]), this class of semigroups is generalized in various aspects; for example, left (right) inverse semigroup (see [1, 29]), (left; right) adequate semigroup (see [8]), (left; right) ample semigroup (see [8]), (left; right) Ehresmann semigroup (see, [3–5,12]), (left; right) restriction semigroup (see [10,15,16,23]), and so on.

Left (right) inverse semigroups may be regarded as generalizations of inverse semigroups in the range of regular semigroups. Precisely, a left (right) inverse semigroup is a regular semigroup with left (right) regular bands of idempotents. In [17], Guo weakened the regularity of left inverse semigroup into a left p.p. semigroup and defined left GC-lpp semigroups; for concrete definition, the reader can be found in Sect. 2. Indeed, left GC-lpp semigroups can be thought of a common generalization of left inverse semigroups and left ample semigroups because a left inverse semigroup is just a regular left GC-lpp semigroup and a left ample semigroup is just a left GC-lpp semigroup with semilattice of idempotents. There are a series of papers on left GC-lpp semigroups (see [16–21]).

Left (right) restriction semigroups have been studied from various points of view under different names; for example, weakly left (right) E-ample semigroup [12], since the 1960s. Concretely, from universal algebraic point of view, a left restriction semigroup is an algebra of type (2, 1) which satisfies certain identities. In particular, each inverse semigroup is a left (right) restriction semigroup if the semigroup is possessed the unary operation which assign the idempotent $a^{-1}a$ (aa^{-1}) to any element a. For a historical overview of (left; right) restriction semigroups, the reader can be referred to [10,15,23]. In a similar way as in generalizing left GC-lpp semigroups, Branco, Gomes and Gould in [2] replaced the identities of left restriction semigroups such that the set of projections becomes a semilattice by those making the set of projections to become a left regular band and call such an algebra of type (2, 1) a glrac semigroup. In [7], Ding, J. Guo, X. Guo and Shum have probed glrac semigroups. We shall find that any left GC-lpp semigroup (left inverse semigroup) is a glrac semigroup and that a semigroup is a left restriction semigroup if and only if its set of projections is a semilattice. Thus, glrac semigroups are non-regular generalizations of left inverse semigroups (of course, inverse semigroups). The semigroups mentioned in the above have the inclusion relationships as follows:



where | stands for the upper class of semigroups including the next one, and

- $GLRAC$ = {glrac semigroups},
- LGC = {left GC-lpp semigroups},
- LI = {left inverse semigroups},
- LR = {left restriction semigroups},
- LA = {left ample semigroups},
- R = {restriction semigroups},
- A = {ample semigroups}, and
- I = {inverse semigroups}.

It is worthy to point out that glrac semigroups become a kind of special perfect restriction semigroups (see [25]).

Factorizability and almost factorizability of weakly ample semigroups are considered in [13]. In the same reference, the authors defined the W-product of a semilattice by a monoid by adapted a semidirect product of a semilattice by a group, which is due to [7] and further extended the fundamental results on factorizable inverse monoids and almost factorizable inverse semigroups to the weakly ample case. Based on W-product of a semilattice by a monoid, in [28], Szendrei proved that these results can be adapted for restriction semigroups and that every restriction semigroup can be embeddable in an almost left factorizable restriction semigroup. Especially, it is verified that each restriction semigroup is isomorphic to some (2, 1, 1)-subsemigroup of a projection-separating homomorphic image of a W-product of a semilattice by a monoid. Inspired by Szendrei's results in [28], we define permissible subsets of a glrac semigroup, which may be regarded as a generalization of permissible subsets of a restriction semigroup, and moreover introduce (uniquely) factorizable glrac monoids and (uniquely) almost factorizable glrac semigroups. It is proved that the set of permissible subsets of a glrac semigroup is a left restriction semigroup under the power product (Theorem 1). In Sect. 4, we introduce the notion of weakly semidirect product and further obtain the weakly semidirect product structures of uniquely almost factorizable glrac semigroups (Theorem 2) and of uniquely factorizable glrac monoids (Theorem 3). It is proved that a glrac semigroup is almost factorizable if and only if it is a projection-separating (2, 1)-homomorphic image of some full (2, 1)-subsemigroup of a weakly semidirect product of a left regular band and a left reduced monoid (Theorem 4). And, the similar result is valid for a factorizable glrac monoid (Theorem 5). Finally, the special case for left GC-lpp semigroups is considered. The weakly semidirect product structures of uniquely almost factorizable left GC-lpp semigroups (left inverse semigroups) and of uniquely factorizable left GC-lpp monoids (left inverse monoids) are obtained (Theorems 14, 15, 16 and 17).

2 Preliminaries

We recall first some concepts and notations, which are used in the sequel without mentions.

Definition 1 A *glrac semigroup* is defined to be an algebra of type (2, 1), more precisely, an algebra $S = (S, \cdot, +)$ where $(S, \cdot)$ is a semigroup and $^+$ is a unary operator such that the following identities are satisfied:

$$\begin{aligned} &(x^+)^+ = x^+,\ x^+x = x,\ x^+y^+ = x^+y^+x^+, \\ &(x^+y)^+ = x^+y^+,\ (xy)^+ = (xy^+)^+,\ xy^+ = (xy^+)^+x. \end{aligned} \tag{1}$$

The identities in the notion of glrac semigroups are somewhat different to that defined by Branco, Gomes and Gould in [2]. We may check that our identities are equivalent to those of Branco, Gomes and Gould. We here omit the detail.

Recall that a *left regular band* is a semigroup satisfying identities: $ab = aba$ and $a^2 = a$. Equivalently, a semigroup is a left regular band if and only if it is a semilattice of left zero semigroups. Let B be a left regular band and $B = \bigcup_{\alpha \in Y} B_\alpha$ be the semilattice decomposition of B into left zero semigroups B_α. We shall call the semilattice Y the structure semilattice of B. If $e \in B_\alpha \subseteq B$, we shall write $B_\alpha = E(e)$; if $B_\alpha B_\beta \subseteq B_\beta$, then we denote $B_\beta \leq B_\alpha$. It is easy to see that any left regular band with respect to the identity unary operation is a glrac semigroup. For left regular bands, the reader can be referred to the textbook [26].

By definition, we have the following corollary:

Corollary 1 *Let S be a glrac semigroup. Then,*

(i) *$P(S) = \{a^+ : a \in S\}$ is a left regular sub-band of S.*
(ii) *S is a left restriction semigroup if and only if $P(S)$ is a semilattice.*

Proof (i) Because

$$a^+ = (a^+a)^+ = a^+a^+,$$

we know that $P(S)$ is a left regular band.

(ii) It is obvious and we omit the detail. □

We call $P(S)$ in Corollary 1 the *set of projections* of S; and any element of $P(S)$ a *projection* of S.

A *left restriction semigroup* is defined to be an algebra of type (2, 1), more precisely, an algebra $S = (S, \cdot, +)$ where $(S, \cdot)$ is a semigroup and $^+$ is a unary operator such that the following identities are satisfied:

$$\begin{aligned}&(x^+)^+ = x^+,\ x^+x = x,\ x^+y^+ = y^+x^+,\\&(x^+y)^+ = x^+y^+\ (xy)^+ = (xy^+)^+,\ xy^+ = (xy^+)^+x.\end{aligned} \tag{2}$$

A *right restriction semigroup* is dually defined, that is, it is an algebra $(S, \cdot, *)$ satisfying the duals of the identities (2). If $S = (S, \cdot, +, *)$ is an algebra of type (2, 1, 1) where $S = (S, \cdot, +)$ is a left restriction semigroup and $S = (S, \cdot, *)$ is a right restriction semigroup and the identities

$$(x^+)^* = x^+,\ (x^*)^+ = x^* \tag{3}$$

hold, then it is called a *restriction semigroup*. By definition, the defining properties of a restriction semigroup are left-right dual. Therefore, in the sequel dual definitions and statements will not be explicitly formulated. It is well known that in a restriction semigroup, we always have

$$(xy)^+ = (xy^+)^+ \text{ and } (xy)^* = (x^*y)^*. \tag{4}$$

(for example, see [15]). Obviously, a left restriction semigroup is just a glrac semigroup whose projection set is a semilattice.

Example 1 Let S be a left inverse semigroup. Then, by a result in [1,29], each $\mathcal{R}$-class of S contains exactly one idempotent. We denote by $a^\dagger$ the unique idempotent of the $\mathcal{R}$-class of S containing $a \in S$. Note that the set $E(S)$ of idempotents of S become a left regular band. Of course, any left inverse semigroup is an orthodox semigroup; for orthodox semigroups, see [24]. Let a' be an inverse of a, then by [24, Theorem 3.5 (i), p.46], $aa' = a^\dagger$. Then, we have the following facts:

(RI1) For any $a \in S$, we have $a = aa'a = a^\dagger a$ and $(a^\dagger)^\dagger = a^\dagger$.
(RI2) For any $a, b \in S$, from that $E(S)$ is a left regular band under the multiplication of S, it follows that $a^\dagger b^\dagger = a^\dagger b^\dagger a^\dagger$.
(RI3) For any $a, b \in S$, since S is an orthodox semigroup, $b'a'$ and $bb'a'$ are inverses of ab and abb', respectively, so that $(ab)^\dagger = abb'a' = (abb')(bb'a')\mathcal{R}(ab^\dagger)^\dagger$, and as each $\mathcal{R}$-class of S contains exactly one idempotent, $(ab)^\dagger = (ab^\dagger)^\dagger$.
(RI4) By $E(S)$ is a left regular band, we have

$$\begin{aligned}(ab^\dagger a')(ab^\dagger a') &= a\cdot b^\dagger(a'a)b^\dagger\cdot a' = a\cdot b^\dagger(a'a)\cdot a' = ab^\dagger a'\\ ab^\dagger &= a\cdot(a'a)b^\dagger = a\cdot(a'a)b^\dagger(a'a) = ab^\dagger a'\cdot a\\ ab^\dagger a' &= ab^\dagger\cdot a'.\end{aligned}$$

The second equality and the third one derive that $ab^\dagger a'\mathcal{R}ab^\dagger$. But each $\mathcal{R}$-class of S contains a unique idempotent, so $ab^\dagger a' = (ab^\dagger)^\dagger$. Therefore, $ab^\dagger = (ab^\dagger)^\dagger a$.

(RI5) Notice that $b^\dagger \mathcal{R} b$ and since $\mathcal{R}$ is a left congruence, we observe that

$$a^\dagger b^\dagger \mathcal{R} a^\dagger b \mathcal{R} (a^\dagger b)^\dagger.$$

But each $\mathcal{R}$-class of S contains exactly one idempotent, now $(a^\dagger b)^\dagger = a^\dagger b^\dagger$.

Define

$$^+: \ S \to S; \ a \mapsto a^\dagger.$$

By the foregoing proof, $(S, \cdot, +)$ is a glrac semigroup whose projection set is $E(S)$ and in which for any $a \in S$, $a^+ \mathcal{R} a$.

The Green's $*$-relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are generalizations of usual Green's $\mathcal{R}$ and $\mathcal{L}$, respectively. For elements a and b of S, $(a, b) \in \mathcal{R}^*$ ($\mathcal{L}^*$) if and only if $(a, b) \in \mathcal{R}$ ($\mathcal{L}$) in some oversemigroup of S. Equivalently, $(a, b) \in \mathcal{R}^*$ if and only if for any $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb.$$

A semigroup S is a *left GC-lpp semigroup* [17] if

(i) the set $E(S)$ of idempotents of S is a left regular band;
(ii) each $\mathcal{R}^*$-class of S contains uniquely one idempotent;
(iii) for any $a \in S$ and $e \in E(S)$, $ae = (ae)^+ a$, where a^+ is the unique idempotent related to a by $\mathcal{R}^*$.

