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Interpolations of Mixed-Norm Function Spaces

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Abstract
This article is devoted to presenting a general interpolation result on mixed-norm function spaces generated by quasi-Banach lattices. Under certain conditions, the authors show that such mixed-norm function spaces are closed under the Calderón product and the $\pm$ interpolation method. As applications, the authors obtain some new interpolation results for mixed-norm variable Lebesgue spaces, mixed-norm Lorentz spaces, and mixed-norm Morrey spaces.

Keywords Calderón product · Gagliardo–Peetre method · $\pm$ Method · Mixed-norm function space · Ball quasi-Banach function space

Mathematics Subject Classification Primary 42B35; Secondary 46B70 · 46E30 · 46B42

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1 Introduction

As a generalization of the classical Lebesgue space, the mixed-norm Lebesgue space $L^{\mathbf{p}}(\mathbb{R}^n)$ with $\mathbf{p} := (p_1, \ldots, p_n) \in (0,\infty)^n$, which is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\mathbf{p}}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1}\, dx_1\right]^{p_2/p_1} \cdots dx_n\right\}^{1/p_n} < \infty,$$

was introduced by Benedek and Panzone [3]. After that, on the mixed-norm Lebesgue spaces as well as their weighted counterparts, the boundedness of some classical operators such as fractional integrals and Calderón–Zygmund operators (see, for instance, [3,8,10,46]), various inequalities such as Hardy–Littlewood–Sobolev inequalities and geometric inequalities (see, for instance, [7,9,65]), interpolations (see, for instance, [37,53]) and convergence problems of summability means (see, for instance, [35]) were considered. In recent years, there also exists an increasing interest in various mixed-norm function spaces based on mixed-norm Lebesgue spaces, including mixed-norm weak Lebesgue spaces (see, for instance, [7–10]), mixed-norm Morrey spaces (see, for instance, [56,61]), mixed-norm Hardy spaces (see, for instance, [12,13,30–34]), mixed-norm Besov spaces (see, for instance, [20]), and mixed-norm Triebel–Lizorkin spaces (see, for instance, [38,39]). Moreover, Bennett and Sharpley [4] considered a more general class of mixed-norm function spaces generated by Banach function spaces (see also [5]), which was further studied and developed in [1,2,11,28,50,62]. For more developments of mixed-norm function spaces, we refer the reader to the survey [36].

The main purpose of this article is to give a general result on the interpolation properties of mixed-norm function spaces generated by quasi-Banach lattices. Under certain conditions, we show that many mixed-norm function spaces, such as mixed-norm Lebesgue spaces with variable exponent, mixed-norm Lorentz spaces (and hence mixed-norm weak Lebesgue spaces), and mixed-norm Morrey spaces, are closed under the Calderón product and the $\pm$ interpolation method. These interpolation results might find potential applications in the study on the boundedness of operators on those mixed-norm function spaces.

Let $(\mathcal{X}, \mu)$ be a measure space, namely, $\mathcal{X}$ is a non-empty set and μ a non negative measure. Let $M(\mathcal{X})$ be the set of all extended complex-valued measurable functions on $\mathcal{X}$ and $L^0(\mathcal{X})$ the space of all equivalence classes of almost everywhere finite measurable functions on $\mathcal{X}$ with the topology of convergence in measure on sets of finite measure. A quasi-Banach space $X \subset L^0(\mathcal{X})$ is called a *quasi-Banach lattice of functions* (in short, *quasi-Banach lattice*) if, for any $f \in X$ and any $g \in M(\mathcal{X})$ satisfying $|g| \le |f|$ almost everywhere, it holds true that $g \in X$ and $\|g\|_X \le \|f\|_X$. Let (X_0, X_1) be a pair of quasi-Banach lattices on $(\mathcal{X}, \mu)$ and $\theta \in (0, 1)$. Then, their

Calderón product $X_0^{1-\theta} X_1^{\theta}$ is defined by setting

$$X_0^{1-\theta} X_1^{\theta} := \Big\{ f \in M(\mathcal{X}) : \ |f| \le |f_0|^{1-\theta} |f_1|^{\theta} \ \text{ almost everywhere with } f_i \in X_i, \ i \in \{0,1\} \Big\} \tag{1.1}$$

and, for any $f \in X_0^{1-\theta} X_1^{\theta}$, its norm $\|f\|_{X_0^{1-\theta} X_1^{\theta}}$ is defined by setting

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf \left\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} \right\}, \tag{1.2}$$

where the infimum is taken over all the presentations as in (1.1). Calderón [6] proved that, if X_0 and X_1 are two quasi-Banach lattices, then $X_0^{1-\theta} X_1^{\theta}$ is a quasi-Banach lattice as well.

Let $X := X(\mathcal{X}, \mu)$ and $Y := Y(\mathcal{Y}, \nu)$ be two quasi-Banach lattices. Then, the *mixed-norm function space* $Y(X)$, in the sense of Benedek and Panzone [3] (see also [5]), is defined to be the set of all measurable functions f on the product space $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$ satisfying

$$\|f\|_{Y(X)} := \Big\| \|f(x,y)\|_{X_x} \Big\|_{Y_y} < \infty. \tag{1.3}$$

Here and thereafter, $\|f(x,y)\|_{X_x}$ denotes the norm in X of $f(x,y)$ on the variable x and $\|f(x,y)\|_{Y_y}$ the norm in Y of $f(x,y)$ on the variable y. Observe that, if $f \in Y(X)$, then, for almost every $y \in \mathcal{Y}$, $f(\cdot, y) \in X$. It is also known that the mixed-norm function space $Y(X)$ generated by quasi-Banach lattices is still a quasi-Banach lattice with the quasi-norm $\|\cdot\|_{Y(X)}$ (see, for instance, [5]).

Recall that a Banach space E on $(\mathcal{X}, \mu)$ is called a *Banach ideal* if there exists a function $f \in E$ such that $f \ge 0$ almost everywhere on $\mathcal{X}$, and $\|g\|_E \le \|h\|_E$ whenever $g, \ h \in E$ satisfying $|g| \le |h|$ almost everywhere on $\mathcal{X}$ (see, for instance, [50, p. 282]). Clearly, any Banach ideal is a quasi-Banach lattice.

In [50], Maligranda studied the Calderón–Lozanovskiĭ construction for mixed-norm function spaces generated by Banach ideals, which also describes their Calderón product. However, it is unknown how the Calderón product behaves on mixed-norm function spaces generated by quasi-Banach lattices. The first main result of this article answers this question as follows.

Theorem 1.1 *Let* $\theta \in (0,1)$. *Assume that* (X_0, X_1) *and* (Y_0, Y_1) *are two pairs of quasi-Banach lattices, respectively, on* $(\mathcal{X}, \mu)$ *and* $(\mathcal{Y}, \nu)$. *Then,*

$$[Y_0(X_0)]^{1-\theta} [Y_1(X_1)]^{\theta} = Y(X)$$

with equivalent quasi-norms, where $X := X_0^{1-\theta} X_1^{\theta}$ *and* $Y := Y_0^{1-\theta} Y_1^{\theta}$.

The proof of Theorem 1.1 is given in Sect. 2. Applying Theorem 1.1, we further obtain some interpolation properties of mixed-norm function spaces generated by quasi-Banach lattices via some different interpolation methods.

To see this, we let X_0 and X_1 be two quasi-Banach spaces on $(\mathcal{X}, \mu)$, which are continuously embedded into a larger Hausdorff topological space Y. Let

$$\|f\|_{X_0\cap X_1} := \max\left\{\|f\|_{X_0}, \|f\|_{X_1}\right\}, \qquad \forall\, f \in X_0 \cap X_1.$$

By [6, p. 114], we know that $(X_0 \cap X_1, \|\cdot\|_{X_0\cap X_1})$ is complete. The space $X_0 + X_1$ is defined by setting

$$X_0 + X_1 := \{h \in Y : \ h = h_0 + h_1 \text{ with } h_i \in X_i,\ i \in \{0, 1\}\} \tag{1.4}$$

equipped with the norm

$$\|h\|_{X_0+X_1} := \inf\left\{\|h_0\|_{X_0} + \|h_1\|_{X_1}\right\}, \qquad \forall\, h \in X_0 + X_1, \tag{1.5}$$

where the infimum is taken over all the representations as in (1.4). Notice that X_0+X_1 is a quasi-Banach space; see [6, p. 114].

