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Positive Solutions for an Iterative System of Nonlinear Elliptic Equations

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Abstract

This paper deals with the existence of positive radial solutions to the iterative system of nonlinear elliptic equations of the form

$$\triangle \mathtt{z}_{\mathtt{j}} + \frac{(\mathtt{N}-2)^2 r_0^{2\mathtt{N}-2}}{|x|^{2\mathtt{N}-2}} \mathtt{z}_{\mathtt{j}} + \varphi(|x|)\mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1}) = 0, \ \mathtt{R}_1 < |x| < \mathtt{R}_2,$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{z}_1 = \mathtt{z}_{\ell+1}$, $\triangle \mathtt{z} = \mathtt{div}(\nabla \mathtt{z})$, $\mathtt{N} > 2$, $0 < r_0 < \pi/2$, $\varphi = \prod_{i=1}^{n} \varphi_i$, each $\varphi_i : (r_0, +\infty) \to (0, +\infty)$ is continuous, $r^{\mathtt{N}-1}\varphi$ is integrable, and $\mathtt{g}_{\mathtt{j}} : [0, +\infty) \to \mathbb{R}$ is continuous, by an application of various fixed point theorems in a Banach space. Further, we also establish uniqueness of the solution for the addressed system by using Rus's theorem in a complete metric space.

Keywords Nonlinear elliptic equation · Annulus · Positive radial solution · Fixed point theorem · Banach space · Rus's theorem · Metric space · Continuous functions

Mathematics Subject Classification 35J66 · 35J60 · 34B18 · 47H10

1 Introduction

The semilinear elliptic equation of the form

$$\triangle \mathtt{z} + \mathtt{g}(|x|)\mathtt{z} + \mathtt{h}(|x|)\mathtt{z}^{\mathtt{p}} = 0 \tag{1}$$

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arises in various fields of pure and applied mathematics such as Riemannian geometry, nuclear physics, astrophysics and so on. For more details of the background of (1), see [9,10,19,24]. Study of nonlinear elliptic system of equations,

$$\left.\begin{aligned} \triangle \mathtt{z}_{\mathtt{j}} + \mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1}) = 0 \ \text{ in } \ \Omega, \\ \mathtt{z}_{\mathtt{j}} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}\right\} \tag{2}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{z}_1 = \mathtt{z}_{\ell+1}$, and Ω is a bounded domain in $\mathbb{R}^N$, has an important applications in population dynamics, combustion theory and chemical reactor theory. For the recent literature for the existence, multiplicity and uniqueness of positive solutions for (2), see [1,3,7,12–14] and references therein.

In [6], Chrouda and Hassine established the uniqueness of positive radial solutions to the following Dirichlet boundary value problem for the semilinear elliptic equation in an annulus,

$$\begin{gathered} \triangle \mathtt{z} = \mathtt{g}(\mathtt{z}) \ \text{ on } \ \Omega = \{x \in \mathbb{R}^{\mathtt{d}} : a < |x| < b\}, \\ \mathtt{z} = 0 \ \text{ on } \ \mathtt{z} \in \partial\Omega, \end{gathered}$$

for any dimension $\mathtt{d} \geq 1$. In [8], Dong and Wei established the existence of radial solutions for the following nonlinear elliptic equations with gradient terms in annular domains,

$$\begin{gathered} \triangle \mathtt{z} + \mathtt{g}\Big(|x|, \mathtt{z}, \frac{x}{|x|} \cdot \nabla \mathtt{z}\Big) = 0 \ \text{ in } \ \Omega_a^b, \\ \mathtt{z} = 0 \ \text{ on } \ \partial\Omega_a^b, \end{gathered}$$

by using Schauder's fixed point theorem and contraction mapping theorem. In [15], R. Kajikiya and E. Ko established the existence of positive radial solutions for a semipositone elliptic equation of the form,

$$\begin{gathered} \triangle \mathtt{z} + \lambda \mathtt{g}(\mathtt{z}) = 0 \ \text{ in } \ \Omega, \\ \mathtt{z} = 0 \ \text{ on } \ \partial\Omega, \end{gathered}$$

where Ω is a ball or an annulus in $\mathbb{R}^{\mathrm{N}}$. Recently, Son and Wang [22] established positive radial solutions to the nonlinear elliptic systems,

$$\begin{gathered} \triangle \mathtt{z}_{\mathtt{j}} + \lambda \mathtt{K}_{\mathtt{j}}(|x|)\mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1}) = 0 \text{ in } \Omega_{\mathrm{E}}, \\ \mathtt{z}_{\mathtt{j}} = 0 \text{ on } |x| = r_0, \\ \mathtt{z}_{\mathtt{j}} \to 0 \text{ as } |x| \to +\infty, \end{gathered}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{z}_1 = \mathtt{z}_{\ell+1}$, $\lambda > 0$, $N > 2$, $r_0 > 0$, and Ω_{E} is an exterior of a ball. Motivated by the above works, in this paper we study the existence of infinitely many positive radial solutions for the following iterative system of nonlinear elliptic equations in an annulus,

$$\triangle \mathtt{z_j} + \frac{(\mathrm{N}-2)^2 r_0^{2\mathrm{N}-2}}{|x|^{2\mathrm{N}-2}} \mathtt{z_j} + \varphi(|x|)\mathtt{g_j}(\mathtt{z_{j+1}}) = 0,\ \mathrm{R}_1 < |x| < \mathrm{R}_2, \tag{3}$$

with one of the following sets of boundary conditions:

$$\left.\begin{aligned} \mathtt{z_j} = 0 \ \text{ on } \ |x| = \mathrm{R}_1 \text{ and } |x| = \mathrm{R}_2, \\ \mathtt{z_j} = 0 \ \text{ on } \ |x| = \mathrm{R}_1 \text{ and } \frac{\partial \mathtt{z_j}}{\partial r} = 0 \text{ on } |x| = \mathrm{R}_2, \\ \frac{\partial \mathtt{z_j}}{\partial r} = 0 \ \text{ on } \ |x| = \mathrm{R}_1 \text{ and } \mathtt{z_j} = 0 \text{ on } |x| = \mathrm{R}_2, \end{aligned}\right\} \tag{4}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{z}_1 = \mathtt{z}_{\ell+1}$, $\triangle \mathtt{z} = \mathtt{div}(\nabla \mathtt{z})$, $N > 2$, $0 < r_0 < \pi/2$, $\varphi = \prod_{i=1}^{n} \varphi_i$, each $\varphi_i : (\mathrm{R}_1, \mathrm{R}_2) \to (0, +\infty)$ is continuous, $r^{\mathrm{N}-1}\varphi$ is integrable, by an application of various fixed point theorems in a Banach space. Further, we also study existence of unique solution by using Rus's theorem in a complete metric space.

The study of positive radial solutions to (3) reduces to the study of positive solutions to the following iterative system of two-point boundary value problems,

$$\mathtt{z}_\mathtt{j}''(\tau) + r_0^2 \mathtt{z_j}(\tau) + \varphi(\tau)\mathtt{g_j}(\mathtt{z_{j+1}}(\tau)) = 0,\ 0 < \tau < 1, \tag{5}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{z}_1 = \mathtt{z}_{\ell+1}$, $0 < r_0 < \pi/2$, and $\varphi(\tau) = \frac{r_0^2}{(\mathrm{N}-2)^2} \tau^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}} \prod_{i=1}^{n} \varphi_i(\tau)$, $\varphi_i(\tau) = \varphi_i(r_0 \tau^{\frac{1}{2-\mathrm{N}}})$ by a Kelvin-type transformation through the change of variables $r = |x|$ and $\tau = \left(\frac{r}{r_0}\right)^{2-\mathrm{N}}$. For the detailed explanation of the transformation from equations (7) to (5), see [2,16,17]. By suitable choices of nonnegative real numbers α, β, γ and δ with $r_0^2 \le \frac{\alpha\gamma}{\beta\delta}$, the set of boundary conditions (5) reduces to

$$\begin{cases} \alpha \mathtt{z_j}(0) - \beta \mathtt{z}_\mathtt{j}'(0) = 0, \\ \gamma \mathtt{z_j}(1) + \delta \mathtt{z}_\mathtt{j}'(1) = 0, \end{cases} \tag{6}$$

we assume that the following conditions hold throughout the paper:

($\mathcal{H}_1$) $\mathtt{g_j} : [0, +\infty) \to [0, +\infty)$ is continuous.
($\mathcal{H}_2$) $\varphi_i \in L^{\mathrm{p}_i}[0, 1]$, $1 \le \mathrm{p}_i \le +\infty$ for $1 \le i \le n$.
($\mathcal{H}_3$) There exists $\varphi_i^\star > 0$ such that $\varphi_i^\star < \varphi_i(\tau) < \infty$ a.e. on $[0, 1]$.

The rest of the paper is organized in the following fashion. In Sect. 2, we convert the boundary value problem (5)–(6) into equivalent integral equation which involves the kernel. Also, we estimate bounds for the kernel which are useful in our main results. In Sect. 3, we develop criteria for the existence of at least one positive radial solution by applying Krasnoselskii's cone fixed point theorem in a Banach space. In Sect. 4, we derive necessary conditions for the existence of at least two positive radial solution by an application of Avery–Henderson cone fixed point theorem in a Banach space. In

Sect. 5, we establish the existence of at least three positive radial solution by utilizing Leggett-William cone fixed point theorem in a Banach space. Further, we also study uniqueness of solution in the final section.

2 Kernel and Its Bounds

In order to study BVP (5), we first consider the corresponding linear boundary value problem,

$$-(\mathrm{z}_1''(\tau)+r_0^2\mathrm{z}_1(\tau))=y(\tau),\ 0<\tau<1, \tag{7}$$

$$\begin{cases}\alpha \mathrm{z}_1(0)-\beta \mathrm{z}_1'(0)=0,\\ \gamma \mathrm{z}_1(1)+\delta \mathrm{z}_1'(1)=0,\end{cases} \tag{8}$$

where $y\in\mathcal{C}[0,1]$ is a given function.

Lemma 1 *Let* $\wp=r_0^2(\alpha\delta+\beta\gamma)\cos(r_0)+r_0(\alpha\gamma-\beta\delta r_0^2)\sin(r_0)$. *For every* $y\in\mathcal{C}(0,1)$, *the linear boundary value problem* (7)–(8) *has a unique solution*

$$\mathrm{z}_1(\tau)=\int_0^1\aleph_{r_0}(\tau,\mathrm{s})y(\mathrm{s})d\mathrm{s}, \tag{9}$$

where

$$\aleph_{r_0}(\tau,\mathrm{s})=\frac{1}{\wp}\begin{cases}\big(\alpha\sin(r_0\tau)+\beta r_0\cos(r_0\tau)\big)\big(\gamma\sin(r_0(1-\mathrm{s}))+\delta r_0\cos(r_0(1-\mathrm{s}))\big), & 0\le\tau\le\mathrm{s}\le1,\\ \big(\alpha\sin(r_0\mathrm{s})+\beta r_0\cos(r_0\mathrm{s})\big)\big(\gamma\sin(r_0(1-\tau))+\delta r_0\cos(r_0(1-\tau))\big), & 0\le\mathrm{s}\le\tau\le1.\end{cases}$$

Lemma 2 *Let* $\sigma=\max\left\{\dfrac{\alpha+\beta r_0}{\beta r_0\cos(r_0)},\dfrac{\gamma+\delta r_0}{\delta r_0\cos(r_0)}\right\}$. *The kernel* $\aleph_{r_0}(\tau,\mathrm{s})$ *has the following properties:*

(i) $\aleph_{r_0}(\tau,\mathrm{s})$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,
(ii) $\aleph_{r_0}(\tau,\mathrm{s})\le\sigma\aleph_{r_0}(\mathrm{s},\mathrm{s})$ *for* $\tau,\mathrm{s}\in[0,1]$,
(iii) $\frac{1}{\sigma}\aleph_{r_0}(\mathrm{s},\mathrm{s})\le\aleph_{r_0}(\tau,\mathrm{s})$ *for* $\tau,\mathrm{s}\in[0,1]$.