Dually, right GC-rpp semigroup can be defined. By definition, any left GC-lpp semigroup can be viewed as a glrac semigroup S with unary operation

$$^+: \ S \to S; \ a \mapsto a^+$$

and in which $a \mathcal{R}^* a^+$ for any $a \in S$. Indeed,

(1) Because every a^+ is an idempotent and $a^+ \mathcal{R}^* (a^+)^+$, we get $a^+ = (a^+)^+$. And, by $a^+ a^+ = 1 \cdot a^+$ and since $a \mathcal{R}^* a^+$, we have $a^+ a = a$;
(2) Notice that $\mathcal{R}^*$ is a left congruence, we observe that $a^+ b^+ \mathcal{R}^* a^+ b \mathcal{R}^* (a^+ b)^+$, so that $a^+ b^+ = (a^+ b)^+$. Also, $ab \mathcal{R}^* ab^+$, so that $(ab)^+ = (ab^+)^+$;
(3) By definition, $E(S)$ is a left regular band. It follows that $E(S)$ satisfies the identity: $xy = xyx$. This shows that $a^+ b^+ = a^+ b^+ a^+$ for all $a, b \in S$;
(4) Consider that b^+ is an idempotent, by definition, we always have $ab^+ = (ab^+)^+ a$.

Consequently, $(S, \cdot, ^+)$ is a glrac semigroup.

Definition 2 (i) A semigroup S is said to be a *left reduced semigroup* if S is a glrac semigroup in which $|P(S)| = 1$. Dually, right reduced semigroup is defined.
(ii) A semigroup S is said to be a *reduced semigroup* if S is both a left reduced semigroup and a right reduced semigroup.
(iii) A glrac semigroup S is called a *glrac monoid* if $(S, \cdot)$ is a monoid with identity 1 and $1^+ = 1$.

By definition, we have the following observations:

(i) Any left reduced semigroup is a left restriction semigroup.
(ii) Any reduced semigroup is a (glrac) monoid.
(iii) Any monoid is a reduced monoid with a suitable unary operation; for, if T is a monoid with identity 1, a routine check that $(T, \cdot, ^+, ^*)$ is a reduced semigroup with unary operations:

$$^+ : \; S \to S; \; a \mapsto 1$$

and

$$^* : \; S \to S; \; x \mapsto 1.$$

So, in what follows, we shall view a monoid as a reduced semigroup in the above sense.

Proposition 1 *Let S be a glrac semigroup. If S is reduced, then S is a glrac monoid.*

Proof Because $P(S)$ is a singleton, we may let $P(S) = \{e\}$, and so $e = a^+$ for any $a \in S$. It follows that $a = a^+a = ea$. This shows that e is a left identity of S. On the other hand, since $ae = (ae)^+a = ea = a$, we know that e is a right identity of S. Consequently, S is a glrac monoid since $e^+ = e$. □

Among glrac semigroups, the notions of subalgebra, homomorphism, congruence and factor algebra are understood in type (2, 1), which is emphasized by using the expressions (2, 1)-subsemigroup, (2, 1)-morphism, (2, 1)-congruence and (2, 1)-factor semigroup, respectively.

Let S be a glrac semigroup and ρ a (2, 1)-congruence on S. On S/ρ, define

$$^+ : S/\rho \to S/\rho; \; x\rho \mapsto x^+\rho.$$

It is routine to check that $(S/\rho, \cdot, +)$ is a glrac semigroup, called the *quotient of S over* ρ and still denoted by S/ρ.

Definition 3 Let S be a glrac semigroup and ρ a (2, 1)-congruence on S. Then, ρ is said to be a *reduced* (2, 1)-*congruence* on S if S/ρ is left reduced.

Proposition 2 *Let S be a glrac semigroup. Then, the relation*

$$\sigma_S = \{(a, b) \in S \times S : (\exists f \in P(S)) \; fa = fb\}$$

is the smallest reduced (2, 1)*-congruence on S.*

Proof We first show that σ_S is an equivalence relation on S. Clearly, σ_S is reflexive and symmetric. To prove the transitivity of σ_S, we let $ex = ey$ and $fy = fz$, where $e, f \in P(S)$. Then, since $P(S)$ is a left regular band, we have

$$(ef)x = ef(ex) = ef(ey) = efy = (ef)z,$$

where $ef \in P(S)$, and so $x\sigma_S z$.

If $x\sigma_S y$ and $z \in S$, then it is obvious that $xz\ \sigma_S\ yz$. If $ex = ey$ for some $e \in P(S)$, then $(ze)^+zx = zex = zey = (ze)^+zy$ and $zx\sigma_S zy$. So, σ_S is a congruence on S.

If $a, b \in S$, then $a^+\sigma_S b^+$ since $(a^+b^+)a^+ = (a^+b^+)b^+$. This means that $|P(S/\sigma_S)| = 1$. Therefore, σ_S is a reduced (2, 1)-congruence on S.

Finally, we suppose that ρ is a reduced (2, 1)-congruence on S. Then, $|P(S/\rho)| = 1$ and $x^+\rho = y^+\rho$ for any $x, y \in S$. If $x\sigma_S y$, then we have $ex = ey$ for some $e \in P(S)$. This implies that

$$x\rho = (x^+\rho)(x\rho) = (e\rho)(x\rho) = (ex)\rho = (ey)\rho = (e\rho)(y\rho) = (y^+\rho)(y\rho) = y\rho.$$

Hence, $\sigma_S \subseteq \rho$, and whence σ_S is the smallest reduced (2, 1)-congruence on S. □

Let V be a semigroup and let U be a monoid with 1. Denote by $End(V)$ the monoid of endomorphisms of V. Let

$$\phi : U \to End(V);\ v \mapsto \phi_v$$

be a monoid homomorphism. In what follows, we write $a\phi_v$ as $v \cdot a$. Notice that ϕ_1 is the identity mapping on V. So, $1 \cdot a = a$. On the set $V \times U$, define

$$(x, u) \circ (y, v) = (x(u \cdot y), uv).$$

It may be checked that $(V \times U, \circ)$ is a semigroup, called the *semidirect product* of V and U over ϕ and denoted by $V \ltimes_\phi U$.

Proposition 3 *Let U be a left reduced monoid with identity* 1 *and Y a left regular band. If ϕ is a monoid homomorphism of U into $End(Y)$, then $Y \ltimes_\phi U$ is a glrac semigroup with unary operator*

$$^+ : Y \ltimes_\phi U \to Y \ltimes_\phi U;\ \ (y, u) \mapsto (y, 1).$$

Proof Let $(x, u), (y, v) \in Y \ltimes_\phi U$. We have

(i) $(x, u)^+ \circ (x, u) = (x, 1) \circ (x, u) = (x(1 \cdot x), u) = (xx, u) = (x, u)$.
(ii) $(x, u)^+ \circ (y, v)^+ \circ (x, u)^+ = (x, 1) \circ (y, 1) \circ (x, 1) = (xyx, 1) = (xy, 1) = (x, 1) \circ (y, 1) = (x, u)^+ \circ (y, v)^+$.
(iii) $((x, u) \circ (y, v))^+ = (x(u \cdot y), uv)^+ = (x(u \cdot y), 1) = ((x, u) \circ (y, 1))^+ = ((x, u) \circ (y, v)^+)^+$.
(iv) $(x, u) \circ (y, v)^+ = (x, u) \circ (y, 1) = (x(u \cdot y), u) = (x(u \cdot y)x, u) = (x(u \cdot y), 1) \circ (x, u) = (x(u \cdot y), u)^+ \circ (x, u) = ((x, u) \circ (y, v)^+)^+ \circ (x, u)$.
(v) $(x, u)^+ = (x, 1) = (x, 1)^+ = ((x, u)^+)^+$.
(vi) $((x, u)^+ \circ (y, v))^+ = ((x, 1) \circ (y, v))^+ = (xy, v)^+ = (xy, 1) = (x(1 \cdot y), 1) = (x, 1) \circ (y, 1) = (x, u)^+ \circ (y, v)^+$.

By definition, $Y \ltimes_\phi U$ is a glrac semigroup. □

3 Permissible Subsets

In this section, we introduce the notion of permissible subsets in a glrac semigroup and establish some properties of permissible subsets.

To begin with, we define permissible subsets.

Definition 4 Let S be a glrac semigroup. A nonempty subset P of S is said to be *permissible* if

(PS1) P is a subset of some σ_S-class of S, where σ_S has the same meaning as in Proposition 2;
(PS2) for any $a \in S$ and $b \in P$, we have $a^+b, ba^+ \in P$; and
(PS3) for any $a, b \in P$, if $a^+ = b^+$, then $a = b$.

Corollary 2 *Let S be a glrac semigroup. Then,*

(i) *$P(S)$ is a permissible subset of S.*
(ii) *The set $(a] = \{x \in S : x = ea, e \in P(S)\}$ is a permissible subset of S.*
(iii) *Let P be a permissible subset of S. If $a, b \in P$, then $a^+b = a^+b^+a$.*

Proof (i) It is obvious.

(ii) If $x \in (a]$, then $x = ea$ for some $e \in P(S)$, so that $x \in a\sigma_S$ since $ex = ea$, it follows that $(a] \subseteq a\sigma_S$. For any $s \in S$ and $x = ea$ with $e \in P(S)$, we have

$$s^+x = s^+ea \in (a] \text{ and } xs^+ = eas^+ = e(as)^+a \in (a],$$

yielding $(PS2)$.

Let $x, y \in (a]$ and $x^+ = y^+$, then there exist $e, f \in P(S)$ such that $x = ea$ and $y = fa$, hence

$$ea^+ = (ea)^+ = x^+ = y^+ = (fa)^+ = fa^+.$$

So,

$$x = ea = (ea^+)a = (fa^+)a = fa = y.$$

It results (PS3). Therefore, $(a]$ is a permissible subset of S.

(iii) Note that

$$(a^+b)^+ = a^+b^+ = a^+b^+a^+ = (a^+b^+a)^+$$

and $a^+b, a^+b^+a \in P$, we can observe that $a^+b = a^+b^+a$ (by (PS3)). □

Let S be a glrac semigroup. If A and B are subsets of S, then we write AB to mean $\{ab : a \in A, b \in B\}$. Obviously, $(AB)C = A(BC)$. As usual, AB is called the set multiplication of A and B. We consider the set multiplication on $C(S) = \{P \subseteq S : P \text{ is a permissible subset of } S\}$.

Lemma 1 *If S is a glrac semigroup, then $C(S)$ is a semigroup with left identity $P(S)$.*

Proof We need only to verify that $C(S)$ is closed under set multiplication and $P(S)$ is a left identity of $C(S)$.

Note that σ_S is a (2, 1)-congruence on S. To see that $C(S)$ is closed under set multiplication, it suffices to verify that AB satisfies Condition $(PS3)$, for any $A, B \in C(S)$. Indeed, if $a, b \in A$, $x, y \in B$ and $(ax)^+ = (by)^+$, then $(ax^+)^+ = (by^+)^+$. Together with $ax^+, by^+ \in A$, this shows that $ax^+ = by^+$ by $(PS3)$. It follows that $ax = ax^+x = by^+x$. Notice that

$$(y^+x)^+ = y^+x^+ = y^+x^+y^+ = (y^+x^+y)^+$$

and $y^+x, y^+x^+y \in B$, (PS3) results that $y^+x = y^+x^+y$. Therefore,

$$ax = by^+x = by^+x^+y = (by^+x^+)^+by = (ax)^+(by^+x^+)^+(by)$$

and $(ax)^+ = (ax)^+(by^+x^+)^+(by)^+$. Also, we have

$$ax = (ax)^+(by^+x^+)^+(by)^+ \cdot by = (ax)^+(by) = (by)^+(by) = by.$$

So, AB is a permissible subset of S.

