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An Iterative Explicit Algorithm for Solving Equilibrium Problems in Banach Spaces

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Abstract
In this paper, a new extragradient algorithm is presented to solve the pseudomonotone equilibrium problem with a Bregman–Lipschitz-type condition. The superiority of this algorithm is that it can be performed without any precedent information about the Bregman–Lipschitz coefficients. The weak convergence of the algorithm is determinate under mild assumption, and the strong convergence will be established as the bifunction equilibrium is satisfied in different additional assumptions. In conclusion, we can use the algorithm to find a solution of the variational inequality problem. At the end, several numerical examples are exhibited that demonstrate the efficiency of our method compared to the related methods in the studies.

Keywords Bregman distance · Equilibrium problem · Bregman–Lipschitz-type condition · Bregman monotone

Mathematics Subject Classification 47H05 · 47H09 · 47H10

1 Introduction

Throughout this paper, we assume that X is a reflexive real Banach space and C is a nonempty closed and convex subset of X unless otherwise stated. We shall denote the dual space of X by X^*. The norm and the duality pairing between X and X^* are,

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respectively, denoted by $\|.\|$ and $\langle ., . \rangle$, and $\mathbb{R}$ stands for the set of real numbers. The equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$ is stated as follows:

$$\text{Find} \quad x^* \in C \qquad \text{such that} \quad f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(f)$.

In some nonlinear analysis problems such as complementarity, fixed point, Nash equilibria, optimization, saddle point and variational inequality, it is common that these interesting and confusing problems can be reformulated as the equilibrium problems (see, for instance, [7]), which caused the problem (EP) to become an interesting area in recent years. Also, the approximation methods to solutions of problem (EP) plus the theoretical results of solution are interesting. The proximal point method [27,33] is a well-known method for solving the problem (EP) that substitutes the original problem with a family of regularization equilibrium subproblems which can easily be solved. Using Bregman distance, Reich and Sabach [40] presented two algorithms for approximating a point of (EP) in reflexive Banach spaces.

The variational inequality problem is a particular part of the equilibrium problem. Korpelevich [28] presented an extragradient method for solving the variational inequality in Euclidean space that two metric projections on feasible sets must be found at each iterative step. This method was extended in Hilbert spaces by Nadezhkina and Takahashi [34]. In this direction, see also [18,19,21,27,42,43,45,46].

In this paper, we are interested in the proximal-like method [16] which is also called the extragradient method [36] due to the early contributions on the saddle point problems in [28]. The convergence of the extragradient method is established in [36] under the assumptions that the bifunction is pseudomonotone and satisfies a Lipschitz-type condition presented in [31]. In the extragradient method, at each iterative step two strongly convex minimization problems on a closed convex constrained set must be solved. In 2018, a hybrid extragradient method to find a common element of the set of solutions of pseudomonotone equilibrium problem in reflexive Banach spaces was proposed by Eskandani et al. [14]. In [14], Bregman–Lipschitz coefficients are necessary to be determined; however, there are some challenges in their estimation. Some of the iterative algorithms were introduced by Hieu et al. [22] to solve the pseudomonotone and Lipschitz-type equilibrium problem in a Hilbert space. The performance of suggested algorithms is made without the prior information of the Lipschitz-type constants. In the studies [1,17,19,21,23–26,31,32,37–39], there are some other methods to solve problem (EP).

In this paper, motivated and inspired by the above results, a new extragradient algorithm is introduced to solve the pseudomonotone equilibrium problem with a Bregman–Lipschitz-type condition. The superiority of this algorithm is that it can be performed without any precedent information about the Bregman–Lipschitz coefficients. The weak convergence of the algorithm is determinate under mild assumption and the strong convergence will be established as the bifunction equilibrium is satisfied in different additional assumptions. In conclusion, we can use the algorithm to find a solution of the variational inequality problem. At the end, several numerical examples are exhibited that demonstrate the efficiency of our method compared to the related methods in the studies.