Proof Since $r_0^2\le\frac{\alpha\gamma}{\beta\delta}$, it follows that $\wp>0$. So, from the definition of kernel, $\aleph_{r_0}(\mathrm{s},\mathrm{s})>0$ and continuous on $[0,1]\times[0,1]$. This proves (i). To prove (ii), consider

$$\frac{\aleph_{r_0}(\tau,\mathrm{s})}{\aleph_{r_0}(\mathrm{s},\mathrm{s})}=\begin{cases}\dfrac{\alpha\sin(r_0\tau)+\beta r_0\cos(r_0\tau)}{\alpha\sin(r_0\mathrm{s})+\beta r_0\cos(r_0\mathrm{s})}, & 0\le\tau\le\mathrm{s}\le 1,\\ \dfrac{\gamma\sin(r_0(1-\tau))+\delta r_0\cos(r_0(1-\tau))}{\gamma\sin(r_0(1-\mathrm{s}))+\delta r_0\cos(r_0(1-\mathrm{s}))}, & 0\le\mathrm{s}\le\tau\le 1,\end{cases}$$
$$\le\begin{cases}\dfrac{\alpha+\beta r_0}{\beta r_0\cos(r_0)}, & 0\le\tau\le\mathrm{s}\le 1,\\ \dfrac{\gamma+\delta r_0}{\delta r_0\cos(r_0)}, & 0\le\mathrm{s}\le\tau\le 1,\end{cases}$$

which proves (ii). Finally for (iii), consider

$$\frac{\aleph_{r_0}(\tau,\mathrm{s})}{\aleph_{r_0}(\mathrm{s},\mathrm{s})}=\begin{cases}\dfrac{\alpha\sin(r_0\tau)+\beta r_0\cos(r_0\tau)}{\alpha\sin(r_0\mathrm{s})+\beta r_0\cos(r_0\mathrm{s})}, & 0\le\tau\le\mathrm{s}\le 1,\\ \dfrac{\gamma\sin(r_0(1-\tau))+\delta r_0\cos(r_0(1-\tau))}{\gamma\sin(r_0(1-\mathrm{s}))+\delta r_0\cos(r_0(1-\mathrm{s}))}, & 0\le\mathrm{s}\le\tau\le 1,\end{cases}$$
$$\ge\begin{cases}\dfrac{\beta r_0\cos(r_0)}{\alpha+\beta r_0}, & 0\le\tau\le\mathrm{s}\le 1,\\ \dfrac{\delta r_0\cos(r_0)}{\gamma+\sigma r_0}, & 0\le\mathrm{s}\le\tau\le 1.\end{cases}$$

This completes the proof. □

From Lemma 1, we note that an ℓ-tuple $(\mathrm{z}_1,\mathrm{z}_2,\cdots,\mathrm{z}_\ell)$ is a solution of the boundary value problem (5)–(6) if and only, if

$$\mathrm{z}_1(\tau)=\int_0^1\aleph_{r_0}(\tau,\mathrm{s}_1)\varphi(\mathrm{s}_1)\mathrm{g}_1\Big[\int_0^1\aleph_{r_0}(\mathrm{s}_1,\mathrm{s}_2)\varphi(\mathrm{s}_2)\mathrm{g}_2\Big[\int_0^1\aleph_{r_0}(\mathrm{s}_2,\mathrm{s}_3)\varphi(\mathrm{s}_3)\mathrm{g}_4\cdots$$
$$\mathrm{g}_{\ell-1}\Big[\int_0^1\aleph_{r_0}(\mathrm{s}_{\ell-1},\mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\Big]\cdots\Big]d\mathrm{s}_3\Big]d\mathrm{s}_2\Big]d\mathrm{s}_1.$$

In general,

$$\mathrm{z}_\mathrm{j}(\tau)=\int_0^1\aleph_{r_0}(\tau,s)\varphi(s)\mathrm{g}_\mathrm{j}\big(\mathrm{z}_{\mathrm{j}+1}(s)\big)ds,\ \ \mathrm{j}=1,2,3,\cdots,\ell,$$
$$\mathrm{z}_1(\tau)=\mathrm{z}_{\ell+1}(\tau).$$

We denote the Banach space $\mathrm{C}((0,1),\mathbb{R})$ by B with the norm $\|\mathrm{z}\|=\max_{\tau\in[0,1]}|\mathrm{z}(\tau)|$. The cone $\mathrm{E}\subset\mathrm{B}$ is defined by

$$\mathrm{E}=\Big\{\mathrm{z}\in\mathrm{B}:\mathrm{z}(\tau)\ge 0\text{ on }[0,1]\text{ and }\min_{\tau\in[0,1]}\mathrm{z}(\tau)\ge\frac{1}{\sigma^2}\|\mathrm{z}\|\Big\}.$$

For any $\mathrm{z}_1\in\mathrm{E}$, define an operator $\mathcal{P}:\mathrm{E}\to\mathrm{B}$ by

$$(\mathcal{P}z_1)(\tau) = \int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\Bigg[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\Bigg[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4 \cdots$$
$$g_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1. \tag{10}$$

Lemma 3 *$\mathcal{P}(E) \subset E$ and $\mathcal{P} : E \to E$ is completely continuous.*

Proof Since $g_j(z_{j+1}(\tau))$ is nonnegative for $\tau \in [0, 1]$, $z_1 \in E$. Since $\aleph_{r_0}(\tau, s)$, is nonnegative for all $\tau, s \in [0, 1]$, it follows that $\mathcal{P}(z_1(\tau)) \geq 0$ for all $\tau \in [0, 1]$, $z_1 \in E$ Now, by Lemmas 1 and 2, we have

$$\begin{aligned}
&\min_{\tau\in[0,1]} (\mathcal{P}z_1)(\tau)\\
&= \min_{\tau\in[0,1]} \Bigg\{ \int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\Bigg[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\Bigg[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4 \cdots\\
&\qquad g_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(s_1, s_1)\varphi(s_1)g_1\Bigg[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\Bigg[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4 \cdots\\
&\qquad g_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\\
&\geq \frac{1}{\sigma^2}\Bigg\{\int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\Bigg[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\Bigg[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4 \cdots\\
&\qquad g_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \frac{1}{\sigma^2}\max_{\tau\in[0,1]} |\mathcal{P}z_1(\tau)|.
\end{aligned}$$

Thus, $\mathcal{P}(E) \subset E$. Therefore, by the means of Arzela–Ascoli theorem, the operator $\mathcal{P}$ is completely continuous. □

3 Existence of at Least One Positive Radial Solution

In this section, we establish the existence of at least one positive radial solution for the system (5)–(6) by an application of following theorems.

Theorem 1 [11] *Let E be a cone in a Banach space B and let G, F be open sets with $0 \in G$, $\overline{G} \subset F$. Let $\mathcal{P} : E \cap (\overline{F}\backslash G) \to E$ be a completely continuous operator such that*

(i) $\|\mathcal{P}z\| \leq \|z\|$, $z \in E \cap \partial G$, *and* $\|\mathcal{P}z\| \geq \|z\|$, $z \in E \cap \partial F$, *or*
(ii) $\|\mathcal{P}z\| \geq \|z\|$, $z \in E \cap \partial G$, *and* $\|\mathcal{P}z\| \leq \|z\|$, $z \in E \cap \partial F$.

Then, $\mathcal{P}$ has a fixed point in $E \cap (\overline{F}\backslash G)$.

Theorem 2 (Hölder's) *Let* $\mathtt{f} \in L^{\mathtt{p}_i}[0,1]$ *with* $\mathtt{p}_i > 1$, *for* $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1$. *Then,* $\prod_{i=1}^{n} \mathtt{f}_i \in L^1[0,1]$ *and* $\left\|\prod_{i=1}^{n} \mathtt{f}_i\right\|_1 \leq \prod_{i=1}^{n} \|\mathtt{f}_i\|_{\mathtt{p}_i}$. *Further, if* $\mathtt{f} \in L^1[0,1]$ *and* $\mathtt{g} \in L^{\infty}[0,1]$. *Then,* $\mathtt{fg} \in L^1[0,1]$ *and* $\|\mathtt{fg}\|_1 \leq \|\mathtt{f}\|_1 \|\mathtt{g}\|_\infty$.

Consider the following three possible cases for $\varphi_i \in L^{\mathtt{p}_i}[0,1]$:

$$\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} > 1.$$

Firstly, we seek positive radial solutions for the case $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1$.

Theorem 3 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold. Further, assume that there exist two positive constants* $a_2 > a_1 > 0$ *such that*

$(\mathcal{H}_4)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}(\tau)) \leq \mathtt{Q}_2 a_2$ *for all* $0 \leq \tau \leq 1$, $0 \leq \mathtt{z} \leq a_2$, *where* $\mathtt{Q}_2 = \left[\frac{\sigma r_0^2}{(\mathtt{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\mathtt{q} \prod_{i=1}^{n} \|\varphi_i\|_{\mathtt{p}_i}\right]^{-1}$ *and* $\widehat{\aleph}_{r_0}(\mathtt{s}) = \aleph_{r_0}(\mathtt{s},\mathtt{s})\mathtt{s}^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}}$.

$(\mathcal{H}_5)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}(\tau)) \geq \mathtt{Q}_1 a_1$ *for all* $0 \leq \tau \leq 1$, $0 \leq \mathtt{z} \leq a_1$, *where* $\mathtt{Q}_1 = \left[\frac{r_0^2}{\sigma(\mathtt{N}-2)^2} \prod_{i=1}^{n} \varphi_i^\star \int_0^1 \aleph_{r_0}(\mathtt{s},\mathtt{s})\mathtt{s}^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}} d\mathtt{s}\right]^{-1}$.

Then, iterative system (5)–(6) *has at least one positive radial solution* $(\mathtt{z}_1, \mathtt{z}_2, \cdots, \mathtt{z}_\ell)$ *such that* $a_1 \leq \|\mathtt{z}_\mathtt{j}\| \leq a_2$, $\mathtt{j} = 1, 2, \cdots, \ell$.