Also, by $(PS2)$, we have that $P(S)A \subseteq A$. Notice that $a^+a = a$. We have $A \subseteq P(S)A$. So, $P(S)A = A$ and $P(S)$ is a left identity of $C(S)$. □

Lemma 2 *Let S be a glrac semigroup and $A \in C(S)$. Then, $A^+ := \{a^+ : a \in A\}$ is a permissible subset of S, and $(A^+)^2 = A^+$.*

Proof By Proposition 2, $\sigma_e \supseteq P(S)$ for $e \in P(S)$. Because $A^+ \subseteq P(S)$, A^+ is a subset of a σ_S-class of S. Notice that $ea^+ = (ea)^+$ and $a^+e = a^+ea^+ = (a^+ea)^+$. So, $P(S)A^+, A^+P(S) \subseteq A^+$. For $a, b \in A$, if $(a^+)^+ = (b^+)^+$, then we have $a^+ = (a^+)^+ = (b^+)^+ = b^+$, hence A^+ satisfies Condition $(PS3)$. Therefore, $A^+ \in C(S)$.

Notice that $a^+a^+ = (a^+a)^+ = a^+$, we have $A^+ \subseteq (A^+)^2$. But $A^+A^+ \subseteq P(S)A^+ \subseteq A^+$, now $A^+ = (A^+)^2$. □

Theorem 1 *Let S be a glrac semigroup. On $C(S)$, define a unary operation as follows:*

$$^+ : \ C(S) \to C(S); \ \ A \mapsto A^+.$$

The semigroup $C(S) = (C(S), \cdot, +)$ is a left restriction semigroup.

Proof Let $A, B \in C(S)$.

(i) $A^+A = A$: By $a = a^+a$, it is clear that $A \subseteq A^+A$. Notice that $A^+A \subseteq P(S)A \subseteq A$. Hence, $A^+A = A$.
(ii) $A^+B^+ = B^+A^+$: By $(PS2)$, $A^+B^+ \subseteq P(S)B^+ \subseteq B^+$ and similarly, $A^+B^+ \subseteq A^+$. It follows that $A^+B^+ \subseteq A^+ \sqcap B^+$. Obviously, $A^+ \sqcap B^+ \subseteq (A^+ \sqcap B^+)^2 \subseteq A^+B^+$. Thus, $A^+B^+ = A^+ \sqcap B^+$. So, $A^+B^+ = A^+ \sqcap B^+ = B^+A^+$.
(iii) $(AB)^+ = (AB^+)^+$: is straightforward by the same condition in S.

(iv) $AB^+ = (AB^+)^+A$: For $a, c \in A, b \in B$, since $ab^+ = (ab^+)^+a$, we have $AB^+ \subseteq (AB^+)^+A$. On the other hand, by Corollary 2, we have $a^+c = a^+c^+a$ and further

$$\begin{aligned}(ab^+)^+c &= (a^+ab^+)^+c = a^+(ab^+)^+c \\ &= a^+(ab^+)^+ \cdot a^+c = a^+(ab^+)^+ \cdot a^+c^+a \\ &= a^+(ab^+)^+a^+c^+ \cdot (ab^+)^+a = a^+(ab^+)^+a^+c^+ \cdot ab^+ \\ &= a^+(ab^+)^+a^+c^+a \cdot b^+ \\ &\in AB^+\end{aligned}$$

so that $(AB^+)^+A \subseteq AB^+$. Therefore, $AB^+ = (AB^+)^+A$.

(v) $(A^+)^+ = A^+$: is straightforward by the same condition in S.

(vi) $A^+B^+ = (A^+B)^+$: is immediate by the same condition in S.

Consequently, $C(S)$ is a left restriction semigroup. □

Corollary 3 *Let S be a glrac monoid. Then, $C(S)$ is a left restriction monoid with identity $P(S)$.*

Proof By Lemma 1, it suffices to verify that $AP(S) = A$ for any $A \in C(S)$. Indeed, by definition, $AP(S) \subseteq A$, and the reverse inclusion follows from that $a = a1 \in AP(S)$ for any $a \in A$. □

For a glrac semigroup S, we denote

$$\mathbb{R}_{a^+} = \{x \in S : x^+ = a^+\}\ (a \in S).$$

If, in addition, S is a monoid with identity 1, then $\mathbb{R}_1$ is a left reduced monoid; for, for any $a, b \in \mathbb{R}_1$, we know $a^+ = 1 = b^+$, so that $(ab)^+ = (ab^+)^+ = a^+ = 1$, it follows that $ab \in \mathbb{R}_1$, thus $\mathbb{R}_1$ is a left reduced monoid.

Proposition 4 *Let S be a glrac monoid. Then,*

$$\mathbb{R}_{P(S)} = \{(a] \in C(S) : a \in \mathbb{R}_1\}.$$

Proof If $a \in \mathbb{R}_1$, then $a^+ = 1$ and $(a]^+ = (1] = P(S)$. It follows that $(a] \in \mathbb{R}_{P(S)}$. Conversely, if $A \in \mathbb{R}_{P(S)}$, then $A^+ = P(S)$. Notice that $P(S) = (1]$, we have $a \in A$ such that $a^+ = 1$. So, $(a] \subseteq A$. On the other hand, for any $x \in A$, we have

$$x^+ = x^+1 = x^+a^+ = (x^+a)^+,$$

further by $(PS3)$, $x = x^+a$, therefore $x \in (a]$. Now, $A = (a]$. We complete the proof. □

Proposition 5 *Let S be a glrac semigroup. Then,*

$$\mathfrak{R}_S := \{A \in \mathbb{R}_{P(S)} : AP(S) = A\}$$

is a left reduced $(2, 1)$*-subsemigroup of $C(S)$ with identity $P(S)$.*

Proof We first verify that $\mathfrak{R}_S$ is closed under set multiplication. To the end, we let $A, B \in \mathfrak{R}_S$, then by Lemma 1,

$$AB = AP(S) \cdot BP(S) = A \cdot P(S)B \cdot P(S) = AB \cdot P(S),$$

as required. Therefore, $\mathfrak{R}_S$ is a subsemigroup of $C(S)$. Furthermore, because $A^+ = P(S)$ for any $A \in \mathfrak{R}_S$, we know that $P(S)A = A = AP(S)$. It follows that $P(S)$ is the identity of $\mathfrak{R}_S$. Again by $A^+ = P(S)$, we have $P(\mathfrak{R}_S) = \{P(S)\}$. Consequently, $\mathfrak{R}_S$ is a left reduced (2, 1)-subsemigroup of $C(S)$. We complete the proof. □

Let T be a restriction semigroup. Recall that a nonempty subset U of T is a permissible subset of T (in sense of Gomes and Szendrei in [13]) if, for every $a, b \in U$, the following conditions are satisfied:

(i) $a \leq b \in U \Rightarrow a \in U$, where $a \leq b$ if and only if $a = a^+b$;
(ii) $a, b \in U \Rightarrow a^+b = b^+a$;
(iii) $a, b \in U \Rightarrow ab^* = ba^*$.

We shall find that our permissible subsets coincide with permissible subsets in sense of Gomes and Szendrei for a restriction semigroup. The following remark can illustrate this version.

Remark 1 Let S be a left restriction semigroup. Then, $A \in C(S)$ if and only if for any $a, b \in A$,

(P1) $a \leq b \in A \Rightarrow a \in A$, where $a \leq b$ if and only if $a = a^+b$;
(P2) $a, b \in A \Rightarrow a^+b = b^+a$.

Indeed, assume that A satisfies Conditions $(P1)$ and $(P2)$. For $x, y \in A$,

(1) By hypothesis, we have $x^+y = y^+x$, so that

$$(x^+y^+)y = x^+y = y^+x = y^+x^+x = (x^+y^+)x.$$

This means that $x\sigma_S y$, therefore A is contained in some σ_S-class of S.
(2) If $e \in P(S)$, then $ex \leq x$ and so $ex \in A$. Also, $xe = (xe)^+x$, accordingly $xf \leq x$, so that $xf \in A$. It results $(PS2)$.
(3) Assume now that $x^+ = y^+$. By $(P2)$, $x^+y = y^+x$. Obviously,

$$y = y^+y = x^+y = y^+x = x^+x = x.$$

So, A satisfies $(PS3)$.

We have verified that A is a permissible subset of S. Conversely, assume that $B \in C(S)$. Let $a, b \in B$ and $x \in S$. By $(PS2)$, it follows that B satisfies $(P1)$. Notice that $(a^+b)^+ = a^+b^+ = b^+a^+ = (b^+a)^+$ and $a^+b, b^+a \in B$. We observe that $a^+b = b^+a$ and whence B satisfies $(P2)$.

4 Almost Factorizable Glrac Semigroups

To begin with, we give two notions.

Definition 5 (i) A glrac monoid S is said to be *(uniquely) factorizable* if for any $s \in S$, there exist (uniquely) $e \in P(S), a \in \mathbb{R}_1$ such that $s = ea$.
(ii) A glrac semigroup S is said to be *(uniquely) almost factorizable* if for every $a \in S$, there exists (uniquely) $A \in \mathfrak{R}_S$ such that $a \in A$.

The next propositions give the relationships between the above two notions.

Proposition 6 *Let S be a glrac monoid with identity* 1. *Then, S is (uniquely) factorizable if and only if S is (uniquely) almost factorizable.*

Proof (6.1) If S is factorizable, then for any $a \in S$, there exist $e \in P(S), r \in \mathbb{R}_1$ such that $a = er$. It follows that $a \in (r] \in \mathbb{R}_{P(S)}$. But by Corollary 3, $\mathbb{R}_{P(S)} = \mathfrak{R}_S$, now S is almost factorizable. Conversely, support that S is almost factorizable. By Proposition 4, for any $A \in \mathbb{R}_{P(S)}$, there exists $a \in \mathbb{R}_1$ such that $A = (a]$. By definition, for any $x \in S$, there exists $a \in \mathbb{R}_1$ such that $x \in (a]$. So $x = fa$ for some $f \in P(S)$. Therefore, S is factorizable.

(6.2) With notations in (6.1), if S is uniquely factorizable, then by (6.1), S is factorizable. Now, let $A \in \mathfrak{R}_S$ such that $a \in A$, then as $\mathbb{R}_{P(S)} = \mathfrak{R}_S$, we see that $A = (s]$ for some $s \in \mathbb{R}_1$. It follows that $a = fs$ for some $f \in P(S)$. By S is uniquely factorizable, we get $r = s, e = f$. This means that $A = (s] = (r]$. Therefore, S is uniquely almost factorizable. Conversely, support that S is uniquely almost factorizable, then by (6.1), S is almost factorizable. We let $t \in \mathbb{R}_1, h \in P(S)$ such that $a = ht$, then $a \in (r], a \in (t]$. But $(r], (t] \in \mathbb{R}_{P(S)} = \mathfrak{R}_S$, now $(r] = (t]$ by hypothesis that S is uniquely almost factorizable. By Corollary 2, $(r]$ is a permissible subset of S. Together with $s^+ = 1 = t^+$ and by (PS3), we know that $s = t$. Further,

$$e = e1 = er^+ = (er)^+ = (ht)^+ = ht^+ = h1 = h.$$

So, S is uniquely factorizable. □

Proposition 7 *Let S be a factorizable glrac monoid with identity* 1. *If for any* $a \in S' := S\backslash\mathbb{R}_1$, *there exists* $e \in P(S)\backslash\{1\}$ *such that* $a = ae$, *then* S' *is almost factorizable.*

Proof We first prove that S' is a glrac semigroup. We claim: S' is a subsemigroup of S; for, if $a, b \in S'$, then $ab \notin \mathbb{R}_1$. Indeed, if $ab \in \mathbb{R}_1$, we have $1 = (ab)^+ = a^+(ab)^+$, so that $a^+ = 1$, contrary to $a \notin \mathbb{R}_1$. Clearly, S' is closed under $^+$-operation. Therefore, S' is a glrac (2, 1)-subsemigroup of S. Obviously, $P(S') = P(S)\backslash\{1\}$.