2 Preliminaries and Lemmas

Let $\{x_n\}$ be a sequence in X and $x \in X$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence is denoted by $x_n \to x$. In this paper, we assume that $g : X \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function. We denote by dom g, the domain of g, that is, the set $\{x \in X : g(x) < \infty\}$. Let $x \in$ int dom g, the subdifferential of g at x is the convex set defined by:

$$\partial g(x) =: \{x^* \in X^* : g(x) + \langle y - x, x^* \rangle \leq g(y), \quad \forall y \in X\},$$

and the Fenchel conjugate of g is the convex function

$$g^* : X^* \to (-\infty, \infty], \quad g^*(x^*) = \sup\{\langle x, x^* \rangle - g(x) : x \in X\}.$$

It is well known that $x^* \in \partial g(x)$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{2}$$

It is easy to see that g^* is proper convex and lower semicontinuous function. The function g is called to be cofinite if dom $g^* = X^*$.

Let $g^\circ(x, y)$ be the right-hand derivative of g at $x \in$ int dom g in the direction y, that is,

$$g^\circ(x, y) := \lim_{t \downarrow 0} \frac{g(x + ty) - g(x)}{t}. \tag{3}$$

If the limit as $t \to 0$ in (3) exists for each y, then the function g is said to be G$\hat{a}$teaux differentiable at x. In this case, the gradient of g at x is the linear function $\nabla g(x)$, which is defined by $\langle y, \nabla g(x) \rangle := g^\circ(x, y)$ for all $y \in X$. The function g is said to be G$\hat{a}$teaux differentiable if it is G$\hat{a}$teaux differentiable at each $x \in$ int dom g. When the limit as $t \to 0$ in (3) is obtained uniformly for each $y \in X$ with $\|y\| = 1$, we say that g is Fréchet differentiable at x. At the end, g is said to be uniformly Fréchet differentiable on a subset C of X if the limit is obtained uniformly for $x \in C$ and $\|y\| = 1$.

The function g is said to be Legendre if it satisfies the following two conditions:

(L1) int dom $g \neq \emptyset$ and ∂g is single-valued on its domain.
(L2) int dom $g^* \neq \emptyset$ and ∂g^* is single-valued on its domain.

Since X is reflexive, we have $(\partial g)^{-1} = \partial g^*$ (see [8]). This fact, when joined together conditions (L1) and (L2), intimates the following equalities:

$$\nabla g = (\nabla g^*)^{-1}, \quad \text{ran} \quad (\nabla g) = \text{dom} \quad (\nabla g^*) = \text{int} \quad \text{dom} \quad g^*,$$
$$\text{ran} \quad (\nabla g^*) = \text{dom} \quad (\nabla g) = \text{int} \quad \text{dom} \quad g.$$

It is well known that if g is Legendre function, then the functions g and g^* are G$\hat{a}$teaux differentiable and strictly convex in the interior of their respective domains [3]. When the Banach space X is smooth and strictly convex, in particular, a Hilbert space, the function $g(.) = \frac{1}{p}\|.\|^p$ with $p \in (1, +\infty)$, is Legendre [3]. Suppose that $g : X \to (-\infty, +\infty]$ is Gâteaux differentiable. The function D_g: dom $g\times$ int dom $g \to [0, +\infty)$ defined by:

$$D_g(y, x):=g(y) - g(x) - \langle y - x, \nabla g(x)\rangle,$$

is called the Bregman distance with respect to g. It should be mentioned that D_g is not a distance in the usual sense of the term. Obviously, $D_g(x, x) = 0$, but $D_g(y, x) = 0$ may not intimate $x = y$. In our case, when g is Legendre this indeed holds (see [3], Theorem 7.3(vi), p. 642). In general, D_g is not symmetric and does not satisfy the triangle inequality. However, D_g satisfies the three-point identity

$$D_g(x, y) + D_g(y, z) - D_g(x, z) = \langle x - y, \nabla g(z) - \nabla g(y)\rangle,$$

and four-point identity

$$D_g(x, y) + D_g(w, z) - D_g(x, z) - D_g(w, y) = \langle x - w, \nabla g(z) - \nabla g(y)\rangle,$$

for any $x, w \in$ dom g and $y, z \in$ int dom g.