Proof Let $\mathtt{G} = \{\mathtt{z} \in \mathtt{B} : \|\mathtt{z}\| < a_2\}$. For $\mathtt{z}_1 \in \partial\mathtt{G}$, we have $0 \leq \mathtt{z} \leq a_2$ for all $\tau \in [0,1]$. It follows from $(\mathcal{H}_4)$ that for $\mathtt{s}_{\ell-1} \in [0,1]$,

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell \leq \sigma \int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell \\
&\leq \sigma \mathtt{Q}_2 a_2 \int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)d\mathtt{s}_\ell \\
&\leq \sigma \mathtt{Q}_2 a_2 \frac{r_0^2}{(\mathtt{N}-2)^2} \int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\mathtt{s}_\ell^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}} \prod_{i=1}^{n} \varphi_i(\mathtt{s}_\ell)d\mathtt{s}_\ell.
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} + \frac{1}{\mathtt{q}} = 1$. By the first part of Theorem 2, we have

$$\begin{aligned}
\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell &\leq \mathtt{Q}_2 a_2 \frac{\sigma r_0^2}{(\mathtt{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\mathtt{q} \prod_{i=1}^{n} \|\varphi_i\|_{\mathtt{p}_i} \\
&\leq a_2.
\end{aligned}$$

It follows in similar manner for $0 < \mathtt{s}_{\ell-2} < 1$,

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-2}, \mathtt{s}_{\ell-1})\varphi(\mathtt{s}_{\ell-1})\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell\right]d\mathtt{s}_{\ell-1}\\
&\leq \sigma\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_{\ell-1})\varphi(\mathtt{s}_{\ell-1})\mathtt{g}_{\ell-1}(a_2)d\mathtt{s}_{\ell-1}\\
&\leq \mathtt{Q}_2 a_2 \frac{\sigma r_0^2}{(\mathtt{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_{\mathtt{q}}\prod_{i=1}^{n}\|\varphi_i\|_{\mathtt{p}_i}\\
&\leq a_2.
\end{aligned}$$

Continuing with this bootstrapping argument, we reach

$$\begin{aligned}
(\mathcal{P}\mathtt{z}_1)(t) = &\int_0^1 \aleph_{r_0}(\tau, \mathtt{s}_1)\varphi(\mathtt{s}_1)\mathtt{g}_1\left[\int_0^1 \aleph_{r_0}(\mathtt{s}_1, \mathtt{s}_2)\varphi(\mathtt{s}_2)\mathtt{g}_2\left[\int_0^1 \aleph_{r_0}(\mathtt{s}_2, \mathtt{s}_3)\varphi(\mathtt{s}_3)\mathtt{g}_4\cdots\right.\right.\\
&\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell\right]\cdots\right]d\mathtt{s}_3\right]d\mathtt{s}_2\right]d\mathtt{s}_1\\
\leq &\, a_2.
\end{aligned}$$

Since $\mathtt{G} = \|\mathtt{z}_1\|$ for $\mathtt{z}_1 \in \mathtt{E} \cap \partial\mathtt{G}$, we get

$$\|\mathcal{P}\mathtt{z}_1\| \leq \|\mathtt{z}_1\|. \tag{11}$$

Next, let $\mathtt{F} = \{\mathtt{z} \in \mathtt{B} : \|\mathtt{z}\| < a_1\}$. For $\mathtt{z}_1 \in \partial\mathtt{F}$, we have $0 \leq \mathtt{z} \leq a_1$ for all $\tau \in [0, 1]$. It follows from $(\mathcal{H}_5)$ that for $\mathtt{s}_{\ell-1} \in [0, 1]$,

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell \geq \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell\\
&\geq \frac{\mathtt{Q}_1 a_1}{\sigma}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)d\mathtt{s}_\ell\\
&\geq \mathtt{Q}_1 a_1 \frac{r_0^2}{\sigma(\mathtt{N}-2)^2}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\mathtt{s}_\ell^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}}\prod_{i=1}^{n}\varphi_i(\mathtt{s}_\ell)d\mathtt{s}_\ell\\
&\geq \mathtt{Q}_1 a_1 \frac{r_0^2}{\sigma(\mathtt{N}-2)^2}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\mathtt{s}_\ell^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}}d\mathtt{s}_\ell\\
&\geq a_1.
\end{aligned}$$

It follows in similar manner for $0 < s_{\ell-2} < 1$,

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(s_{\ell-2}, s_{\ell-1})\varphi(s_{\ell-1})g_{\ell-1}\left[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_{\ell})\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\right]ds_{\ell-1}\\
&\geq \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_{\ell-1})\varphi(s_{\ell-1})g_{\ell-1}(a_1)ds_{\ell-1}\\
&\geq \frac{Q_1 a_1}{\sigma}\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_{\ell-1})\varphi(s_{\ell-1})ds_{\ell-1}\\
&\geq Q_1 a_1 \frac{r_0^2}{\sigma(N-2)^2}\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_{\ell-1})s_{\ell-1}^{\frac{2(N-1)}{2-N}}\prod_{i=1}^{n}\varphi_i(s_{\ell-1})ds_{\ell-1}\\
&\geq Q_1 a_1 \frac{r_0^2}{\sigma(N-2)^2}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_{\ell-1})s_{\ell-1}^{\frac{2(N-1)}{2-N}}ds_{\ell-1}\\
&\geq a_1.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
(\mathcal{P}z_1)(\tau) = &\int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\left[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\left[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4\cdots\right.\right.\\
&\left.\left.g_{\ell-1}\left[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\
\geq &\, a_1.
\end{aligned}$$

Thus, for $z_1 \in E \cap \partial F$, we have

$$\|\mathcal{P}z_1\| \geq \|z_1\|. \tag{12}$$

It is clear that $0 \in F \subset \overline{F} \subset G$ and by Lemma 3, $\mathcal{P} : E \cap (\overline{F}\backslash G) \to E$ is completely continuous operator. Also from (11) and (12) that $\mathcal{P}$ satisfies (i) of Theorem 1. Hence, from Theorem 1, $\mathcal{P}$ has a fixed point $z_1 \in E \cap \big(\overline{F}\backslash G\big)$ such that $z_1(\tau) \geq 0$ on $(0, 1)$. Next setting $z_{\ell+1} = z_1$, we obtain infinitely many positive solutions $(z_1, z_2, \cdots, z_\ell)$ of (5)–(6) given iteratively by

$$\begin{aligned}
z_j(\tau) &= \int_0^1 \aleph_{r_0}(\tau, s)\varphi(s)g_j(z_{j+1}(s))ds,\ j = 1, 2, \cdots, \ell-1, \ell,\\
z_{\ell+1}(\tau) &= z_1(\tau),\ \tau \in (0, 1).
\end{aligned}$$

This completes the proof. □

For $\sum_{i=1}^{n}\frac{1}{p_i} = 1$ and $\sum_{i=1}^{n}\frac{1}{p_i} > 1$, we have following results.

Theorem 4 *Suppose* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold. Further, assume that there exist two positive constants* $b_2 > b_1 > 0$ *such that* g_j $(j = 1, 2, \cdots, \ell)$ *satisfies* $(\mathcal{H}_5)$ *and*

($\mathcal{H}_6$) $\mathrm{g_j}(\mathrm{z}(\tau)) \le \mathfrak{N}_2 b_2$ *for all* $0 \le \tau \le 1,\ 0 \le \mathrm{z} \le b_2$,

$$\textit{where}\ \mathfrak{N}_2 = \left[\frac{\sigma r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\infty \prod_{i=1}^{n}\|\varphi_i\|_{\mathrm{p}_i}\right]^{-1} \textit{and}\ \widehat{\aleph}_{r_0}(\mathrm{s}) = \aleph_{r_0}(\mathrm{s},\mathrm{s})\mathrm{s}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}.$$

Then, iterative system (5)–(6) *has at least one positive radial solution* $(\mathrm{z}_1, \mathrm{z}_2, \cdots, \mathrm{z}_\ell)$ *such that* $b_1 \le \|\mathrm{z_j}\| \le b_2,\ \mathrm{j} = 1, 2, \cdots, \ell$.

Theorem 5 *Suppose* ($\mathcal{H}_1$)–($\mathcal{H}_3$) *hold. Further, assume that there exist two positive constants* $c_2 > c_1 > 0$ *such that* $\mathrm{g_j}$ $(\mathrm{j} = 1, 2, \cdots, \ell)$ *satisfies* ($\mathcal{H}_5$) *and*

($\mathcal{H}_7$) $\mathrm{g_j}(\mathrm{z}(\tau)) \le \mathfrak{M}_2 c_2$ *for all* $0 \le \tau \le 1,\ 0 \le \mathrm{z} \le c_2$,

$$\textit{where}\ \mathfrak{M}_2 = \left[\frac{\sigma r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\infty \prod_{i=1}^{n}\|\varphi_i\|_1\right]^{-1} \textit{and}\ \widehat{\aleph}_{r_0}(\mathrm{s}) = \aleph_{r_0}(\mathrm{s},\mathrm{s})\mathrm{s}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}.$$

Then, iterative system (5)–(6) *has at least one positive radial solution* $(\mathrm{z}_1, \mathrm{z}_2, \cdots, \mathrm{z}_\ell)$ *such that* $c_1 \le \|\mathrm{z_j}\| \le c_2,\ \mathrm{j} = 1, 2, \cdots, \ell$.

Example 1 Consider the following nonlinear elliptic system of equations,

$$\triangle \mathrm{z_j} + \frac{(\mathrm{N}-2)^2 r_0^{2\mathrm{N}-2}}{|x|^{2\mathrm{N}-2}}\mathrm{z_j} + \varphi(|x|)\mathrm{g_j}(\mathrm{z_{j+1}}) = 0,\ 1 < |x| < 2, \tag{13}$$

$$\left.\begin{aligned} \mathrm{z_j} = 0 \ \text{ on } \ |x| = 1 \text{ and } |x| = 2,\\ \mathrm{z_j} = 0 \ \text{ on } \ |x| = 1 \text{ and } \frac{\partial \mathrm{z_j}}{\partial r} = 0 \text{ on } |x| = 2,\\ \frac{\partial \mathrm{z_j}}{\partial r} = 0 \ \text{ on } \ |x| = 1 \text{ and } \mathrm{z_j} = 0 \text{ on } |x| = 2, \end{aligned}\right\} \tag{14}$$

where $r_0 = 1,\ \mathrm{N} = 3,\ \mathrm{j} \in \{1, 2\},\ \mathrm{z}_3 = \mathrm{z}_1,\ \varphi(\tau) = \frac{1}{\tau^4}\prod_{i=1}^{2}\varphi_i(\tau),\ \varphi_i(\tau) = \varphi_i\left(\frac{1}{\tau}\right)$, in which

$$\varphi_1(t) = \frac{1}{t^2+2} \quad \text{and} \quad \varphi_2(t) = \frac{1}{\sqrt{t+2}},$$

then it is clear that

$$\varphi_1, \varphi_2 \in L^{\mathrm{p}}[0, 1] \quad \text{and} \quad \prod_{i=1}^{2}\varphi_i^* = 2\sqrt{2}.$$

Let $\mathrm{g}_1(\mathrm{z}) = 1 + \frac{1}{3}\sin(1+\mathrm{z}) + \frac{1}{1+\mathrm{z}}$, $\mathrm{g}_2(\mathrm{z}) = 1 + \frac{2}{5}\cos(\sqrt{1+\mathrm{z}}) + \frac{1}{1+\mathrm{z}^2}$. Let $\alpha = \beta = \gamma = 1, \delta = \frac{1}{2}$, then $1 = r_0^2 < 2 = \frac{\alpha\gamma}{\beta\delta}$, $\wp = \frac{3}{2}\cos(1) + \frac{1}{2}\sin(1) \approx 1.231188951$,

$$\aleph_{r_0}(\tau, \mathrm{s}) = \frac{2}{3\cos(1)+\sin(1)} \begin{cases} \big(\sin(\tau)+\cos(\tau)\big)\big(\sin(1-\mathrm{s})+\frac{1}{2}\cos(1-\mathrm{s})\big), & 0 \le \tau \le \mathrm{s} \le 1,\\ \big(\sin(\mathrm{s})+\cos(\mathrm{s})\big)\big(\sin(1-\tau)+\frac{1}{2}\cos(1-\tau)\big), & 0 \le \mathrm{s} \le \tau \le 1, \end{cases}$$

and $\sigma = \frac{3}{\cos(1)}$. Also,

$$Q_1 = \left[\frac{r_0^2}{\sigma(N-2)^2} \prod_{i=1}^{n} \varphi_i^\star \int_0^1 \aleph_{r_0}(s,s) s^{\frac{2(N-1)}{2-N}} ds \right]^{-1} \approx 0.4811486562 \times 10^{-2}.$$