Let $r \in \mathbb{R}_1$, then it is easy to know that $(r] \in C(S)$ implies that $A_r = (r] \setminus \{r\} \in C(S')$. Obviously, $A_r^+ = P(S) \setminus \{1\} = P(S')$ and $A_r \in \mathbb{R}_{P(S')}$. Moreover, by Corollary 3, $P(S')A_r = A_r$. By hypothesis, for any $x \in A_r$, there exists $e \in P(S')$ such that $x = xe$. This shows that $A_r P(S') = A_r$. Therefore, $A_r \in \mathfrak{R}_{S'}$. Notice that S is factorizable, we observe that for any $a \in S'$, there is $e \in P(S), r \in \mathbb{R}_1$ such that $a = er \in (r]$. Thus, $a \in A_r$. Consequently, S' is almost factorizable. □

Example 2 Let U be a monoid with identity 1_U, and Y a left regular band with identity 1_Y. Let $\phi :\ u \mapsto \phi_u$ be a monoid homomorphism of U into $End(Y)$ such that every ϕ_u is a monoid endomorphism. Then, the semigroup $Y \ltimes_\phi U$ is a uniquely factorizable glrac monoid with unary operation:

$$^+ :\ Y \ltimes_\phi U \to Y \ltimes_\phi U;\ (y, u) \mapsto (y, 1_U).$$

Proof It is routine to check that $(1_Y, 1_U)$ is the identity of $Y \ltimes_\phi U$ and that $P(Y \ltimes_\phi U) = Y \times \{1_U\}$. So, $Y \ltimes_\phi U$ is a glrac monoid. For $(x, u) \in Y \ltimes_\phi U$, since ϕ is a monoid homomorphism, we have

$$(x, u) = (x(1_U \cdot 1_Y), u) = (x, 1_U) \circ (1_Y, u).$$

Also, we have $(1_Y, u) \in \mathbb{R}_{(1_Y, 1_U)}$ since $(1_Y, u)^+ = (1_Y, 1_U)$, it follows that

$$\{(1_Y, u) \in Y \ltimes_\phi U : u \in U\} \subseteq \mathbb{R}_{(1_Y, 1_U)},$$

and the converse inclusion is clear. Thus,

$$\mathbb{R}_{(1_Y, 1_U)} = \{(1_Y, u) \in Y \ltimes_\phi U : u \in U\}.$$

Therefore, $Y \ltimes_\phi U \subseteq P(Y \ltimes_\phi U) \cdot \mathbb{R}_{(1_Y, 1_U)}$. The uniqueness of $(x, 1_U)$ and $(1_Y, u)$ is obvious. Consequently, $Y \ltimes_\phi U$ is a uniquely factorizable glrac monoid. □

Example 3 Let U be a monoid with identity 1 and Y a left regular band. Let $\phi : u \to End(Y)$ be a homomorphism of U into $End(Y)$ such that

(UFA) for any $y \in Y$ and $u \in U$, there is $z \in Y$ such that $y(u \cdot z) = y$.

Then, the semigroup $Y \ltimes_\phi U$ is an almost factorizable glrac semigroup with unary operation:

$$^+ : Y \ltimes_\phi U \to Y \ltimes_\phi U;\ (y, u) \mapsto (y, 1).$$

Proof Let i be a symbol not in Y. On $Y^i := Y \sqcup \{i\}$, define a multiplication by: for any $x, y \in Y^i$,

$$x \circ y = \begin{cases} xy \text{ if } x, y \in Y; \\ x \text{ if } x \in Y, y = i; \\ y \text{ if } x = i, y \in Y; \\ i \text{ if } x = i = y, \end{cases}$$

where xy is the product of x and y in Y. It is easy to check that Y^i is a left regular band with identity i.

For any $u \in U$, we define φ_u as the monoid homomorphism of Y^i into itself in which the restriction of φ_u to Y is ϕ_u. Now, define

$$\varphi : U \to End(Y^i);\ u \mapsto \varphi_u.$$

By Example 2, $Y^i \ltimes_\varphi U$ is a uniquely factorizable glrac monoid with identity $(i, 1)$. Evidently, $\mathbb{R}_{(i,1)} = \{i\} \times U$ and $Y \ltimes_\phi U = (Y^i \ltimes_\varphi U) \setminus \mathbb{R}_{(i,1)}$.

Let $(y, u) \in Y \ltimes_\phi U$. By hypothesis, $y(u \cdot z) = y$ for some $z \in Y$. It follows that $(y, u) \circ (z, 1) = (y(u \cdot z), u) = (y, u)$. Notice that $(z, 1) \in P(Y \ltimes_\phi U)$. Now, by Proposition 7, $Y \ltimes_\phi U$ is almost factorizable. □

By definition, $V \times U = V \ltimes_\phi U$, where ϕ_u is the identity mapping of V onto itself, for all $u \in U$. In this case, if V is a left regular band and U is a monoid, then ϕ satisfies the condition (UFA), so that by Example 3, $V \times U$ is an almost factorizable glrac semigroup.

Let S and T be glrac semigroups. A $(2, 1)$-homomorphism ϕ from S into T is said to be *projection-separating* if for any $e, f \in P(S)$, $e\phi = f\phi$ implies that $e = f$. Moreover, the $(2, 1)$-congruence induced by a projection-separating $(2, 1)$-homomorphism is called to be *projection-separating*.

Proposition 8 *Any projection-separating* $(2, 1)$*-homomorphic image of an almost factorizable glrac semigroup is still an almost factorizable glrac semigroup.*

Proof Let S and T be glrac semigroup, and let ϕ be a projection-separating $(2, 1)$-homomorphism of S onto T. Then, $P(T) = P(S)\phi$. Indeed, for any $f \in P(T)$, there is $a \in S$ such that $f = a\phi$, it follows that $f = (a\phi)^+ = a^+\phi \in P(S)\phi$, whence $P(T) \subseteq P(S)\phi$. The reverse inclusion follows from the computation: $a^+\phi = (a\phi)^+ \in P(T)$, for $a \in S$. Therefore, $P(T) = P(S)\phi$. By Lemma 2, this shows that

$(*)$ *For any* $a, b \in S$, *if* $(a, b) \in \sigma_S$, *then* $(a\phi, b\phi) \in \sigma_T$.

We next prove that $C(S)\phi := \{A\phi : A \in C(S)\} \subseteq C(T)$. To the end, we let $A \in C(S)$. To show that $A\phi \in C(T)$, by $(*)$, it suffices to verify that for any $x, y \in A$, if

$$x^+\phi = (x\phi)^+ = (y\phi)^+ = y^+\phi,$$

then $x\phi = y\phi$. Indeed, since ϕ is projection-separating, $x^+\phi = y^+\phi$ implies that $x^+ = y^+$ and further by Definition 4 (PS3), $x = y$. Obviously, $x\phi = y\phi$. It follows that $A\phi \in C(T)$ and whence $C(S)\phi \subseteq C(T)$.

By definition, it is easy to see that $\mathfrak{R}_S\phi = \{A\phi : A \in \mathfrak{R}_S\} \subseteq \mathfrak{R}_T$. For any $a\phi \in T$ with $a \in S$, by S is almost factorizable, there is $A \in C(S)$ such that $a \in A$. It follows that $a\phi \in A\phi$. But $A\phi \in \mathfrak{R}_T$, so T is almost factorizable. □

Corollary 4 *Any projection-separating* $(2, 1)$*-homomorphic image of a factorizable glrac monoid is still a factorizable glrac monoid.*

Proof With notations in the proof of Proposition 8, we have $P(S)\phi = P(T)$. If 1 is the identity of S and $S = P(S) \cdot \mathbb{R}_1$, it is easy to check that 1ϕ is the identity of T and

$$T = S\phi = P(S)\phi \cdot \mathbb{R}_1\phi = P(T) \cdot \mathbb{R}_1\phi.$$

Because ϕ is a (2, 1)-homomorphism, we have that $(a\phi)^+ = a^+\phi$ for any $a \in S$. It follows that $\mathbb{R}_1\phi \subseteq \mathbb{R}_{1\phi}$. For any $a\phi \in T$ with $a \in S$, by S is factorizable, there are $r \in \mathbb{R}_1$, $e \in P(S)$ such that $a = (e\phi)(r\phi)$. But $r\phi \in \mathbb{R}_{1\phi}$, so T is factorizable. □

5 Weakly Semidirect Products and Construction

We first introduce the definition of weakly semidirect products.

Let U be a semigroup and V a monoid with 1. Denote by $End(U)$ the monoid of endomorphisms of U. Let

$$\eta : U \times V \to End(U); (u, v) \mapsto \eta_{u,v}$$

be a mapping. In what follows, we write $a\eta_{u,v}$ as $(u, v)^{\#}a$. Suppose that the following condition holds:

(WSD) For any $(u, x), (v, y) \in U \times V$, $\eta_{u,x}\eta_{v,y} = \eta_{u\cdot(u,x)^{\#}v,xy}$.
(ID) For any $u \in U$, $\eta_{u,1} = \lambda_u$, where

$$\lambda_u : U \to U; v \mapsto uv.$$

(I) For any $(u, x) \in U \times V$, $u \in Im(\eta_{u,x})$, where $Im(\eta_{u,x})$ is the image of $\eta_{u,x}$.

On the set $U \times V$, define

$$(u, x) \otimes (v, y) = (u \cdot (u, x)^{\#}v, xy).$$

Lemma 3 $(U \times V, \otimes)$ *is a semigroup.*

Proof Obviously, $\otimes$ is well defined and closed. For any $(u, x), (v, y), (w, z) \in U \times V$, we have

$$\begin{aligned}((u, x) \otimes (v, y)) \otimes (w, z) &= (u \cdot (u, x)^{\#}v, xy) \otimes (w, z)\\ &= (u \cdot (u, x)^{\#}v \cdot (u \cdot (u, x)^{\#}v, xy)^{\#}w, xyz).\end{aligned}$$

By (WSD), we have

$$\begin{aligned}(u, x)^{\#}((v, y)^{\#}w) = \eta_{u,x}\eta_{v,y}(w) &= \eta_{u\cdot(u,x)^{\#}v,xy}(w)\\ &= (u \cdot (u, x)^{\#}v, xy)^{\#}w,\end{aligned}$$

so that

$$((u, x) \otimes (v, y)) \otimes (w, z) = (u \cdot (u, x)^{\#}v \cdot (u, x)^{\#}((v, y)^{\#}w), xyz).$$

Thus,

$$\begin{aligned}(u, x) \otimes ((v, y) \otimes (w, z)) &= (u, x) \otimes (v \cdot (v, y)^{\#} w, yz) \\ &= (u \cdot (u, x)^{\#} (v \cdot (v, y)^{\#} w), xyz) \\ &= (u \cdot (u, x)^{\#} v \cdot (u, x)^{\#} ((v, y)^{\#} w), xyz) \\ &= ((u, x) \otimes (v, y)) \otimes (w, z).\end{aligned}$$

Consequently, $(U \times V, \otimes)$ is a semigroup. □

Definition 6 The above semigroup $(U \times V, \otimes)$ is called the *weakly semidirect product* of V and U via η and denoted by $U \ltimes_\eta^w V$.