Next, we recall the definitions of the Gagliardo–Peetre interpolation method (see, for instance, [19]) and the $\pm$ interpolation method (see, for instance, [55,60] and also [22,23]).

Definition 1.2 Let (X_0, X_1) be a pair of quasi-Banach lattices and $\theta \in (0, 1)$.

(i) (The *Gagliardo–Peetre method*) We say $a \in \langle X_0, X_1\rangle_\theta$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$, $\sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i$ for any bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$ converges in X_j with $j \in \{0, 1\}$, and, moreover, there exists a nonnegative constant C such that, for any $j \in \{0, 1\}$,

$$\left\|\sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i\right\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|, \tag{1.6}$$

where C is independent of F $\{a_i\}_{i\in\mathbb{Z}}$ and $\{\varepsilon_i\}_{i\in\mathbb{Z}}$. Let

$$\|a\|_{\langle X_0,X_1\rangle_\theta} := \inf\{C\},$$

where the infimum is taken over all the nonnegative constants satisfying (1.6).

(ii) (The $\pm$ *method*) We say $a \in \langle X_0, X_1, \theta\rangle$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$, $\sum_{i\in F} \varepsilon_i 2^{i(j-\theta)} a_i$ for any finite subset $F \subset \mathbb{Z}$ and any bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$ converges in X_j with $j \in \{0, 1\}$, and, moreover, there exists a nonnegative constant C such that, for any $j \in \{0, 1\}$,

$$\left\|\sum_{i\in F} \varepsilon_i 2^{i(j-\theta)} a_i\right\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|, \tag{1.7}$$

where C is independent of $\{a_i\}_{i\in\mathbb{Z}}$ and $\{\varepsilon_i\}_{i\in\mathbb{Z}}$. Let

$$\|a\|_{\langle X_0,X_1,\theta\rangle} := \inf\{C\},$$

where the infimum is taken over all the nonnegative constants satisfying (1.7).

Below, for any two spaces X and Y, $X \subset Y$ means that X is a subset of Y, and $X \hookrightarrow Y$ means that X is continuously embedded into Y. It is easy to see that $\langle X_0, X_1\rangle_\theta \hookrightarrow \langle X_0, X_1, \theta\rangle$ for any $\theta \in (0, 1)$ and any pair (X_0, X_1) of quasi-Banach lattices.

Definition 1.3 Let (X_0, X_1) be a pair of quasi-Banach lattices. Then, a quasi-Banach space X is called an *intermediate space* with respect to (X_0, X_1) if and only if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

Assume that X is an intermediate space with respect to (X_0, X_1) and X° denotes the closure of $X_0 \cap X_1$ in X. The *Gagliardo closure* of X with respect to $X_0 + X_1$, denoted by $X^\sim$, is defined in the following sense: $a \in X^\sim$ if and only if there exists a sequence $\{a_i\}_{i\in\mathbb{N}} \subset X$ such that $a_i \to a$ in $X_0 + X_1$ as $i \to \infty$ and there exists a positive constant λ such that, for any $i \in \mathbb{N}$,

$$\|a_i\|_X \le \lambda. \tag{1.8}$$

The norm $\|a\|_{X^\sim}$ of a in $X^\sim$ is defined by setting $\|a\|_{X^\sim} := \inf\{\lambda\}$, where the infimum is taken over all $\lambda \in (0, \infty)$ satisfying (1.8) (see, for instance, [19]).

Let X be a quasi-Banach lattice and $p \in (0, \infty)$. The *p-convexification* of X (see, for instance, [55, p. 140]), denoted by $X^{(p)}$, is defined by setting $f \in X^{(p)}$ if and only if $|f|^p \in X$. Moreover, for any $f \in X^{(p)}$, let

$$\|f\|_{X^{(p)}} := \left\||f|^p\right\|_X^{1/p}. \tag{1.9}$$

The following definition is just [55, Definition 1.7].

Definition 1.4 A quasi-Banach lattice $X := (X, \|\cdot\|_X)$ is said to *be of type* $\mathfrak{C}$ if there exists an equivalent lattice quasi-norm $|||\cdot|||_X$ on X such that, for some $p \in [1, \infty)$, $X^{(p)}$ is a Banach lattice with respect to the norm defined as in (1.9) with $\|\cdot\|_X$ replaced by $|||\cdot|||_X$.

Let $(\mathcal{X}, \mu)$ be a σ-finite measure space. For any sequence $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ and $a \in \mathbb{R}$, $a_j \uparrow a$ as $j \to \infty$ means that $\{a_j\}_{j\in\mathbb{N}}$ is a non-decreasing sequence and $\lim_{j\to\infty} a_j = a$. Denote by $\mathbf{1}_E$ the *characteristic function* of E for any set $E \subset \mathcal{X}$.

The following definition of the quasi-Banach function space is from [4, Definitions 1.1 and 1.3] (see also [5, Subsection 3.1]).

Definition 1.5 A quasi-Banach space $X \subset M(\mathcal{X})$ with the quasi-norm $\|\cdot\|_X$ is called a *quasi-Banach function space* if it satisfies the following assertions:

(i) for any $f \in X$, $\|f\|_X = 0$ implies $f = 0$ almost everywhere on $\mathcal{X}$;

(ii) $g, f \in X$ and $|g| \le |f|$ almost everywhere imply $\|g\|_X \le \|f\|_X$;
(iii) if $\{f_j\}_{j\in\mathbb{N}} \subset M(\mathcal{X})$ is a sequence of nonnegative functions and $f \in M(\mathcal{X})$, then $f_j \uparrow f$ almost everywhere as $j \to \infty$ implies $\|f_j\|_X \uparrow \|f\|_X$ as $j \to \infty$;
(iv) for any measurable set $E \subset \mathcal{X}$ with finite measure, $\mathbf{1}_E \in X$;
(v) for any measurable set $E \subset \mathcal{X}$ with finite measure, there exists a positive constant $C_{(E)}$, depending on E, such that, for any $f \in X$,

$$\int_E |f(x)|\, d\mu(x) \le C_{(E)} \|f\|_X. \tag{1.10}$$

Additionally, if $\mathcal{X}$ is a product space, we require $E \subset I$ for some rectangle $I \subset \mathcal{X}$ with finite measure in (iv) and (v).

In some situations, the condition (1.10) is a bit restrictive because some classical function spaces, such as Morrey spaces, may violate it (see, for instance, [63]). To overcome this and unify more function spaces, Sawano et al. [64] introduced the following ball quasi-Banach function spaces.

Definition 1.6 A quasi-Banach space $X \subset M(\mathcal{X})$ with the quasi-norm $\|\cdot\|_X$ is called a *ball quasi-Banach function space* if it satisfies the following assertions:

(i) for any $f \in X$, $\|f\|_X = 0$ implies $f = 0$ almost everywhere on $\mathcal{X}$;
(ii) $g, f \in X$ and $|g| \le |f|$ almost everywhere imply $\|g\|_X \le \|f\|_X$;
(iii) if $\{f_j\}_{j\in\mathbb{N}} \subset M(\mathcal{X})$ is a sequence of non negative functions and $f \in M(\mathcal{X})$, then $f_j \uparrow f$ almost everywhere as $j \to \infty$ implies $\|f_j\|_X \uparrow \|f\|_X$ as $j \to \infty$;
(iv) for any ball $B \subset \mathcal{X}$ with finite radius, $\mathbf{1}_B \in X$ (if $\mathcal{X}$ is a product space, we require $B \subset I$ with I being the product of two balls with finite radius).

In addition, a ball quasi-Banach function space X is called a *ball Banach function space* if its norm satisfies the triangle inequality and, for any ball $B \subset \mathcal{X}$ with finite radius, there exists a positive constant $C_{(B)}$, depending on B, such that, for any $f \in X$,

$$\int_B |f(x)|\, d\mu(x) \le C_{(B)} \|f\|_X$$

(if $\mathcal{X}$ is a product space, we require $B \subset I$ with I being the product of two balls with finite radius).

