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Oscillation of Second-Order Functional Differential Equations with Superlinear Neutral Terms

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Abstract

This work deals with the study on the oscillatory behavior of solutions to a class of nonlinear second-order functional differential equation with superlinear neutral terms. It presents new sufficient conditions that ensure the oscillation of all solutions under the assumptions that allow applications to differential equations with delayed and/or advanced arguments. Illustrative examples are also provided to show applicability of the results.

Keywords Oscillation · Second-order · Neutral differential equations · Superlinear

Mathematics Subject Classification 34K11 · 34K40 · 39A21

1 Introduction

In this article, we restrict our attention to oscillation of a class of the second-order functional differential equations of the form:

$$\left(m(t)\omega'(t)\right)' + q(t)f\left(x(\xi(t))\right) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

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where the neutral part $\omega\big(t\big)$ is defined by

$$\omega\big(t\big) = x(t) + \tilde{r}(t)x^{\theta}\big(\nu(t)\big) + r(t)x^{\theta}\big(\mu(t)\big).$$

Without further mention, we assume that the following hypotheses hold throughout the whole paper:

(i) θ is a quotient of two odd positive integers such that $\theta \geq 1$;
(ii) $m \in C\big([t_0, \infty), (0, \infty)\big)$, $q \in C\big([t_0, \infty), [0, \infty)\big)$ and q does not vanish identically on any half-line of the form $[t_x, \infty)$, $t_x \geq t_0$;
(iii) $\nu, \mu, \xi \in C([t_0, \infty), \mathbb{R})$, $\nu(t) \leq t \leq \mu(t)$, ν and μ are strictly increasing functions and $\lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} \xi(t) = \infty$;
(iv) $f \in C\big(\mathbb{R}, \mathbb{R}\big)$ and, there exists $\kappa > 0$ such that $f(\upsilon)/\upsilon^{\lambda} \geq \kappa$ for all $\upsilon \neq 0$, where λ is a ratio of odd positive integers;
(v) $\tilde{r}, r \in C\big([t_0, \infty), [0, \infty)\big)$ with $r(t) \to \infty$ as $t \to \infty$.

The study of the oscillatory behavior of solutions of various classes of the second-order or higher order neutral differential equations and neutral dynamic equations on time scales is an active area that has been extensively studied in the literature, and we refer the reader to the papers [1,2,4,5,7–10,13,14,16–22,24,25,29,31–34,36,38–43] and the references therein as examples of recent results on this topic. However, oscillation results for neutral differential equations with a nonlinearity in the neutral term are relatively scarce; some results can be found, for example, in [3,11,12,27,30,35,37] and the references contained therein. Meanwhile, in reviewing the related literature, it is clearly seen that most of such results are concerned with the sublinear case, i.e., under the assumption that $0 < \theta \leq 1$.

Recently, Bohner et al. [6] considered the second-order neutral delay differential equation

$$\Big(m\big(t\big)\big(x(t) + p(t)x^{\alpha}\big(\tau(t)\big)\big)'\Big)' + q(t)f\big(x(\sigma(t))\big) = 0, \quad t \geq t_0 > 0, \qquad \text{(E1)}$$

under the conditions with $\tau(t) \leq t$, $\tau'(t) > 0$, and $\sigma(t) \leq t$ where $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, and they established some nice oscillation criteria for Eq. (E1), for superlinear case $\alpha \geq 1$. One should also note that results reported by Bohner et al. [6] do not apply to Eq. (E1) when $\tau(t) \geq t$ and/or $\sigma(t) \geq t$.

The main objective of this paper is to establish some new sufficient conditions for the oscillatory behavior of solutions of the second-order mixed neutral differential equation (1.1) in the case $\theta \geq 1$. Note that Eq. (1.1) contains both delayed and advanced arguments in the neutral term, so obtained results in this paper extend and generalize related results reported in the literature, please see Remark 4.1. It should be also pointed out that our results allow applications to differential equations in the case when $\sigma(t)$ is an advanced argument as well as when $\sigma(t)$ is a delayed argument, please see Remark 4.2.

By a solution of Eq. (1.1), we mean a function $x : [T_x, \infty) \to \mathbb{R}$ which has the properties $\omega \in C^1([T_x, \infty), \mathbb{R})$, $m(\omega') \in C^1([T_x, \infty), \mathbb{R})$ and satisfies (1.1) on $[T_x, \infty)$ where $T_x \geq t_0$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[T_x, \infty)$, for some $T_x \geq t_0$, and

$$\sup\{|x(t)| : t \geq T\} > 0 \quad \text{for all } T \geq T_x.$$

We make the standing hypothesis that (1.1) admits such solutions. Such a solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is called *nonoscillatory*. Equation (1.1) is said to be *oscillatory* if all its solutions are *oscillatory*.

We establish sufficient conditions for oscillation of all solutions in which both canonical and noncanonical cases, that is, in the cases

$$\int_t^{\infty} m^{-1}(\eta)\, \mathrm{d}\eta = \infty; \tag{1.2}$$

and

$$\int_t^{\infty} m^{-1}(\eta)\, \mathrm{d}\eta < \infty. \tag{1.3}$$

respectively.