Proposition 9 *If U is a left regular band and V is a left reduced monoid with identity* 1*, then $U \ltimes_\eta^w V$ is a uniquely almost factorizable glrac semigroup with unary operator*

$$^+ : U \times V \to U \times V;\ (u, x) \mapsto (u, 1).$$

Proof Let $(u, x), (v, y) \in U \ltimes_\eta^w V$. We have

(i) $(u, x)^+ \otimes (u, x) = (u, 1) \otimes (u, x) = (u \cdot (u, 1)^{\#} u, x) = (uuu, x) = (u, x)$.
(ii) $(u, x)^+ \otimes (v, y)^+ \otimes (u, x)^+ = (u, 1) \otimes (v, 1) \otimes (u, 1) = (uuv, 1) \otimes (u, 1) = (uvuvu, 1) = (uv, 1) = (u, 1) \otimes (v, 1) = (u, x)^+ \otimes (v, y)^+$.
(iii) $((u, x) \otimes (v, y))^+ = (u \cdot (u, x)^{\#} v, xy)^+ = (u \cdot (u, x)^{\#} v, 1) = ((u, x) \otimes (v, 1))^+ = ((u, x) \otimes (v, y)^+)^+$.
(iv) $(u, x) \otimes (v, y)^+ = (u, x) \otimes (v, 1) = (u \cdot (u, x)^{\#} v, x) = (u \cdot (u, x)^{\#} v \cdot u, x) = (u \cdot (u, x)^{\#} v, 1) \otimes (u, x) = (u \cdot (u, x)^{\#} v, x)^+ \otimes (u, x) = ((u, x) \otimes (v, y)^+)^+ \otimes (u, x)$.
(v) $((u, x)^+)^+ = (u, 1)^+ = (u, 1) = (u, x)^+$.
(vi) $((u, x)^+ \otimes (v, y))^+ = ((u, 1) \otimes (v, y))^+ = (u \cdot (u, 1)^{\#} v, y)^+ = (uuv, 1) = (u, 1) \otimes (v, 1) = (u, x)^+ \otimes (v, y)^+$.

Consequently, $U \ltimes_\eta^w V$ is a glrac semigroup. It is routine to check that $P(U \ltimes_\eta^w V) = U \times \{1\}$.

We next show that $U \times \{x\}$ is a permissible subset of $U \ltimes_\eta^w V$, for any $x \in V$. Let $u_0 \in U$ and $(u, x) \in U \times \{x\}$, then $(uu_0, 1) \in P(U \ltimes_\eta^w V)$ and

$$(uu_0, 1) \otimes (u, x) = (uu_0uu_0u, x) = (uu_0, 1) \otimes (u_0, x).$$

This shows that $U \times \{x\} \subseteq (u_0, x)\sigma_{U \ltimes_\eta^w V}$. It follows that $U \times \{x\}$ satisfied (PS1). A routine computation shows that $U \times \{x\}$ satisfies (PS2).

Let $(e, x), (f, x) \in U \times \{x\}$. If $(e, 1) = (e, x)^+ = (f, x)^+ = (f, 1)$, then $e = f$ and further $(e, x) = (f, x)$. It results (PS3).

Consequently, $U \times \{x\}$ is a permissible subset of $U \ltimes_\eta^w V$.

By the foregoing proof, $P(U \ltimes_\eta^w V) = U \times \{1\}$ and $\mathbb{R}_{P(U \ltimes_\eta^w V)} = \{U \times \{v\} : v \in V\}$. For any $(u, x) \in U \times \{x\}$, by (I), we have $v \in U$ such that $(u, x)^{\#} v = u$. So, $(u, x) = (u, x) \otimes (v, 1)$. It follows that $U \times \{x\} \subseteq (U \times \{x\}) \otimes P(U \ltimes_\eta^w V)$. The reverse inclusion is evident. Therefore, $\mathfrak{R}_{U \ltimes_\eta^w V} = \{U \times \{x\} : x \in V\}$.

For any $(v, y) \in U \ltimes_\eta^w V$, we know that $U \times \{y\} \in \mathfrak{R}_{U \ltimes_\eta^w V}$ and $(v, y) \in \mathfrak{R}_{U \ltimes_\eta^w V}$. If $(v, y) \in U \times \{z\}$ $(\in \mathfrak{R}_{U \ltimes_\eta^w V})$, then $y = z$ and so $U \times \{y\} = U \times \{z\}$. Therefore, $U \ltimes_\eta^w V$ is a uniquely almost factorizable glrac semigroup. □

We arrive now at the structure theorem of uniquely almost factorizable glrac semigroups.

Theorem 2 *If U is a left regular band and V is a left reduced monoid with identity* 1*, then $U \rtimes_\eta^w V$ is a uniquely almost factorizable glrac semigroup with unary operator*

$$^+ : U \times V \to U \times V;\ (u, x) \mapsto (u, 1).$$

Conversely, any uniquely almost factorizable glrac semigroup can be constructed in this way.

Proof By Proposition 9, we need only to verify the converse part. To the end, we let S be a uniquely almost factorizable glrac semigroup, then $P(S)$ is a left regular band and by Proposition 5, $\mathfrak{R}_S$ is a left reduced monoid.

For any $(u, A) \in P(S) \times \mathfrak{R}_S$, we know that A is a permissible subset of S and further by Definition 4, there exists uniquely $a \in A$ such that $a^+ = u$. We define

$$\zeta_{u,A} : P(S) \to P(S);\ v \mapsto (av)^+.$$

Then, $\zeta_{u,A}$ is a homomorphism. Indeed, if $x, y \in P(S)$, then

$$\begin{aligned}\zeta_{u,A}(xy) &= (axy)^+ = ((ax)^+ay)^+ = (ax)^+(ay)^+ \\ &= (\zeta_{u,A}x)(\zeta_{u,A}y)\end{aligned}$$

so that $\zeta_{u,A}$ is a homomorphism, as required.

We next prove that the mapping

$$P(S) \times \mathfrak{R}_S \to End(P(S));\ (u, A) \mapsto \zeta_{u,A}$$

satisfies Condition (WSD). For any $(u, A), (v, B) \in P(S) \times \mathfrak{R}_S$, if $t \in P(S)$, then

$$\zeta_{u,A}\zeta_{v,B}(t) = \zeta_{u,A}(bt)^+ = (a(bt)^+)^+ = (abt)^+,$$

where $b \in B$ and $b^+ = v$. Also,

$$\begin{aligned}\zeta_{u\cdot(u,A)^\# v, AB} &= \zeta_{u(av)^+, AB} = \zeta_{a^+(av)^+, AB} \\ &= \zeta_{(av)^+, AB} = \zeta_{(ab^+)^+, AB} \\ &= \zeta_{(ab)^+, AB}\end{aligned}$$

since $\zeta_{u,A}(v) = (av)^+$. Notice that

$$\begin{aligned} u \cdot (u, A)^{\#} v = a^+ \cdot (u, A)^{\#} b^+ &= a^+(ab^+)^+ \\ &= (a^+ab^+)^+ = (ab^+)^+ \\ &= (ab)^+, \end{aligned}$$

we observe that $\zeta_{u\cdot(u,A)^{\#}v,AB}(t) = (abt)^+ = \zeta_{(ab)^+,AB}(t)$. Therefore, $\zeta_{u,A}\zeta_{v,B} = \zeta_{u\cdot(u,A)^{\#}v,AB}$ and (WSD) is valid.

Because $AP(S) = A$, there exist $b \in A$ and $f \in P(S)$ such that $bf = a$. Hence, $af = a$ and $\zeta_{u,A}(f) = (af)^+ = a^+ = u$. Therefore, (I) is valid.

It remains to show that the mapping

$$\phi : S \to P(S) \ltimes^w_\zeta \mathfrak{R}_S; \;\; s \mapsto (s^+, P),$$

where $s \in P$, is a (2, 1)-isomorphism. By definition, ϕ is well-defined. For any $(u, P) \in P(S) \ltimes^w_\zeta \mathfrak{R}_S$, by definition, there is uniquely $a \in P$ such that $a^+ = u$. So $\phi(a) = (a^+, P) = (u, P)$, and whence ϕ is surjective. If $(s^+, P) = (t^+, Q)$ where $s \in P$ and $t \in Q$, then $s^+ = t^+, P = Q$. By (PS3), we have $s = t$. Then, ϕ is injective.

To see that ϕ is a homomorphism, we let $s, t \in S$ and further $P, Q \in \mathfrak{R}_S$ with $s \in P$ and $t \in Q$. Then,

$$\begin{aligned} \phi(s)\phi(t) = (s^+, P) \otimes (t^+, Q) &= (s^+(s^+, P)^{\#}t^+, PQ) \\ &= (s^+(st^+)^+, PQ) = ((st)^+, PQ) \\ &= \phi(st), \end{aligned}$$

as required.

Finally, we have

$$(\phi(s))^+ = (s^+, P)^+ = (s^+, P(S)) = \phi(s^+)$$

and ϕ is a (2, 1)-isomorphism. We complete the proof. □

Based on Theorem 2, we may give the structure theorem for uniquely factorizable glrac monoids.

Theorem 3 *Let U be a left regular band with identity e and V a left reduced monoid with identity 1. If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$, then $U \ltimes^w_\eta V$ is a uniquely factorizable glrac monoid with unary operator*

$$^+ : U \times V \to U \times V; \;\; (u, x) \mapsto (u, 1).$$

Conversely, any uniquely factorizable glrac monoid can be constructed in this way.

Proof For the direct part, by Proposition 6 and Theorem 2, we need only to verify that $(e, 1)$ is the identity of $U \ltimes_{\eta}^{w} V$. For any $(u, x) \in U \ltimes_{\eta}^{w} V$, we have

$$(u, x) \otimes (e, 1) = (u \cdot (u, x)^{\#} e, x) = (uu, x) = (u, x)$$

since $\eta_{u,x}(e) = u$. Also,

$$(e, 1) \otimes (u, x) = (e \cdot (e, 1)^{\#} u, 1 \cdot x) = (e \cdot eu, x) = (u, x).$$

Therefore, $(e, 1)$ is the identity of $U \ltimes_{\eta}^{w} V$.

To see the converse part, with notations in the proof of Theorem 2, it suffices to verify that $\zeta_{u,A}(e) = u$. Indeed, $\zeta_{u,A}(e) = (ae)^{+} = a^{+} = u$. We complete the proof.□

A $(2, 1)$-subsemigroup T of a glrac semigroup S is *full* if $P(S) \subseteq T$. In this case, it is obvious that $P(T) = P(S)$ and T is a glrac semigroup.

Based on Proposition 8 and Theorem 2, we may obtain the following theorem.

Theorem 4 *Let S be a glrac semigroup. Then, S is an almost factorizable glrac semigroup if and only if S is a projection-separating* $(2, 1)$*-homomorphic image of some full* $(2, 1)$*-subsemigroup of a weakly semidirect product of a left regular band and a left reduced monoid.*

Proof By Propositions 8 and 9 , we need only to verify the direct part. For the direct part, by Theorem 2, $P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S$ is a uniquely almost factorizable glrac semigroup. So, it suffices to verify that S is a projection-separating $(2, 1)$-homomorphic image of $P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S$.

Let $(u, A) \in P(S) \times \mathfrak{R}_S$ and since $A^{+} = P(S)$, we have uniquely $a \in A$ such that $a^{+} = u$. Define

$$\psi : P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S \to S;\ (u, A) \mapsto a.$$

Obviously, ψ is well defined. For any $s \in S$, by S is an almost factorizable glrac semigroup, there exists $A \in \mathfrak{R}_S$ such that $s \in A$. So by Definition 4, $\psi(s^{+}, A) = s$ and whence ψ is surjective.

For $(u, A), (v, B) \in P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S$ with $\psi(u, A) = a$, $\psi(v, B) = b$, we know that $b^{+} = v$ and $a^{+} = u$, accordingly

$$u(av)^{+} = a^{+}(av)^{+} = (av)^{+} = (ab^{+})^{+} = (ab)^{+}.$$

It follows that

$$\begin{aligned}\psi((u, A) \otimes (v, B)) &= \psi(u(u, A)^{\#} v, AB) = \psi(u(av)^{+}, AB)\\ &= \psi((ab)^{+}, AB) = ab\\ &= \psi(u, A) \cdot \psi(v, B).\end{aligned}$$

Also, $\psi((u, A)^{+}) = \psi(u, P(S)) = u = a^{+} = (\psi(u, A))^{+}$. So ψ is a $(2, 1)$-homomorphism.