Remark 1.7 Observe that, in Definition 1.6, if we replace any ball B by any bounded measurable set E, we obtain its another equivalent formulation of ball quasi-Banach function spaces.

Compared with quasi-Banach function spaces, ball quasi-Banach function spaces contain more function spaces. For instance, Morrey spaces, mixed-norm Lebesgue spaces, weighted Lebesgue spaces, and Orlicz-slice spaces are all ball quasi-Banach function spaces, but not quasi-Banach function spaces (see, for instance, [64]).

Bennett and Sharpley established several basic properties of quasi-Banach function spaces in [4, Chapter 1]. With some arguments similar to those used in [4, Chapter 1], we obtain the following properties of ball quasi-Banach function spaces and we omit the details.

Proposition 1.8 *Let X be a ball quasi-Banach function space on $(\mathcal{X}, \mu)$. Then, for any f, $g \in M(\mathcal{X})$ and sequence $\{f_j\}_{j\in\mathbb{N}} \subset M(\mathcal{X})$, the following statements hold true:*

(i) (the lattice property) *If $|g| \le |f|$ almost everywhere and $f \in X$, then $g \in X$ and $\|g\|_X \le \|f\|_X$.*
(ii) (the Fatou property) *Assume that $\{f_j\}_{j\in\mathbb{N}} \subset X$ is a sequence of nonnegative functions satisfying $f_j \uparrow f$ almost everywhere as $j \to \infty$. If $f \in X$, then $\|f_j\|_X \uparrow \|f\|_X$ as $j \to \infty$; whereas, if $f \notin X$, then $\|f_j\|_X \uparrow \infty$ as $j \to \infty$.*
(iii) (the Fatou lemma) *Assume that $\{f_j\}_{j\in\mathbb{N}} \subset X$ satisfy $\lim_{j\to\infty} f_j = f$ almost everywhere and $\liminf_{j\to\infty} \|f_j\|_X < \infty$. Then, $f \in X$ and $\|f\|_X \le \liminf_{j\to\infty} \|f_j\|_X$.*

Moreover, if X is a ball Banach function space and $f_j \to f$ in X as $j \to \infty$, then there exists a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{j_k} = f$ almost everywhere on $(\mathcal{X}, \mu)$.

Remark 1.9 Let $X := X(\mathcal{X}, \mu)$ and $Y := Y(\mathcal{Y}, \nu)$ be two ball quasi-Banach function spaces and $Y(X)$ the mixed-norm function space generated by X and Y defined as in (1.3). Using iteration, we find that the mixed-norm function space $Y(X)$ is also a ball quasi-Banach function space as well (see [5, p.158] for the case of Banach function spaces).

The following theorem is the second main result of this article.

Theorem 1.10 *Let $\theta \in (0, 1)$. Assume that, for any $i \in \{0, 1\}$, X_i and Y_i are ball quasi-Banach function spaces defined, respectively, on $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$, and $Y_i(X_i)$ is of type $\mathfrak{C}$. Let $X := X_0^{1-\theta} X_1^{\theta}$ and $Y := Y_0^{1-\theta} Y_1^{\theta}$. Then*

$$\langle Y_0(X_0), Y_1(X_1)\rangle_\theta = \left([Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta}\right)^{\circ} = (Y(X))^{\circ} \tag{1.11}$$

with equivalent quasi-norms. Moreover, if (X_0, X_1) and (Y_0, Y_1) are two pairs of ball Banach function spaces, then

$$\langle Y_0(X_0), Y_1(X_1), \theta\rangle = [Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta} = Y(X) \tag{1.12}$$

with equivalent norms.

The remainder of this article is organized as follows.

In Sect. 2, we give the proofs of Theorems 1.1 and 1.10. In the proof of Theorem 1.10, we need two key tools: one is the relation between the Calderón product and the ±-interpolation space established in [55, Theorem 2.1], and the other one is the behavior of limits of pointwise convergent sequences in ball quasi-Banach function spaces mentioned in Proposition 1.8. In Sect. 3, we show several examples of function spaces falling into the scope of Theorem 1.10, including mixed-norm Lebesgue spaces with variable exponent, mixed-norm Lorentz spaces (and hence mixed-norm weak Lebesgue spaces), and mixed-norm Morrey spaces. Some of them are new even for the classical function spaces, for instance, the Calderón product results for mixed-norm

Lebesgue spaces with variable exponent (see Proposition 3.3 below) and mixed-norm Lorentz spaces (see Propositions 3.7 and 3.9 below).

Throughout this article, we always adopt the following notation. Let $\mathbb{N} := \{1, 2, \ldots\}$. We always use C to denote a positive constant which is independent of the main parameters. The symbol $f \lesssim g$ means that there exists a positive constant C such that $f \le Cg$. The symbol $f \sim g$ is used as an abbreviation of $f \lesssim g \lesssim f$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \le h$. For any set E, let $\mathbf{1}_E$ be its *characteristic function*. For any $p \in [1, \infty]$, let p' be the *conjugate index* of p, namely, $1/p + 1/p' = 1$.

2 Proofs of Theorems 1.1 and 1.10

We begin with proving Theorem 1.1.

Proof (Proof of Theorem 1.1) We first show that

$$[Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta} \hookrightarrow Y(X). \tag{2.1}$$

Indeed, by the definition of the Calderón product, we know that, for any $f \in [Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta}$, there exist functions $f_0 \in Y_0(X_0)$ and $f_1 \in Y_1(X_1)$ such that, for almost every $(x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$|f(x, y)| \le |f_0(x, y)|^{1-\theta}|f_1(x, y)|^{\theta} \tag{2.2}$$

and

$$\|f_0\|_{Y_0(X_0)}^{1-\theta}\|f_1\|_{Y_1(X_1)}^{\theta} \sim \|f\|_{[Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta}} < \infty. \tag{2.3}$$

Since two functions differing only on a set of measure zero are the same in the Calderón product, without loss of generality, we may assume that (2.2) holds true for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Notice that, for almost every $y \in \mathcal{Y}$, $f_0(\cdot, y) \in X_0$ and $f_1(\cdot, y) \in X_1$. From this, the definition of the Calderón product, and (2.2), we deduce that, for almost every $y \in \mathcal{Y}$, $f(\cdot, y) \in X_0^{1-\theta}X_1^{\theta} = X$ and

$$\|f(\cdot, y)\|_X = \|f(\cdot, y)\|_{X_0^{1-\theta}X_1^{\theta}} \le \|f_0(\cdot, y)\|_{X_0}^{1-\theta}\|f_1(\cdot, y)\|_{X_1}^{\theta}; \tag{2.4}$$

furthermore, by the fact that $f_0 \in Y_0(X_0)$ and $f_1 \in Y_1(X_1)$, we conclude that $\|f_0(\cdot, y)\|_{X_0} \in Y_0$ and $\|f_1(\cdot, y)\|_{X_1} \in Y_1$, which, together with (2.4) and (2.3), implies that $\|f(\cdot, y)\|_X \in Y_0^{1-\theta}Y_1^{\theta} = Y$ and

$$\|f\|_{Y(X)} = \|\|f(\cdot, y)\|_X\|_Y \le \|f_0\|_{Y_0(X_0)}^{1-\theta}\|f_1\|_{Y_1(X_1)}^{\theta} \sim \|f\|_{[Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta}} < \infty.$$

This shows that (2.1) holds true.