For simplicity in what follows, we define the functions:

$$\chi(t) := \int_t^{\infty} \frac{\mathrm{d}\eta}{m(\eta)}, \quad A(t) := \int_{t_1}^{t} \frac{\mathrm{d}\eta}{m(\eta)},$$

where t_1 is large enough. We also define that

$$\psi_1(t) = \frac{1}{r(\mu^{-1}(t))}\left[1 - \frac{\tilde{r}(\mu^{-1}(t))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))} - \frac{1}{\theta r^{1/\theta}(\mu^{-1}(\mu^{-1}(t)))}\left(1 + \frac{\theta - 1}{\delta(\mu^{-1}(t))}\right)\right]$$

and

$$\psi_2(t) := \frac{1}{r(\mu^{-1}(t))}\left[1 - \frac{\tilde{r}(\mu^{-1}(t))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))}\frac{\chi(\mu^{-1}(\nu(\mu^{-1}(t))))}{\chi(\mu^{-1}(t))} - \frac{1}{\theta r^{1/\theta}(\mu^{-1}(\mu^{-1}(t)))}\left(\frac{\chi(\mu^{-1}(\mu^{-1}(t)))}{\chi(\mu^{-1}(t))} + \frac{\theta - 1}{\chi^2(\mu^{-1}(t))}\right)\right]$$

where μ^{-1} denotes the inverse function of μ and the function δ is to be specified later. Meanwhile, it is assumed that $\psi_1(t) > 0$ and $\psi_2(t) > 0$ for all sufficiently large t.

2 Preliminary Lemmas

First of all, we present some lemmas that will be used to prove our main results.

Lemma 2.1 [15, Theorem 41] *If* ε *is positive and* $0 < \beta \leq 1$, *then*

$$\varepsilon^{\beta} \leq \beta\varepsilon + (1-\beta)\,. \tag{2.1}$$

Lemma 2.2 *Assume that* (1.2) *holds. If* $x(t)$ *is a positive solution of Eq.* (1.1) *on* $[t_0, \infty)$, *then the corresponding* $\omega(t)$ *satisfies*

$$\omega(t) > 0,\ \omega'(t) > 0 \quad \text{and} \quad (m\omega')'(t) \leq 0 \tag{2.2}$$

for $t \geq t_1 \in [t_0, \infty)$.

The proof of the above lemma is straightforward; hence, we omit the details.

Lemma 2.3 *Let* $x(t)$ *be an eventually positive solution* (1.1) *such that corresponding* $\omega(t)$ *satisfies* (2.2). *If there exists a positive decreasing function* $\delta(t)$ *which is tending to zero, then*

$$x^{\theta}(t) \geq \psi_1(t)\omega(\mu^{-1}(t)) \tag{2.3}$$

for $t \geq t_2 \geq t_1$.

Proof From condition (v) and the definition of $\omega(t)$, we have $\omega(t) \geq x(t)$ and $\omega(t) \geq x(t) + \tilde{r}(t)x^{\theta}(\nu(t))$ for all $t \geq t_1 \geq t_0$. Meanwhile, we have

$$\begin{aligned}
x^{\theta}(\mu(t)) &= \frac{\omega(t)}{r(t)} - \frac{x(t)}{r(t)} - \frac{\tilde{r}(t)x^{\theta}(\nu(t))}{r(t)} \\
&= \frac{\omega(t)}{r(t)} - \frac{1}{r(t)}\left[\frac{\omega(\mu^{-1}(t))}{r(\mu^{-1}(t))} - \frac{x(\mu^{-1}(t))}{r(\mu^{-1}(t))} - \frac{\tilde{r}(\mu^{-1}(t))x^{\theta}(\nu(\mu^{-1}(t)))}{r(\mu^{-1}(t))}\right]^{1/\theta} \\
&\quad - \frac{\tilde{r}(t)}{r(t)}\left[\frac{\omega(\mu^{-1}(\nu(t)))}{r(\mu^{-1}(\nu(t)))} - \frac{x(\mu^{-1}(\nu(t)))}{r(\mu^{-1}(\nu(t)))}\right. \\
&\quad \left. - \frac{\tilde{r}(\mu^{-1}(\nu(t)))x^{\theta}(\nu(\mu^{-1}(\nu(t))))}{r(\mu^{-1}(\nu(t)))}\right] \\
&\geq \frac{\omega(t)}{r(t)} - \frac{1}{r(t)}\frac{\omega^{1/\theta}(\mu^{-1}(t))}{r^{1/\theta}(\mu^{-1}(t))}\left[1 - \frac{x(\mu^{-1}(t)) + \tilde{r}(\mu^{-1}(t))x^{\theta}(\nu(\mu^{-1}(t)))}{\omega(\mu^{-1}(t))}\right]^{1/\theta} \\
&\quad - \frac{\tilde{r}(t)\omega(\mu^{-1}(\nu(t)))}{r(t)r(\mu^{-1}(\nu(t)))}.
\end{aligned} \tag{2.4}$$

If we apply Lemma 2.1 in (2.4), we obtain

$$\begin{aligned}
x^{\theta}(\mu(t)) \geq{}& \frac{\omega(t)}{r(t)} - \frac{\omega^{1/\theta}(\mu^{-1}(t))}{r(t)r^{1/\theta}(\mu^{-1}(t))}\left[1 - \frac{1}{\theta}\frac{x(\mu^{-1}(t)) + \tilde{r}(\mu^{-1}(t))x^{\theta}(\nu(\mu^{-1}(t)))}{\omega(\mu^{-1}(t))}\right] \\
&- \frac{\tilde{r}(t)\omega(\mu^{-1}(\nu(t)))}{r(t)r(\mu^{-1}(\nu(t)))}
\end{aligned}$$

$$\geq \frac{1}{r(t)}\left[\omega(t) - \frac{\omega^{1/\theta}(\mu^{-1}(t))}{r^{1/\theta}(\mu^{-1}(t))} - \frac{\tilde{r}(t)\omega(\mu^{-1}(\nu(t)))}{r(\mu^{-1}(\nu(t)))}\right].$$

