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Permutation Groups and Set-Orbits on the Power Set

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Abstract

A permutation group G acting on a set Ω induces a permutation group on the power set $\mathscr{P}(\Omega)$ (the set of all subsets of Ω). Let G be a finite permutation group of degree n and $s(G)$ denote the number of orbits of G on $\mathscr{P}(\Omega)$. It is an interesting problem to determine the lower bound $\inf\left(\frac{\log_2 s(G)}{n}\right)$ over all groups G that do not contain any alternating group A_ℓ (where $\ell > t$ for some fixed $t \geq 4$) as a composition factor. The second author obtained the answer for the case $t = 4$ in Yang (J Algebra Appl 19:2150005, 2020). In this paper, we continue this investigation and study the cases when $t \geq 5$, and give the explicit lower bounds $\inf\left(\frac{\log_2 s(G)}{n}\right)$ for each positive integer $5 \leq t \leq 166$.

Keywords Finite groups · Orbits · Permutation groups · Power set

Mathematics Subject Classification 20B05

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1 Introduction

A permutation group G acting on a set Ω induces a permutation group on the power set $\mathscr{P}(\Omega)$. It is an important subject in representation theory to study this particular action. For example, Gluck [4] showed that if the group G is solvable and the action is primitive, then G always has a regular orbit on the power set $\mathscr{P}(\Omega)$ except for a few exceptional cases. Later, Seress generalized this to arbitrary groups in [8]. We define the orbits of this action to be set-orbits, and let $s(G)$ denote the number of set-orbits of G. Since sets of different cardinalities belong to different orbits, it is clear that $s(G) \geq |\Omega| + 1$. Groups with the property $s(G) = |\Omega| + 1$ (i.e., set transitive groups) have been classified by Beaumont and Peterson in [2]. They showed that apart from a few exceptional cases of degree at most 9, a set-transitive group of degree n always contains the alternating group A_n.

Let G be a permutation group of degree n. In [1], Babai and Pyber showed that if G has no large alternating composition factors then $s(G)$ is exponential in n. More precisely they proved the following result [1, Theorem 1]. Let G be a permutation group of degree n. If G does not contain any A_ℓ (where $\ell > t$ for some fixed $t \geq 4$) as a composition factor, then $\frac{\log_2 s(G)}{n} \geq \frac{c}{t}$ for some positive constant c (unspecified). It appears that this result has many applications. In [5], Keller applied this result to find a lower bound for the number of conjugacy classes of a solvable group. In [7], Nguyen used this result to study the multiplicities of conjugacy class sizes of finite groups.

In [1], Babai and Pyber also raised the following question: what is $\inf\left(\frac{\log_2 s(G)}{n}\right)$ over all solvable groups G? This question was answered in a recent paper of the second author in [9]. Clearly, a more interesting question is to answer the following: what is $\inf\left(\frac{\log_2 s(G)}{n}\right)$ over all groups G that do not contain any A_ℓ (where $\ell > t$ for some fixed $t \geq 4$) as a composition factor? The second author studied a special case of this question in [10], and showed that when $t = 4$, the answer is

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\mapsto\infty} \frac{\log_2 s(\mathrm{M}_{24} \wr \mathrm{M}_{12} \wr \overbrace{S_4 \wr \cdots \wr S_4}^{\text{k terms}})}{24 \cdot 12 \cdot 4^k}.$$

In this paper, we continue this investigation and study the cases when $t \geq 5$, and give the explicit lower bounds $\inf\left(\frac{\log_2 s(G)}{n}\right)$ for each positive integer $5 \leq t \leq 166$. In fact, we show that

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\mapsto+\infty} \frac{\log_2 s(\mathrm{M}_{24} \wr \overbrace{\mathrm{S}_t \wr \cdots \wr \mathrm{S}_t}^{\text{kterms}})}{24 \cdot t^k} if t \in [5, 16], \quad \text{and}$$

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\mapsto+\infty} \frac{\log_2 s(\overbrace{\mathrm{S}_t \wr \cdots \wr \mathrm{S}_t}^{\text{kterms}})}{t^k} if t \in [17, 166].$$

We also ask some related questions at the end of the paper. Our main results are the following.

Theorem 1.1 *Let t be an integer with* $5 \leq t \leq 16$. *Then*

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k \mapsto +\infty} \frac{\log_2 s(\mathrm{M}_{24} \wr \overbrace{\mathrm{S}_t \wr \cdots \wr \mathrm{S}_t}^{k\ terms})}{24 \cdot t^k},$$

where the infimum is taken over all permutation groups G not containing any composition factor A_ℓ *with* $\ell > t$, *and n denotes the degree of G.*

Theorem 1.2 *Let t be an integer with* $17 \leq t \leq 166$. *Then*

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k \mapsto +\infty} \frac{\log_2 s(\overbrace{\mathrm{S}_t \wr \cdots \wr \mathrm{S}_t}^{k\ terms})}{t^k},$$

where the infimum is taken over all permutation groups G not containing any composition factor A_ℓ *with* $\ell > t$, *and n denotes the degree of G.*

2 Preliminary Results

Some of the following definitions and lemmas appear in [10], we include those here for the convenience of the reader.

Let T be a finite group and S a permutation group. We denote by $T \wr S$ the wreath product of T with S. Let G be a permutation group of degree n. We use $s(G)$ to denote the number of set-orbits of G and we denote $rs(G) = \frac{\log_2 s(G)}{n}$.

We recall some basic facts about the decompositions of transitive groups. Let G be a transitive permutation group acting on a set Ω, where $|\Omega| = n$. A *system of imprimitivity* is a partition of Ω, invariant under G. A *primitive group* has no non-trivial system of imprimitivity. Let $\{\Omega_1, \ldots, \Omega_m\}$ denote a system of imprimitivity of G with maximal block-size b ($1 \leq b < n$; $b = 1$ if and only if G is primitive; $bm = n$). Let N be the intersection of the stabilizers of the blocks. Then G/N is a primitive group of degree m acting upon the set of blocks Ω_i. If G_i denotes the permutation group of degree b induced on Ω_i by the set-wise stabilizer of Ω_i in G, then the groups G_i are permutationally equivalent transitive groups and $N \leq G_1 \times \cdots \times G_m \leq \mathrm{Sym}(\Omega)$.

Let G be a transitive group of degree n and assume that G is not primitive. Let us consider a system of imprimitivity of G that consists of $m \geq 2$ blocks of size b, where b is maximal. Thus G can be embedded in $K \wr P_1$, written $G \lesssim K \wr P_1$, where K is a permutation group of degree n/m and P_1 is the primitive quotient group of G that acts upon the m blocks. We may keep doing this, and after re-index for convenience, and we can get that $G \lesssim H \wr P_1 \wr \cdots \wr P_k$, where H is a permutation group and the P_i are all primitive groups. If this happens, we say that G is *induced* from the permutation group H.

In what follows, we need to make some preparations for the proof of Theorems 1.1 and 1.2, and we begin with two important lemmas.