The modulus of total convexity at $x \in$ int dom g is the function $\upsilon_g(x, .) : [0, +\infty) \to [0, \infty]$ defined by:

$$\upsilon_g(x, t) := \inf\{D_g(y, x) : y \in \text{dom} \quad g, \quad \|y - x\| = t\}.$$

If for any $t > 0$, $\upsilon_g(x, t)$ is positive, then the function g is called totally convex at x. This concept was first presented by Butnariu and Iusem in [10]. Let C be a nonempty subset of X. The modulus of total convexity of g on C is defined by:

$$\upsilon_g(C, t) = \inf\{\upsilon_g(x, t) : \quad x \in C \cap \text{int} \quad \text{dom} \quad g\}.$$

The function g is termed totally convex on bounded subsets if $\upsilon_g(C, t)$ is positive for any $t > 0$ and for any nonempty and bounded subset C.

Lemma 1 *[39] If $g : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇g is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Lemma 2 *[10] The function $g : X \to (-\infty, +\infty]$ is totally convex on bounded subsets of X if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in int dom g and dom g, respectively, such that the first one is bounded,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 3 *[41] Let the function $g : X \to \mathbb{R}$ be Gâteaux differentiable and totally convex at a point $x \in$ int dom g. Let $\{x_n\} \subset$ dom g. If $\{D_g(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 4 *[41] Let $g : X \to \mathbb{R}$ be a Gâteaux differentiable function such that ∇g^* is bounded on bounded subsets of dom g^*. Let $x_0 \in X$ and $\{x_n\} \subset X$. If $\{D_g(x_0, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

A Bregman projection [9,15] of $x \in$ int dom g onto the nonempty closed convex set $C \subset$ int dom g is the unique vector $\overleftarrow{proj}_C^g(x) \in C$ satisfying

$$D_g(\overleftarrow{proj}_C^g(x), x) = \inf\{D_g(y, x) : y \in C\}.$$

Lemma 5 *[13] Let the function $g : X \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on $int \quad dom \quad g$. Let $x \in int \quad dom \quad g$ and $C \subset int \quad dom \quad g$ be a nonempty closed convex set. If $\hat{x} \in C$, then the following statements are equivalent:*

(i) The vector $\hat{x} \in C$ is the Bregman projection of x onto C.
(ii) The vector $\hat{x} \in C$ is the unique solution of the variational inequality

$$\langle z - y, \nabla g(x) - \nabla g(z) \rangle \geq 0, \quad \forall y \in C.$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_g(y, z) + D_g(z, x) \leq D_g(y, x), \quad \forall y \in C.$$

Let B and S be the closed unit ball and the unit sphere of a Banach space X. Let $rB = \{z \in X : \|z\| \leq r\}$ for all $r > 0$. Then, the function $g : X \to \mathbb{R}$ is said to be uniformly convex on bounded subsets (see [48]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by:

$$\rho_r(t) = \inf_{x, y \in rB, \|x - y\| = t, \alpha \in (0,1)} \frac{\alpha g(x) + (1 - \alpha) g(y) - g(\alpha x + (1 - \alpha) y)}{\alpha(1 - \alpha)},$$

for all $t \geq 0$. The function ρ_r is called the gauge of uniform convexity of g. It is known that ρ_r is a nondecreasing function.

Lemma 6 *[35] Let X be a Banach space, $r > 0$ be a fixed number and $g : X \to \mathbb{R}$ be a uniformly convex function on bounded subsets of X. Then,*

$$g\left(\sum_{k=0}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k g(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $\alpha_k \in (0, 1)$, $x_k \in rB$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of g.