Applying Lemma 2.1 in the last inequality again, we conclude that

$$x^\theta(t) \geq \frac{1}{r(\mu^{-1}(t))}\left[\omega(\mu^{-1}(t)) - \frac{\tilde{r}(\mu^{-1}(t))\omega(\mu^{-1}(\nu(\mu^{-1}(t))))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))} - \frac{1}{r^{1/\theta}(\mu^{-1}(\mu^{-1}(t)))}\left(\frac{1}{\theta}\omega(\mu^{-1}(\mu^{-1}(t))) + \frac{\theta-1}{\theta}\right)\right]. \tag{2.5}$$

On the other hand, from condition (iii), we see that

$$\mu^{-1}(\mu^{-1}(t))) \leq \mu^{-1}(t) \tag{2.6}$$

and

$$\mu^{-1}(\nu(\mu^{-1}(t))) \leq \mu^{-1}(t). \tag{2.7}$$

Since $\omega(t)$ is increasing, we obtain from (2.6) and (2.7) that

$$\omega\left(\mu^{-1}(\mu^{-1}(t)))\right) \leq \omega\left(\mu^{-1}(t)\right) \tag{2.8}$$

and

$$\omega\left(\mu^{-1}(\nu(\mu^{-1}(t)))\right) \leq \omega\left(\mu^{-1}(t)\right), \tag{2.9}$$

respectively. Using (2.8) and (2.9) in (2.5) gives

$$x^\theta(t) \geq \frac{1 - \frac{\tilde{r}(\mu^{-1}(t))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))} - \frac{1}{\theta r^{1/\theta}(\mu^{-1}(\mu^{-1}(t)))}\left[1 + \frac{\theta-1}{\omega(\mu^{-1}(t))}\right]}{r(\mu^{-1}(t))}\omega(\mu^{-1}(t)). \tag{2.10}$$

Since $\omega(t)$ is increasing and $\delta(t)$ is decreasing and tending to zero, there exists a $t_2 \geq t_1$ such that

$$\omega(t) \geq \delta(t)$$

for all $t \geq t_2$. Substituting the latter inequality in (2.10) and rearranging we obtain (2.3), which completes the proof. □

Lemma 2.4 *Assume that* (1.3) *holds. If $x(t)$ is a positive solution of* (1.1)*, then the corresponding $\omega(t)$ satisfies eventually one of the following two cases:*

(I) $\omega > 0$, $\omega' > 0$, $(m\omega')' \leq 0$;
(II) $\omega > 0$, $\omega' < 0$, $(m\omega')' \leq 0$.

The proof of the above lemma is straightforward; hence, we omit the details.

Lemma 2.5 *Let $x(t)$ be an eventually positive solution of* (1.1) *and suppose that* (1.3) *holds and $\omega(t)$ satisfies Case* (II) *of Lemma* 2.4. *Then, there exists $t_2 \geq t_1 \geq t_0$ such that*

$$x^{\theta}(t) \geq \psi_2(t)\omega(\mu^{-1}(t)) \tag{2.11}$$

for all $t \geq t_2$.

Proof Proceeding exactly as in the proof of Lemma 2.3, we again arrive at (2.5), (2.6) and (2.7). Moreover, for $\eta \geq t$, we have

$$\omega'(\eta) \leq \frac{m(t)\omega'(t)}{m(\eta)}$$

and integrating the above inequality from t to ℓ, we have

$$\omega(\ell) \leq \omega(t) + m(t)\omega'(t)\int_t^{\ell} \frac{\mathrm{d}\eta}{m(\eta)}.$$

Letting $\ell \to \infty$, we obtain

$$0 \leq \omega(t) + \chi(t)m(t)\omega'(t),$$

and we conclude from the last inequality that

$$\left(\frac{\omega(t)}{\chi(t)}\right)' \geq 0 \tag{2.12}$$

for $t \geq t_1$. In view of (2.6) and (2.7), we obtain from (2.12) that

$$\omega(\mu^{-1}(\mu^{-1}(t))) \leq \frac{\chi(\mu^{-1}(\mu^{-1}(t))]}{\chi(\mu^{-1}(t))}\omega(\mu^{-1}(t)) \tag{2.13}$$

and

$$\omega(\mu^{-1}(\nu(\mu^{-1}(t)))) \leq \frac{\chi(\mu^{-1}(\nu(\mu^{-1}(t))))}{\chi(\mu^{-1}(t))}\omega(\mu^{-1}(t)) \tag{2.14}$$

respectively. Using (2.13) and (2.14) in (2.5) yields

$$\begin{aligned} x^{\theta}(t) \geq \frac{1}{r(\mu^{-1}(t))}\Bigg[&\omega(\mu^{-1}(t)) - \frac{\tilde{r}(\mu^{-1}(t))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))}\frac{\chi(\mu^{-1}(\nu(\mu^{-1}(t))))}{\chi(\mu^{-1}(t))}\omega(\mu^{-1}(t)) \\ &- \frac{1}{\theta r^{1/\theta}(\mu^{-1}(\mu^{-1}(t))}\left[\frac{\chi(\mu^{-1}(\mu^{-1}(t)))}{\chi(\mu^{-1}(t))}\omega(\mu^{-1}(t)) + \theta - 1\right]\Bigg] \\ = \frac{\omega(\mu^{-1}(t))}{r(\mu^{-1}(t))}\Bigg[&1 - \frac{\tilde{r}(\mu^{-1}(t))}{r(\mu^{-1}(\nu(\mu^{-1}(t))))}\frac{\chi(\mu^{-1}(\nu(\mu^{-1}(t))))}{\chi(\mu^{-1}(t))} \end{aligned}$$

$$-\frac{1}{\theta r^{1/\theta}(\mu^{-1}(\mu^{-1}(t))}\left[\frac{\chi(\mu^{-1}(\mu^{-1}(t)))}{\chi(\mu^{-1}(t))}+\frac{\theta-1}{\omega(\mu^{-1}(t))}\right]\bigg]. \tag{2.15}$$