Lemma 2.1 [1, Prop. 1] *If $H \leq G \leq \mathrm{Sym}(\Omega)$, then $s(G) \leq s(H) \leq s(G) \cdot |G : H|$.*

Lemma 2.2 [1, Prop. 2] *Assume that G is intransitive on Ω and has orbits $\Omega_1, \ldots, \Omega_m$. Let G_i be the restriction of G to Ω_i. Then*

$$s(G) \geq s(G_1) \times \cdots \times s(G_m).$$

In view of Lemma 2.2 and the fact that $\frac{a+b}{c+d} \geq \min\{\frac{a}{c}, \frac{b}{d}\}$ for positive integers a, b, c and d, it suffices to consider transitive permutation groups in order to find $\inf\left(\frac{\log_2 s(G)}{n}\right)$. To see this, assume that the action of G on Ω is not transitive, then we may assume G has two blocks Ω_1 and Ω_2, where Ω is a disjoint union of Ω_1 and Ω_2. We assume $|\Omega_1| = m_1$ and $|\Omega_2| = m_2$. Let G_i be the restriction of G to Ω_i for $i = 1, 2$. Then $s(G) \geq s(G_1) \cdot s(G_2)$. We see that

$$rs(G) = \frac{\log_2 s(G)}{m_1 + m_2} \geq \frac{\log_2 s(G_1) + \log_2 s(G_2)}{m_1 + m_2} \geq \min\{rs(G_1), rs(G_2)\}.$$

Lemma 2.3 is basic and also very useful to find $\inf\left(\frac{\log_2 s(G)}{n}\right)$. This lemma and its proof are almost identical to [1, Lemma 1].

Lemma 2.3 *Let G be a transitive permutation group acting on a set Ω where $|\Omega| = n$. Let $\{\Omega_1, \ldots, \Omega_m\}$ denote a system of imprimitivity of G with maximal block-size b, where $1 \leq b < n$ and $bm = n$. In particular, $b = 1$ if and only if G is primitive. Let N denote the normal subgroup of G stabilizing each of the blocks Ω_i. Let $G_i = \mathrm{Stab}_G(\Omega_i)$ and $s = s(G_1)$. Then*

(i) $s(G) \geq s^m/|G/N|$.
(ii) $s(G) \geq \binom{s+m-1}{s-1}$. *Moreover, the equality holds if $G/N \cong \mathrm{S}_m$.*

Proof By Lemmas 2.1 and 2.2, it is easy to verify that part (i) follows.

Let A be a subset of Ω and let α_j $(0 \leq j \leq s)$ denote the number of intersections of A with Ω_i that lie in the j-th orbit of G_i on the powerset of Ω_i. Let B be another subset of Ω with the number β_j defined similarly. If A and B are in the same orbit of G, then $\alpha_j = \beta_j$ for $(0 \leq j \leq s)$. Therefore, $s(G)$ is at least the number of partitions of m into s nonnegative integers (where the order of the summands is taken into consideration). It is well-known that this number is

$$\binom{s+m-1}{s-1}$$

which proves the first part of (ii). Since S_m is set-transitive on all the subsets of the same size, we know the equation will hold. □

We need the following estimates of the order of the primitive permutation groups.

Table 1 The lower bounds and corresponding primitive groups

Degree n	The lower bound of $s(G)$	Corresponding primitive group
14	35	The bound 35 is attained by PrimitiveGroup(14,2)$\cong$ PGL(2, 13)
15	46	The bound 46 is attained by PrimitiveGroup(15,4) $\cong$ PSL(4, 2)
16	32	The bound 32 is attained by PrimitiveGroup(16,11) $\cong 2^4$.PSL(4, 2)
17	48	The bound 48 is attained by PrimitiveGroup(17,8) $\cong$ PGL(2, 2^4)
21	158	The bound 158 is attained by PrimitiveGroup(21,7) $\cong$ PGL(3, 4)
22	105	The bound 105 is attained if $G \cong M_{22}$ or $M_{22}.2$
23	72	The bound 72 is attained if $G \cong M_{23}$
24	49	The bound 49 is attained if $G \cong M_{24}$
32	361	The bound 361 is attained by PrimitiveGroup(32,3)$\cong$ ASL(5, 2)

Lemma 2.4 *Let G be a primitive permutation group of degree n where G does not contain A_n. Then*

(i) $|G| < 50 \cdot n^{\sqrt{n}}$.
(ii) $|G| < 3^n$. *Moreover, if* $n > 24$, *then* $|G| < 2^n$.
(iii) $|G| \leq 2^{0.76n}$ *when* $n \geq 25$ *and* $n \neq 32$.

Proof Part (i) is from [6, Corollary 1.1 (ii)], and part (ii) is from [6, Corollary 1.2]. Part (iii) follows from part (i) for $n \geq 89$, and we may check the remaining results using GAP [3] for $25 \leq n \leq 88$. □

Lemma 2.5 *Let G be a primitive permutation group of degree n where G does not contain any A_ℓ ($\ell > 4$) as a composition factor. If $n \leq 24$ or $n = 32$, then the lower bound of $s(G)$ and corresponding primitive permutation group can be determined.*

Proof The results can be easily checked by GAP [3]. For convenience, we list the results in Table 1. We also need to mention the following cases which will be used in Sects. 3 and 4.

If $n = 23$, the group with the largest order is $G \cong M_{23}$, and $s(G) = 72$. However, the group with the second largest order has order 506 and $s(G) \geq 16770$.

If $n = 24$, the group with the second largest order has order 12144 and $s(G) \geq 1674$. The group with the largest order is $G \cong M_{24}$ and $s(G) = 49$.

If $n = 32$, the group with the second largest order has order 29760 and $s(G) \geq 144321$. The group with the largest order is $G \cong$ PrimitiveGroup(32,3) $\cong$ ASL(5, 2), and $s(G) \geq 361$. We remark here that 361 is a lower bound obtained by GAP [3] using random search, but we do not get the best possible lower bound since the current one works for our purpose. □

Remark 2.6. Using GAP [3], we can easily obtain the maximum order of primitive groups of degree n which do not contain the simple group A_n as a composition factor. For convenience, we list some results in Table 2.

Table 2 The maximal order of primitive groups not containing A_n

Degree	Maximal order	Degree	Maximal order	Degree	Maximal order	Degree	Maximal order
5	20	14	2184	23	10,200,960	31	9,999,360
6	120	15	20,160	24	244,823,040	32	319,979,520
7	168	16	322,560	25	28,800	33	163,680
8	1344	17	16,320	26	31,200	34	Does not exist
9	1512	18	4896	27	303,264	35	40,320
10	1440	19	342	28	1,451,520	36	1,451,520
11	7920	20	6840	29	812	37	1332
12	95,040	21	120,960	30	24,360	38	50,616
13	5616	22	887,040				

3 Proof of Theorem 1.1

In this section, we shall give the explicit lower bounds $\inf\left(\frac{\log_2 s(G)}{n}\right)$ where G does not contain A_ℓ with $\ell > t$ for $t \in [5, 16]$) as a composition factor. In what follows, we only provide the detailed proof for the case $t = 5$. The proof for the remaining cases is analogous up to replacing a few numbers appropriately.