Let $p_1 = 2$, $p_2 = 3$ and $q = 6$, then $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q} = 1$ and

$$Q_2 = \left[\frac{\sigma r_0^2}{(N-2)^2} \|\widehat{\aleph}_{r_0}\|_q \prod_{i=1}^{n} \|\varphi_i\|_{p_i} \right]^{-1} \approx 0.996201 \times 10^{-5}.$$

Choose $a_1 = 0.5$ and $a_2 = 10^6$. Then,

$$\begin{aligned} g_1(z) &= 1 + \frac{1}{3}\sin(1+z) + \frac{1}{1+z} \le 2.34 \le 9.96201 = Q_2 a_2, \quad z \in [0, 10^6], \\ g_1(z) &= 1 + \frac{1}{3}\sin(1+z) + \frac{1}{1+z} \ge 0.6 \ge 0.00240574 = Q_1 a_1, \quad z \in [0, 0.5], \end{aligned}$$

and

$$\begin{aligned} g_2(z) &= 1 + \frac{2}{5}\cos(\sqrt{1+z}) + \frac{1}{1+z^2} \le 2.4 \le 9.96201 = Q_2 a_2, \quad z \in [0, 10^6], \\ g_2(z) &= 1 + \frac{2}{5}\cos(\sqrt{1+z}) + \frac{1}{1+z^2} \ge 0.6 \ge 0.00240574 = Q_1 a_1, \quad z \in [0, 0.5]. \end{aligned}$$

Therefore, by Theorem 3, the boundary value problem (13)–(14) has at least one positive solution (z_1, z_2) such that $0.5 \le \|z_j\| \le 10^6$ for $j = 1, 2$.

4 Existence of at Least Two Positive Radial Solutions

In this section, we establish the existence of at least two positive radial solutions for the system (5)–(6) by an application of following Avery–Henderson fixed point theorem.

Let ψ be a nonnegative continuous functional on a cone E of the real Banach space $\mathcal{B}$. Then, for a positive real numbers a' and c', we define the sets

$$E(\psi, c') = \{z \in E : \psi(z) < c'\},$$

and

$$E_{a'} = \{z \in E : \|z\| < a'\}.$$

Theorem 6 (Avery–Henderson [5]) *Let* $\mathtt{E}$ *be a cone in a real Banach space* $\mathtt{B}$. *Suppose* $ß_1$ *and* $ß_2$ *are increasing, nonnegative continuous functionals on* $\mathtt{E}$ *and* $ß_3$ *is nonnegative continuous functional on* $\mathtt{E}$ *with* $ß_3(0) = 0$ *such that, for some positive numbers* c' *and* k, $ß_2(\mathtt{z}) \leq ß_3(\mathtt{z}) \leq ß_1(\mathtt{z})$ *and* $\|\mathtt{z}\| \leq kß_2(\mathtt{z})$, *for all* $\mathtt{z} \in \overline{\mathtt{E}(ß_2, c')}$. *Suppose that there exist positive numbers* a' *and* b' *with* $a' < b' < c'$ *such that* $ß_3(\lambda\mathtt{z}) \leq \lambda ß_3(\mathtt{z})$, *for all* $0 \leq \lambda \leq 1$ *and* $\mathtt{z} \in \partial\mathtt{E}(ß_3, b')$. *Further, let* $\mathcal{P} : \overline{\mathtt{E}(ß_2, c')} \to \mathtt{E}$ *be a completely continuous operator such that*

(*a*) $ß_2(\mathcal{P}\mathtt{z}) > c'$, *for all* $\mathtt{z} \in \partial\mathtt{E}(ß_2, c')$,
(*b*) $ß_3(\mathcal{P}\mathtt{z}) < b'$, *for all* $\mathtt{z} \in \partial\mathtt{E}(ß_3, b')$,
(*c*) $\mathtt{E}(ß_1, a') \neq \emptyset$ *and* $ß_1(\mathcal{P}\mathtt{z}) > a'$, *for all* $\partial\mathtt{E}(ß_1, a')$.

Then, $\mathcal{P}$ *has at least two fixed points* ${}^1\mathtt{z}, {}^2\mathtt{z} \in \mathtt{P}(ß_2, c')$ *such that* $a' < ß_1({}^1\mathtt{z})$ *with* $ß_3({}^1\mathtt{z}) < b'$ *and* $b' < ß_3({}^2\mathtt{z})$ *with* $ß_2({}^2\mathtt{z}) < c'$.

Define the nonnegative, increasing, continuous functional $ß_2$, $ß_3$, and $ß_1$ by

$$ß_2(\mathtt{z}) = \min_{\tau\in[0,1]} \mathtt{z}(\tau),\ ß_3(\mathtt{z}) = \max_{\tau\in[0,1]} \mathtt{z}(\tau),\ ß_1(\mathtt{z}) = \max_{\tau\in[0,1]} \mathtt{z}(\tau).$$

It is obvious that for each $\mathtt{z} \in \mathtt{E}$,

$$ß_2(\mathtt{z}) \leq ß_3(\mathtt{z}) = ß_1(\mathtt{z}).$$

In addition, by Lemma 1, for each $\mathtt{z} \in \mathtt{P}$,

$$ß_2(\mathtt{z}) \geq \frac{1}{\sigma^2}\|\mathtt{z}\|.$$

Thus,

$$\|\mathtt{z}\| \leq \sigma^2 ß_2(\mathtt{z}) \text{ for all } \mathtt{z} \in \mathtt{E}.$$

Finally, we also note that

$$ß_3(\lambda\mathtt{z}) = \lambda ß_3(\mathtt{z}),\ \ 0 \leq \lambda \leq 1 \text{ and } \mathtt{z} \in \mathtt{E}.$$

Theorem 7 *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold and Suppose there exist real numbers* a', b' *and* c' *with* $0 < a' < b' < c'$ *such that* $\mathtt{g}_\mathtt{j}$ $(\mathtt{j} = 1, 2, \cdots, \ell)$ *satisfies*

$(\mathcal{H}_8)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) > \frac{c'}{\mathtt{C}_1}$, *for all* $c' \leq \mathtt{z} \leq \sigma^2 c'$,
where $\mathtt{C}_1 = \frac{r_0^2}{\sigma(\mathrm{N}-2)^2} \prod_{i=1}^n \varphi_i^\star \int_0^1 \aleph_{r_0}(\mathtt{s}, \mathtt{s})\mathtt{s}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}} d\mathtt{s}$,
$(\mathcal{H}_9)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) < \frac{b'}{\mathtt{C}_2}$, *for all* $0 \leq \mathtt{z} \leq \sigma^2 b'$, *where* $\mathtt{C}_2 = \frac{\sigma r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\mathtt{q} \prod_{i=1}^n \|\varphi_i\|_{\mathtt{p}_i}$,
$(\mathcal{H}_{10})$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) > \frac{a'}{\mathtt{C}_1}$, *for all* $a' \leq \mathtt{z} \leq \sigma^2 a'$.

Then, the boundary value problem (5)–(6) *has at least two positive radial solutions* $\{({}^1z_1, {}^1z_2, \cdots, {}^1z_\ell)\}$ *and* $\{({}^2z_1, {}^2z_2, \cdots, {}^2z_\ell)\}$ *satisfying*

$$a' < ß_1\big({}^1z_j\big) \ \textit{with} \ ß_3\big({}^1z_j\big) < b', \ j = 1, 2, \cdots, \ell,$$

and

$$b' < ß_3\big({}^2z_j\big) \ \textit{with} \ ß_2\big({}^2z_j\big) < c', \ j = 1, 2, \cdots, \ell.$$

Proof We begin by defining the completely continuous operator $\mathcal{P}$ by (10). So it is easy to check that $\mathcal{P} : \overline{E(ß_2, c')} \to E$. Firstly, we shall verify that condition (a) of Theorem 6 is satisfied. So, let us choose $z_1 \in \partial E(ß_2, c')$. Then, $ß_2(z_1) = \min_{\tau\in[0,1]} z_1(\tau) = c'$ this implies that $c' \le z_1(\tau)$ for $\tau \in [0, 1]$. Since $\|z_1\| \le \sigma^2 ß_2(z_1) = \sigma^2 c'$. So we have

$$c' \le z_1(\tau) \le \sigma^2 c', \ \tau \in [0, 1].$$

Let $s_{\ell-1} \in [0, 1]$. Then, by $(\mathcal{H}_8)$, we have

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell \ge \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell \\
&\ge \frac{c'}{\sigma C_1}\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)\varphi(s_\ell)ds_\ell \\
&\ge \frac{c' r_0^2}{\sigma(N-2)^2 C_1}\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}\prod_{i=1}^{n}\varphi_i(s_\ell)ds_\ell \\
&\ge \frac{c' r_0^2}{\sigma(N-2)^2 C_1}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}ds_\ell \\
&\ge c'.
\end{aligned}$$

It follows in similar manner for $0 < \mathrm{s}_{\ell-2} < 1$,

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-2}, \mathrm{s}_{\ell-1})\varphi(\mathrm{s}_{\ell-1})\mathrm{g}_{\ell-1}\left[\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\right]d\mathrm{s}_{\ell-1}\\
&\geq \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_{\ell-1})\varphi(\mathrm{s}_{\ell-1})\mathrm{g}_{\ell-1}(c')d\mathrm{s}_{\ell-1}\\
&\geq \frac{c'}{\sigma\mathtt{C}_1}\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_{\ell-1})\varphi(\mathrm{s}_{\ell-1})d\mathrm{s}_{\ell-1}\\
&\geq \frac{c' r_0^2}{\sigma(\mathrm{N}-2)^2\mathtt{C}_1}\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_{\ell-1})\mathrm{s}_{\ell-1}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{n}\varphi_i(\mathrm{s}_{\ell-1})d\mathrm{s}_{\ell-1}\\
&\geq \frac{c' r_0^2}{\sigma(\mathrm{N}-2)^2\mathtt{C}_1}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_{\ell-1})\mathrm{s}_{\ell-1}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}d\mathrm{s}_{\ell-1}\\
&\geq c'.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
&\text{ß}_2\,(\mathcal{P}\mathrm{z}_1) = \min_{\tau\in[0,1]}\int_0^1 \aleph_{r_0}(\tau, \mathrm{s}_1)\varphi(\mathrm{s}_1)\mathrm{g}_1\\
&\left[\int_0^1 \aleph_{r_0}(\mathrm{s}_1, \mathrm{s}_2)\varphi(\mathrm{s}_2)\mathrm{g}_2\left[\int_0^1 \aleph_{r_0}(\mathrm{s}_2, \mathrm{s}_3)\varphi(\mathrm{s}_3)\mathrm{g}_4\cdots\right.\right.\\
&\quad \mathrm{g}_{\ell-1}\left[\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\right]\cdots\Bigg]\\
&\left.\left. d\mathrm{s}_3\right]d\mathrm{s}_2\right]d\mathrm{s}_1\\
&\geq c'.
\end{aligned}$$