Since $\frac{\omega(t)}{\chi(t)}$ is positive and increasing and $\chi(t)$ is decreasing and tending to zero, there exists a $t_2 \geq t_1$ such that

$$\frac{\omega(t)}{\chi(t)} \geq \chi(t)$$

for all $t \geq t_2$. Substituting this last inequality in (2.15) gives (2.11) which completes the proof. □

3 Main Results

Now, we can give our first oscillation criterion.

Theorem 3.1 *Assume that* (1.2) *holds,* $\lambda \geq \theta$ *and* $\xi(t) \leq \mu(t)$. *If there exists a positive and nondecreasing function* $\phi \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left(\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)}-\frac{m(\eta)\left(\phi'(\eta)\right)^2}{4\kappa_1\phi(\eta)}\right)\mathrm{d}\eta=\infty \tag{3.1}$$

for every constant $\kappa_1 > 0$ *and for all* $t > T \geq t_1 \in [t_0, \infty)$, *then all solutions of equation* (1.1) *are oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. It follows from (1.1) and (2.3) that

$$\left(m(t)\omega'(t)\right)'+\kappa q(t)\psi_1^{\lambda/\theta}(\xi(t))\,\omega^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)\leq 0, \tag{3.2}$$

for $t \geq t_2 \in [t_1, \infty)$. Since $m(t)\omega'(t)$ is decreasing on $[t_1, \infty)$, we obtain

$$\omega(t)=\omega(t_1)+\int_{t_1}^t \frac{m(\eta)\omega'(\eta)}{m(\eta)}\mathrm{d}\eta \geq m(t)\omega'(t)A(t). \tag{3.3}$$

In view of (3.3), we see that

$$\left(\frac{\omega(t)}{A(t)}\right)' \leq 0 \tag{3.4}$$

for all $t \geq t_2 \in [t_1, \infty)$. Using (3.4) and the fact that $\mu^{-1}(\xi(t)) \leq t$, we get

$$\frac{\omega\left(\mu^{-1}(\xi(t))\right)}{A\left(\mu^{-1}(\xi(t))\right)} \geq \frac{\omega(t)}{A(t)} \tag{3.5}$$

for $t \geq t_2$. In view of (3.5), inequality (3.2) can be written as:

$$\left(m(t)\omega'(t)\right)' + \kappa q(t)\psi_1^{\lambda/\theta}(\xi(t))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)}{A^{\lambda/\theta}(t)}\omega^{\lambda/\theta}(t) \leq 0, \tag{3.6}$$

for $t \geq t_2$. Define the Riccati-type substitution by

$$\Psi(t) = \phi(t)\frac{m(t)\omega'(t)}{\omega(t)} \quad \text{for} \quad t \geq t_2. \tag{3.7}$$

Obviously $\Psi(t) > 0$ and from (3.6), we obtain

$$\begin{aligned}\Psi'(t) &= \frac{\phi'(t)}{\phi(t)}\Psi(t) + \frac{\phi(t)\left(m(t)\omega'(t)\right)'}{\omega(t)} - \frac{\Psi^2(t)}{\phi(t)m(t)} \\ &\leq \frac{\phi'(t)}{\phi(t)}\Psi(t) - \kappa\phi(t)q(t)\psi_1^{\lambda/\theta}(\sigma(t))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)}{A^{\lambda/\theta}(t)}\frac{\omega^{\lambda/\theta}(t)}{\omega(t)} - \frac{\Psi^2(t)}{\phi(t)m(t)}\end{aligned} \tag{3.8}$$

for $t \geq t_2$. Since $\omega(t)$ is positive and strictly increasing, there exists a $C > 0$ such that $\omega(t) \geq C > 0$ for all $t \geq t_2$. Completing the square with respect to Ψ, it follows from (3.8) that

$$\Psi'(t) \leq -\kappa_1\phi(t)q(t)\psi_1^{\lambda/\theta}(\xi(t))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)}{A^{\lambda/\theta}(t)} + \frac{m(t)\left(\phi'(t)\right)^2}{4\phi(t)}, \tag{3.9}$$

where $\kappa_1 = \kappa C^{(\lambda-\theta)/\theta}$. Integrating (3.9) from t_2 to t yields

$$\int_{t_2}^{t}\left(\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)} - \frac{m(\eta)\left(\phi'(\eta)\right)^2}{4\kappa_1\phi(\eta)}\right)\mathrm{d}\eta \leq \Psi(t_2),$$

which contradicts (3.1) and completes the proof. □

From Theorem 3.1, we can establish different sufficient conditions for oscillation of (1.1), using different choices of the function $\phi(t)$. For instance, letting $\phi(t) = 1$ and $\phi(t) = t^{\rho}$ with $\rho \geq 1$, we obtain the following corollaries, respectively.