Proposition 3.1 *We have the following equality*

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\mapsto+\infty} \frac{\log_2 s(M_{24} \wr \overbrace{S_5 \wr \cdots \wr S_5}^{k\ terms})}{24 \cdot 5^k},$$

where the infimum is taken over all permutation groups G not containing any composition factor A_ℓ *with* $\ell > 5$*, and n denotes the degree of G.*

To complete the proof of Proposition 3.1, we need a number of preparatory lemmas. Set

$$a_k = \frac{\log_2 s(M_{24} \wr S_5 \wr \cdots \wr S_5)}{24 \cdot 5^k}\ with\ k \geq 0 \text{ and } M = \lim_{k\to+\infty} a_k.$$

For convenience of notation, we first define a sequence $\{s_k\}_{k\geq 0}$, where

$$s_0 = s(M_{24}) = 49 \text{ and } s_{k+1} = \binom{s_k+4}{5}\ for\ k \geq 0.$$

It is clear that the sequence $\{s_k\}_{k\geq 0}$ is strictly increasing. By the definition, $a_k = \frac{\log_2 s_k}{24\cdot 5^k}$. It is easy to see that $a_k > 0$. Now, we consider the sequence $\{a_k\}_{k\geq 0}$. Since

$$a_{k+1} = \frac{\log_2 \binom{s_k+4}{5}}{24 \cdot 5^{k+1}} = \frac{\log_2\left(\frac{(s_k+4)(s_k+3)(s_k+2)(s_k+1)s_k}{120}\right)}{24 \cdot 5^{k+1}} < \frac{\log_2 (s_k)^5}{24 \cdot 5^{k+1}} = a_k,$$

the sequence $\{a_k\}_{k\geq 0}$ is strictly decreasing, and so the $\lim\limits_{k\mapsto\infty} a_k$ exists.

Applying Lemma 2.3, we calculate that

$$a_0 \approx 0.233946243505,\ a_1 \approx 0.178770507941,\ a_2 \approx 0.167259031994.$$

Lemma 3.2 *Let G be a primitive permutation group of degree n. If G does not contain any A_ℓ ($\ell > 5$) as a composition factor, then $rs(G) \geq a_2$.*

Proof If $n \geq 25$ and $n \neq 32$, then by Lemma 2.4 (ii), we have $s(G) \geq 2^n/|G| \geq 2^n/2^{0.76n} = 2^{0.24n}$, and so $rs(G) = \frac{\log_2 s(G)}{n} \geq 0.24 \geq a_2$.

If $n \leq 24$, then $s(G) \geq n+1$. Thus $rs(G) \geq 0.193494007907 \geq a_2$.

If $n = 32$, then by Table 1, we have $s(G) \geq 361$. In this case, one has

$$rs(G) \geq 0.265495469577 \geq a_2.$$

This completes the proof. □

Lemma 3.3 *Let G be a transitive permutation group of degree n induced from a permutation group H of degree m, where G does not contain any A_ℓ ($\ell > 5$) as a composition factor. Let $\alpha_1 = 120^{\frac{1}{4}}$. If $\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m} \geq \beta$, then $rs(G) \geq \beta$.*

Proof We may assume that $G \lesssim H \wr P_1 \wr \cdots \wr P_z$, where the P_i are primitive permutation groups and $\deg(P_i) = k_i$ for $1 \leq i \leq z$. Using GAP [3] and Lemma 2.4, it is straightforward to verify that $|P_1| \leq \alpha_1^{k_1-1}$. Since $\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m} \geq \beta$, we have $s(H) \geq \alpha_1 \cdot 2^{m\beta}$. On the other hand, Lemma 2.3 implies

$$s(H \wr P_1) \geq s(H)^{k_1}/\alpha_1^{k_1-1} \geq \alpha_1 \cdot 2^{mk_1\beta}.$$

Consequently,

$$\frac{\log_2 s(H \wr P_1)}{mk_1} - \frac{\log_2 \alpha_1}{mk_1} \geq \frac{\log_2 \alpha_1 \cdot 2^{mk_1\beta}}{mk_1} - \frac{\log_2 \alpha_1}{mk_1} = \beta.$$

Now, let $K = H \wr P_1$. Then by Lemma 2.3, one has

$$s(K \wr P_2) \geq s(K)^{k_2}/\alpha_1^{k_2-1} \geq \alpha_1 \cdot 2^{m(k_1k_2)\beta}.$$

This implies that

$$\frac{\log_2 s(H \wr P_1 \wr P_2)}{mk_1k_2} - \frac{\log_2 \alpha_1}{mk_1k_2} \geq \frac{\log_2 \alpha_1 \cdot 2^{m(k_1k_2)\beta}}{mk_1k_2} - \frac{\log_2 \alpha_1}{mk_1k_2} = \beta.$$

In a similar fashion, one can prove $\frac{\log_2 s(H\wr P_1\wr\cdots\wr P_z)}{mk_1\cdots k_z} - \frac{\log_2 \alpha_1}{mk_1\cdots k_z} \geq \beta$, where $n = mk_1\cdots k_z$. Since $G \lesssim H \wr P_1 \wr \cdots \wr P_z$, we derive from Lemma 2.2 that $s(G) \geq$

Table 3 $f(H, m, \alpha_1)$ $(12 \le m \le 24$ or $m = 32)$

m	$s(H)$	$f(H, m, \alpha_1)$	m	s(H)	$f(H, m, \alpha_1)$
32	361	0.211535386812	24	1674	0.374265048485
23	72	0.193182710980	22	105	0.226705584944
21	158	0.265574195204	20	260	0.314782258206
19	1610	0.469795911795	18	113	0.282969795195
17	48	0.227955285401	16	32	0.204579834444
15	46	0.253122620477	14	35	0.243040026289
13	30	0.244628303593	12	14	0.173386022763

$s(H \wr P_1 \wr \cdots \wr P_z)$, and thereby $rs(G) \ge \frac{\log_2 s(H \wr P_1 \wr \cdots \wr P_z)}{n} \ge \beta$. This completes the proof. □

Lemma 3.4 *Let G be a transitive permutation group of degree n, which is induced from a primitive permutation group H of degree m. If G does not contain any A_ℓ $(\ell > 5)$ as a composition factor, and $H \not\cong \mathrm{M}_{24}$, then $rs(G) \ge a_2$.*

Proof In view of Proposition 3.3, it suffices to prove that:

$$\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m} \ge a_2. \tag{1}$$

Suppose that $m \ge 25$ and $m \ne 32$. By Lemma 2.4, we have $|H| \le 2^{0.76m}$. Since $s(H) \ge 2^m/|H|$, it follows that

$$\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m} \ge \frac{\log_2(2^m/2^{0.76m})}{m} - \frac{\log_2 120}{4 \cdot m} \ge 0.170931094044 \ge a_2.$$

In this case, inequality (1) follows.

Now, we denote by $f(H, m, \alpha_1)$ the number $\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m}$. If $12 \le m \le 24$ or $m = 32$, then by Table 1, we can check all the minimum values of $f(H, m, \alpha_1)$ using Table 3.