Next, we prove

$$Y(X) \hookrightarrow [Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^\theta. \tag{2.5}$$

Observe that, for any $f \in Y(X)$, there exists a set $E \subset \mathcal{Y}$ with $\nu(E) = 0$ such that, for any given $y \in \mathcal{Y}\backslash E$, $f(\cdot, y) \in X = X_0^{1-\theta} X_1^\theta$. Then, by the definition of $X_0^{1-\theta} X_1^\theta$, we know that, for any $y \in \mathcal{Y}\backslash E$, there exist two functions $g_0(\cdot, y) \in X_0$ and $g_1(\cdot, y) \in X_1$ such that, for almost every $x \in \mathcal{X}$,

$$|f(x, y)| \le |g_0(x, y)|^{1-\theta} |g_1(x, y)|^\theta, \tag{2.6}$$

and

$$\|g_0(\cdot, y)\|_{X_0}^{1-\theta} \|g_1(\cdot, y)\|_{X_1}^\theta \le 2\|f(\cdot, y)\|_X. \tag{2.7}$$

Again, since two functions differing only on a set of measure zero are the same in X_0 and X_1, we may assume that (2.6) holds true for any $(x, y) \in \mathcal{X} \times (\mathcal{Y}\backslash E)$. Similarly, from the fact that $\|f(\cdot, y)\|_X \in Y = Y_0^{1-\theta} Y_1^\theta$, it follows that there exist two functions $h_0 \in Y_0$ and $h_1 \in Y_1$ such that, for any $y \in \mathcal{Y}$,

$$\|f(\cdot, y)\|_X \le |h_0(y)|^{1-\theta} |h_1(y)|^\theta, \tag{2.8}$$

and

$$\|h_0\|_{Y_0}^{1-\theta} \|h_1\|_{Y_1}^\theta \sim \|f\|_{Y(X)}. \tag{2.9}$$

Let

$$A_0 := \left\{y \in \mathcal{Y}\backslash E : \ \|g_0(\cdot, y)\|_{X_0} > 0\right\} \text{ and } A_1 := \left\{y \in \mathcal{Y}\backslash E : \ \|g_1(\cdot, y)\|_{X_1} > 0\right\}.$$

Define

$$f_0(x, y) := 2\frac{g_0(x, y)h_0(y)}{\|g_0(\cdot, y)\|_{X_0}} \mathbf{1}_{A_0}(y) \quad \text{and} \quad f_1(x, y) := 2\frac{g_1(x, y)h_1(y)}{\|g_1(\cdot, y)\|_{X_1}} \mathbf{1}_{A_1}(y)$$

for any $(x, y) \in \mathcal{X} \times (\mathcal{Y}\backslash E)$, and $f_0(x, y) := 0$ and $f_1(x, y) := 0$ for any $(x, y) \in \mathcal{X} \times E$. Clearly, $f_0 \in Y_0(X_0)$, $f_1 \in Y_1(X_1)$, and, moreover,

$$\|f_0\|_{Y_0(X_0)} \le 2\|h_0\|_{Y_0} \text{ and } \|f_1\|_{Y_1(X_1)} \le 2\|h_1\|_{Y_1}.$$

We claim

$$|f| \le |f_0|^{1-\theta} |f_1|^\theta \tag{2.10}$$

holds true almost everywhere on $\mathcal{X} \times \mathcal{Y}$. Assume for the moment that the claim holds true. Then, from the definition of the Calderón product and (2.9), we deduce that

$$\|f\|_{[Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^\theta} \leq \|f_0\|_{Y_0(X_0)}^{1-\theta}\|f_1\|_{Y_1(X_1)}^\theta \leq 2\|h_0\|_{Y_0}^{1-\theta}\|h_1\|_{Y_1}^\theta \sim \|f\|_{Y(X)} < \infty,$$

which implies $f \in [Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^\theta$ and hence (2.5) holds true.

It remains to show (2.10). By (2.8) and (2.7), we know that, for any $(x, y) \in \mathcal{X} \times (\mathcal{Y}\backslash E)$,

$$\begin{aligned}|f_0(x,y)|^{1-\theta}|f_1(x,y)|^\theta &= 2\frac{|g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta|h_0(y)|^{1-\theta}|h_1(y)|^\theta}{\|g_0(\cdot,y)\|_{X_0}^{1-\theta}\|g_1(\cdot,y)\|_{X_1}^\theta}\mathbf{1}_{A_0}(y)\mathbf{1}_{A_1}(y)\\ &\geq 2\frac{|g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta\|f(\cdot,y)\|_X}{\|g_0(\cdot,y)\|_{X_0}^{1-\theta}\|g_1(\cdot,y)\|_{X_1}^\theta}\mathbf{1}_{A_0}(y)\mathbf{1}_{A_1}(y)\\ &\geq |g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta\mathbf{1}_{A_0}(y)\mathbf{1}_{A_1}(y). \end{aligned} \tag{2.11}$$

Observe that, by Definition 1.6(i), we may assume that, for any $y \in \mathcal{Y}\backslash E$ and $y \notin A_0$,

$$g_0(x,y) = 0, \qquad \forall x \in \mathcal{X}; \tag{2.12}$$

similarly, for any $y \in \mathcal{Y}\backslash E$ and $y \notin A_1$,

$$g_1(x,y) = 0, \qquad \forall x \in \mathcal{X}.$$

From this, (2.12), and (2.11), it follows that

$$\begin{aligned}&|f_0(x,y)|^{1-\theta}|f_1(x,y)|^\theta \geq |g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta\mathbf{1}_{A_0}(y)\mathbf{1}_{A_1}(y)\\ &= \begin{cases}|g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta, & \forall y \in A_0\cap A_1\cap(\mathcal{Y}\backslash E),\ \forall x \in \mathcal{X},\\ 0 = |g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta, & \forall y \in (\mathcal{Y}\backslash A_0)\cap(\mathcal{Y}\backslash E),\ \forall x \in \mathcal{X},\\ 0 = |g_0(x,y)|^{1-\theta}|g_1(x,y)|^\theta, & \forall y \in (\mathcal{Y}\backslash A_1)\cap(\mathcal{Y}\backslash E),\ \forall x \in \mathcal{X}.\end{cases}\end{aligned}$$

Thus, by this and (2.6), we conclude that

$$|f_0(x,y)|^{1-\theta}|f_1(x,y)|^\theta \geq |f(x,y)|, \qquad \forall (x,y) \in \mathcal{X} \times (\mathcal{Y}\backslash E),$$

and hence (2.10) holds true almost everywhere on $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$. This finishes the proof of Theorem 1.1. □

To prove Theorem 1.10, we first recall the following result from [55, Theorem 2.1].

Lemma 2.1 *Let (X_0, X_1) be a pair of quasi-Banach lattices which are both of type $\mathfrak{C}$. Then, for any $\theta \in (0, 1)$,*

$$\langle X_0, X_1\rangle_\theta = \left(X_0^{1-\theta}X_1^\theta\right)^\circ$$

and

$$X_0^{1-\theta}X_1^{\theta} \hookrightarrow \langle X_0, X_1, \theta\rangle \hookrightarrow \left(X_0^{1-\theta}X_1^{\theta}\right)^{\sim}$$

hold true.

Now, we can show Theorem 1.10.

Proof (*Proof of Theorem* 1.10.) Notice that (1.11) is a direct consequence of Proposition 1.8(i), Theorem 1.1, and Lemma 2.1, and we omit its details.

Next, we prove (1.12). By Proposition 1.8(i), Theorem 1.1, and Lemma 2.1, we find that

$$\begin{aligned} Y(X) = [Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta} &\hookrightarrow \langle Y_0(X_0), Y_1(X_1), \theta\rangle \\ &\hookrightarrow \left([Y_0(X_0)]^{1-\theta}[Y_1(X_1)]^{\theta}\right)^{\sim} = (Y(X))^{\sim}. \end{aligned}$$

Thus, to show (1.12), it suffices to prove

$$(Y(X))^{\sim} \hookrightarrow Y(X). \tag{2.13}$$

From Theorem 1.1 and the discussion in [55, p. 139], we deduce that $Y(X)$ is an intermediate space with respect to $(Y_0(X_0), Y_1(X_1))$. By this and the definition of $(Y(X))^{\sim}$, we conclude that, for any $f \in (Y(X))^{\sim}$, there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset Y(X)$ such that

$$\lim_{j\to\infty} \|f_j - f\|_{Y_0(X_0)+Y_1(X_1)} = 0 \tag{2.14}$$

and, for any $j \in \mathbb{N}$,

$$\|f_j\|_{Y(X)} \le 2\|f\|_{(Y(X))^{\sim}}. \tag{2.15}$$

Observe that (2.14) implies that, for any $j \in \mathbb{N}$, $f_j - f \in Y_0(X_0) + Y_1(X_1)$. Thus, by (1.4) and (1.5), we find that, for any $j \in \mathbb{N}$, there exist functions $f_j^{(0)} \in Y_0(X_0)$ and $f_j^{(1)} \in Y_1(X_1)$ satisfying

$$f_j - f = f_j^{(0)} + f_j^{(1)} \tag{2.16}$$

and

$$\left\|f_j^{(0)}\right\|_{Y_0(X_0)} + \left\|f_j^{(1)}\right\|_{Y_1(X_1)} \lesssim \|f_j - f\|_{Y_0(X_0)+Y_1(X_1)} \to 0 \ \text{as}\ j \to \infty. \tag{2.17}$$