For any $(u, P(S)), (v, P(S)) \in P(P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S)$, if $\psi(u, P(S)) = \psi(v, P(S))$, then by the foregoing proof, $a = b$ and so $u = a^+ = b^+ = v$. It follows that $(u, P(S)) = (v, P(S))$. Therefore, ψ is projection-separating. Notice that $P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S$ is a full (2, 1)-subsemigroup of itself. We complete the proof. □

Based on Corollary 4, and Theorems 3 and 4, we may obtain the following theorem.

Theorem 5 *Let S be a glrac monoid. Then, S is a factorizable glrac monoid if and only if S is a projection-separating* (2, 1)*-homomorphic image of some full* (2, 1)*-subsemigroup of a weakly semidirect product of a left regular band with identity and a left reduced monoid.*

Proof If S is a factorizable glrac monoid, then by Proposition 6, S is an almost factorizable semigroup. Further by Theorem 4, S is a projection-separating (2, 1)-homomorphic image of some full (2, 1)-subsemigroup of a weakly semidirect product of a left regular band with identity and a left reduced monoid. Conversely, if S is a projection-separating (2, 1)-homomorphic image of some full (2, 1)-subsemigroup of a weakly semidirect product of a left regular band with identity and a left reduced monoid, then by Theorem 4, S is an almost factorizable semigroup. But S is a monoid, so by Proposition 6, S is a factorizable monoid. □

6 Left Restriction Semigroups

Let us turn back to the weakly semidirect product $U \ltimes_{\eta}^{w} V$. By Proposition 9, it is easy to know that

- $P(U \ltimes_{\eta}^{w} V) = U \times \{1\}$, *isomorphic to* U.

So, $U \ltimes_{\eta}^{w} V$ is a left restriction semigroup if and only if U is a semilattice. Based on this observation, the following theorems are easy consequences of Theorems 2, 3, 4 and 5 , respectively.

Theorem 6 *If U is a semilattice and V is a left reduced monoid with identity* 1*, then $U \ltimes_{\eta}^{w} V$ is a uniquely almost factorizable left restriction semigroup with unary operator*

$$^+ : U \times V \to U \times V;\ (u, x) \mapsto (u, 1).$$

Conversely, any uniquely almost factorizable left restriction semigroup can be constructed in this way.

Theorem 7 *Let U be a semilattice with identity e and V a left reduced monoid with identity* 1*. If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$, then $U \ltimes_{\eta}^{w} V$ is a uniquely factorizable left restriction monoid with unary operator*

$$^+ : U \times V \to U \times V;\ (u, x) \mapsto (u, 1).$$

Conversely, any uniquely factorizable left restriction monoid can be constructed in this way.

Theorem 8 *Let S be a left restriction semigroup. Then, S is an almost factorizable left restriction semigroup if and only if S is a projection-separating* $(2, 1)$*-homomorphic image of some full* $(2, 1)$*-subsemigroup of a weakly semidirect product of a semilattice and a left reduced monoid.*

Theorem 9 *Let S be a left restriction monoid. Then, S is a factorizable left restriction monoid if and only if S is a projection-separating* $(2, 1)$*-homomorphic image of some full* $(2, 1)$*-subsemigroup of a weakly semidirect product of a semilattice with identity and a left reduced monoid.*

7 Restriction Semigroups

In this section, we consider restriction semigroups. Let S be a restriction semigroup. By Proposition 2, it is easy to see that

Observation 1 $P(S) \times P(S) \sqsubseteq \sigma_S$.

This shows that

Observation 2 σ_S *is a* $(2, 1, 1)$*-congruence on* S.

Based these observations, we know that S/σ_S is a reduced monoid. On the other hand, by Remark 1, $C(S)$ is a restriction monoid (see [28, Lemma 3.3]), hence

$$\mathfrak{R}_S = \{A \in C(S) : A^+ = P(S)\}.$$

So, we have the following observations:

Observation 3 *S is a left factorizable restriction semigroup in sense of Szandrei in [28] if and only if S is a factorizable glrac semigroup.*

Observation 4 *S is an almost left factorizable restriction semigroup in sense of Szandrei in [28] if and only if S is an almost factorizable glrac semigroup.*

We now arrive at the structure theorem of uniquely almost factorizable restriction semigroups.

Theorem 10 *Let U be a semilattice and V a reduced monoid with identity* 1*. If for any* $(u, x), (v, y) \in U \ltimes_\eta^w V$*, there exists an element* $(u, x)^\circ \in U$ *such that*

(RS1) $u \cdot (u, x)^\# (u, x)^\circ = u$;
(RS2) $(u \cdot (u, x)^\# v, x)^\circ = (u, x)^\circ v$ *for all* $v \in U$;
(RS3) $(u, x)^\circ v = v \cdot (v, y)^\# ((u, x)^\circ v, y)^\circ$;
(RS4) $(u \cdot (u, x)^\# v, xy)^\circ = ((u, x)^\circ v, y)^\circ$;
(RS5) $(u, 1)^\circ = u$ *for any* $u \in U$,

then $U \ltimes_\eta^w V$ is a uniquely almost left factorizable restriction semigroup with unary operators

$$^+ : U \times V \to U \times V;\ (u, x) \mapsto (u, 1)$$

and

$$^* : U \times V \to U \times V;\ (u, x) \mapsto ((u, x)^\circ, 1).$$

Conversely, any uniquely almost left factorizable restriction semigroup can be constructed in this way.

Proof *Necessity*. By the arguments before Theorem 6, $U \ltimes_\eta^w V$ is a left restriction semigroup. Compute

(i) $(u, x) \otimes (u, x)^* = (u, x) \otimes ((u, x)^\circ, 1) = (u \cdot (u, x)^\#(u, x)^\circ, x) = (u, x)$;

(ii) Indeed, we have

$$\begin{aligned}(u, x)^* \otimes (v, y)^* &= ((u, x)^\circ, 1) \otimes ((v, y)^\circ, 1) = ((u, x)^\circ (v, y)^\circ, 1)\\ &= ((v, y)^\circ (u, x)^\circ, 1) = ((v, y)^\circ, 1) \otimes ((u, x)^\circ, 1)\\ &= (v, y)^* \otimes (u, x)^*;\end{aligned}$$

(iii) Applying (RS2), we may obtain that

$$\begin{aligned}((u, x) \otimes (v, 1))^* &= (u \cdot (u, x)^\# v, x)^* = ((u \cdot (u, x)^\# v, x)^\circ, 1)\\ &= ((u, x)^\circ v, 1) = ((u, x)^\circ, 1) \otimes (v, 1)\\ &= (u, x)^* \otimes (v, 1);\end{aligned}$$

(iv) For any $(u, x), (v, y) \in U \ltimes_\eta^w V$, we have

$$\begin{aligned}(u, x)^* \otimes (v, y) &= ((u, x)^\circ, 1) \otimes (v, y) = ((u, x)^\circ \cdot ((u, x)^\circ, 1)^\# v, y)\\ &= ((u, x)^\circ (u, x)^\circ v, y) = ((u, x)^\circ v, y)\\ &= (v \cdot (v, y)^\#((u, x)^\circ v), y)^\circ, y) \text{ (by } (RS3))\\ &= (v, y) \otimes ((u, x)^\circ v, y)^\circ, 1)\\ &= (v, y) \otimes ((u, x)^\circ v, y)^*\\ &= (v, y) \otimes ((u, x)^* \otimes (v, y))^*.\end{aligned}$$

(v) By $(RS4)$, we have

$$\begin{aligned}((u, x) \otimes (v, y))^* &= (u \cdot (u, x)^\# v, xy)^* = ((u \cdot (u, x)^\# v, xy)^\circ, 1)\\ &= ((u, x)^\circ v, y)^\circ, 1) = ((u, x)^\circ v, y)^*\\ &= ((u, x)^\circ (u, x)^\circ v, y)^*\\ &= (((u, x)^\circ, 1) \otimes (v, y))^*\\ &= ((u, x)^* \otimes (v, y))^*.\end{aligned}$$

(vi) $(u, x)^* = ((u, x)^\circ, 1) = (((u, x)^\circ, 1)^\circ, 1) = ((u, x)^\circ, 1)^* = ((u, x)^*)^*$.

So, $U \ltimes_\eta^w V$ is a right restriction semigroup. Again together with

(A) $((u, x)^*)^+ = ((u, x)^\circ, 1)^+ = ((u, x)^\circ, 1) = (u, x)^*$;

(B) $((u, x)^+)^* = (u, 1)^* = ((u, 1)^\circ, 1) = (u, 1) = (u, x)^+$,

we know that $U \ltimes_\eta^w V$ is a restriction semigroup, and further by Observation 4, a uniquely almost left factorizable restriction semigroup.

Sufficiency. Assume that S is a uniquely almost left factorizable restriction semigroup. By Theorem 2, S is isomorphic to some $U \ltimes_\eta^w V$. For convenience, we shall identify S with $U \ltimes_\eta^w V$. So, $U \ltimes_\eta^w V$ is a restriction semigroup with

$$P(U \ltimes_\eta^w V) = \{(u, 1) : u \in U\}.$$

For any $(u, x), (v, y) \in U \ltimes_\eta^w V$, if $(u, x)\sigma_S(v, y)$, then there exists $w \in U$ such that $(w, 1) \otimes (u, x) = (w, 1) \otimes (v, y)$ and by comparing the second components, $x = y$. Conversely, if $x = y$, then

$$(uv, 1) \otimes (u, x) = (uvu, x) = (uv, y) = (uv, 1) \otimes (v, y),$$

and whence $(u, x)\sigma_S(v, y)$. We have now verified that $(u, x)\sigma_S(v, y)$ if and only if $x = y$. This shows that the mapping defined by: $(u, x)\sigma_S \mapsto x$ is a (2, 1, 1)-isomorphism of S/σ_S onto V. Notice that S/σ_S is a reduced monoid, we obtain that V is a reduced monoid.

To the end, for any $(u, x) \in U \ltimes_\eta^w V$ and $v \in U$, we let $(u, x)^* = ((u, x)^\circ, 1)$.

- Because $U \ltimes_\eta^w V$ is a restriction semigroup, we have

$$(u, x) = (u, x) \otimes (u, x)^* = (u, x) \otimes ((u, x)^\circ, 1) = (u \cdot (u, x)^\# (u, x)^\circ, x),$$

  so that $u = u \cdot (u, x)^\# (u, x)^\circ$, resulting (RS1).
- Let $v \in U$. We have $(v, 1) \in P(U \ltimes_\eta^w V)$ and so

$$\begin{aligned} ((u \cdot (u, x)^\# v, x)^\circ, 1) &= (u \cdot (u, x)^\# v, x)^* = ((u, x) \otimes (v, 1))^* \\ &= (u, x)^* \otimes (v, 1) = ((u, x)^\circ, 1) \otimes (v, 1) \\ &= ((u, x)^\circ \cdot ((u, x)^\circ, 1)^\# v, 1) \\ &= ((u, x)^\circ v, 1). \end{aligned}$$

  It follows that $(u \cdot (u, x)^\# v, x)^\circ = (u, x)^\circ v$, and whence (RS2) holds.
- Let $(v, y) \in U \ltimes_\eta^w V$. Then,

$$\begin{aligned} ((u, x)^\circ v, y) &= ((u, x)^\circ \cdot ((u, x)^\circ, 1)^\# v, y) = ((u, x)^\circ, 1) \otimes (v, y) \\ &= (u, x)^* \otimes (v, y) = (v, y) \otimes ((u, x)^* \otimes (v, y))^* \\ &= (v, y) \otimes ((u, x)^\circ v, y)^* \\ &= (v, y) \otimes (((u, x)^\circ v, y)^\circ, 1) \\ &= (v \cdot (v, y)^\# ((u, x)^\circ v, y)^\circ, y), \end{aligned}$$

so that $(u, x)^\circ v = v \cdot (v, y)^\#((u, x)^\circ v, y)^\circ$. So, $(RS3)$ holds.