If $4 \le m \le 11$, then $s(H) \ge m + 1$, and thereby

$$\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_1}{m} \ge \frac{\log_2(m+1)}{m} - \frac{\log_2 120}{4m} \ge 0.168930895619 \ge a_2.$$

If $m = 3$ or 2, then by Lemma 3.2, we may assume that

$$G \lesssim H \wr P_1 \wr P_2 \cdots \wr P_z, \, where \deg(P_1) = m_1.$$

Let $K = H \wr P_1$. By Proposition 3.3, it is sufficient to check that inequality (2) holds.

$$\frac{\log_2 s(K)}{mm_1} - \frac{\log_2 \alpha_1}{mm_1} \geq a_2. \tag{2}$$

Denote by $f(K, m, m_1, \alpha_1)$ the number $\frac{\log_2 s(K)}{mm_1} - \frac{\log_2 \alpha_1}{mm_1}$. Then we need to check the inequality $f(K, m, m_1, \alpha_1) \geq a_2$ holds. In what follows, we distinguish two cases.

Case 1 $m = 3$. In this case, we know that $s(H) \geq 4$. By Lemma 2.3, one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 4^{m_1}/|P_1|$. Thus

$$f(K, 3, m_1, \alpha_1) \geq \frac{m_1 \log_2 4 - \log_2 |P_1|}{3m_1} - \frac{\log_2 120}{12m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 3, m_1, \alpha_1) \geq \frac{m_1 \log_2 4 - \log_2 2^{0.76m_1}}{3m_1} - \frac{\log_2 120}{12m_1} \geq 0.390310364681 \geq a_2.$$

If $m_1 \leq 16$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{3+m_1}{3}$, and thereby

$$f(K, 3, m_1, \alpha_1) \geq \frac{\log_2 \binom{3+m_1}{3}}{3m_1} - \frac{\log_2 120}{12m_1} \geq 0.170700629302 \geq a_2.$$

If $17 \leq m_1 \leq 24$ or $m_1 = 32$, then by Table 2, we can calculate the minimum values of $f(K, 3, m_1, \alpha_1)$ as follows (Table 4).

Case 2 $m = 2$. In this case, we know that $s(H) \geq 3$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 3^{m_1}/|P_1|$. Consequently,

$$f(K, 2, m_1, \alpha_1) \geq \frac{m_1 \log_2 3 - \log_2 |P_1|}{2m_1} - \frac{\log_2 120}{8m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 2, m_1, \alpha_1) \geq \frac{m_1 \log_2 3 - \log_2 2^{0.76m_1}}{2m_1} - \frac{\log_2 120}{8m_1} \geq 0.277946797383 \geq a_2.$$

Table 4 $f(K, 3, m_1, \alpha_1)$ $(17 \leq m_1 \leq 24$ or $m_1 = 32)$

m_1	$\lvert P_1\rvert$	$f(K, 3, m_1, \alpha_1)$	m_1	$\lvert P_1\rvert$	$f(K, 3, m_1, \alpha_1)$
32	319979520	0.354373677316	24	244823040	0.255640461907
23	10200960	0.304218489247	22	887040	0.341130874456
21	120960	0.371255664000	20	6840	0.425558279022
19	342	0.488691663793	18	4896	0.407701657563
17	16320	0.358410272828			

If $m_1 \leq 17$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{2+m_1}{2}$. Consequently

$$f(K, 2, m_1, \alpha_1) \geq \frac{\log_2 \binom{2+m_1}{2}}{8m_1} - \frac{\log_2 120}{8m_1} \geq 0.167386172529 \geq a_2.$$

If $18 \leq m_1 \leq 24$ or $m_1 = 32$, then by Table 2, we can calculate the minimum values of $f(K, 2, m_1, \alpha_1)$ as follows (Table 5).

Lemma 3.5 *Let $G \lesssim H \wr P_1 \wr \cdots \wr P_z$ be a transitive permutation group of degree n which does not contain any A_ℓ ($\ell > 5$) as a composition factor, where $H \cong \mathrm{M}_{24}$, and all the P_i are primitive groups. If $\deg(P_1) \neq 5$, then $rs(G) \geq a_2$.*

Proof By Table 1, we have $s(H) = 49$. Let $K = H \wr P_1$, where $\deg(P_1) \neq 5$. By Proposition 3.3, it suffices to prove that

$$\frac{\log_2 s(K)}{24m_1} - \frac{\log_2 \alpha_1}{24m_1} \geq a_2.$$

Denote by $f(K, m_1, \alpha_1)$ the number $\frac{\log_2 s(K)}{24m_1} - \frac{\log_2 \alpha_1}{24m_1}$. By Lemma 2.3 (i), we have that $s(K) \geq 49^{m_1}/|P_1|$, and thereby

$$f(K, m_1, \alpha_1) \geq \frac{m_1 \log_2 49 - \log_2 |P_1|}{24m_1} - \frac{\log_2 120}{96m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, m_1, \alpha_1) \geq \frac{m_1 \log_2 49 - \log_2 2^{0.76m_1}}{24m_1} - \frac{\log_2 120}{96m_1} \geq 0.199401705757 \geq a_2.$$

If $m_1 = 2$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{50}{48}$, and so

$$f(K, 2, \alpha_1) \geq \frac{\log_2 \binom{50}{48}}{48} - \frac{\log_2 120}{192} \geq 0.177746737187 \geq a_2.$$

If $3 \leq m_1 \leq 24$ or $m_1 = 32$, then we apply Table 2 to calculate the minimum values of $f(K, m_1, \alpha_1)$ as follows (Table 6).

Table 5 $f(K, 2, m_1, \alpha_1)$ ($18 \leq m_1 \leq 24$ or $m_1 = 32$)

m_1	$\lvert P_1\rvert$	$f(K, 2, m_1, \alpha_1)$	m_1	$\lvert P_1\rvert$	$f(K, 2, m_1, \alpha_1)$
32	319,979,520	0.323829325064	24	244,823,040	0.175941943221
23	10,200,960	0.248808984230	22	887,040	0.304177562044
21	120,960	0.349364746360	20	6840	0.430818668894
19	342	0.525518746050	18	4896	0.404033736705

Table 6 $f(K, m_1, \alpha_1)$ $(3 \leq m_1 \leq 24$ or $m_1 = 32)$

m_1	$\|P_1\|$	$f(K, m_1, \alpha_1)$	m_1	$\|P_1\|$	$f(K, m_1, \alpha_1)$
32	319,979,520	0.194909619836	24	244,823,040	0.182567967910
23	10,200,960	0.188640221327	22	887,040	0.193254269478
21	120,960	0.197019861714	20	6840	0.203807695049
19	342	0.211699368146	18	4896	0.201575617367
17	16,320	0.195414194275	16	322,560	0.181795382392
15	20,160	0.189429769429	14	2184	0.195793022984
13	5616	0.188490955462	12	95,040	0.170533153797
11	7920	0.178347730763	10	1440	0.183035511233
9	1512	0.177052886684	8	1344	0.170826243131
7	168	0.179666243078	6	120	0.173990595973
5	20	0.183540820640	4	20	0.168199523196
3	6	0.174061727538			

This completes the proof of this proposition. □

Lemma 3.6 *Let* $H \cong \mathrm{M}_{24} \wr \mathrm{S}_5 \wr \cdots \wr \mathrm{S}_5$ *be a transitive permutation group of degree* $24 \cdot 5^k$ *with* $k \geq 0$. *Let* $P_1, \ldots, P_\ell$ *be primitive permutation groups. If* $\deg(P_1) \neq 5$, *then* $rs(H \wr \mathrm{S}_5 \wr \mathrm{S}_5) \leq rs(H \wr P_1 \wr \cdots \wr P_\ell)$.