On the other hand, by Remark 1.9, we know that $Y_i(X_i)$ with $i \in \{0, 1\}$ is a ball Banach function space on the product space $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$. From this, Proposition 1.8, and (2.17), it follows that there exists a subsequence $\{f_{j_k}^{(0)}\}_{k\in\mathbb{N}} \subset \{f_j^{(0)}\}_{j\in\mathbb{N}}$

such that $\lim_{k\to\infty} f_{j_k}^{(0)} = 0$ almost everywhere on $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$. Repeating this argument on $\{f_{j_k}^{(1)}\}_{k\in\mathbb{N}}$, we obtain a subsequence $\{f_{j_{k_\ell}}^{(1)}\}_{\ell\in\mathbb{N}} \subset \{f_{j_k}^{(1)}\}_{k\in\mathbb{N}}$ such that $\lim_{\ell\to\infty} f_{j_{k_\ell}}^{(1)} = 0$ almost everywhere on $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$. Thus, for any $i \in \{0, 1\}$,

$$\lim_{\ell\to\infty} f_{j_{k_\ell}}^{(i)} = 0 \text{ almost everywhere on } (\mathcal{X} \times \mathcal{Y}, \mu \times \nu),$$

which, together with (2.16), implies that $\lim_{\ell\to\infty} f_{j_{k_\ell}} = f$ almost everywhere on $(\mathcal{X} \times \mathcal{Y}, \mu \times \nu)$. Combining this with Proposition 1.8(iii) and (2.15), we conclude that

$$\|f\|_{Y(X)} = \Big\| \lim_{\ell\to\infty} f_{j_{k_\ell}} \Big\|_{Y(X)} \le \liminf_{\ell\to\infty} \|f_{j_{k_\ell}}\|_{Y(X)} \lesssim \|f\|_{(Y(X))^\sim} < \infty.$$

This shows $f \in Y(X)$, which, combined with the arbitrariness of f, implies that (2.13) holds true and hence finishes the proof of Theorem 1.10. □

3 Several Examples

In this section, we give several examples of function spaces which fall into the scope of Theorem 1.10.

3.1 Mixed-Norm Lebesgue Spaces with Variable Exponent

The mixed-norm Lebesgue space with variable exponent, which was originally introduced and studied by Ho [29], is a generalization of Lebesgue spaces with variable exponent. First, we recall some notation related to Lebesgue spaces with variable exponent (see, for instance, [15,17,44]).

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ satisfying

$$0 < p_- := \operatorname*{essinf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ \le \infty.$$

For any given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)} & \text{if } p(x) \in (0, \infty), \\ 0 & \text{if } p(x) = \infty \text{ and } t \in [0, 1], \\ \infty & \text{if } p(x) = \infty \text{ and } t \in (1, \infty). \end{cases}$$

Moreover, the *variable exponent modular* $\varrho_{p(\cdot)}(f)$ of a measurable function f on $\mathbb{R}^n$ is given by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|)\, dx.$$

Now, we recall the definition of Lebesgue spaces with variable exponent.

Definition 3.1 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. The *Lebesgue space with variable exponent*, $L^{p(\cdot)}(E)$, is defined to be the set of all measurable functions f on E such that

$$\|f\|_{L^{p(\cdot)}(E)} := \inf\left\{\lambda \in (0,\infty) : \varrho_{p(\cdot)}(f\mathbf{1}_E/\lambda) \le 1\right\} < \infty. \tag{3.1}$$

The following definition of mixed-norm Lebesgue spaces with variable exponent (see, for instance, [66]) generalizes [29, Definition 3.1], where a restriction $p_1(\cdot),\ p_2(\cdot) : \mathbb{R} \to [1,\infty)$ was needed.

Definition 3.2 Let $p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The *mixed-norm Lebesgue space with variable exponent*, $L^{p_2(\cdot)}(L^{p_1(\cdot)})(\mathbb{R}^n \times \mathbb{R}^n)$, is defined to be the set of all measurable functions f on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\|f\|_{L^{p_2(\cdot)}(L^{p_1(\cdot)})(\mathbb{R}^n\times\mathbb{R}^n)} := \left\|\|f(x,y)\|_{L_x^{p_1(\cdot)}(\mathbb{R}^n)}\right\|_{L_y^{p_2(\cdot)}(\mathbb{R}^n)} < \infty.$$

Recall that, in [44], Kopaliani considered the complex interpolation between Lebesgue spaces with variable exponent and Hardy spaces or BMO; furthermore, in [45, Proposition 3.1], Kopaliani and Chelidze obtained the Calderón product of $L^{p_0(\cdot)}(\mathbb{R}^n)$ and $L^{p_1(\cdot)}(\mathbb{R}^n)$ with $1 \le (p_i)_- \le (p_i)_+ < \infty$ for any $i \in \{0,1\}$. Via an argument different from that used in the proof of [45, Proposition 3.1], we get rid of the restriction $1 \le (p_i)_- \le (p_i)_+ < \infty$ and prove the following result.

Proposition 3.3 *Let* $\theta \in (0,1)$ *and* $p(\cdot),\ p_0(\cdot),\ p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy* $\frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}$. *Then,*

$$\left[L^{p_0(\cdot)}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1(\cdot)}(\mathbb{R}^n)\right]^{\theta} = L^{p(\cdot)}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Proof First, we show that

$$\left[L^{p_0(\cdot)}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1(\cdot)}(\mathbb{R}^n)\right]^{\theta} \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n). \tag{3.2}$$

For any $f \in [L^{p_0(\cdot)}(\mathbb{R}^n)]^{1-\theta}[L^{p_1(\cdot)}(\mathbb{R}^n)]^{\theta}$, there exist $f_0 \in L^{p_0(\cdot)}(\mathbb{R}^n)$ and $f_1 \in L^{p_1(\cdot)}(\mathbb{R}^n)$ such that, for almost every $x \in \mathbb{R}^n$,

$$|f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^{\theta} \tag{3.3}$$

and

$$\|f_0\|_{L^{p_0(\cdot)}(\mathbb{R}^n)}^{1-\theta}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\theta} \lesssim \|f\|_{[L^{p_0(\cdot)}(\mathbb{R}^n)]^{1-\theta}[L^{p_1(\cdot)}(\mathbb{R}^n)]^{\theta}}. \tag{3.4}$$

Let

$$\lambda_0 := \|f_0\|_{L^{p_0(\cdot)}(\mathbb{R}^n)} \text{ and } \lambda_1 := \|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Then, by (3.3) and the Young inequality, namely, for any $a,\ b \in [0, \infty)$ and $q,\ q' \in (1, \infty)$ with $1/q + 1/q' = 1$,

$$ab \le \frac{a^q}{q} + \frac{b^{q'}}{q'}, \tag{3.5}$$

we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda_0^{1-\theta}\lambda_1^{\theta}}\right|^{p(x)} dx &\le \int_{\mathbb{R}^n} \left|\frac{f_0(x)}{\lambda_0}\right|^{(1-\theta)p(x)} \left|\frac{f_1(x)}{\lambda_1}\right|^{\theta p(x)} dx \\ &\le \int_{\mathbb{R}^n} \frac{(1-\theta)p(x)}{p_0(x)} \left|\frac{f_0(x)}{\lambda_0}\right|^{p_0(x)} dx + \int_{\mathbb{R}^n} \frac{\theta p(x)}{p_1(x)} \left|\frac{f_1(x)}{\lambda_1}\right|^{p_1(x)} dx \\ &\le \frac{(1-\theta)p_+}{(p_0)_-} \int_{\mathbb{R}^n} \left|\frac{f_0(x)}{\lambda_0}\right|^{p_0(x)} dx + \frac{\theta p_+}{(p_1)_-} \int_{\mathbb{R}^n} \left|\frac{f_1(x)}{\lambda_1}\right|^{p_1(x)} dx \\ &\le \frac{(1-\theta)p_+}{(p_0)_-} + \frac{\theta p_+}{(p_1)_-}. \end{aligned} \tag{3.6}$$

We claim that, for any $\lambda,\ a \in (0, \infty)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and any measurable function f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{q(x)}\, dx \le a \quad \text{implies} \quad \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \lambda \max\left\{1, a^{1/q_-}\right\}. \tag{3.7}$$

Assume for the moment that (3.7) holds true. Then, from (3.6), (3.4), and the definition of λ_0 and λ_1, we deduce that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \lambda_0^{1-\theta}\lambda_1^{\theta} \sim \|f_0\|^{1-\theta}_{L^{p_0(\cdot)}(\mathbb{R}^n)} \|f_1\|^{\theta}_{L^{p_1(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{[L^{p_0(\cdot)}(\mathbb{R}^n)]^{1-\theta}[L^{p_1(\cdot)}(\mathbb{R}^n)]^{\theta}} < \infty.$$

Thus, $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Combining this with the arbitrariness of f, we obtain (3.2).