The function g is also said to be uniformly smooth on bounded subsets (see [48]) if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, \infty) \to [0, \infty]$ is defined by:

$$\sigma_r(t) = \sup_{x \in rB, y \in S, \alpha \in (0,1)} \frac{\alpha g(x + (1 - \alpha) t y) + (1 - \alpha) g(x - \alpha t y) - g(x)}{\alpha(1 - \alpha)},$$

for all $t \geq 0$. A function g is said to be supercoercive if $\lim_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = +\infty$.

Theorem 1 *[48] Let $g : X \to \mathbb{R}$ be a supercoercive and convex function. Then, the following are equivalent:*

(i) g is bounded on bounded subsets and uniformly smooth on bounded subsets of X;
(ii) g is Fréchet differentiable and ∇g is uniformly norm-to-norm continuous on bounded subsets of X;
(iii) dom $g^* = X^*$, g^* is supercoercive and uniformly convex on bounded subsets of X^*.

Theorem 2 *[48] Let $g : X \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of X. Then, the following are equivalent:*

(i) g is supercoercive and uniformly convex on bounded subsets of X;
(ii) dom $g^* = X^*$, g^* is bounded on bounded subsets and uniformly smooth on bounded subsets of X^*;
(iii) dom $g^* = X^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of X^*.

Theorem 3 *[3] (Supercoercivity) Let $g : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, the following are equivalent:*

(i) g is supercoercive;
(ii) g^* is bounded above on bounded sets;
(iii) dom $g^* = X^*$ and ∂g^* maps bounded sets to bounded sets.

Theorem 4 *[11] Suppose that $g : X \to (-\infty, +\infty]$ is a Legendre function. The function g is uniformly convex on bounded subsets of X if and only if g is totally convex on bounded subsets of X.*

A bifunction f satisfies a Bregman–Lipschitz-type condition [14] if there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 D_g(y, x) - c_2 D_g(z, y), \quad \forall x, y, z \in C,$$

where $g : X \to (-\infty, +\infty]$ is a Legendre function. The constants c_1 and c_2 are called Bregman–Lipschitz coefficients.

Let $g : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function, recall that the proximal mapping of a proper convex and lower semicontinuous function $f : C \to (-\infty, +\infty]$ with respect to g is defined by:

$$prox_f^g(x) := argmin\{f(y) + D_g(y, x) : y \in C\}, \quad x \in X.$$

Applying the tools used in [12], we can state the following lemma.

Lemma 7 *Let $g : X \to (-\infty, +\infty]$ be a Legendre and supercoercive function. Let $x \in int\,dom\,g$, $C \subset int\,dom\,g$ and $f : C \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then,*

$$\left\langle prox_f^g(x) - y, \nabla g(x) - \nabla g(prox_f^g(x))\right\rangle + f(y) - f(prox_f^g(x)) \geq 0, \quad \forall y \in C. \tag{4}$$

3 Weak Convergence

In this section, we first establish some crucial lemmas and then we introduce a new extragradient algorithm (Algorithm 1) for solving pseudomonotone equilibrium problem with the Bregman–Lipschitz-type conditions. The algorithm is explicit in the sense that it is done without previously knowing the Bregman–Lipschitz coefficients of bifunction. This is quite interesting in the case where these coefficients are unknown or even demanding to approximate. We prove a weak convergence theorem (Theorem 5) for approximating a point of EP. For the sake of simplicity in the presentation, we will apply the symbol $[t]_+ = \max\{0, t\}$ and assume that $\frac{0}{0} = +\infty$ and $\frac{a}{0} = +\infty$ $(a > 0)$.

Motivated by Proposition 2.5 of [29], we prove the following lemma.