Corollary 3.1 *Suppose that* (1.2) *holds,* $\lambda \geq \theta$ *and* $\xi(t) \leq \mu(t)$*. If*

$$\limsup_{t\to\infty}\int_T^t q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)}\mathrm{d}\eta = \infty \tag{3.10}$$

for all $t > T \geq t_1 \in [t_0, \infty)$*, then Eq.* (1.1) *is oscillatory.*

Corollary 3.2 *Suppose that* (1.2) *holds,* $\lambda \geq \theta$ *and* $\xi(t) \leq \mu(t)$*. If*

$$\limsup_{t\to\infty} \int_T^t \left(\eta^\rho q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta)) \frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)} - \frac{\rho^2 m(\eta)\eta^{\rho-2}}{4\kappa_1} \right) \mathrm{d}\eta = \infty \tag{3.11}$$

for every constant $\kappa_1 > 0$ *and for all* $t > T \geq t_1 \in [t_0, \infty)$*, then Eq.* (1.1) *is oscillatory.*

In the next theorem, we give an oscillation criterion for (1.1) by using the integral averaging technique due to Philos [26]. First we need to introduce the function class $\mathcal{P}$.

Let $S_0 \equiv \{(t, \eta) \in \mathbb{R}^2 : t > \eta \geq t_0\}$ and $S \equiv \{(t, \eta) \in \mathbb{R}^2 : t \geq \eta \geq t_0\}$. We say that the function $J \in C(S, \mathbb{R})$ belongs to the class $\mathcal{P}$, denoted by $J \in \mathcal{P}$ if

($\mathcal{P}_1$) $J(t, t) = 0$ for $t \geq t_0$ and $J(t, \eta) > 0$ on S_0;
($\mathcal{P}_2$) $J(t, \eta)$ has a continuous and non-positive partial derivative on S_0 with respect to the second variable.

Theorem 3.2 *Assume that* (1.2) *be fulfilled,* $\lambda \geq \theta$ *and* $\xi(t) \leq \mu(t)$*. Let* $j, J : S \to \mathbb{R}$ *be continuous functions such that* $J \in \mathcal{P}$ *and*

$$-\frac{\partial J}{\partial \eta}(t, \eta) = j(t, \eta)\sqrt{J(t, \eta)} \quad \text{forall} \quad (t, \eta) \in S_0. \tag{3.12}$$

If there exists a positive and nondecreasing function $\phi \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{J(t, T)} \int_T^t \left[J(t, \eta)\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta)) \frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)} \right.$$
$$\left. - \frac{\phi(\eta)m(\eta)\Phi^2(t, \eta)}{4\kappa_1} \right] \mathrm{d}\eta = \infty \tag{3.13}$$

for every constant $\kappa_1 > 0$ *and for all* $t > T \geq t_1 \in [t_0, \infty)$*, where*

$$\Phi(t, \eta) = -j(t, \eta) + \frac{\phi'(\eta)}{\phi(\eta)}\sqrt{J(t, \eta)}, \tag{3.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.8). It follows from (3.8) that

$$\int_{t_2}^t J(t, \eta)\kappa_1\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta)) \frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)} \mathrm{d}\eta \leq -\int_{t_2}^t J(t, \eta)\Psi'(\eta)\, d\eta$$

$$+\int_{t_2}^{t} J(t,\eta)\frac{\phi'(\eta)}{\phi(\eta)}\Psi(\eta)\,\mathrm{d}\eta - \int_{t_2}^{t} J(t,\eta)\frac{\Psi^2(\eta)}{\phi(\eta)m(\eta)}\,\mathrm{d}\eta \tag{3.15}$$

for $t > t_2 \in [t_1, \infty)$. Using the integration by parts formula, we obtain

$$\begin{aligned}\int_{t_2}^{t} J(t,\eta)\Psi'(\eta)\,d\eta &= J(t,\eta)\Psi(\eta)\Big|_{t_2}^{t} - \int_{t_2}^{t} \frac{\partial J}{\partial \eta}(t,\eta)\Psi(\eta)\,\mathrm{d}\eta \\ &= -J(t,t_2)\Psi(t_2) - \int_{t_2}^{t} \frac{\partial J}{\partial \eta}(t,\eta)\Psi(\eta)\,\mathrm{d}\eta \end{aligned} \tag{3.16}$$

In view of (3.12) and (3.16), we have from (3.15) that

$$\begin{aligned}\int_{t_2}^{t} J(t,\eta)\kappa_1\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)}\,\mathrm{d}\eta &\le J(t,t_2)\Psi(t_2) \\ +\int_{t_2}^{t}\left[-j(t,\eta)\sqrt{J(t,\eta)} + J(t,\eta)\frac{\phi'(\eta)}{\phi(\eta)}\right]\Psi(\eta)\,\mathrm{d}\eta \\ -\int_{t_2}^{t} J(t,\eta)\frac{\Psi^2(\eta)}{\phi(\eta)m(\eta)}\,\mathrm{d}\eta. \end{aligned} \tag{3.17}$$

By completing the square with respect to Ψ, it follows from (3.17) that

$$\begin{aligned}\int_{t_2}^{t} J(t,\eta)\kappa_1\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)}\,\mathrm{d}\eta \\ \le J(t,t_2)\Psi(t_2) + \frac{1}{4}\int_{t_2}^{t}\phi(\eta)m(\eta)\Phi^2(t,\eta)\,\mathrm{d}\eta. \end{aligned} \tag{3.18}$$

So, for all $t > t_2$, we conclude that

$$\begin{aligned}\frac{1}{J(t,t_2)}\int_{t_2}^{t}\Bigg[J(t,\eta)\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\,\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A^{\lambda/\theta}(\eta)} \\ -\frac{\phi(\eta)m(\eta)\Phi^2(t,\eta)}{4\kappa_1}\Bigg]\,\mathrm{d}\eta \le \Psi(t_2),\end{aligned}$$

which contradicts (3.13) and completes the proof. □

Next, we give some oscillation results in the case when $\lambda < \theta$.