Proof Let $s(H) = A$ and $n = \deg(H) = 24 \cdot 5^k$. Then $A \geq s(\mathrm{M}_{24}) = 49$. Let $B = s(H \wr \mathrm{S}_5)$. By Lemma 2.3 (ii) we have $B = \binom{A+4}{5} = \frac{(A+4)(A+3)(A+2)(A+1)A}{120}$, and thus $B + 4 \leq \frac{(A+4)^5}{120}$. Again using Lemma 2.3 (ii), one has

$$s(H \wr \mathrm{S}_5 \wr \mathrm{S}_5) = \binom{B+4}{5} = \frac{(B+4)(B+3)(B+2)(B+1)B}{120},$$

and so $s(H \wr \mathrm{S}_5 \wr \mathrm{S}_5) \leq \frac{[\frac{(A+4)^5}{120}]^5}{120}$. Thus, we have the following inequality

$$\begin{aligned} rs(H \wr \mathrm{S}_5 \wr \mathrm{S}_5) &= \frac{\log_2 s(H \wr \mathrm{S}_5 \wr \mathrm{S}_5)}{n \cdot 5^2} \leq \frac{\log_2 \frac{[\frac{(A+4)^5}{120}]^5}{120}}{n \cdot 5^2} \\ &= \frac{\log_2 (A+4)}{n} - \frac{\log_2 120}{5n} - \frac{\log_2 120}{25n}. \end{aligned}$$

On the other hand, let $\deg(P_1) = m_1$. By Lemma 2.3 (i), we have $s(H \wr P_1) \geq s(H)^{m_1}/|P_1| = A^{m_1}/|P_1|$, and hence

$$\frac{\log_2 s(H \wr P_1)}{nm_1} - \frac{\log_2 \alpha_1}{nm_1} \geq \frac{\log_2 (A)}{n} - \frac{\log_2 |P_1|}{nm_1} - \frac{\log_2 \alpha_1}{nm_1}.$$

Table 7 $f(P_1, m_1, \alpha_1)$ $(2 \leq m_1 \leq 24)$

m_1	$\|P_1\|$	$f(K, m_1, \alpha_1)$	m_1	$\|P_1\|$	$f(K, m_1, \alpha_1)$
24	244,823,040	1.089185060204	23	10,200,960	1.087344532260
22	887,040	1.976607376633	21	120,960	0.886233008001
20	6840	0.723325166934	19	342	0.533992500862
18	4896	0.776895027311	17	16,320	0.924769181515
16	322,560	1.251620666706	15	20,160	1.068395377819
14	2184	0.915677127844	13	5616	1.090926913016
12	95,040	1.521914155383	11	7920	1.334364305806
10	1440	1.221857574523	9	1512	1.345440563680
8	1344	1.514880008960	7	168	1.302720010240
6	120	1.438935540752	5	20	1.209730148758
4	20	1.512162685947	3	6	1.437228383208
2	2	1.363361324451			

Let $G = H \wr P_1 \wr \cdots \wr P_\ell$. Then it is enough to verify the following inequality:

$$\frac{\log_2(A+4)}{n} - \frac{\log_2 120}{5n} - \frac{\log_2 120}{25n} \leq \frac{\log_2(A)}{n} - \frac{\log_2 |P_1|}{nm_1} - \frac{\log_2 \alpha_1}{nm_1}. \tag{3}$$

Consequently, (3) holds if and only if

$$\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_1}{m_1} \leq \frac{\log_2 120}{5} + \frac{\log_2 120}{25} - \log_2\left(\frac{A+4}{A}\right).$$

Since $A \geq 49$, we have $\log_2(\frac{A+7}{A}) \leq 0.113210611045$. Thus, we only need to check the following inequality:

$$\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_1}{m_1} \leq 1.544443132498.$$

Denote by $f(P_1, m_1, \alpha_1)$ the number $\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_1}{m_1}$. If $m_1 > 24$, then by Lemma 2.4 (ii), we have $|P_1| < 2^{m_1}$, and so

$$f(P_1, m_1, \alpha_1) < \frac{\log_2 2^{m_1}}{m_1} + \frac{\log_2 \alpha_1}{m_1} < 1 + \frac{\log_2 120}{24 \times 4} \approx 1.071946777038.$$

If $m_1 \leq 24$, then by Table 2, we can check each minimum value of function $f(P_1, m_1, \alpha_1)$ using Table 7.

This completes the proof of this lemma. □

We are now ready to complete the proof of proposition 3.1.

Proof of Proposition 3.1: Note that $a_k = \frac{\log_2 s(M_{24} \wr S_5 \wr \cdots \wr S_5)}{24 \cdot 5^k}$ and $M = \lim_{k \to +\infty} a_k$. Since the sequence $\{a_k\}_{k \geq 0}$ is strictly decreasing, this implies that $M < a_2$. If G is primitive, then by Lemma 3.2, we have $rs(G) \geq a_2$. Now, if G is not primitive, then G is induced from M_{24} by Lemma 3.4. Combining Lemma 3.5 and Lemma 3.6, one has

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\to+\infty} \frac{\log_2 s(M_{24} \wr \overbrace{S_5 \wr \cdots \wr S_5}^{\text{k } terms})}{24 \cdot 5^k}.$$

This completes the proof of this proposition. □

4 Proof of Theorem 1.2

In this section, we shall give the explicit lower bounds $\inf\left(\frac{\log_2 s(G)}{n}\right)$ where G does not contain any alternating group A_ℓ with $\ell > t$ for $t \in [17, 166]$ as a composition factor. We give the detailed proof for the case where $t = 17$. Essentially the identical proof works in the remaining cases up to replacing a few numbers appropriately.

Proposition 4.1 *We have*

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k\mapsto+\infty} \frac{\log_2 s(\overbrace{S_{17} \wr S_{17} \wr \cdots \wr S_{17}}^{k\ terms})}{17^k}.$$

where the infimum is taken over all permutation groups G not containing any composition factor A_ℓ with $\ell > 17$, and n denotes the degree of G.

For the remaining of section, we prove Proposition 4.1 by a series of lemmas. Let

$$b_k = \frac{\log_2 s(S_{17} \wr S_{17} \wr \cdots \wr S_{17})}{17^k} and M = \lim_{k\to+\infty} b_k, where k \geq 1.$$

We first define a sequence $\{s_k\}_{k\geq 1}$ where $s_1 = s(S_{17}) = 18$ and $s_{k+1} = \binom{s_k+16}{17}$ for $k \geq 1$. Clearly the sequence $\{s_k\}_{k\geq 1}$ is strictly increasing. By the definition, we have $b_k = \frac{\log_2 s_k}{17^k}$. It is easy to see $b_k > 0$. Noting that

$$b_{k+1} = \frac{\log_2 \binom{s_k+16}{17}}{17^{k+1}} = \frac{\log_2(\frac{(s_k+16)(s_k+15)\cdots(s_k+2)(s_k+1)s_k}{17!})}{17^{k+1}} < \frac{\log_2(s_k)^{17}}{17^{k+1}} = b_k,$$

the sequence $\{b_k\}_{k\geq 1}$ is strictly decreasing, and hence M exists.