- Compute

$$\begin{aligned}(((u, x)^\circ v, y)^\circ, 1) &= ((u, x)^\circ v, y)^* = ((u, x)^\circ \cdot ((u, x)^\circ, 1)^\# v, y)^* \\ &= ((u, x)^\circ, 1) \otimes (v, y))^* = ((u, x)^* \otimes (v, y))^* \\ &= ((u, x) \otimes (v, y))^* = (u \cdot (u, x)^\# v, xy)^* \\ &= ((u \cdot (u, x)^\# v, xy)^\circ, 1),\end{aligned}$$

so that $(u \cdot (u, x)^\# v, xy)^\circ = ((u, x)^\circ v, y)^\circ$. This means that (RS4) is true.

- Because $(u, 1) \in P(U \ltimes_\eta^w V)$, we have $(u, 1) = (u, 1)^* = ((u, 1)^\circ, 1)$. It follows that $u = (u, 1)^\circ$, resulting (RS5).

Notice that $P(U \ltimes_\eta^w V) = \{(u, 1) : u \in U\}$, we easily know that the mapping: $(u, 1) \to u$ is an isomorphism of $P(U \ltimes_\eta^w V)$ onto U. Therefore, U is a semilattice. We complete the proof. ◻

By the proof of Theorem 10, $U \ltimes_\eta^w V$ is a restriction semigroup whenever Conditions (RS1)-(RS5) are satisfied. We shall denote such a semigroup by $RS(U \ltimes_\eta^w V)$. On the other hand, by the proof of Theorem 3, $U \ltimes_\eta^w V$ is a monoid whenever $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$. So, the following theorem is immediate from Theorem 3 and Observation 3.

Theorem 11 *Let U be a semilattice with identity e and V a reduced monoid with identity* 1*. If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$, then $RS(U \ltimes_\eta^w V)$ is a uniquely left factorizable restriction monoid.*

Conversely, any uniquely left factorizable restriction monoid is (2, 1, 1)*-isomorphic to some $RS(U \ltimes_\eta^w V)$.*

Theorem 12 *Let S be a restriction semigroup. Then, S is an almost left factorizable restriction semigroup if and only if S is a projection-separating* (2, 1, 1)*-homomorphic image of some full* (2, 1, 1)*-subsemigroup of some $RS(U \ltimes_\eta^w V)$.*

Proof By Proposition 8, Observation 4 and Theorem 10, we need only to verify the necessity. By Theorem 4, S is a projection-separating (2, 1)-homomorphic image of $P(S) \ltimes_\zeta^w \mathfrak{R}_S$. On the other hand, $P(S)$ is a semilattice since S is a restriction semigroup. So, it suffices to verify that there exists an element $(u, A)^\circ \in P(S)$ such that $(RS1) - (RS5)$ are satisfied for any $(u, A) \in P(S) \ltimes_\zeta^w \mathfrak{R}_S$.

Let $(u, A)^\circ = a^*$, where a is the unique element of A with $a^+ = u$.

(i) $u \cdot (u, A)^\#(u, A)^\circ = u \cdot (u, A)^\# a^* = u \cdot (aa^*)^+ = u \cdot a^+ = u$. It results $(RS1)$.
(ii) for all $v \in P(S)$, we have

$$(u \cdot (u, A)^\# v, A)^\circ = (u \cdot (av)^+, A)^\circ = ((av)^+, A)^\circ = (av)^* = a^* v = (u, A)^\circ v,$$

and $(RS2)$ holds.

(iii) Notice that $(u, A)^\circ = a^*$, $(u, P(S))^\circ = u$ and since S is a restriction semigroup, we observe that

$$\begin{aligned} v \cdot (v, B)^{\#}((u, A)^\circ v, B)^\circ &= v \cdot (v, B)^{\#}(a^* v, B)^\circ = v \cdot (v, B)^{\#}(a^* b^+, B)^\circ \\ &= v \cdot (v, B)^{\#}(a^* b)^* = v(b(a^* b)^*)^+ = b^+(b(a^* b)^*)^+ \\ &= (b^+ b(a^* b)^*)^+ = (b(a^* b)^*)^+ = (a^* b)^+ \\ &= a^* b^+ = a^* v \\ &= (u, A)^\circ v \end{aligned}$$

and $(RS3)$ is satisfied.

(iv) For any $(v, B) \in P(S) \ltimes_\zeta^w \mathfrak{R}_S$,

$$\begin{aligned} (u \cdot (u, A)^{\#} v, AB)^\circ &= ((av)^+, AB)^\circ = ((ab^+)^+, AB)^\circ \\ &= ((ab)^+, AB)^\circ = (ab)^* = (a^* b)^* \\ &= ((a^* b)^+, B)^\circ \\ &= ((u, A)^\circ v, B)^\circ, \end{aligned}$$

where $b^+ = v$. This shows $(RS4)$.

(v) For any $u \in P(S)$, we know that u is the unique element in $P(S)$ such that $u^* = u$. Thus, $(u, P(S))^\circ = u$, resulting $(RS5)$.

Also, $\psi((u, A)^*) = \psi((u, A)^\circ, P(S)) = \psi(a^*, P(S)) = a^* = (\psi(u, A))^*$. Therefore, ψ is a $(2, 1, 1)$-homomorphism of $RS(P(S) \ltimes_\zeta^w \mathfrak{R}_S)$ onto S. Consequently, S is a projection-separating $(2, 1, 1)$-homomorphic image of $RS(P(S) \ltimes_\zeta^w \mathfrak{R}_S)$. □

Theorem 13 *Let S be a restriction monoid. Then, S is a left factorizable restriction monoid if and only if S is a projection-separating* $(2, 1, 1)$*-homomorphic image of some full* $(2, 1, 1)$*-subsemigroup of some* $RS(U \ltimes_\eta^w V)$.

Proof It is immediate from Proposition 6 and Theorem 12. The proof is similar as Theorem 5. □

8 Left GC-lpp Semigroups

In this section, we shall establish the structure theorems of uniquely factorizable left GC-lpp monoids and of uniquely almost factorizable left GC-lpp semigroups. Pointed out as in Sect. 2, a left GC-lpp semigroup is just a glrac semigroup S in which $a\mathcal{R}^*a^+$ for any $a \in S$. So, the following lemma is the key to establishing the structure of (uniquely) almost factorizable left GC-lpp semigroups.

Lemma 4 *If U is a left regular band and V a right cancellative monoid with identity* 1*, then $U \ltimes_\eta^w V$ is a left GC-lpp semigroup.*

Proof By Proposition 9, $U \ltimes_\eta^w V$ is a glrac semigroup. So, $U \ltimes_\eta^w V$ is a left GC-lpp semigroup if and only if for any $(u, v) \in U \times V$, $(u, v)\mathcal{R}^*(u, v)^+ = (u, 1)$. So, it suffices to verify that $(u, v)\mathcal{R}^*(u, 1)$. Obviously,

$$(u, 1) \otimes (u, v) = (u, v)^+ \otimes (u, v) = (u, v).$$

We first prove that for $(u_1, v_1), (u_2, v_2) \in U \ltimes_\eta^w V$, if

$$\begin{aligned}(u_1 \cdot (u_1, v_1)^\# u, v_1 v) = (u_1, v_1) \otimes (u, v) = (u_2, v_2) \otimes (u, v)\\ = (u_2 \cdot (u_2, v_2)^\# u, v_2 v),\end{aligned} \tag{5}$$

that is, $u_1 \cdot (u_1, v_1)^\# u = u_2 \cdot (u_2, v_2)^\# u$, then $(u_1, v_1) \otimes (u, 1) = (u_2, v_2) \otimes (u, 1)$. Indeed, if (5) holds, we have $v_1 v = v_2 v$, so that $v_1 = v_2$ since V is a right cancellative monoid, it follows that

$$\begin{aligned}(u_1 \cdot (u_1, v_1)^\# u, v_1) = (u_1, v_1) \otimes (u, 1) = (u_2, v_2) \otimes (u, 1)\\ = (u_2 \cdot (u_2, v_2)^\# u, v_2),\end{aligned}$$

as required. If $(u_1, v_1) \otimes (u, v) = (u, v)$, then

$$(u_1, v_1) \otimes (u, v) = (u, v) = (u, 1) \otimes (u, v),$$

and further by the foregoing proof,

$$(u_1, v_1) \otimes (u, 1) = (u, 1) \otimes (u, 1) = (u, 1).$$

We have now proved that for $(u_1, v_1), (u_2, v_2) \in (U \rtimes_\eta^w V)^1$, if

$$(u_1, v_1) \otimes (u, v) = (u_2, v_2) \otimes (u, v),$$

then $(u_1, v_1) \otimes (u, 1) = (u_2, v_2) \otimes (u, 1)$. Therefore, $(u, v)\mathcal{R}^*(u, 1)$. □

By Lemma 4 and Theorem 2, we have the following theorem:

Theorem 14 *Let U be a left regular band and V a right cancellative monoid. Then, $U \ltimes_\eta^w V$ is a uniquely almost factorizable left GC-lpp semigroup.*

Conversely, any uniquely almost factorizable left GC-lpp semigroup can be constructed in this way.

Proof By Lemma 4 and Theorem 2, we need only to verify the converse part. To the end, we assume that S is a uniquely almost factorizable left GC-lpp semigroup. By Theorem 2, S is isomorphic to some $U \ltimes_\eta^w V$ where U is a left regular band and V is a left reduced monoid. Notice that σ_S is a right cancellative monoid congruence whenever S is a left GC-lpp semigroup (see [17]). We get that $(U \ltimes_\eta^w V)/\sigma_{U \ltimes_\eta^w V}$ is a right cancellative monoid. By the foregoing proof of the sufficiency of Theorem 10, we have verified that $(u, v)\sigma_{U \ltimes_\eta^w V} = U \times \{v\}$ for any $(u, v) \in U \ltimes_\eta^w V$.

Define a mapping

$$\xi : (U \ltimes_{\eta}^{w} V)/\sigma_{U \ltimes_{\eta}^{w} V} \to V; (u, v)\sigma_{U \ltimes_{\eta}^{w} V} \mapsto v.$$

It is easy to see that ξ is a surjective mapping. Also,

$$\begin{aligned}((u, v)\sigma_{U \ltimes_{\eta}^{w} V} \cdot (x, y)\sigma_{U \ltimes_{\eta}^{w} V})\xi &= ((u \cdot (u, v)^{\#}x, vy)\sigma_{U \ltimes_{\eta}^{w} V})\xi = vy \\ &= ((u, v)\sigma_{U \ltimes_{\eta}^{w} V})\xi((x, y)\sigma_{U \ltimes_{\eta}^{w} V})\xi.\end{aligned}$$

Thus, ξ is an isomorphism. This implies that V is a right cancellative monoid. □

Let us turn back to the proof of Theorems 2 and 14 . By the proof of Theorem 2, S is isomorphic to $P(S) \ltimes_{\zeta}^{w} \mathfrak{R}_S$. Moreover, by the proof of Theorem 14, $\mathfrak{R}_S$ is a right cancellative monoid with identity $P(S)$. So, we have proved that $\mathfrak{R}_S$ is a right cancellative monoid whenever S is a uniquely almost factorizable left GC-lpp semigroup. Now, the following theorem is an immediate consequence of Theorems 14 and 3 .

Theorem 15 *Let U be a left regular band with identity e and V a right cancellative monoid with identity 1. If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$, then $U \ltimes_{\eta}^{w} V$ is a uniquely factorizable left GC-lpp monoid.*

Conversely, any uniquely factorizable left GC-lpp monoid can be constructed in this way.