Theorem 3.3 *Assume that* (1.2) *holds,* $\lambda < \theta$ *and* $\xi(t) \le \mu(t)$. *If there exists a positive and nondecreasing function* $\phi \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left(\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\,\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A(\eta)} - \frac{m(\eta)\left(\phi'(\eta)\right)^2}{4\kappa_2\phi(\eta)}\right)\mathrm{d}\eta = \infty \tag{3.19}$$

for every constant $\kappa_2 > 0$ *and for all* $t > T \geq t_1 \in [t_0, \infty)$*, then all solutions of equation* (1.1) *are oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.2)–(3.4) and (3.5). Since $\lambda < \theta$ and the function $\omega(t)/A(t)$ is non-increasing on $[t_1, \infty)$, there exists a $C_1 > 0$ such that

$$\left(\frac{\omega(t)}{A(t)}\right)^{\frac{\lambda-\theta}{\theta}} \geq \frac{1}{C_1^{1-\frac{\lambda}{\theta}}} \tag{3.20}$$

for $t \geq t_2 \in [t_1, \infty)$. Using (3.20) and (3.5) in (3.2), we obtain

$$\left(m(t)\omega'(t)\right)' + \kappa_2 q(t)\psi_1^{\lambda/\theta}(\xi(t))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)}{A(t)}\omega(t) \leq 0, \tag{3.21}$$

where $\kappa_2 = \frac{\kappa}{C_1^{1-\frac{\lambda}{\theta}}}$. The remainder of the proof is similar to that of Theorem 3.1, so we omit the details here. The proof of this theorem is complete. □

Theorem 3.4 *Assume that* (1.2) *holds,* $\lambda < \theta$ *and* $\xi(t) \leq \mu(t)$*. Let* $j, J : S \to \mathbb{R}$ *be continuous functions such that* $J \in \mathcal{P}$ *and* (3.12) *holds. If there exists a positive and nondecreasing function* $\phi \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\frac{1}{J(t,T)}\int_T^t\left[J(t,\eta)\phi(\eta)q(\eta)\psi_1^{\lambda/\theta}(\xi(\eta))\frac{A^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)}{A(\eta)}\right.$$
$$\left.-\frac{\phi(\eta)m(\eta)\Phi^2(t,\eta)}{4\kappa_2}\right]\mathrm{d}\eta = \infty \tag{3.22}$$

for every constant $\kappa_2 > 0$ *and for all* $t > T \geq t_1 \in [t_0, \infty)$*, where* $\Phi(t, \eta)$ *is as in* (3.14)*, then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from Theorems 3.2 and 3.3. □

Next, we give following oscillation theorem for noncanonical case, i.e., when (1.3) holds.

Theorem 3.5 *Assume that* (1.3) *holds and* $\xi(t) < \mu(t)$*. If the first-order delay differential inequality*

$$z'(t) + \kappa q(t)\psi_1^{\lambda/\theta}(\xi(t))A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right)z^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right) \leq 0 \tag{3.23}$$

has no positive solution, and

$$\int_T^\infty \frac{1}{m(t)}\left(\int_T^t q(\eta)\psi_2^{\lambda/\theta}(\xi(\eta))\chi^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right)\mathrm{d}\eta\right)\mathrm{d}t = \infty \tag{3.24}$$

for all $t > T \geq t_1 \in [t_0, \infty)$, then all solutions of equation (1.1) are oscillatory.

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Then, corresponding $\omega(t)$ satisfies either Case (I) or Case (II) of Lemma 2.4. We will consider each case separately.

Case I: Assume first that $\omega'(t) > 0$ on $[t_1, \infty)$. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.2) and (3.3). Define $z(t) = m(t)\omega'(t)$ for $t \geq t_2$. It is clear to see that $z(t) > 0$. Using (3.3), it follows from (3.2) that

$$z'(t) + \kappa q(t)\psi_1^{\lambda/\theta}\left(\xi(t)\right) A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right) z^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right) \leq 0$$

for $t \geq t_2$. Then, z is a positive solution of the inequality (3.23), which is a contradiction.

Case II: Suppose now $\omega'(t) < 0$ on $[t_1, \infty)$. Proceeding as in the proof of Lemma 2.5, we again arrive at (2.12). From (1.1) and (2.11), we have

$$\left(m(t)\omega'(t)\right)' + \kappa q(t)\psi_2^{\lambda/\theta}\left(\xi(t)\right) \omega^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right) \leq 0,$$

for $t \geq t_2$. Integrating the latter inequality from t_2 to t, we obtain

$$\kappa \int_{t_2}^{t} q(\eta)\psi_2^{\lambda/\theta}\left(\xi(\eta)\right) \omega^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right) \mathrm{d}\eta \leq -m(t)\omega'(t). \tag{3.25}$$

On the other hand, since (2.12) indicates that the function $\omega(t)/\chi(t)$ is non-decreasing on $[t_1, \infty)$, there exists a $C_2 > 0$ such that

$$\frac{\omega(t)}{\chi(t)} \geq C_2 > 0 \quad \text{for all } t \geq t_2 \in [t_1, \infty).$$