Now, it remains to prove (3.7). Indeed, if $a \in (0, 1]$, then (3.1) directly leads to $\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \lambda$. If $a \in (1, \infty)$, then, for any $x \in \mathbb{R}^n$, $(1/a)^{1/q_-} \le (1/a)^{1/q(x)}$. Thus, we have

$$\int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda a^{1/q_-}}\right|^{q(x)} dx \le \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda a^{1/q(x)}}\right|^{q(x)} dx = \frac{1}{a} \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{q(x)} dx \le 1,$$

which, together with (3.1), implies $\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \lambda a^{1/q_-}$ and hence (3.7) holds true.

Now, we show that

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow \left[L^{p_0(\cdot)}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1(\cdot)}(\mathbb{R}^n)\right]^{\theta}. \tag{3.8}$$

Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Without loss of generality, we may assume $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$. Let

$$f_0(\cdot) := |f(\cdot)|^{p(\cdot)/p_0(\cdot)} \text{ and } f_1(\cdot) := |f(\cdot)|^{p(\cdot)/p_1(\cdot)}.$$

Clearly, $|f| = |f_0|^{1-\theta}|f_1|^{\theta}$. Notice that

$$\int_{\mathbb{R}^n} |f_0(x)|^{p_0(x)}\,dx = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \le 1$$

implies that $f_0 \in L^{p_0(\cdot)}(\mathbb{R}^n)$ and $\|f_0\|_{L^{p_0(\cdot)}(\mathbb{R}^n)} \le 1$. Similarly, we know that $f_1 \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \le 1$. Thus, by (1.1) and (1.2), we have

$$\|f\|_{[L^{p_0(\cdot)}(\mathbb{R}^n)]^{1-\theta}[L^{p_1(\cdot)}(\mathbb{R}^n)]^{\theta}} \le \|f_0\|^{1-\theta}_{L^{p_0(\cdot)}(\mathbb{R}^n)}\|f_1\|^{\theta}_{L^{p_1(\cdot)}(\mathbb{R}^n)} \le 1,$$

which implies $f \in [L^{p_0(\cdot)}(\mathbb{R}^n)]^{1-\theta}[L^{p_1(\cdot)}(\mathbb{R}^n)]^{\theta}$ and hence (3.8) holds true. This, combined with (3.2), then finishes the proof of Proposition 3.3. □

By Theorem 1.10 and Proposition 3.3, we obtain the following interpolation properties of mixed-norm Lebesgue spaces with variable exponent.

Corollary 3.4 *Let $\theta \in (0,1)$ and $p(\cdot),\ q(\cdot),\ p_i(\cdot),\ q_i(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $i \in \{0,1\}$ satisfy*

$$\frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)} \text{ and } \frac{1}{q(\cdot)} = \frac{1-\theta}{q_0(\cdot)} + \frac{\theta}{q_1(\cdot)}.$$

Then

$$\left\langle L^{q_0(\cdot)}\left(L^{p_0(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n), L^{q_1(\cdot)}\left(L^{p_1(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right\rangle_{\theta}$$
$$= \left(\left[L^{q_0(\cdot)}\left(L^{p_0(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right]^{1-\theta}\left[L^{q_1(\cdot)}\left(L^{p_1(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right]^{\theta}\right)^{\circ} = \left(L^{q(\cdot)}\left(L^{p(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right)^{\circ}.$$

Moreover, if, additionally, $p_-,\ q_-,\ (p_i)_-,\ (q_i)_- \in [1,\infty]$ with $i \in \{0,1\}$, then

$$\left\langle L^{q_0(\cdot)}\left(L^{p_0(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n), L^{q_1(\cdot)}\left(L^{p_1(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n),\theta\right\rangle$$
$$= \left[L^{q_0(\cdot)}\left(L^{p_0(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right]^{1-\theta}\left[L^{q_1(\cdot)}\left(L^{p_1(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n)\right]^{\theta}$$
$$= L^{q(\cdot)}\left(L^{p(\cdot)}\right)(\mathbb{R}^n\times\mathbb{R}^n).$$

Proof By [15, Theorems 2.17 and 2.59, Lemma 2.39, and Proposition 2.41] (see also [17, p. 77]), we know that, for any $p(\cdot): \mathbb{R}^n \to [1,\infty]$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a ball Banach function space, which, together with the fact that, for any measurable function f on $\mathbb{R}^n$,

$$\left\||f|^r\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^r_{L^{rp(\cdot)}(\mathbb{R}^n)}, \qquad \forall\, r \in (0,\infty),$$

implies that, for any $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\delta \in (0, \min\{1, p_-, q_-\})$, the $1/\delta$-convexification of the mixed-norm variable Lebesgue space $L^{q(\cdot)}(L^{p(\cdot)})(\mathbb{R}^n \times \mathbb{R}^n)$ is a Banach space. Namely, the mixed-norm Lebesgue space with variable exponent is of type $\mathfrak{C}$. Combining these with Proposition 3.3 and Theorem 1.10, we obtain the desired conclusions and hence complete the proof of Corollary 3.4. □

Remark 3.5 (i) As a special case of Corollary 3.4, we obtain the following interpolation properties of classical mixed-norm Lebesgue spaces: for any $\theta \in (0, 1)$ and $p,\ q,\ p_i,\ q_i \in (0, \infty]$ with $i \in \{0, 1\}$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, it holds true that

$$\begin{aligned}&\langle L^{q_0}\left(L^{p_0}\right)(\mathbb{R}^n \times \mathbb{R}^n), L^{q_1}\left(L^{p_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\rangle_\theta\\ &\quad = \left(\left[L^{q_0}\left(L^{p_0}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta}\left[L^{q_1}\left(L^{p_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta}\right)^{\circ} = \left(L^{q}\left(L^{p}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right)^{\circ};\end{aligned}$$

in particular, when $p,\ q,\ p_i,\ q_i \in [1, \infty]$ with $i \in \{0, 1\}$, one has

$$\begin{aligned}&\left\langle L^{q_0}\left(L^{p_0}\right)(\mathbb{R}^n \times \mathbb{R}^n), L^{q_1}\left(L^{p_1}\right)(\mathbb{R}^n \times \mathbb{R}^n), \theta\right\rangle\\ &\quad = \left[L^{q_0}\left(L^{p_0}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta}\left[L^{q_1}\left(L^{p_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta} = L^{q}\left(L^{p}\right)(\mathbb{R}^n \times \mathbb{R}^n).\end{aligned}$$

(ii) We also recall that, recently, Tan [66] studied the off-diagonal extrapolation on Lebesgue spaces with variable exponent.

3.2 Mixed-Norm Lorentz Spaces

The mixed-norm Lorentz spaces were originally introduced by Fernandez [18]. Later, Milman [51,52] studied the properties of interpolations of operators on mixed-norm Lorentz spaces. Recently, Chen and Sun studied the mixed-norm weak Lebesgue spaces intensively (see, for instance, [7,8,10]), which are special cases of mixed-norm Lorentz spaces; in particular, Chen and Sun considered the real interpolation properties of mixed-norm (weak) Lebesgue spaces via the K-functional. In this subsection, we consider the interpolation properties on mixed-norm Lorentz spaces via the Calderón product, the Gagliardo–Peetre method, and the $\pm$ method.