Applying Lemma 2.3, we calculate that

$$b_1 \approx 0.245289705967 \text{ and } b_2 \approx 0.107681363290.$$

Lemma 4.2 *Let G be a primitive permutation group of degree n. If G does not contain any A_ℓ ($\ell > 17$) as a composition factor, then $rs(G) \geq b_2$.*

Proof If $n \geq 25$ and $n \neq 32$, then by Theorem 2.4 (ii), we have $s(G) \geq 2^n/|G| \geq 2^{0.24n}$, and so $rs(G) = \frac{\log_2 s(G)}{n} \geq 0.24 \geq b_2$.

If $n \leq 24$, then $s(G) \geq n+1$, and thus

$$rs(G) = \frac{\log_2 s(G)}{n} \geq \frac{\log_2(n+1)}{n} \geq 0.193494007907 \geq b_2.$$

If $n = 32$, then by Table 1, we have $s(G) \geq 361$. In this case, one has

$$rs(G) = \frac{\log_2 s(G)}{32} \geq 0.265495469577 \geq b_2.$$

This completes the proof. □

Arguing as in Proposition 3.3, we have the following conclusion.

Lemma 4.3 *Let G be a transitive permutation group of degree n induced from a permutation group H of degree m, where G does not contain any* A_ℓ ($\ell > 17$) *as a composition factor. Let* $\alpha_2 = (17!)^{\frac{1}{16}}$. *If* $\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_2}{m} \geq \beta$, *then* $rs(G) \geq \beta$.

Lemma 4.4 *Let G be a transitive permutation group of degree n, which be induced from a primitive permutation group H of degree m. If G does not contain any* A_ℓ ($\ell > 17$) *as a composition factor, and* $H \ncong S_{17}$, *then* $rs(G) \geq b_2$.

Proof In view of Lemma 4.3, it suffices to prove the following inequality:

$$\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_2}{m} \geq b_2. \tag{4}$$

Suppose that $m \geq 25$ and $m \neq 32$. By Lemma 2.4 (ii), we have $|H| \leq 2^{0.76m}$. Since $s(H) \geq 2^m/|H|$, one has

$$\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_2}{m} \geq \frac{\log_2(2^m/2^{0.76m})}{m} - \frac{\log_2 17!}{16m} \geq 0.119155991722 > b_2.$$

In this case, inequality (4) follows.

Now, we use $f(H, m, \alpha_2)$ to denote $\frac{\log_2 s(H)}{m} - \frac{\log_2 \alpha_2}{m}$. If $13 \leq m \leq 24$ or $m = 32$, then by Table 2, we can verify $f(H, m, \alpha_2) \geq 0.108067068215$, and so (4) holds.

If $2 \leq m \leq 12$, then by Lemma 4.2, we may assume that $G \lesssim H \wr P_1 \wr P_2 \wr \cdots \wr P_k$, where $\deg(P_1) = m_1$. Let $K = H \wr P_1$. By Lemma 3.3, it is enough to check the following inequality:

$$\frac{\log_2 s(K)}{mm_1} - \frac{\log_2 \alpha_2}{mm_1} \geq b_2. \tag{5}$$

Denote by $f(K, m, m_1, \alpha_2)$ the number $\frac{\log_2 s(K)}{mm_1} - \frac{\log_2 \alpha_2}{mm_1}$. Then we only need to check the following inequality:

$$f(K, m, m_1, \alpha_2) = \frac{\log_2 s(K)}{mm_1} - \frac{\log_2 17!}{16mm_1} \geq b_2.$$

In what follows, we separate the argument into eleven cases.

Case 1 $m = 9$. In this case, we know that $s(H) \geq 10$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 10^{m_1}/|P_1|$, and thereby

$$f(K, 9, m_1, \alpha_2) \geq \frac{m_1 \log_2 10 - \log_2 |P_1|}{9m_1} - \frac{\log_2 17!}{144m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 9, m_1, \alpha_2) \geq 0.27123156517 \geq b_2.$$

If $m_1 \leq 21$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{9+m_1}{9}$, and thereby

$$f(K, 9, m_1, \alpha_2) \geq 0.109783771278 \geq b_2.$$

If $22 \leq m_1 \leq 24$ or $m_1 = 32$, then by Table 2, we can check all the values of $f(K, 9, m_1, \alpha_2) \geq 0.226101898040$, and so (5) follows.

Case 2 $m = 8$. In this case, we know that $s(H) \geq 10$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 10^{m_1}/|P_1|$, and so

$$f(K, 8, m_1, \alpha_2) \geq \frac{m_1 \log_2 10 - \log_2 |P_1|}{8m_1} - \frac{\log_2 17!}{128m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 8, m_1, \alpha_2) \geq 0.305135510826 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{9+m_1}{9}$, and so

$$f(K, 8, m_1, \alpha_2) \geq 0.115519620447 \geq b_2.$$

Case 3 $m = 7$. In this case, we know $s(H) \geq 10$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 10^{m_1}/|P_1|$, and thus

$$f(K, 7, m_1, \alpha_2) \geq \frac{m_1 \log_2 10 - \log_2 |P_1|}{7m_1} - \frac{\log_2 17!}{112m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 7, m_1, \alpha_2) \geq 0.348726298087 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{9+m_1}{9}$, and thereby

$$f(K, 7, m_1, \alpha_2) \geq 0.132022423368 \geq b_2.$$

Case 4 $m = 12$. For this case, we know that $s(H) \geq 14$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 14^{m_1}/|P_1|$, and consequently

$$f(K, 12, m_1, \alpha_2) \geq \frac{m_1 \log_2 14 - \log_2 |P_1|}{12m_1} - \frac{\log_2 17!}{192m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 12, m_1, \alpha_2) \geq 0.243875909482 \geq b_2.$$

If $m_1 \leq 20$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{13+m_1}{13}$, and so

$$f(K, 12, m_1, \alpha_2) \geq 0.108638661215 \geq b_2.$$

If $21 \leq m_1 \leq 24$ or $m_1 = 32$, then by Table 2, we can calculate

$$f(K, 9, m_1, \alpha_2) = 0.210028659127 \geq b_2.$$

Case 5 $m = 11$. In this case, we have $s(H) \geq 14$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 14^{m_1}/|P_1|$, and thus

$$f(K, 11, m_1, \alpha_2) \geq \frac{m_1 \log_2 14 - \log_2 |P_1|}{11m_1} - \frac{\log_2 17!}{176m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 11, m_1, \alpha_2) \geq 0.266046446707 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{13+m_1}{13}$, and so

$$f(K, 11, m_1, \alpha_2) \geq 0.108746702344 \geq b_2.$$

Case 6 $m = 10$. In this case, we know that $s(H) \geq 14$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 14^{m_1}/|P_1|$, and thus

$$f(K, 10, m_1, \alpha_2) \geq \frac{m_1 \log_2 14 - \log_2 |P_1|}{10m_1} - \frac{\log_2 17!}{160m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 11, m_1, \alpha_2) \geq 0.292651091378 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{13+m_1}{13}$, and so