Hence, from (3.25), we conclude that

$$\frac{\kappa C_2^{\lambda/\theta}}{m(t)} \int_{t_2}^{t} q(\eta)\psi_2^{\lambda/\theta}\left(\xi(\eta)\right) \chi^{\lambda/\theta}\left(\mu^{-1}(\xi(\eta))\right) \mathrm{d}\eta \leq -\omega'(t). \tag{3.26}$$

Integrating (3.26) from t_2 to t, and passing to the limit as $t \to \infty$, we obtain a contradiction to (3.24). This contradiction completes the proof. □

Corollary 3.3 *Suppose that (1.3) holds, $\lambda = \theta$ and $\xi(t) < \mu(t)$. If*

$$\liminf_{t\to\infty} \int_{\mu^{-1}(\xi(t))}^{t} q(\eta)\psi_1\left(\xi(\eta)\right) A\left(\mu^{-1}(\xi(\eta))\right) \mathrm{d}\eta > \frac{1}{e\kappa} \tag{3.27}$$

and

$$\int_T^\infty \frac{1}{m(t)} \left(\int_T^t q(\eta)\psi_2(\xi(\eta))\,\chi\left(\mu^{-1}(\xi(\eta))\right) \mathrm{d}\eta \right) \mathrm{d}t = \infty, \tag{3.28}$$

then all solutions of equation (1.1) *are oscillatory.*

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Then, corresponding $\omega(t)$ satisfies either Case (I) or Case (II) of Lemma 2.4. The proof when Case (II) holds is exactly the same as that of Case (II) of Theorem 3.5 with $\lambda = \theta$.

Case I: Assume that $\omega'(t) > 0$ on $[t_1, \infty)$. Then, z is a positive solution of the inequality (3.23). On the other hand, applying condition (3.27) to (3.23), one concludes that (3.23) cannot have positive solutions by [18, Theorem 2.1.1]. This contradiction completes the proof. □

Corollary 3.4 *Suppose that (1.3) holds,* $\lambda < \theta$ *and* $\xi(t) < \mu(t)$*. If*

$$\int_T^\infty q(t)\psi_1^{\lambda/\theta}(\xi(t))\,A^{\lambda/\theta}\left(\mu^{-1}(\xi(t))\right) \mathrm{d}t = \infty, \tag{3.29}$$

and (3.24) *holds, then Eq.* (1.1) *is oscillatory.*

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Then, corresponding $\omega(t)$ satisfies either Case (I) or Case (II) of Lemma 2.4. The proof when Case (II) holds is exactly the same as that of Case (II) of Theorem 3.5.

Case I: Assume that $\omega'(t) > 0$ on $[t_1, \infty)$. Then, z is a positive solution of the inequality (3.23). On the other hand, if (3.29) holds, Eq. (3.23) cannot have positive solutions by virtue of Kitamura and Kusano [23, Theorem 2]. The proof is complete in view of this contradiction. □

Corollary 3.5 *Assume that* (1.3) *holds,* $\lambda > \theta$, $\xi(t) = t - b$, $\nu(t) = t - e$ *and* $\mu(t) = t + d$*, where* $b, e, d > 0$ *are constants. If*

$$\liminf_{t\to\infty} \left[\left(\frac{\lambda}{\theta}\right)^{-\frac{t}{b+d}} \log\left(q(t)\psi_1^{\lambda/\theta}(t-b)\,A^{\lambda/\theta}(t-b-d) \right) \right] > 0, \tag{3.30}$$

and (3.24) *holds, then Eq.* (1.1) *is oscillatory.*

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\nu(t)) > 0$, $x(\mu(t)) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_1$. Then, corresponding $\omega(t)$ satisfies either Case (I) or Case (II) of Lemma 2.4. If Case (II) holds, the proof is exactly the same as that of Case (II) of Theorem 3.5.

Case I: Assume that $\omega'(t) > 0$ on $[t_1, \infty)$. Then, from Sakamoto and Tanaka [28, Lemma 2.2], the inequality (3.23) and the equation

$$z'(t) + \kappa q(t)\psi_1^{\lambda/\theta}(t-b)\, A^{\lambda/\theta}(t-b-d)\, z^{\lambda/\theta}(t-b-d) = 0 \tag{3.31}$$

have a positive solution, for $t \geq t_2$. On the other hand, if (3.30) holds, Eq. (3.31) cannot have positive solutions by virtue of Sakamoto and Tanaka [28, Corollary 1.2]. This contradiction completes the proof. □

4 Examples and Remarks

Remark 4.1 If we take $\tilde{r}(t) = 0$ in Theorems 3.1–3.5 and Corollaries 3.1–3.5, then we obtain sufficient conditions for the oscillation of all solutions of the equation

$$\left(m(t)\left(x(t) + r(t)x^{\theta}\left(\mu(t)\right)\right)'\right)' + q(t)f\left(x(\xi(t))\right) = 0, \quad t \geq t_0 > 0, \tag{4.1}$$

in the case when $\theta \geq 1$, $\mu(t) \geq t$ and $\xi(t) \leq \mu(t)$. It should be pointed out that obtained results in this paper are new even for the second-order advanced neutral differential equation (4.1) under conditions $(i) - (v)$.

Remark 4.2 We point out versatility of the obtained results in this paper with respect to the behavior of the functions $\mu(t)$ and $\xi(t)$. Note that our conditions on the deviating arguments are $\mu(t) \geq t$ and $\xi(t) \leq \mu(t)$. Therefore, $\xi(t)$ can be a delayed or an appropriate advanced argument.