Lemma 8 *Let the function $g : X \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on bounded subset of X and let $\{x_n\}$ and $\{y_n\}$ be two sequences in int dom g and dom g, respectively. If $\{x_n\}$ and $\{D_g(y_n, x_n)\}$ are bounded, then the sequence $\{y_n\}$ is bounded too.*

Proof Assume that the sequence $\{y_n\}$ is not bounded. Then, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\lim_{k\to\infty} \|y_{n_k}\| = +\infty$. Consequently, $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = +\infty$ and there exists some $k_0 > 0$ such that $\|x_{n_k} - y_{n_k}\| > 1$ for all $k > k_0$. So, using [10, Proposition 1.2.2], for all $k > k_0$ we get

$$\upsilon_g(x_{n_k}, \|x_{n_k} - y_{n_k}\|) \geq \|x_{n_k} - y_{n_k}\|\upsilon_g(x_{n_k}, 1). \tag{5}$$

Since g is totally convex on bounded subset of X, letting $k \to \infty$ in (5), we get that $\{\upsilon_g(x_n, \|x_n - y_n\|)\}_{n\in\mathbb{N}}$ is not bounded. Since, by definition,

$$\upsilon_g(x_n, \|x_n - y_n\|) \leq D_g(y_n, x_n),$$

for all $n \in \mathbb{N}$, this implies that the sequence $\{D_g(y_n, x_n)\}_{n\in\mathbb{N}}$ cannot be bounded which is a contradiction. □

Lemma 9 *[30] Let $g : X \to \mathbb{R}$ be a Legendre function such that ∇g is weakly sequentially continuous and ∇g^* is bounded on bounded subsets of dom g^*. Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that for every $x \in C$, $\{D_g(x, x_n)\}$ converges and every weak cluster point of $\{x_n\}$ belongs to C. Then, $\{x_n\}$ converges weakly to a point in C.*

Proof It is suffice to show that there is exactly one weak subsequential limit of $\{x_n\}$. Using Lemma 4, we get that $\{x_n\}$ is bounded. So there is at least one weak subsequential limit of $\{x_n\}$. Suppose that x and y are two weak subsequential limits of $\{x_n\}$ in C, say $x_{k_n} \rightharpoonup x$ and $x_{l_n} \rightharpoonup y$. Since ∇g is weakly sequentially continuous, we have $\nabla g(x_{k_n}) \rightharpoonup \nabla g(x)$ and $\nabla g(x_{l_n}) \rightharpoonup \nabla g(y)$. Since x and y belong to C, the sequences $\{D_g(x, x_n)\}$ and $\{D_g(y, x_n)\}$ converge. In turn, since

$$D_g(x, y) + D_g(y, x_n) - D_g(x, x_n) = \langle x - y, \nabla g(x_n) - \nabla g(y)\rangle,$$

passing to the limit along x_{k_n} and along x_{l_n}, respectively, yields

$$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = \langle x - y, \nabla g(y) - \nabla g(y)\rangle = 0.$$

Thus, $D_g(x, y) + D_g(y, x) = 0$ and hence $x = y$. □

Definition 1 [2] Let $g : X \to (-\infty, +\infty]$ be a G$\hat{a}$teaux differentiable function and C be a nonempty subset of X. A sequence $\{x_n\}$ in X is called Bregman monotone with respect to C if the following conditions hold:

(i) $C \cap dom \quad g \neq \emptyset$.
(ii) $\{x_n\} \subset int \quad dom \quad g$.
(iii) $D_g(x, x_{n+1}) \leq D_g(x, x_n), \quad \forall x \in C \cap dom \quad g, \forall n \in \mathbb{N}$.