By a *left (right) ample semigroup*, we mean a semigroup S satisfying the following conditions:

(i) Every $\mathcal{R}^*$- ($\mathcal{L}^*$-)class of S contains exactly one idempotent.
(ii) The set $E(S)$ of idempotents of S form a semilattice under the multiplication.
(iii) For any $e \in E(S)$, $a \in S$, $ae = (ae)^{\dagger}a$ ($ea = a(ea)^*$), where $a^{\dagger}$ (a^*) denotes the unique idempotent in the $\mathcal{R}^*$- ($\mathcal{L}^*$-)class of S containing a.

If S is both a left ample semigroup and a right ample semigroup, we call S an *ample semigroup*. (Left; Right) ample semigroup is formerly called (left; right) type-A semigroup. Of course, a left ample semigroup S may be regarded as a (2, 1)-algebra with unary operator defined by

$$^{+} : \ S \to S; \ a \mapsto a^{\dagger}.$$

By definition, any left ample semigroup is a glrac semigroup satisfying the conditions:

- $a^+b^+ = b^+a^+$ for any $a, b \in S$;
- $a\mathcal{R}^*a^+$ for any $a \in S$.

Moreover, S is a left ample semigroup if and only if S is a left restriction semigroup in which $a\mathcal{R}^*a^+$ for any $a \in S$; if and only if S is a left GC-lpp semigroup in which $P(S)$ is a semilattice. Dually, we know that a right ample semigroup may be regarded as a right restriction semigroup with unary operator defined by:

$$^{*} : \ S \to S; \ a \mapsto a^*$$

and in which $a\mathcal{L}^*a^*$ for any element a. Therefore, an ample semigroup may be regarded as a restriction semigroup in which $a^+\mathcal{R}^*a\mathcal{L}^*a^*$ for any element a.

Based on the fact that a semigroup is a left ample semigroup if and only if it is a left GC-lpp semigroup whose set of projections is a semilattice, it is easy to see that whenever U is a semilattice, Theorem 14 is the structure theorem for uniquely almost factorizable left ample semigroups and that whenever U is a semilattice with identity, Theorem 15 is the structure theorem for uniquely factorizable left ample monoids. We here omit the details.

Lemma 5 *Let U be a semilattice and V a reduced monoid with identity* 1. *If for any $(u, x) \in U \ltimes_\eta^w V$, there exists an element $(u, x)^\circ \in U$ satisfying Conditions* (RS1)-(RS5)*, then $U \ltimes_\eta^w V$ is an ample semigroup if and only if*

(C) *V is a cancellative monoid;*
(L) *for any $(u, x) \in U \ltimes_\eta^w V$ and $v, w \in U$, $u \cdot (u, x)^\# v = u \cdot (u, x)^\# w$ implies that $(u, x)^\circ v = (u, x)^\circ w$.*

Proof Suppose that $U \ltimes_\eta^w V$ is an ample semigroup. Fountain in [8] pointed out that in the semigroup $U \ltimes_\eta^w V$, the congruence σ is a cancellative monoid congruence, so that $U \ltimes_\eta^w V/\sigma$ is a cancellative monoid. But by the proof of Theorem 10, $U \ltimes_\eta^w V/\sigma \cong V$, now V is a cancellative monoid.

Let $(u, x) \in U \ltimes_\eta^w V$ and $v, w \in U$, $u \cdot (u, x)^\# v = u \cdot (u, x)^\# w$. By Theorem 10, $U \ltimes_\eta^w V/\sigma$ is a restriction semigroup. Notice that

$$(u, x) \otimes (v, 1) = (u \cdot (u, x)^\# v, x) = (u \cdot (u, x)^\# w, x) = (u, x) \otimes (w, 1)$$

and since $(u, x)\mathcal{L}^*(u, x)^* = ((u, x)^\circ, 1)$ (by the argument before the lemma), we can observe that

$$\begin{aligned}((u, x)^\circ v, 1) &= ((u, x)^\circ, 1) \otimes (v, 1) = (u, x)^* \otimes (v, 1) = (u, x)^* \otimes (w, 1)\\ &= ((u, x)^\circ, 1) \otimes (w, 1)\\ &= ((u, x)^\circ w, 1),\end{aligned}$$

thereby $(u, x)^\circ v = (u, x)^\circ w$, resulting (L).

Conversely, if the given conditions hold, then by Theorem 10, $U \ltimes_\eta^w V/\sigma$ is a restriction semigroup. Again by the proof of Lemma 4, $(u, x)\mathcal{R}^*(u, 1) = (u, x)^+$. On the other hand, by the proof of Theorem 10, we have proved that $(u, x)^* = ((u, x)^\circ, 1)$. So, $(u, x) \otimes ((u, x)^\circ, 1) = (u, x)$. For any $(v, y), (w, z) \in U \ltimes_\eta^w V$, if

$$(u \cdot (u, x)^\# v, xy) = (u, x) \otimes (v, y) = (u, x) \otimes (w, z) = (u \cdot (u, x)^\# w, xz),$$

then $u \cdot (u, x)^\# v = u \cdot (u, x)^\# w$ and $xy = xz$. The first equality implies that $(u, x)^\circ v = (u, x)^\circ w$; the second one implies that $y = z$ since V is cancellative. Therefore,

$$\begin{aligned}(u, x)^* \otimes (v, y) &= ((u, x)^\circ, 1) \otimes (v, y) = ((u, x)^\circ v, y) = ((u, x)^\circ w, z)\\ &= ((u, x)^\circ, 1) \otimes (w, z)\\ &= (u, x)^* \otimes (w, z).\end{aligned} \tag{6}$$

And, if $(u, x) \otimes (v, y) = (u, x)$, then

$$(u, x) \otimes (v, y) = (u, x) = (u, x) \otimes ((u, x)^\circ, 1)$$

and by (6),

$$(u, x)^* \otimes (v, y) = (u, x)^* \otimes ((u, x)^\circ, 1) = (u, x)^* \otimes (u, x)^* = (u, x)^*.$$

We have verified that for any $(v, y), (w, z) \in (U \ltimes_\eta^w V)^1$, if $(u, x) \otimes (v, y) = (u, x) \otimes (w, z)$, then $(u, x)^* \otimes (v, y) = (u, x)^* \otimes (w, z)$. Therefore, $(u, x)\mathcal{L}^*(u, x)^*$. Consequently, $U \ltimes_\eta^w V$ is an ample semigroup. □

The following theorem is immediate from Lemma 5 and Theorem 10.

Theorem 16 *Let U be a semilattice and V a cancellative monoid. If for any $(u, x), (v, y) \in U \ltimes_\eta^w V$, there exists an element $(u, x)^\circ \in U$ such that Conditions $(RS1)$-$(RS5)$, (C) and (L) are satisfied, then $U \ltimes_\eta^w V$ is a uniquely almost left factorizable ample semigroup.*

Conversely, any uniquely almost left factorizable ample semigroup can be constructed in this way.

By Proposition 6, Lemma 5 and Theorem 11, we have

Theorem 17 *Let U be a semilattice with identity e and V a cancellative monoid with identity 1. If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$ and if for any $(u, x) \in U \ltimes_\eta^w V$, there exists an element $(u, x)^\circ \in U$ such that Conditions $(RS1)$-$(RS5)$, (C) and (L) are satisfied, then $U \ltimes_\eta^w V$ is a uniquely left factorizable ample monoid.*

Conversely, any uniquely left factorizable ample monoid can be constructed in this way.

9 Left Inverse Semigroups

In this section, we establish the structures of uniquely factorizable left inverse monoids and uniquely almost factorizable left inverse semigroups.

To begin with, we give a characterization of weakly semidirect products.

Lemma 6 *Let U be a left regular band and V a left reduced monoid. Then, $U \ltimes_\eta^w V$ is a left inverse semigroup if and only if*

(RI1) *V is a group;*
(RI2) *for any $(u, v) \in U \times V$, there exists $z \in U$ such that $u \cdot (u, v)^\# z = u$.*

Proof Let $U \ltimes_\eta^w V$ be a left inverse semigroup, then $U \ltimes_\eta^w V$ is a left GC-lpp semigroup and $U \ltimes_\eta^w V/\sigma$ is a right cancellative monoid. Again by Theorem 10, $U \ltimes_\eta^w V/\sigma$ is isomorphic to V. Therefore, V is a right cancellative monoid. On the other hand, by $U \ltimes_\eta^w V$ is a regular semigroup, we know that for any $(u, v) \in U \times V$, there exists

$(u', v') \in U \times V$ such that $(u, v) \otimes (u', v') \otimes (u, v) = (u, v)$ and $(u', v') \otimes (u, v) \otimes (u', v') = (u', v')$. Denote $z = u' \cdot (u', v')^{\#}u$. Compute

$$\begin{aligned}(u, v) \otimes (u', v') \otimes (u, v) &= (u, v) \otimes (u' \cdot (u', v')^{\#}u, v'v)\\ &= (u, v) \otimes (z, v'v)\\ &= (u \cdot (u, v)^{\#}z, vv'v)\\ &= (u, v),\end{aligned}$$

hence $u \cdot (u, v)^{\#}z = u$ and $vv'v = v$. Also,

$$\begin{aligned}(u', v') \otimes (u, v) \otimes (u', v') &= (u' \cdot (u', v')^{\#}u, v'v) \otimes (u', v')\\ &= (z, v'v) \otimes (u', v')\\ &= (z \cdot (z, v'v)^{\#}u', v'vv')\\ &= (u', v').\end{aligned} \tag{7}$$

Hence, $z \cdot (z, v'v)^{\#}u' = u'$ and $v'vv' = v'$. Therefore, vv' and $v'v$ are idempotents. Hence, $vv' = v'v = 1_V$. This means that any element of V is a unit. Then, V is a group.

To see the converse part, for any $(u, v) \in U \times V$, we have

$$\begin{aligned}(u, v) \otimes (z, v^{-1}) \otimes (u, v) &= (u \cdot (u, v)^{\#}z, 1_V) \otimes (u, v)\\ &= ((u \cdot (u, v)^{\#}z) \cdot (u \cdot (u, v)^{\#}z, 1_V)^{\#}u, 1_V v)\\ &= ((u \cdot (u, v)^{\#}z)(u \cdot (u, v)^{\#}z), v)\\ &= (u, v)\end{aligned}$$

where v^{-1} is an inverse of v in V and 1_V is the identity of V. Hence, $U \ltimes_\eta^w V$ is regular. □

Notice that a glrac semigroup is left inverse if and only if it is regular. So, by Lemma 6 and Theorem 2, the following theorem is immediate.

Theorem 18 *Let U be a left regular band and V a group. If Conditions* (RI1) and (RI2) *are satisfied, then $U \ltimes_\eta^w V$ is a uniquely almost factorizable left inverse semigroup.*

Conversely, any uniquely almost factorizable left inverse semigroup can be constructed in this way.

The following theorem is an immediate consequence of Lemma 6 and Theorem 3.

Theorem 19 *Let U be a left regular band with identity e and V a group with identity* 1. *If $\eta_{u,x}(e) = u$ for any $(u, x) \in U \times V$, and Conditions* (RI1) and (RI2) *are satisfied, then $U \ltimes_\eta^w V$ is a uniquely factorizable left inverse monoid*

Conversely, any uniquely factorizable left inverse monoid can be constructed in this way.

Remark 2 By definition, a left inverse semigroup is an inverse semigroup if and only if its set of idempotents forms a semilattice under the multiplication. So, replacing "Let U be a left regular band" by "Let U be a semilattice" in Theorem 18, we can obtain the structure theorem of uniquely almost factorizable inverse semigroups. And, replacing "Let U be a left regular band with identity" by "Let U be a semilattice with identity" in Theorem 19, we can obtain the structure theorem of uniquely factorizable inverse monoids. We here omit the details.

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