This proves (i) of Theorem 6. We next address (ii) of Theorem 6. So, we choose $\mathrm{z}_1 \in \partial\mathbb{E}(\text{ß}_3, b')$. Then, $\text{ß}_3(\mathrm{z}_1) = \max_{\tau\in[0,1]}\mathrm{z}_1(\tau) = b'$ this implies that $0 \leq \mathrm{z}_1(\tau) \leq b'$ for $\tau \in [0, 1]$. Since $\|\mathrm{z}_1\| \leq \sigma^2\text{ß}_2(\mathrm{z}_1) \leq \sigma^2\text{ß}_3(\mathrm{z}_1) = \sigma^2 b'$. So we have

$$0 \leq \mathrm{z}_1(\tau) \leq \sigma^2 b',\ \ \tau \in [0, 1].$$

Let $0 < \mathrm{s}_{\ell-1} < 1$. Then, by $(\mathcal{H}_9)$, we have

$$\begin{aligned}
&\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell \leq \sigma\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\\
&\leq \frac{\sigma b'}{\mathtt{C}_2}\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)d\mathrm{s}_\ell\\
&\leq \frac{\sigma b' r_0^2}{(\mathrm{N}-2)^2\mathtt{C}_2}\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell, \mathrm{s}_\ell)\mathrm{s}_\ell^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{n}\varphi_i(\mathrm{s}_\ell)d\mathrm{s}_\ell.
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} + \frac{1}{\mathtt{q}} = 1$. By the first part of Theorem 2, we have

$$\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell \le \frac{\sigma b' r_0^2}{(\mathtt{N}-2)^2 \complement_2}\|\widehat{\aleph}_{r_0}\|_{\mathtt{q}} \prod_{i=1}^{n} \|\varphi_i\|_{\mathtt{p}_i} \le b'.$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned} \text{ß}_3(\mathcal{P}\mathtt{z}_1) = \max_{\tau\in[0,1]} & \int_0^1 \aleph_{r_0}(\tau, \mathtt{s}_1)\varphi(\mathtt{s}_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph_{r_0}(\mathtt{s}_1, \mathtt{s}_2)\varphi(\mathtt{s}_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph_{r_0}(\mathtt{s}_2, \mathtt{s}_3)\varphi(\mathtt{s}_3)\mathtt{g}_4 \cdots \\ & \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell\Bigg]\cdots\Bigg]d\mathtt{s}_3\Bigg]d\mathtt{s}_2\Bigg]d\mathtt{s}_1 \\ \le b'. \end{aligned}$$

Hence, condition (b) is satisfied. Finally, we verify that (c) of Theorem 6 is also satisfied. We note that $\mathtt{z}_1(\tau) = a'/4,\ \tau \in [0, 1]$ is a member of $\mathtt{E}(\text{ß}_1, a')$ and $a'/4 < a'$. So $\mathtt{E}(\text{ß}_1, a') \neq \emptyset$. Now let $\mathtt{z}_1 \in \mathtt{E}(\text{ß}_1, a')$. Then, $a' = \text{ß}_1(\mathtt{z}_1) = \max_{\tau\in[0,1]} \mathtt{z}_1(\tau) = \|\mathtt{z}_1\| = \sigma^2\text{ß}_2(\mathtt{z}_1) \le \sigma^2\text{ß}_3(\mathtt{z}_1) = \sigma^2\text{ß}_1(\mathtt{z}_1) = \sigma^2 a'$, i.e., $a' \le \mathtt{z}_1(\tau) \le \sigma^2 a'$ for $\tau \in [0, 1]$. Let $0 < \mathtt{s}_{\ell-1} < 1$. Then, by $(\mathcal{H}_{10})$, we have

$$\begin{aligned} \int_0^1 \aleph_{r_0}(\mathtt{s}_{\ell-1}, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell &\ge \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)\mathtt{g}_\ell\big(\mathtt{z}_1(\mathtt{s}_\ell)\big)d\mathtt{s}_\ell \\ &\ge \frac{a'}{\sigma\complement_1}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\varphi(\mathtt{s}_\ell)d\mathtt{s}_\ell \\ &\ge \frac{a' r_0^2}{\sigma(\mathtt{N}-2)^2\complement_1}\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\mathtt{s}_\ell^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}}\prod_{i=1}^{n}\varphi_i(\mathtt{s}_\ell)d\mathtt{s}_\ell \\ &\ge \frac{a' r_0^2}{\sigma(\mathtt{N}-2)^2\complement_1}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(\mathtt{s}_\ell, \mathtt{s}_\ell)\mathtt{s}_\ell^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}}d\mathtt{s}_\ell \\ &\ge a'. \end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
&\beta_1(\mathcal{P}z_1) = \max_{\tau\in[0,1]}\int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\\
&\left[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\left[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4\cdots\right.\right.\\
&g_{\ell-1}\left[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\right]\cdots\Big]ds_3\Big]ds_2\Big]ds_1\\
&\geq \min_{\tau\in[0,1]}\int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1\left[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\left[\int_0^1\right.\right.\\
&\aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4\cdots\\
&g_{\ell-1}\left[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\right]\cdots\Big]ds_3\Big]ds_2\Big]ds_1\\
&\geq \quad a'.
\end{aligned}$$

Thus, condition (c) of Theorem 6 is satisfied. Since all hypotheses of Theorem 6 are satisfied, the assertion follows. ☐

For $\sum_{i=1}^{n}\frac{1}{p_i}=1$ and $\sum_{i=1}^{n}\frac{1}{p_i}>1$, we have following results.

Theorem 8 *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold and Suppose there exist real numbers* a', b' *and* c' *with* $0<a'<b'<c'$ *such that* $g_j\,(j=1,2,\cdots,\ell)$ *satisfies* $(\mathcal{H}_8)$, $(\mathcal{H}_{10})$ *and*

$(\mathcal{H}_9')$ $g_j(z)<\frac{b'}{C_3}$, *for all* $0\leq z\leq\sigma^2 b'$, *where* $C_3=\frac{\sigma r_0^2}{(N-2)^2}\|\widehat{\aleph}_{r_0}\|_\infty\prod_{i=1}^{n}\|\varphi_i\|_{p_i}$.

Then, the boundary value problem (5)–(6) *has at least two positive radial solutions* $\{({}^1z_1, {}^1z_2,\cdots,{}^1z_\ell)\}$ *and* $\{({}^2z_1, {}^2z_2,\cdots,{}^2z_\ell)\}$ *satisfying*

$$a'<\beta_1\big({}^1z_j\big)\ \textit{with}\ \beta_3\big({}^1z_j\big)<b',\ j=1,2,\cdots,\ell,$$

and

$$b'<\beta_3\big({}^2z_j\big)\ \textit{with}\ \beta_2\big({}^2z_j\big)<c',\ j=1,2,\cdots,\ell.$$

Theorem 9 *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold and Suppose there exist real numbers* a', b' *and* c' *with* $0<a'<b'<c'$ *such that* $g_j\,(j=1,2,\cdots,\ell)$ *satisfies* $(\mathcal{H}_8)$, $(\mathcal{H}_{10})$ *and*

$(\mathcal{H}_9'')$ $g_j(z)<\frac{b'}{C_4}$, *for all* $0\leq z\leq\sigma^2 b'$, *where* $C_4=\frac{\sigma r_0^2}{(N-2)^2}\|\widehat{\aleph}_{r_0}\|_\infty\prod_{i=1}^{n}\|\varphi_i\|_1$.

Then, the boundary value problem (5)–(6) *has at least two positive radial solutions* $\{({}^1z_1, {}^1z_2,\cdots,{}^1z_\ell)\}$ *and* $\{({}^2z_1, {}^2z_2,\cdots,{}^2z_\ell)\}$ *satisfying*

$$a'<\beta_1\big({}^1z_j\big)\ \textit{with}\ \beta_3\big({}^1z_j\big)<b',\ j=1,2,\cdots,\ell,$$

and

$$b'<\beta_3\big({}^2z_j\big)\ \textit{with}\ \beta_2\big({}^2z_j\big)<c',\ j=1,2,\cdots,\ell.$$

Example 2 Consider the following nonlinear elliptic system of equations,

$$\triangle z_j + \frac{(N-2)^2 r_0^{2N-2}}{|x|^{2N-2}} z_j + \varphi(|x|) g_j(z_{j+1}) = 0, \ 1 < |x| < 2, \tag{15}$$

$$\left.\begin{aligned} z_j = 0 \ \text{ on } \ |x| = 1 \text{ and } |x| = 2, \\ z_j = 0 \ \text{ on } \ |x| = 1 \text{ and } \frac{\partial z_j}{\partial r} = 0 \text{ on } |x| = 2, \\ \frac{\partial z_j}{\partial r} = 0 \ \text{ on } \ |x| = 1 \text{ and } z_j = 0 \text{ on } |x| = 2, \end{aligned}\right\} \tag{16}$$

where $r_0 = 1$, $N = 3$, $j \in \{1, 2\}$, $z_3 = z_1$, $\varphi(\tau) = \frac{1}{\tau^4} \prod_{i=1}^{2} \varphi_i(\tau)$, $\varphi_i(\tau) = \varphi_i\left(\frac{1}{\tau}\right)$, in which

$$\varphi_1(t) = \frac{1}{t+1} \quad \text{ and } \quad \varphi_2(t) = \frac{1}{\sqrt{t^2+9}},$$

then it is clear that

$$\varphi_1, \varphi_2 \in L^p[0, 1] \quad \text{and} \quad \prod_{i=1}^{2} \varphi_i^* = 3.$$

Let $g_1(z) = g_2(z) = 1 + \frac{1}{\sqrt{1+z^2}}$. Let $\alpha = \delta = \gamma = 1$, $\beta = \frac{1}{2}$, then $1 = r_0^2 < 2 = \frac{\alpha\gamma}{\beta\delta}$, $\wp = \frac{3}{2}\cos(1) + \frac{1}{2}\sin(1) \approx 1.231188951$,

$$\aleph_{r_0}(\tau, s) = \frac{2}{3\cos(1) + \sin(1)} \begin{cases} \left(\sin(\tau) + \frac{1}{2}\cos(\tau)\right)\left(\sin(1-s) + \cos(1-s)\right), & 0 \le \tau \le s \le 1, \\ \left(\sin(s) + \frac{1}{2}\cos(s)\right)\left(\sin(1-\tau) + \cos(1-\tau)\right), & 0 \le s \le \tau \le 1, \end{cases}$$

and $\sigma = \frac{3}{\cos(1)}$. Also,

$$C_1 = \frac{r_0^2}{\sigma(N-2)^2} \prod_{i=1}^{n} \varphi_i^\star \int_0^1 \aleph_{r_0}(s, s) s^{\frac{2(N-1)}{2-N}} ds \approx 1.248429695 \times 10^8.$$

Let $p_1 = 6$, $p_2 = 2$ and $q = 3$, then $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q} = 1$ and

$$C_2 = \frac{\sigma r_0^2}{(N-2)^2} \|\widehat{\aleph}_{r_0}\|_q \prod_{i=1}^{n} \|\varphi_i\|_{p_i} \approx 9.113677218 \times 10^6.$$

Choose $a' = 10^3$, $b' = 2 \times 10^7$ and $c' = 10^8$. Then,

$$\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) = 1 + \frac{1}{\sqrt{1+\mathrm{z}^2}} \geq 0.8010062593 = \frac{c'}{\mathfrak{C}_1}, \quad \mathrm{z} \in [10^8, 30.8 \times 10^8],$$
$$\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) = 1 + \frac{1}{\sqrt{1+\mathrm{z}^2}} \leq 2.194503878 = \frac{b'}{\mathfrak{C}_2}, \quad \mathrm{z} \in [0, 61.6 \times 10^7],$$
$$\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) = 1 + \frac{1}{\sqrt{1+\mathrm{z}^2}} \geq 0.000008 = \frac{a'}{\mathfrak{C}_1}, \quad \mathrm{z} \in [10^3, 30.8 \times 10^3].$$