Recall that, for any given $p \in (0, \infty)$ and $q \in (0, \infty]$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \left\{\int_0^\infty \left[\lambda|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p}\right]^q \frac{d\lambda}{\lambda}\right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$ (see, for instance, [21, Section 1.4]). Notice that, when $q \in (0, \infty)$ and $q = \infty$, the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$ is just the weak Lebesgue space.

Also, recall that, for any given $p_0,\ p_1 \in (0, \infty)$ and $q_0,\ q_1 \in (0, \infty]$, the *mixed-norm Lorentz space* $L^{p_0,q_0}(L^{p_1,q_1})(\mathbb{R}^n \times \mathbb{R}^n)$ is defined to be the set of all Lebesgue measurable functions f on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\|f\|_{L^{p_0,q_0}(L^{p_1,q_1})(\mathbb{R}^n\times\mathbb{R}^n)} := \left\| \|f(x,y)\|_{L_x^{p_0,q_0}(\mathbb{R}^n)} \right\|_{L_y^{p_1,q_1}(\mathbb{R}^n)} < \infty.$$

Remark 3.6 Clearly, for any given $p \in (1,\infty)$ and $q \in (1,\infty]$, $L^{p,q}(\mathbb{R}^n)$ is a Banach space (see, for instance, [21, Remark 1.4.12]) and, for any given $p,\ r \in (0,\infty)$ and $q \in (0,\infty]$, and for any measurable function f on $\mathbb{R}^n$,

$$\||f|^r\|_{L^{p,q}(\mathbb{R}^n)} = \|f\|^r_{L^{pr,qr}(\mathbb{R}^n)} \tag{3.9}$$

(see, for instance, [21, Remark 1.4.7]). Thus, for any given $p_0,\ p_1 \in (0,\infty)$, $q_0,\ q_1 \in (0,\infty]$, and $\delta \in (0,\min\{1,p_0,q_0,p_1,q_1\})$, the $1/\delta$-convexification of $L^{p_1,q_1}(L^{p_0,q_0})(\mathbb{R}^n\times\mathbb{R}^n)$ is a Banach space, which means that the mixed-norm Lorentz space is of type $\mathfrak{C}$.

Recall that Calderón [6, p. 124] showed that, for any $\theta \in (0,1)$, and $p_i \in (1,\infty)$ and $q_i \in [1,\infty]$ with $i \in \{0,1\}$ satisfying

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad\text{and}\quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{3.10}$$

it holds true that

$$\left[L^{p_0,q_0}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1,q_1}(\mathbb{R}^n)\right]^{\theta} = L^{p,q}(\mathbb{R}^n)$$

in the sense of equivalent norms. Notice that the argument used in [6] depends heavily on the characterization of Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $q \in [1,\infty]$ that, any measurable function f belongs to $L^{p,q}(\mathbb{R}^n)$ if and only if

$$\|f\|_{L^{(p,q)}(\mathbb{R}^n)} := \left\{\int_0^\infty t^{q/p}\left[\frac{1}{t}\sup_{E\subset\mathbb{R}^n,\ |E|\le t}\int_E |f(x)|\,dx\right]^q \frac{dt}{t}\right\}^{1/q}$$

(with usual modification made when $q=\infty$) is finite, and $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ on $L^{p,q}(\mathbb{R}^n)$ (see, for instance, [4, Lemma 4.5 in p. 219]).

Via an argument different from that used in [6, p. 124], which does not need the aforementioned equivalence, we establish the Calderón products on Lorentz spaces for some more ranges as follows.

Proposition 3.7 *Let $\theta \in (0,1)$ and $p_i,\ q_i \in (0,\infty)$ with $i \in \{0,1\}$ satisfy (3.10). Then,*

$$\left[L^{p_0,q_0}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1,q_1}(\mathbb{R}^n)\right]^{\theta} = L^{p,q}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Proof By [16, Theorem 3] and [14, Theorem 3], one knows that, for any couple of quasi-Banach spaces, (A_0, A_1), and for any $Q_i \in (0, \infty]$ with $i \in \{0, 1\}$,

$$\left[(A_0, A_1)_{\lambda_0, Q_0}, (A_0, A_1)_{\lambda_1, Q_1}\right]_\eta = (A_0, A_1)_{\lambda, Q} \tag{3.11}$$

with $\lambda_0,\ \lambda_1,\ \eta \in (0, 1)$, $\lambda = (1-\eta)\lambda_0 + \eta\lambda_1$, and $\frac{1}{Q} = \frac{1-\eta}{Q_0} + \frac{\eta}{Q_1}$, where, for any two quasi-Banach spaces X_0 and X_1, $(X_0, X_1)_{\lambda, Q}$ denotes their real interpolation and $[X_0, X_1]_\eta$ their complex interpolation. Recall that, for any $\lambda \in (0, 1)$, $P_0 \in (0, \infty)$, and $Q \in (0, \infty]$,

$$\left(L^{P_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\lambda, Q} = L^{P,Q}(\mathbb{R}^n), \tag{3.12}$$

where $1/P = (1-\lambda)/P_0$ (see, for instance, [59] and [43, Theorem 4.1]). Choose $\theta_0,\ \theta_1 \in (0, 1)$ and $r \in (0, \infty)$ such that

$$\frac{1}{p_0} = \frac{1-\theta_0}{r} \quad \text{and} \quad \frac{1}{p_1} = \frac{1-\theta_1}{r}.$$

From this, (3.11), (3.12), and (3.10), we deduce that, for any $\theta \in (0, 1)$, and $p_i \in (0, \infty)$ and $q_i \in (0, \infty]$ with $i \in \{0, 1\}$,

$$\begin{aligned}\left[L^{p_0,q_0}(\mathbb{R}^n), L^{p_1,q_1}(\mathbb{R}^n)\right]_\theta &= \left[\left(L^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\theta_0,q_0}, \left(L^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{\theta_1,q_1}\right]_\theta \\ &= \left(L^r(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right)_{(1-\theta)\theta_0+\theta\theta_1,q} = L^{p,q}(\mathbb{R}^n). \quad (3.13)\end{aligned}$$

On the other hand, by [41, Theorem 7.9] (see also [40, Theorem 3.4]) and the fact that, for any $p,\ q \in (0, \infty)$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is separable (see [42, p. 290]), we find that, for any $\theta \in (0, 1)$, and $p_i, q_i \in (0, \infty)$ with $i \in \{0, 1\}$,

$$\left[L^{p_0,q_0}(\mathbb{R}^n), L^{p_1,q_1}(\mathbb{R}^n)\right]_\theta = \left[L^{p_0,q_0}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1,q_1}(\mathbb{R}^n)\right]^\theta,$$

which, together with (3.13), then completes the proof of Proposition 3.7. □

Remark 3.8 Notice that, in Proposition 3.7, we exclude the case that $q_i = \infty$ (namely, the weak Lebesgue space) for at least one $i \in \{0, 1\}$, because the weak Lebesgue space is not separable (see, for instance, [42, p, 296]). So far, it is still unknown how $\left[L^{p_0,\infty}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1,q}(\mathbb{R}^n)\right]^\theta$ behaves for any $\theta \in (0, 1)$, $p_0,\ p_1 \in (0, \infty)$, and $q \in (0, \infty)$. However, for $q_0 = q_1 = \infty$, we have the following Calderón product for weak Lebesgue spaces via an argument different from that used in [6, p. 124].