$$f(K, 10, m_1, \alpha_2) \geq 0.119621372578 \geq b_2.$$

Case 7 $m = 6$. Then $s(H) \geq 8$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 8^{m_1}/|P_1|$, and consequently

$$f(K, 6, m_1, \alpha_2) \geq \frac{m_1 \log_2 8 - \log_2 |P_1|}{6m_1} - \frac{\log_2 17!}{96m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 6, m_1, \alpha_2) \geq 0.353192665287 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{7+m_1}{7}$, and so

$$f(K, 6, m_1, \alpha_2) \geq 0.127120125055 \geq b_2.$$

Case 8 $m = 5$. In this case, we know that $s(H) \geq 6$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 6^{m_1}/|P_1|$, and thus

$$f(K, 5, m_1, \alpha_2) \geq \frac{m_1 \log_2 6 - \log_2 |P_1|}{5m_1} - \frac{\log_2 17!}{80m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 5, m_1, \alpha_2) \geq 0.340823698489 \geq b_2.$$

If $m_1 \leq 24$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{5+m_1}{5}$, and so

$$f(K, 5, m_1, \alpha_2) \geq 0.115304404373 \geq b_2.$$

Case 9 $m = 4$. In this case, we know that $s(H) \geq 5$. By Lemma 2.3 (i), one has $s(K) \geq s(H)^{m_1}/|P_1| \geq 5^{m_1}/|P_1|$, and so

$$f(K, 4, m_1, \alpha_2) \geq \frac{m_1 \log_2 5 - \log_2 |P_1|}{4m_1} - \frac{\log_2 17!}{64m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, 4, m_1, \alpha_2) \geq 0.360271021652 \geq b_2.$$

If $m_1 \le 24$, then by Lemma 2.3 (ii), we have $s(K) \ge \binom{4+m_1}{4}$, and thereby

$$f(K, 4, m_1, \alpha_2) \ge 0.117713287755 \ge b_2.$$

Case 10 $m = 3$. Then $s(H) \ge 4$. By Lemma 2.3 (i), one has $s(K) \ge s(H)^{m_1}/|P_1| \ge 4^{m_1}/|P_1|$, and so

$$f(K, 3, m_1, \alpha_2) \ge \frac{m_1 \log_2 4 - \log_2 |P_1|}{3m_1} - \frac{\log_2 17!}{48m_1}.$$

If $m_1 \ge 25$ and $m_1 \ne 32$, then by Lemma 2.4 (iii), we have $|P_1| \le 2^{0.76m_1}$, and so

$$f(K, 3, m_1, \alpha_2) \ge 0.373051997241 \ge b_2.$$

If $m_1 \le 24$, then by Lemma 2.3 (ii), we have $s(K) \ge \binom{3+m_1}{3}$, and so

$$f(K, 3, m_1, \alpha_2) \ge 0.117960009756 \ge b_2.$$

Case 11 $m = 2$. Then $s(H) \ge 3$. By Lemma 2.3 (i), one has $s(K) \ge s(H)^{m_1}/|P_1| \ge 3^{m_1}/|P_1|$, and so

$$f(K, 2, m_1, \alpha_2) \ge \frac{m_1 \log_2 3 - \log_2 |P_1|}{2m_1} - \frac{\log_2 17!}{32m_1}.$$

If $m_1 \ge 25$ and $m_1 \ne 32$, then by Lemma 2.4 (iii), we have $|P_1| \le 2^{0.76m_1}$, and so

$$f(K, 2, m_1, \alpha_2) \ge 0.352059246222 \ge b_2.$$

If $m_1 \le 24$, then by Lemma 2.3 (ii), we have $s(K) \ge \binom{2+m_1}{2}$, and consequently,

$$f(K, 2, m_1, \alpha_2) \ge 0.110899910437 \ge b_2.$$

This completes the proof of this lemma. □

Lemma 4.5 *Let* $G \lesssim \mathrm{S}_{17} \wr P_1 \wr \cdots \wr P_k$ *be a transitive permutation group of degree* n *which does not contain any* A_ℓ $(\ell > 17)$ *as a composition factor, where all the* P_i *are primitive groups. If* $\deg(P_1) \ne 17$*, then* $rs(G) \ge b_2$.

Proof By Table 1, we know that $s(H) = 18$. Let $K = \mathrm{S}_{17} \wr P_1$, where $\deg(P_1) \ne 17$. By Lemma 4.3, it suffices to prove that

$$\frac{\log_2 s(K)}{17m_1} - \frac{\log_2 \alpha_2}{17m_1} \ge b_2.$$

Denote by $f(K, m_1, \alpha_2)$ the number $\frac{\log_2 s(K)}{17m_1} - \frac{\log_2 \alpha_2}{17m_1}$. By Lemma 2.3 (i), we have $s(K) \geq 18^{m_1}/|P_1|$, and so

$$f(K, m_1, \alpha_2) \geq \frac{m_1 \log_2 18 - \log_2 |P_1|}{17m_1} - \frac{\log_2 17!}{272m_1}.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then by Lemma 2.4 (iii), we have $|P_1| \leq 2^{0.76m_1}$, and so

$$f(K, m_1, \alpha_2) \geq 0.193475352539 \geq b_2.$$

If $m_1 \leq 13$, then by Lemma 2.3 (ii), we have $s(K) \geq \binom{17+m_1}{17}$, and so

$$f(K, m_1, \alpha_2) \geq 0.107757777700 \geq b_2.$$

If $14 \leq m_1 \leq 24$ or $m_1 = 32$, then by Table 2, we calculate

$$f(K, m_1, \alpha_2) \geq 0.166906219845 \geq b_2.$$

□

Lemma 4.6 *Let $H \cong \mathrm{S}_{17} \wr \mathrm{S}_{17} \wr \cdots \wr \mathrm{S}_{17}$ be a transitive permutation group of degree 17^k with $k \geq 1$. Let $P_1, \ldots, P_\ell$ be primitive permutation groups. If $\deg(P_1) \neq 17$, then $rs(H \wr \mathrm{S}_{17} \wr \mathrm{S}_{17}) \leq rs(H \wr P_1 \wr \cdots \wr P_\ell)$.*

Proof Let $s(H) = A$ and $n = \deg(H) = 17^k$. Then $A \geq s(\mathrm{S}_{17}) = 18$. Let $B = s(H \wr \mathrm{S}_{17})$. By Lemma 2.3 (ii), we have $B = \binom{A+16}{17} = \frac{(A+16)(A+15)\cdots(A+2)(A+1)A}{17!}$, and so

$$B + 16 \leq \frac{(A+16)^{17}}{17!}.$$

Again by Lemma 2.3 (ii), one has

$$s(H \wr \mathrm{S}_{17} \wr \mathrm{S}_{17}) = \binom{B+16}{17} = \frac{(B+16)(B+15)\cdots(B+2)(B+1)B}{17!},$$

and thereby

$$s(H \wr \mathrm{S}_{17} \wr \mathrm{S}_{17}) \leq \frac{[\frac{(A+16)^{17}}{17!}]^{17}}{17!}.$$