Therefore, by Theorem 3, the boundary value problem (15)–(16) has at least two positive radial solutions $({}^{\mathrm{j}}\mathrm{z}_1, {}^{\mathrm{j}}\mathrm{z}_2)$, $\mathrm{j} = 1, 2$ such that

$$10^3 < \max_{\tau \in [0,1]} {}^{\mathrm{j}}\mathrm{z}_1(\tau) \text{ with } \max_{\tau \in [0,1]} {}^{\mathrm{j}}\mathrm{z}_1(\tau) < 2 \times 10^7, \text{ for } \mathrm{j} = 1, 2,$$
$$2 \times 10^7 < \max_{\tau \in [0,1]} {}^{\mathrm{j}}\mathrm{z}_2(\tau) \text{ with } \min_{\tau \in [0,1]} {}^{\mathrm{j}}\mathrm{z}_2(\tau) < 10^8, \text{ for } \mathrm{j} = 1, 2.$$

5 Existence of at Least Three Positive Radial Solutions

In this section, we establish the existence of at least three positive radial solutions for the system (5)–(6) by an application of following Leggett-William fixed point theorem. Let a', b' be two real numbers such that $0 < a' < b'$ and $\mathbb{k}$ a nonnegative, continuous, concave functional on E. We define the following convex sets,

$$\mathrm{E}_{a'} = \{\mathrm{z} \in \mathrm{E} : \|\mathrm{z}\| < a'\},$$
$$\mathrm{E}(\mathbb{k}, a', b') = \{\mathrm{z} \in \mathrm{E} : a' \leq \mathbb{k}(\mathrm{z}), \|\mathrm{z}\| < b'\}.$$

Theorem 10 (Leggett-William [18]) *Let E be a cone in a Banach space B. Let $\mathbb{k}$ a nonnegative, continuous, concave functional on E satisfying for some $c' > 0$ such that $\mathbb{k}(\mathrm{z}) \leq \|\mathrm{z}\|$ for all $\mathrm{z} \in \overline{\mathrm{E}}_{c'}$. Suppose there exists a completely continuous operator $\mathcal{P} : \overline{\mathrm{E}}_{c'} \to \overline{\mathrm{E}}_{c'}$ and $0 < a' < b' < d' \leq c'$ such that*

(*a*) $\{\mathrm{z} \in \mathrm{E}(\mathbb{k}, b', d') : \mathbb{k}(\mathrm{z}) > a'\} \neq \emptyset$ *and* $\mathbb{k}(\mathcal{P}\mathrm{z}) > b'$ *for* $\mathrm{z} \in \mathrm{E}(\mathbb{k}, b', d')$,
(*b*) $\|\mathcal{P}\mathrm{z}\| < a'$ *for* $\|\mathrm{z}\| < a'$,
(*c*) $\mathbb{k}(\mathcal{P}\mathrm{z}) > b'$ *for* $\mathrm{z} \in \mathrm{E}(\mathbb{k}, a', c')$, *with* $\|\mathcal{P}\mathrm{z}\| > d'$

Then, $\mathcal{P}$ has at least three fixed points ${}^1\mathrm{z}, {}^2\mathrm{z}, {}^3\mathrm{z} \in \mathrm{E}_{c'}$ satisfying $\|{}^1\mathrm{z}\| < a'$, $b' < \mathbb{k}({}^2\mathrm{z})$ and $\|{}^3\mathrm{z}\| > a'$ and $\mathbb{k}({}^3\mathrm{z}) < b'$.

Theorem 11 *Assume that $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold. Let $0 < a' < b' < c'$ and suppose that g_{j}, $\mathrm{j} = 1, 2, \cdots, \ell$ satisfies the following conditions,*

$(\mathcal{H}_{11})$ $\mathrm{g}_{\mathrm{j}}(\mathrm{z}) < \frac{a'}{\mathfrak{O}_1}$ *for* $0 \leq \mathrm{z} \leq a'$, *where* $\mathfrak{O}_1 = \frac{\sigma r_0^2}{(\mathrm{N}-2)^2} \|\widehat{\aleph}_{r_0}\|_{\mathrm{q}} \prod_{i=1}^{n} \|\varphi_i\|_{\mathrm{p}_i}$.
$(\mathcal{H}_{12})$ $\mathrm{g}_{\mathrm{j}}(\mathrm{z}) > \frac{b'}{\mathfrak{O}_2}$ *for* $b' \leq \mathrm{z} \leq c'$,
where $\mathfrak{O}_2 = \frac{r_0^2}{\sigma(\mathrm{N}-2)^2} \prod_{i=1}^{n} \varphi_i^{\star} \int_0^1 \aleph_{r_0}(\mathrm{s}, \mathrm{s}) \mathrm{s}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}} d\mathrm{s}$.

$(\mathcal{H}_{13})$ $g_j(z) < \frac{c'}{\mathfrak{O}_1}$ *for* $0 \le z \le c'$.

Then, the iterative system (5)–(6) *has at least three positive radial solutions* $({}^1z_1, {}^1z_2, \cdots, {}^1z_\ell)$, $({}^2z_1, {}^2z_2, \cdots, {}^2z_\ell)$ *and* $({}^3z_1, {}^3z_2, \cdots, {}^3z_\ell)$ *with* $\|{}^jz_1\| < a'$, $b' < \Bbbk({}^jz_2)$, $\|{}^jz_3\| > a'$ *and* $\Bbbk({}^jz_3) < b'$ *for* $j = 1, 2, \cdots, \ell$.

Proof From Lemma 3, $\mathcal{P} : E \to E$ is a completely continuous operator. If $z_1 \in \overline{E}_{c'}$, then $\|z_1\| \le c'$ and for $0 < s_{\ell-1} < 1$ and by $(\mathcal{H}_{13})$, we have

$$\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell \le \sigma \int_0^1 \aleph_{r_0}(s_\ell, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell$$
$$\le \frac{\sigma c'}{\mathfrak{O}_1}\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)\varphi(s_\ell)ds_\ell$$
$$\le \frac{\sigma c' r_0^2}{(N-2)^2\mathfrak{O}_1}\int_0^1 \aleph_{r_0}(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}\prod_{i=1}^n \varphi_i(s_\ell)ds_\ell.$$

There exists a $q > 1$ such that $\sum_{i=1}^n \frac{1}{p_i} + \frac{1}{q} = 1$. By the first part of Theorem 2, we have

$$\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell \le \frac{\sigma c' r_0^2}{(N-2)^2\mathfrak{O}_1}\|\widehat{\aleph}_{r_0}\|_q\prod_{i=1}^n \|\varphi_i\|_{p_i}$$
$$\le c'.$$

Continuing with this bootstrapping argument, we get

$$\|\mathcal{P}z_1\| = \max_{\tau\in[0,1]}\int_0^1 \aleph_{r_0}(\tau, s_1)\varphi(s_1)g_1 \Bigg[\int_0^1 \aleph_{r_0}(s_1, s_2)\varphi(s_2)g_2\Bigg[\int_0^1 \aleph_{r_0}(s_2, s_3)\varphi(s_3)g_4\cdots$$
$$g_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(s_{\ell-1}, s_\ell)\varphi(s_\ell)g_\ell\big(z_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1$$
$$\le c'.$$

Hence, $\mathcal{P} : \overline{E}_{c'} \to \overline{E}_{c'}$. In the same way, if $z_1 \in \overline{E}_{a'}$, then $\mathcal{P} : \overline{E}_{a'} \to \overline{E}_{a'}$. Therefore, condition (b) of Theorem 10 satisfied. To check condition (a) of Theorem 10, choose $z_1(\tau) = (b' + c')/2$, $\tau \in [0, 1]$. It is easy to see that $z_1 \in E(\Bbbk, b', c')$ and $\Bbbk(z_1) = \Bbbk((b' + c')/2) > b'$. So, $\{z_1 \in E(\Bbbk, b', c') : \Bbbk(z_1) > b'\} \neq \emptyset$. Hence, if $z_1 \in E(\Bbbk, b', c')$ then $b' < z_1(\tau) < c'$, $\tau \in [0, 1]$. Let $0 < s_{\ell-1} < 1$. Then, by $(\mathcal{H}_{12})$, we have

$$\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1},\mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell \geq \frac{1}{\sigma}\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell,\mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell$$
$$\geq \frac{b'}{\sigma\mathfrak{O}_2}\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell,\mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)d\mathrm{s}_\ell$$
$$\geq \frac{b' r_0^2}{\sigma(\mathrm{N}-2)^2\mathfrak{O}_2}\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell,\mathrm{s}_\ell)\mathrm{s}_\ell^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^n \varphi_i(\mathrm{s}_\ell)d\mathrm{s}_\ell$$
$$\geq \frac{b' r_0^2}{\sigma(\mathrm{N}-2)^2\mathfrak{O}_2}\prod_{i=1}^n \varphi_i^\star\int_0^1 \aleph_{r_0}(\mathrm{s}_\ell,\mathrm{s}_\ell)\mathrm{s}_\ell^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}d\mathrm{s}_\ell$$
$$\geq b'.$$

Continuing with this bootstrapping argument, we get

$$\min_{\tau\in[0,1]}(\mathcal{P}\mathrm{z}_1) = \min_{\tau\in[0,1]}\int_0^1 \aleph_{r_0}(\tau,\mathrm{s}_1)\varphi(\mathrm{s}_1)\mathrm{g}_1\Bigg[\int_0^1 \aleph_{r_0}(\mathrm{s}_1,\mathrm{s}_2)\varphi(\mathrm{s}_2)\mathrm{g}_2$$
$$\Bigg[\int_0^1 \aleph_{r_0}(\mathrm{s}_2,\mathrm{s}_3)\varphi(\mathrm{s}_3)\mathrm{g}_4\cdots$$
$$\mathrm{g}_{\ell-1}\Bigg[\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1},\mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\Bigg]\cdots\Bigg]d\mathrm{s}_3\Bigg]d\mathrm{s}_2\Bigg]d\mathrm{s}_1$$
$$\geq b'.$$

Therefore, we have

$$\mathbb{k}(\mathcal{P}\mathrm{z}_1) > b', \text{ for } \mathrm{z}_1 \in \mathrm{E}(\mathbb{k}, b', c').$$

This implies that condition (a) of Theorem 10 is satisfied.

Finally, if $\mathrm{z}_1 \in \mathrm{E}(\mathbb{k}, b', c')$, then what we have already proved, $\mathbb{k}(\mathcal{P}\mathrm{z}_1) > b'$, which proves the condition (c) of Theorem 10. To sum up, all the conditions of Theorem 10 are satisfied. Therefore, $\mathcal{P}$ has at least three fixed points, that is, problem (5)–(6) has at least three positive solutions $({}^1\mathrm{z}_1, {}^1\mathrm{z}_2, \cdots, {}^1\mathrm{z}_\ell)$, $({}^2\mathrm{z}_1, {}^2\mathrm{z}_2, \cdots, {}^2\mathrm{z}_\ell)$ and $({}^3\mathrm{z}_1, {}^3\mathrm{z}_2, \cdots, {}^3\mathrm{z}_\ell)$ with $\|{}^\mathrm{j}\mathrm{z}_1\| < a'$, $b' < \mathbb{k}({}^\mathrm{j}\mathrm{z}_2)$, $\|{}^\mathrm{j}\mathrm{z}_3\| > a'$ and $\mathbb{k}({}^\mathrm{j}\mathrm{z}_3) < b'$ for $\mathrm{j} = 1, 2, \cdots, \ell$. □

For $\sum_{i=1}^n \frac{1}{\mathrm{p}_i} = 1$ and $\sum_{i=1}^n \frac{1}{\mathrm{p}_i} > 1$, we have following results.