Example 4.1 Consider the second-order neutral differential equation

$$\left(e^{-t}\left[x(t) + x^3(t/2) + e^{3t}x^3(2t)\right]'\right)' + \Omega e^{3t}x^3(2t-1) = 0. \tag{4.2}$$

for $t \geq 2$. Here, we have

- $\theta = \lambda = 3$, $m(t) = e^{-t}$, $\kappa = 1$ and $q(t) = \Omega e^{3t}$ with Ω is a constant;
- $\nu(t) = t/2$, $\mu(t) = 2t$, $\xi(t) = 2t - 1$, $\tilde{r}(t) = 1$ and $r(t) = e^{3t}$.

It is easy to see that conditions $(i) - (v)$ and (1.2) hold, and $A(t) = e^t - e^2$. If we choose $\delta(t) = 4e^{-t/2}$, then $\delta(t)$ is a positive decreasing function such that tending to zero as $t \to \infty$. So, we see that

$$\psi_1(t) = \frac{5e^{5t/8} - 2e^{3t/8} - 6e^{t/4}}{6e^{17t/8}} > 0$$

for $t \geq T \geq 2$. Noting that $\mu^{-1}(\xi(t)) = t - 1/2 < t$, and taking $\phi(t) = 1$, a direct calculation shows that (3.10) is satisfied with $\Omega > 0$ and $t > T \geq 2$. Therefore, Eq. (4.2) is oscillatory by Corollary 3.1, provided that $\Omega > 0$.

Example 4.2 Consider the second-order neutral differential equation

$$\left(t^2\left[x(t)+tx^{5/3}(t-1)+t^5x^{5/3}(t+2)\right]'\right)'+t^5x(t+1)=0. \tag{4.3}$$

for $t \geq 7$. Here, we have

- $\theta = 5/3$, $\lambda = 1$, $m(t) = t^2$, $\kappa = 1$ and $q(t) = t^5$;
- $\nu(t) = t-1$, $\mu(t) = t+2$, $\xi(t) = t+1$, $\tilde{r}(t) = t$ and $r(t) = t^5$.

It is obvious that conditions $(i) - (v)$ and (1.3) hold. Then,

$$A(t) = \frac{t-7}{7t} \quad \text{and} \quad \chi(t) = \frac{1}{t}.$$

If we choose $\delta(t) = 4(t-2)^{-1}$, then $\delta(t)$ is a positive decreasing function such that tending to zero as $t \to \infty$. On the other hand, we see that

$$\psi_1(t) = \frac{1}{(t-2)^5}\left[1 - \frac{t-2}{(t-5)^5} - \frac{3}{5(t-4)^3} - \frac{1}{10(t-4)^2}\right] > 0$$

and

$$\psi_2(t) = \frac{1}{(t-2)^5}\left[1 - \frac{(t-2)^2}{(t-5)^6} - \frac{3}{5(t-4)^3}\left(\frac{t-2}{t-4} + \frac{2(t-2)^2}{3}\right)\right] > 0$$

for $t \geq 7$. Noting that $\mu^{-1}(\xi(t)) = t-1 < t$, and taking $\phi(t) = 1$, a direct calculation shows that (3.24) and (3.29) are satisfied for $t > T \geq 7$. Hence, Eq. (4.3) is oscillatory by Corollary 3.4.

Remark 4.3 Assume that condition (v) is replaced by

(v^*) $\tilde{r}, r \in C\big([t_0, \infty), [0, \infty)\big)$ with $\tilde{r}(t) \to \infty$ as $t \to \infty$.

In this case, if the functions $\psi_1(t)$ and $\psi_2(t)$ are replaced by

$$\begin{aligned}\psi_3(t) := \frac{1}{\tilde{r}(\nu^{-1}(t))}\Bigg[1 &- \frac{r(\nu^{-1}(t))}{\tilde{r}(\nu^{-1}(\mu(\nu^{-1}(t))))}\frac{A(\nu^{-1}(\mu(\nu^{-1}(t))))}{A(\nu^{-1}(t))} \\ &- \frac{1}{\theta\tilde{r}^{1/\theta}(\nu^{-1}(\nu^{-1}(t)))}\left(\frac{A(\nu^{-1}(\nu^{-1}(t)))}{A(\nu^{-1}(t))} + \frac{\theta-1}{\delta(\nu^{-1}(t))}\right)\Bigg]\end{aligned}$$

and

$$\psi_4(t) = \frac{1}{\tilde{r}(\nu^{-1}(t))}\left[1 - \frac{r(\nu^{-1}(t))}{\tilde{r}(\nu^{-1}(\mu(\nu^{-1}(t))))} - \frac{1}{\theta\tilde{r}^{1/\theta}(\nu^{-1}(\nu^{-1}(t)))}\left(1 + \frac{\theta-1}{\chi^2(\nu^{-1}(t))}\right)\right]$$

respectively, then one can obtain new oscillation results for Eq. (1.1) under assumptions of $(i) - (v^*)$, $\xi(t) \leq \nu(t)$, $\psi_3(t) > 0$ and $\psi_4(t) > 0$. The details are left to the reader.

Remark 4.4 It would also be of interest to study Eq. (1.1) with sub-linear neutral terms and unbounded coefficients, i.e., under conditions of $0 < \theta \leq 1$ and $r(t) \to \infty$ and/or $\tilde{r}(t) \to \infty$ as $t \to \infty$.

Remark 4.5 It would also be of interest to study Eq. (1.1) for the cases where $-\infty < r, \tilde{r} < 0$ or $-\infty < r, \tilde{r} < -1$ or $-1 < r, \tilde{r} < 0$ with superlinear neutral terms $\theta \geq 1$.

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