Proposition 3.9 *Let* $\theta \in (0, 1)$ *and* $p_0,\ p_1 \in (0, \infty)$ *satisfy* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then*

$$\left[L^{p_0,\infty}(\mathbb{R}^n)\right]^{1-\theta}\left[L^{p_1,\infty}(\mathbb{R}^n)\right]^\theta = L^{p,\infty}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Proof The desired conclusion is a consequence of the Young inequality (3.5), (3.9), and the fact that, for any given $\theta \in (0, 1)$, any pair (X_0, X_1) of quasi-Banach lattices, and any $f \in X_0^{1-\theta} X_1^{\theta}$,

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} \sim \inf \Big\{\lambda \in (0,\infty): \ |f| \le \lambda |f_0|^{1-\theta} |f_1|^{\theta} \text{ almost everywhere with } f_i \in X_i \\ \text{and } \|f_i\|_{X_i} \le 1, \ i \in \{0,1\}\Big\}$$

(see, for instance, [6, Section 13.5] and [67, Lemma 2.3(ii)]), and we omit the details. This finishes the proof of Proposition 3.9. □

As a consequence of Theorem 1.10, Remark 3.6, and Propositions 3.7 and 3.9, we obtain the following results, and we omit its details.

Corollary 3.10 *Let $\theta \in (0, 1)$ and, for any $i \in \{0, 1\}$, $p_i, \ r_i \in (0, \infty)$ and, either $q_i = s_i = \infty$ or $q_i, \ s_i \in (0, \infty)$ satisfy*

$$\begin{cases} \dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1} \text{ and } \dfrac{1}{q} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}, \\ \dfrac{1}{r} = \dfrac{1-\theta}{r_0} + \dfrac{\theta}{r_1} \text{ and } \dfrac{1}{s} = \dfrac{1-\theta}{s_0} + \dfrac{\theta}{s_1}. \end{cases}$$

Then,

$$\begin{aligned} &\left\langle L^{s_0,r_0}\left(L^{p_0,q_0}\right)(\mathbb{R}^n \times \mathbb{R}^n), L^{s_1,r_1}(L^{p_1,q_1})(\mathbb{R}^n \times \mathbb{R}^n)\right\rangle_\theta \\ &= \left(\left[L^{s_0,r_0}(L^{p_0,q_0})(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta} \left[L^{s_1,r_1}(L^{p_1,q_1})(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta}\right)^{\circ} \\ &= \left(L^{s,r}(L^{p,q})(\mathbb{R}^n \times \mathbb{R}^n)\right)^{\circ}. \end{aligned}$$

Additionally, if $p_i, \ r_i \in (1, \infty)$ and, either $q_i = s_i = \infty$ or $q_i, \ s_i \in (1, \infty)$ with $i \in \{0, 1\}$, it also holds true that

$$\begin{aligned} &\left\langle L^{s_0,r_0}(L^{p_0,q_0})(\mathbb{R}^n \times \mathbb{R}^n), L^{s_1,r_1}(L^{p_1,q_1})(\mathbb{R}^n \times \mathbb{R}^n), \theta\right\rangle \\ &= \left[L^{s_0,r_0}(L^{p_0,q_0})(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta} \left[L^{s_1,r_1}(L^{p_1,q_1})(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta} \\ &= L^{s,r}(L^{p,q})(\mathbb{R}^n \times \mathbb{R}^n). \end{aligned}$$

3.3 Mixed-Norm Morrey Spaces

The study of the Morrey space can be traced to [54]. The *Morrey space* $\mathcal{M}_p^u(\mathbb{R}^n)$ with $0 < p \le u \le \infty$ is defined to be the set of all locally p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}_p^u(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} |B|^{1/u-1/p} \left[\int_B |f(x)|^p \, dx\right]^{1/p} < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$. Clearly, $\mathcal{M}_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for any $p \in (0, \infty]$.

As is well known, there has been a series of works on the interpolation properties of Morrey type spaces (see, for instance, [24–27,47–49,67] and the references therein). In recent years, there also exist an increasing interest in the mixed-norm Morrey spaces in the sense of (1.3). For instance, Ragusa and Scapellato [61] studied the regularity of the solutions of the non-divergence form parabolic equations via the mixed-norm Morrey spaces, and Nogayama et al. [56–58] studied the boundedness of several operators and commutators on mixed-norm Morrey spaces, such as the maximal operator, the fractional integral operator, and the Calderón–Zygmund operator.

However, the interpolation properties of mixed-norm Morrey spaces are still unknown. In this section, we establish the interpolation results on mixed-norm Morrey spaces in terms of the Gagliardo–Peetre method and the $\pm$ method.

Recall that Lu et al. [49, Proposition 2.1] established the following Calderón product result in Morrey spaces.

Lemma 3.11 *Let $\mathcal{X}$ be a quasi-metric measure space, $\theta \in (0, 1)$ and, for any $i \in \{0, 1\}$, $0 < p_i \le u_i \le \infty$. Assume that $p_0 u_1 = p_1 u_0$,*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}.$$

Then

$$\left[\mathcal{M}_{p_0}^{u_0}(\mathcal{X})\right]^{1-\theta}\left[\mathcal{M}_{p_1}^{u_1}(\mathcal{X})\right]^{\theta} = \mathcal{M}_p^u(\mathcal{X})$$

with equivalent quasi-norms.

From Lemma 3.11 and Theorem 1.10, we immediately deduce the following conclusion.

Corollary 3.12 *Let $\theta \in (0, 1)$ and, for any $i \in \{0, 1\}$, $0 < p_i \le u_i \le \infty$ and $0 < q_i \le v_i \le \infty$. Assume that $v_0 q_1 = v_1 q_0$, $u_0 p_1 = u_1 p_0$,*

$$\begin{cases} \dfrac{1}{u} = \dfrac{1-\theta}{u_0} + \dfrac{\theta}{u_1} \quad \text{and} \quad \dfrac{1}{v} = \dfrac{1-\theta}{v_0} + \dfrac{\theta}{v_1}, \\ \dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1} \quad \text{and} \quad \dfrac{1}{q} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}. \end{cases}$$

Then,

$$\begin{aligned}&\left\langle \mathcal{M}_{q_0}^{v_0}\left(\mathcal{M}_{p_0}^{u_0}\right)(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{M}_{q_1}^{v_1}\left(\mathcal{M}_{p_1}^{u_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right\rangle_\theta \\ &= \left(\left[\mathcal{M}_{q_0}^{v_0}\left(\mathcal{M}_{p_0}^{u_0}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta}\left[\mathcal{M}_{q_1}^{v_1}\left(\mathcal{M}_{p_1}^{u_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta}\right)^{\circ} = \left(\mathcal{M}_q^v\left(\mathcal{M}_p^u\right)(\mathbb{R}^n \times \mathbb{R}^n)\right)^{\circ}.\end{aligned}$$

Moreover, if, additionally, $1 \le p_i \le u_i \le \infty$ *and* $1 \le q_i \le v_i \le \infty$ *with* $i \in \{0, 1\}$, *then*

$$\begin{aligned}&\left\langle \mathcal{M}_{q_0}^{v_0}\left(\mathcal{M}_{p_0}^{u_0}\right)(\mathbb{R}^n \times \mathbb{R}^n), \mathcal{M}_{q_1}^{v_1}\left(\mathcal{M}_{p_1}^{u_1}\right)(\mathbb{R}^n \times \mathbb{R}^n), \theta\right\rangle \\ &= \left[\mathcal{M}_{q_0}^{v_0}\left(\mathcal{M}_{p_0}^{u_0}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{1-\theta}\left[\mathcal{M}_{q_1}^{v_1}\left(\mathcal{M}_{p_1}^{u_1}\right)(\mathbb{R}^n \times \mathbb{R}^n)\right]^{\theta} \\ &= \mathcal{M}_{q}^{v}\left(\mathcal{M}_{p}^{u}\right)(\mathbb{R}^n \times \mathbb{R}^n).\end{aligned}$$

Proof Observe that, for any $1 \le p \le u \le \infty$, $\mathcal{M}_p^u(\mathbb{R}^n)$ is a ball Banach function space (see, for instance, [64, Subsection 7.4]). This, together with the fact that, for any given $r \in (0, \infty)$ and $0 < p \le u \le \infty$, and for any measurable function f, $\||f|^r\|_{\mathcal{M}_p^u(\mathbb{R}^n)} = \|f\|^r_{\mathcal{M}_{pr}^{ur}(\mathbb{R}^n)}$, shows that the mixed-norm Morrey space is of type $\mathfrak{C}$. Thus, from Lemma 3.11 and Theorem 1.10, we immediately deduce the desired conclusions and hence complete the proof of Corollary 3.12. □

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