Theorem 12 *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold. Let* $0 < a' < b' < c'$ *and suppose that* g_j, $\mathrm{j} = 1, 2, \cdots, \ell$ *satisfies* $(\mathcal{H}_{12})$, $(\mathcal{H}_{13})$ *and*

$(\mathcal{H}_{14})$ $\mathrm{g}_\mathrm{j}(\mathrm{z}) < \frac{a'}{\mathfrak{O}_3}$ *for* $0 \leq \mathrm{z} \leq a'$, *where* $\mathfrak{O}_3 = \frac{\sigma r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_\infty \prod_{i=1}^n \|\varphi_i\|_{\mathrm{p}_i}$.

Then, the iterative system (5)–(6) *has at least three positive solutions* $({}^1\mathrm{z}_1, {}^1\mathrm{z}_2, \cdots, {}^1\mathrm{z}_\ell)$, $({}^2\mathrm{z}_1, {}^2\mathrm{z}_2, \cdots, {}^2\mathrm{z}_\ell)$ *and* $({}^3\mathrm{z}_1, {}^3\mathrm{z}_2, \cdots, {}^3\mathrm{z}_\ell)$ *with* $\|{}^\mathrm{j}\mathrm{z}_1\| < a'$, $b' < \mathbb{k}({}^\mathrm{j}\mathrm{z}_2)$, $\|{}^\mathrm{j}\mathrm{z}_3\| > a'$ *and* $\mathbb{k}({}^\mathrm{j}\mathrm{z}_3) < b'$ *for* $\mathrm{j} = 1, 2, \cdots, \ell$.

Theorem 13 *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold. Let* $0 < a' < b' < c'$ *and suppose that* $\mathtt{g_j},\ \mathtt{j} = 1, 2, \cdots, \ell$ *satisfies* $(\mathcal{H}_{12})$, $(\mathcal{H}_{13})$ *and*

$(\mathcal{H}_{15})$ $\mathtt{g_j}(\mathtt{z}) < \frac{a'}{\mathfrak{O}_4}$ *for* $0 \le \mathtt{z} \le a'$, *where* $\mathfrak{O}_4 = \frac{\sigma r_0^2}{(\mathtt{N}-2)^2} \|\widehat{\aleph}_{r_0}\|_\infty \prod_{i=1}^{n} \|\varphi_i\|_1$.

Then, the iterative system (5)–(6) *has at least three positive solutions* $({}^1\mathtt{z}_1, {}^1\mathtt{z}_2, \cdots, {}^1\mathtt{z}_\ell)$, $({}^2\mathtt{z}_1, {}^2\mathtt{z}_2, \cdots, {}^2\mathtt{z}_\ell)$ *and* $({}^3\mathtt{z}_1, {}^3\mathtt{z}_2, \cdots, {}^3\mathtt{z}_\ell)$ *with* $\|{}^\mathtt{j}\mathtt{z}_1\| < a'$, $b' < \Bbbk({}^\mathtt{j}\mathtt{z}_2)$, $\|{}^\mathtt{j}\mathtt{z}_3\| > a'$ *and* $\Bbbk({}^\mathtt{j}\mathtt{z}_3) < b'$ *for* $\mathtt{j} = 1, 2, \cdots, \ell$.

Example 3 Consider the following nonlinear elliptic system of equations,

$$\triangle \mathtt{z_j} + \frac{(\mathtt{N}-2)^2 r_0^{2\mathtt{N}-2}}{|x|^{2\mathtt{N}-2}} \mathtt{z_j} + \varphi(|x|)\mathtt{g_j}(\mathtt{z_{j+1}}) = 0,\ 1 < |x| < 2, \tag{17}$$

$$\left.\begin{aligned} \mathtt{z_j} = 0 \ \text{ on } \ |x| = 1 \text{ and } |x| = 2, \\ \mathtt{z_j} = 0 \ \text{ on } \ |x| = 1 \text{ and } \frac{\partial \mathtt{z_j}}{\partial r} = 0 \text{ on } |x| = 2, \\ \frac{\partial \mathtt{z_j}}{\partial r} = 0 \ \text{ on } \ |x| = 1 \text{ and } \mathtt{z_j} = 0 \text{ on } |x| = 2, \end{aligned}\right\} \tag{18}$$

where $r_0 = 1$, $\mathtt{N} = 3$, $\mathtt{j} \in \{1, 2\}$, $\mathtt{z_3} = \mathtt{z_1}$, $\varphi(\tau) = \frac{1}{\tau^4} \prod_{i=1}^{2} \varphi_i(\tau)$, $\varphi_i(\tau) = \varphi_i\left(\frac{1}{\tau}\right)$, in which

$$\varphi_1(t) = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad \varphi_2(t) = \frac{1}{\sqrt{t^2+16}},$$

then it is clear that

$$\varphi_1, \varphi_2 \in L^{\mathtt{p}}[0,1] \quad \text{and} \quad \prod_{i=1}^{2} \varphi_i^* = 4.$$

Let

$$\mathtt{g_1}(\mathtt{z}) = \mathtt{g_2}(\mathtt{z}) = \begin{cases} 51, & \mathtt{z} \ge 1, \\ 50\mathtt{z}^2 + 1, & \mathtt{z} < 1. \end{cases}$$

Let $\alpha = 2$, $\beta = \gamma = \delta = 1$, then $1 = r_0^2 < 2 = \frac{\alpha\gamma}{\beta\delta}$, $\wp = 2\cos(1) + \sin(1) \approx 1.922075596$,

$$\aleph_{r_0}(\tau, \mathtt{s}) = \frac{1}{2\cos(1) + \sin(1)}$$
$$\begin{cases} \big(2\sin(\tau) + \cos(\tau)\big)\big(\sin(1-\mathtt{s}) + \cos(1-\mathtt{s})\big), & 0 \le \tau \le \mathtt{s} \le 1, \\ \big(2\sin(\mathtt{s}) + \cos(\mathtt{s})\big)\big(\sin(1-\tau) + \cos(1-\tau)\big), & 0 \le \mathtt{s} \le \tau \le 1, \end{cases}$$

and $\sigma = \frac{3}{\cos(1)}$. Also,

$$\mathfrak{D}_1 = \frac{r_0^2}{\sigma(\mathrm{N}-2)^2}\prod_{i=1}^{n}\varphi_i^\star\int_0^1 \aleph_{r_0}(\mathrm{s},\mathrm{s})\mathrm{s}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}d\mathrm{s} \approx 1.732057708\times 10^8.$$

Let $\mathrm{p}_1 = 6, \mathrm{p}_2 = 3$ and $\mathrm{q} = 2$, then $\frac{1}{\mathrm{p}_1} + \frac{1}{\mathrm{p}_2} + \frac{1}{\mathrm{q}} = 1$ and

$$\mathfrak{D}_2 = \frac{\sigma r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\aleph}_{r_0}\|_{\mathrm{q}}\prod_{i=1}^{n}\|\varphi_i\|_{\mathrm{p}_i} \approx 1.266858405\times 10^{12}.$$

Choose $a' = 10^{10}$, $b' = 10^{12}$ and $c' = 10^{13}$. Then,

$$\begin{aligned}
\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) &\le 57.73479691 = \frac{a'}{\mathfrak{D}_1},\ \mathrm{z}\in[0,10^{10}],\\
\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) &\ge 0.789354197 = \frac{b'}{\mathfrak{D}_2},\ \mathrm{z}\in[10^{12},10^{13}],\\
\mathrm{g}_1(\mathrm{z}) = \mathrm{g}_2(\mathrm{z}) &\le 57734.79691 = \frac{c'}{\mathfrak{D}_1},\ \mathrm{z}\in[0,10^{13}].
\end{aligned}$$

Therefore, by Theorem 3, the boundary value problem (15)–(16) has at least two positive radial solutions $({}^{\mathrm{j}}\mathrm{z}_1, {}^{\mathrm{j}}\mathrm{z}_2)$, $\mathrm{j} = 1, 2$ such that

$$\max_{\tau\in[0,1]} {}^{\mathrm{j}}\mathrm{z}_1(\tau) < 10^{10},\ 10^{12} < \min_{\tau\in[0,1]} {}^{\mathrm{j}}\mathrm{z}_2(\tau) < \max_{\tau\in[0,1]} {}^{\mathrm{j}}\mathrm{z}_2(\tau) < 10^{13},\ \text{ for } \mathrm{j} = 1, 2,$$
$$10^{10} < \max_{\tau\in[0,1]} {}^{\mathrm{j}}\mathrm{z}_3(\tau) < 10^{13},\ \min_{\tau\in[0,1]} {}^{\mathrm{j}}\mathrm{z}_3(\tau) < 10^{12},\ \text{ for } \mathrm{j} = 1, 2.$$

6 Existence of Unique Positive Radial Solution

In the next, for the existence of unique solution to the boundary value problem (5)–(6) where we employ two metrics under Rus's theorem (see [4,20,23] for more details). In this regard, consider the set of real valued functions that are defined and continuous on [0, 1] and denote this space by $\mathrm{X} = C([0, 1])$. For functions $\mathrm{y}_1, \mathrm{y}_2 \in \mathrm{X}$, consider the following two metrics on X :

$$\mathrm{d}(\mathrm{y}_1, \mathrm{y}_2) = \max_{t\in[0,1]} |\mathrm{y}_1(t) - \mathrm{y}_2(t)|, \tag{19}$$

$$\varrho(\mathrm{y}_1, \mathrm{y}_2) = \left[\int_0^1 |\mathrm{y}_1(t) - \mathrm{y}_2(t)|^{\mathrm{p}}dt\right]^{\frac{1}{\mathrm{p}}},\ \ \mathrm{p} > 1. \tag{20}$$

For d in (19), the pair $(C([0, 1]), \mathrm{d})$ forms a complete metric space. For ϱ in (20), the pair $(C([0, 1]), \varrho)$ forms a metric space. The relationship between the two metrics on

$\mathtt{X}$ is given by

$$\varrho(\mathtt{y}_1, \mathtt{y}_2) \leq \mathtt{d}(\mathtt{y}_1, \mathtt{y}_2) \text{ for all } \mathtt{y}_1, \mathtt{y}_2 \in \mathtt{X}. \tag{21}$$

Theorem 14 (Rus [21]) *Let* $\mathtt{X}$ *be a nonempty set and let* $\mathtt{d}$ *and* ϱ *be two metrics on* $\mathtt{X}$ *such that* $(\mathtt{X}, \mathtt{d})$ *forms a complete metric space. If the mapping* $\mho : \mathtt{X} \to \mathtt{X}$ *is continuous with respect to* $\mathtt{d}$ *on* $\mathtt{X}$ *and*

$$\mathtt{d}(\mho\mathtt{y}_1, \mho\mathtt{y}_2) \leq \mathtt{c}_1\varrho(\mathtt{y}_1, \mathtt{y}_2), \tag{22}$$

for some $\mathtt{c}_1 > 0$ *and for all* $\mathtt{y}_1, \mathtt{y}_2 \in \mathtt{X}$,

$$\varrho(\mho\mathtt{y}_1, \mho\mathtt{y}_2) \leq \mathtt{c}_2\varrho(\mathtt{y}_1, \mathtt{y}_2), \tag{23}$$

for some $0 < \mathtt{c}_2 < 1$ *for all* $\mathtt{y}_1, \mathtt{y}_2 \in \mathtt{X}$, *then there is a unique* $\mathtt{y}^* \in \mathtt{X}$ *such that* $\mho\mathtt{y}^* = \mathtt{y}^*$.