Then we obtain the following inequality

$$\begin{aligned} rs(H \wr \mathrm{S}_{17} \wr \mathrm{S}_{17}) &= \frac{\log_2 s(H \wr \mathrm{S}_{17} \wr \mathrm{S}_{17})}{n \cdot 17^2} \leq \frac{\log_2 \frac{[\frac{(A+16)^{17}}{17!}]^{17}}{17!}}{n \cdot 17^2} \\ &= \frac{\log_2 (A+16)}{n} - \frac{\log_2 17!}{17n} - \frac{\log_2 17!}{289n} \end{aligned}$$

On the other hand, let $\deg(P_1) = m_1$. By Lemma 2.3 (i), we have $s(H \wr P_1) \geq s(H)^{m_1}/|P_1| = A^{m_1}/|P_1|$, and so

$$\frac{\log_2 s(H \wr P_1)}{nm_1} - \frac{\log_2 \alpha_2}{nm_1} \geq \frac{\log_2 A}{n} - \frac{\log_2 |P_1|}{nm_1} - \frac{\log_2 \alpha_2}{nm_1}$$

Let $G = H \wr P_1 \wr \cdots \wr P_\ell$. It suffices to check the following inequality:

$$\frac{\log_2(A+16)}{n} - \frac{\log_2 17!}{17n} - \frac{\log_2 17!}{289n} \leq \frac{\log_2 A}{n} - \frac{\log_2 |P_1|}{nm_1} - \frac{\log_2 \alpha_2}{nm_1} \tag{6}$$

Consequently, (6) follows if and only if

$$\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_2}{m_1} \leq \frac{\log_2 17!}{17} + \frac{\log_2 17!}{289} - \log_2(\frac{A+16}{A}).$$

Since $A \geq 18$, it follows that $\log_2(\frac{A+16}{A}) \leq 0.917537839808$. Therefore, we only need to check the following inequality:

$$\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_2}{m_1} \leq 2.093108733204.$$

Denote by $f(P_1, m_1, \alpha_2)$ the number $\frac{\log_2 |P_1|}{m_1} + \frac{\log_2 \alpha_2}{m_1}$. If $m_1 > 24$, then by Lemma 2.4 (ii), we know that $|P_1| < 2^{m_1}$, and so

$$f(P_1, m_1, \alpha_2) < 1.120844008278 < 2.093108733204.$$

If $m_1 \leq 24$, then by Table 2, we calculate

$$f(P_1, m_1, \alpha_2) \leq 1.868687569222 < 2.093108733204.$$

□

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1: Note that $b_k = \frac{\log_2 s(S_{17} \wr S_{17} \wr \cdots \wr S_{17})}{17^k}$ and $M = \lim_{k \to +\infty} b_k$. Since the sequence $\{b_k\}_{k \geq 1}$ is strictly decreasing, it follows that $M < b_2$. If G is primitive, then $rs(G) \geq b_2$ by Lemma 4.2. Now, if G is not primitive, then G is induced from S_{17} by Lemma 4.4. Combining Lemma 4.5 and Lemma 4.6, one has

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k \to +\infty} \frac{\log_2 s(\overbrace{S_{17} \wr S_{17} \wr \cdots \wr S_{17}}^{\text{k } terms})}{17^k}.$$

This completes the proof of this proposition. □

5 Further Considerations

In this section, we first make some remarks on Proposition 3.1. For convenience, we use the notation defined in Proposition 3.1.

Remark 5.1 Using the proof of Lemma 3.6, we can obtain a good estimate of the limit. We first notice that

$$rs(\mathrm{M}_{24} \wr \mathrm{S}_5 \wr \mathrm{S}_5 \wr \cdots) \leq \frac{\log_2(\binom{53}{5}+4)}{24 \cdot 5} - \frac{\log_2 120}{24 \cdot 5^2} - \frac{\log_2 120}{24 \cdot 5^3} - \cdots .$$

This implies that

$$\inf\left(\frac{\log_2 s(G)}{n}\right) \leq \frac{\log_2(\binom{53}{5}+4)}{24 \cdot 5} - \frac{\log_2 120}{24 \cdot 5}\left(\frac{1}{5}+\frac{1}{5^2}+\cdots\right) \approx 0.164381169292.$$

On the other hand, by Lemma 3.5 and the value a_2, we obtain

$$\inf\left(\frac{\log_2 s(G)}{n}\right) \geq 0.164381160912.$$

Taking into consideration the possible mistakes in the last two digits, the following bound can be guaranteed,

$$0.164381160900 < \inf\left(\frac{\log_2 s(G)}{n}\right) < 0.164381169300.$$

We compare the bound of $\inf\left(\frac{\log_2 s(G)}{n}\right)$ in Proposition 3.1 with that of Proposition 4.1. Now, we make the following remarks. For convenience, we use the notation defined in Proposition 4.1.

Remark 5.2 By the proof of Lemma 4.6, one can obtain a good estimate of the limit. We first observe that

$$rs(\mathrm{S}_{17} \wr \mathrm{S}_{17} \wr \mathrm{S}_{17} \wr \cdots) \leq \frac{\log_2(\binom{34}{17}+16)}{17^2} - \frac{\log_2 17!}{17^3} - \frac{\log_2 17!}{17^4} - \cdots .$$

This implies that

$$\inf\left(\frac{\log_2 s(G)}{n}\right) \leq \frac{\log_2(\binom{34}{17}+16)}{17^2} - \frac{\log_2 17!}{17^2}\left(\frac{1}{17}+\frac{1}{17^2}+\cdots\right) \approx 0.097227729390.$$

On the other hand, by Lemma 4.3 and the value b_2, we obtain

$$\inf\left(\frac{\log_2 s(G)}{n}\right) \geq 0.097227729356.$$

Taking into consideration the possible mistakes in the last two digits, the bound can be guaranteed

$$0.097227729300 < \inf\left(\frac{\log_2 s(G)}{n}\right) < 0.097227729395.$$

Remark 5.3 One of the main difficulties of determining $\inf\left(\frac{\log_2 s(G)}{n}\right)$ is to identify the group that achieves the lower bound. We found a range of the bound, but we still do not know what the exact value of the above limit is. Although we conjecture this limit is a rational number, we cannot prove this. Therefore, one interesting problem is to determine whether the limit is a rational number or not.

Remark 5.4 A related problem would be to determine what is $\inf\left(\frac{\log_2 s(G)}{n}\right)$ where G does not contain any A_ℓ with $\ell > t \geq 5$ for any positive integer t. The second author [10] answered this question for $t = 4$, and the work in this paper has answered this question for $5 \leq t \leq 166$. We believe the following is true.

Conjecture 5.5 *Let G be a permutation group of degree n. If G does not contain any A_ℓ with $\ell > t$ for $167 \leq t < \infty$ as a composition factor, then*

$$\inf\left(\frac{\log_2 s(G)}{n}\right) = \lim_{k \mapsto +\infty} \frac{\log_2 s(S_t \wr S_t \wr \cdots \wr S_t)}{t^k}.$$

Although it might be possible to go beyond $t = 166$ by tweaking the methods in this paper, it seems that our method is not strong enough for a general proof of this.

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