Denote $\Psi(\mathtt{s}) = \aleph_{r_0}(\mathtt{s}, \mathtt{s})\mathtt{s}^{\frac{2(\mathtt{N}-1)}{2-\mathtt{N}}} \prod_{i=1}^{n} \varphi_i(\mathtt{s})d\mathtt{s}$.

Theorem 15 *Assume that* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ *and the following condition are satisfied.*

$(\mathcal{H}_{14})$ *there exists a number* $\mathtt{K} > 0$ *such that*

$$|\mathtt{g}_\mathtt{j}(\mathtt{z}) - \mathtt{g}_\mathtt{j}(\mathtt{y})| \leq \mathtt{K}|\mathtt{z} - \mathtt{y}| \;\; \textit{for} \;\; \mathtt{z}, \mathtt{y} \in \mathtt{X}.$$

Further, assume that there are constants $\mathtt{p} > 1$ *and* $\mathtt{q} > 1$ *such that* $1/\mathtt{p} + 1/\mathtt{q} = 1$ *with*

$$\left[\frac{\sigma\mathtt{K}r_0^2}{(\mathtt{N}-2)^2}\right]^{\ell+1} \left[\int_0^1 |\Psi(\mathtt{s})|d\mathtt{s}\right]^{\ell} \left[\int_0^1 |\Psi(\mathtt{s})|^{\mathtt{q}}d\mathtt{s}\right]^{\frac{1}{q}} < 1, \tag{24}$$

then the boundary value problem (5)–(6) *has a unique positive radial solution in* $\mathtt{X}$.

Proof Let $\mathrm{z}_1, \mathrm{y}_1 \in C([0,1])$ and $\mathrm{s} \in [0,1]$. Then, by Hölder's inequality, we have

$$\begin{aligned}
&\Big|\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell - \int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{y}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\Big| \\
&\quad \le \int_0^1 |\aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\,\varphi(\mathrm{s}_\ell)||\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big) - \mathrm{g}_\ell\big(\mathrm{y}_1(\mathrm{s}_\ell)\big)|d\mathrm{s}_\ell \\
&\quad \le \sigma \int_0^1 |\aleph_{r_0}(\mathrm{s}_\ell, \mathrm{s}_\ell)\,\varphi(\mathrm{s}_\ell)|\,\mathrm{K}|\mathrm{z}_1(\mathrm{s}_\ell) - \mathrm{y}_1(\mathrm{s}_\ell)|d\mathrm{s}_\ell \\
&\quad \le \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\int_0^1 |\Psi(\mathrm{s}_\ell)||\mathrm{z}_1(\mathrm{s}_\ell) - \mathrm{y}_1(\mathrm{s}_\ell)|d\mathrm{s}_\ell \\
&\quad \le \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1 |\Psi(\mathrm{s}_\ell)|^{\mathrm{q}} d\mathrm{s}_\ell\right]^{\frac{1}{\mathrm{q}}}\left[\int_0^1 |\mathrm{z}_1(\mathrm{s}_\ell) - \mathrm{y}_1(\mathrm{s}_\ell)|^{\mathrm{p}} d\mathrm{s}_\ell\right]^{\frac{1}{\mathrm{p}}} \\
&\quad \le \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1 |\Psi(\mathrm{s}_\ell)|^{\mathrm{q}} d\mathrm{s}_\ell\right]^{\frac{1}{\mathrm{q}}} \varrho(\mathrm{z}_1, \mathrm{y}_1) \\
&\quad \le \mathrm{c}_1^\star \varrho(\mathrm{z}_1, \mathrm{y}_1),
\end{aligned}$$

where

$$\mathrm{c}_1^\star = \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1 |\Psi(\mathrm{s}_\ell)|^{\mathrm{q}}\right]^{\frac{1}{\mathrm{q}}}.$$

It follows in similar manner for $0 < \mathrm{s}_{\ell-2} < 1$,

$$\begin{aligned}
&\Big|\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-2}, \mathrm{s}_{\ell-1})\varphi(\mathrm{s}_{\ell-1})\mathrm{g}_{\ell-1}\Big[\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell\big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\Big]d\mathrm{s}_{\ell-1} \\
&\qquad - \int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-2}, \mathrm{s}_{\ell-1})\varphi(\mathrm{s}_{\ell-1})\mathrm{g}_{\ell-1}\Big[\int_0^1 \aleph_{r_0}(\mathrm{s}_{\ell-1}, \mathrm{s}_\ell)\varphi(\mathrm{s}_\ell)\mathrm{g}_\ell \\
&\qquad \big(\mathrm{z}_1(\mathrm{s}_\ell)\big)d\mathrm{s}_\ell\Big]d\mathrm{s}_{\ell-1}\Big| \\
&\qquad \le \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\int_0^1 |\Psi(\mathrm{s}_{\ell-1})|\mathrm{c}_1\varrho(\mathrm{z}_1, \mathrm{y}_1)d\mathrm{s}_{\ell-1} \\
&\qquad \le \widehat{c}_1 c_1^\star \varrho(\mathrm{z}_1, \mathrm{y}_1),
\end{aligned}$$

where

$$\widehat{c}_1 = \frac{\sigma \mathrm{K} r_0^2}{(\mathrm{N}-2)^2}\int_0^1 |\Psi(\mathrm{s})|d\mathrm{s}.$$

Continuing with bootstrapping argument, we get

$$|\mho\mathrm{z}_1(\mathrm{s}) - \mho\mathrm{y}_1(\mathrm{s})| \le \widehat{\mathrm{c}}_1^\ell \mathrm{c}_1^\star \varrho(\mathrm{z}_1, \mathrm{y}_1).$$

we see that

$$d(\mho z_1, \mho y_1) \leq c_1 \varrho(z_1, y_1), \tag{25}$$

for some $c_1 = \widehat{c}_1^{\ell} c_1^{\star} > 0$ for all $z_1, y_1 \in X$, and so the inequality (22) of Theorem 14 holds. Now, for all $z_1, y_1 \in X$, we may apply (21) to (25) to obtain

$$d(\mho z_1, \mho y_1) \leq c_1 \varrho(z_1, y_1) \leq c_1 d(z_1, y_1).$$

Thus, given any $\varepsilon > 0$ we can choose $\eta = \varepsilon / c_1$ so that $d(\mho z_1, \mho y_1) < \varepsilon$, whenever $d(z_1, y_1) < \eta$. Hence, $\mho$ is continuous on X with respect to the metric d. Finally, we show that $\mho$ is contractive on X with respect to the metric ϱ. From (25), for each $z_1, y_1 \in X$ consider

$$\left[\int_0^1 |(\mho z_1)(s) - (\mho y_1)(s)|^p ds\right]^{\frac{1}{p}} \leq \left[\int_0^1 \left|\widehat{c}_1^{\ell} c_1^{\star} \varrho(z_1, y_1)\right|^p ds\right]^{\frac{1}{p}}$$
$$\leq \left[\frac{\sigma K r_0^2}{(N-2)^2}\right]^{\ell+1} \left[\int_0^1 |\Psi(s)| ds\right]^{\ell} \left[\int_0^1 |\Psi(s)|^q ds\right]^{\frac{1}{q}} \varrho(y_1, y_2).$$

That is

$$\varrho(\mho z_1, \mho y_1) \leq \left[\frac{\sigma K r_0^2}{(N-2)^2}\right]^{\ell+1} \left[\int_0^1 |\Psi(s)| ds\right]^{\ell} \left[\int_0^1 |\Psi(s)|^q ds\right]^{\frac{1}{q}} \varrho(y_1, y_2).$$

From the assumption (24), we have

$$\varrho(\mho y_1, \mho y_2) \leq c_2 \varrho(y_1, y_2)$$

for some $c_2 < 1$ and all $y_1, y_2 \in X$. Thus, Theorem 14, the operator $\mho$ has a unique fixed point in X. Also, we note that the operator $\mho$ is positive from Lemma 3. Therefore, the boundary value problem (2) has a unique positive radial solution. □

Example 4 Consider the following nonlinear elliptic system of equations,

$$\triangle z_j + \frac{(N-2)^2 r_0^{2N-2}}{|x|^{2N-2}} z_j + \varphi(|x|) g_j(z_{j+1}) = 0, \ 1 < |x| < 2, \tag{26}$$

$$\left.\begin{aligned} z_j = 0 \ \text{ on } \ |x| = 1 \text{ and } |x| = 2, \\ z_j = 0 \ \text{ on } \ |x| = 1 \text{ and } \frac{\partial z_j}{\partial r} = 0 \text{ on } |x| = 2, \\ \frac{\partial z_j}{\partial r} = 0 \ \text{ on } \ |x| = 1 \text{ and } z_j = 0 \text{ on } |x| = 2, \end{aligned}\right\} \tag{27}$$

where $r_0 = 1$, $N = 3$, $j \in \{1, 2\}$, $z_3 = z_1$, $\varphi(\tau) = \frac{1}{\tau^4} \prod_{i=1}^{2} \varphi_i(\tau)$, $\varphi_i(\tau) = \varphi_i\left(\frac{1}{\tau}\right)$, in which $\varphi_1(t) = \varphi_2(t) = \frac{1}{t+1}$, then $\prod_{i=1}^{2} \varphi_i^* = 1$. Let $g_1(z) = \frac{1}{10^{10}} \sin(z)$, $g_2(z) =$

$\frac{z}{10^{10}(1+z)}$ and $\alpha = \beta = \delta = 1$, $\gamma = 2$, then $1 = r_0^2 < 2 = \frac{\alpha\gamma}{\beta\delta}$, $\wp = 2\cos(1) + \sin(1) \approx 1.922075596$,

$$\aleph_{r_0}(\tau, s) = \frac{1}{2\cos(1) + \sin(1)} \begin{cases} \big(\sin(\tau) + \cos(\tau)\big)\big(2\sin(1-s) + \cos(1-s)\big), & 0 \le \tau \le s \le 1, \\ \big(\sin(s) + \cos(s)\big)\big(2\sin(1-\tau) + \cos(1-\tau)\big), & 0 \le s \le \tau \le 1, \end{cases}$$

and $\sigma = \frac{3}{\cos(1)}$. Then,

$$|g_1(z) - g_1(y)| = \frac{|\sin(z) - \sin(y)|}{10^{10}} \le \frac{1}{10^{10}}|z - y|,$$

and

$$|g_2(z) - g_2(y)| = \frac{1}{10^{10}}\left|\frac{z}{1+z} - \frac{y}{1+y}\right| \le \frac{1}{10^{10}}|z - y|.$$

So, $K = \frac{1}{10^{10}}$. Let $\ell = 2$ and $p = q = 2$. Then,

$$\left[\frac{\sigma K r_0^2}{(N-2)^2}\right]^{\ell+1}\left[\int_0^1 |\Psi(s)|ds\right]^{\ell}\left[\int_0^1 |\Psi(s)|^q ds\right]^{\frac{1}{q}} \approx 0.8595804542 < 1.$$

Therefore, from Theorem 15, the iterative system of boundary value problems (26)–(27) has a unique positive radial solution.

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