Lemma 10 *Let $g : X \to \mathbb{R}$ be a Legendre function such that ∇g is weakly sequentially continuous and ∇g^* is bounded on bounded subsets of dom g^*. Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $\{x_n\}$ is Bregman monotone with respect to C and every weak cluster point of $\{x_n\}$ belongs to C. Then, $\{x_n\}$ converges weakly to a point in C.*

Proof It deduces immediately from Lemma 9. □

Lemma 11 *Let $g : X \to \mathbb{R}$ be a Legendre function and totally convex on bounded subset of X such that ∇g^* is bounded on bounded subsets of dom g^*. Let $\{x_n\} \subset X$ be a Bregman monotone sequence with respect to C. Then, $\{\overleftarrow{proj}^g_C(x_n)\}$ converges strongly to some $z \in C$. Furthermore, if $x_n \rightharpoonup \overline{x}$ and ∇g is weakly sequentially continuous, then $\overline{x} = z$.*

Proof Let $u_n = \overleftarrow{proj}^g_C(x_n)$ and $m > n$. Since

$$D_g(u_m, x_m) \leq D_g(u_n, x_m) \leq D_g(u_n, x_n),$$

the sequence $\{D_g(u_n, x_n)\}$ is convergent and hence it is bounded. Using Lemmas 4 and 8, the sequence $\{u_n\}$ is bounded too. Set $r = \sup\{\|u_n\|, n \in \mathbb{N}\}$. So, using Lemma 6 and Theorem 4, we have

$$\begin{aligned} D_g(\frac{u_n + u_m}{2}, x_m) =& g(\frac{u_n + u_m}{2}) - g(x_m) - \langle \frac{u_n + u_m}{2} - x_m, \nabla g(x_m)\rangle \\ \leq& \frac{1}{2} g(u_n) + \frac{1}{2} g(u_m) - \frac{1}{4}\rho_r(\|u_n - u_m\|) - g(x_m) \end{aligned}$$

$$
\begin{aligned}
&- \langle \frac{u_n + u_m}{2} - x_m, \nabla g(x_m) \rangle \\
&= \frac{1}{2} D_g(u_n, x_m) + \frac{1}{2} D_g(u_m, x_m) - \frac{1}{4} \rho_r(\|u_n - u_m\|),
\end{aligned}
$$

and then,

$$
\begin{aligned}
\frac{1}{4}\rho_r(\|u_n - u_m\|) &\leq \frac{1}{2} D_g(u_n, x_m) + \frac{1}{2} D_g(u_m, x_m) - D_g(\frac{u_n + u_m}{2}, x_m) \\
&\leq \frac{1}{2} D_g(u_n, x_m) + \frac{1}{2} D_g(u_m, x_m) - D_g(u_m, x_m) \\
&\leq \frac{1}{2} D_g(u_n, x_n) - \frac{1}{2} D_g(u_m, x_m).
\end{aligned}
$$

Therefore, $\rho_r(\|u_n - u_m\|) \to 0$ as $m, n \to \infty$. Now, we show that

$$
\lim_{n,m \to \infty} \|u_n - u_m\| = 0.
$$

If this were not the case, there exist $\epsilon_0 > 0$ and subsequences $\{n_k\}$ and $\{m_k\}$ of $\{n\}$ and $\{m\}$, respectively, such that $\|u_{n_k} - u_{m_k}\| \geq \epsilon_0$. Since ρ_r is nondecreasing, we get

$$
\rho_r(\epsilon_0) \leq \rho_r(\|u_{n_k} - u_{m_k}\|).
$$

Letting $k \to \infty$, we have $\rho_r(\epsilon_0) \leq 0$. But this yields a contradiction to the uniform convexity of g on bounded subsets of X. Therefore, $\{u_n\}$ is a Cauchy sequence and hence converges strongly to some $z \in C$.
Now, let $x_n \rightharpoonup \overline{x}$ and ∇g be weakly sequentially continuous. Using Lemma 5, we have

$$
\left\langle \nabla g(x_n) - \nabla g\left(\overleftarrow{proj}^g_C(x_n)\right), \overline{x} - \overleftarrow{proj}^g_C(x_n) \right\rangle \leq 0.
$$

Letting $n \to \infty$, we get $\langle \nabla g(\overline{x}) - \nabla g(z), \overline{x} - z \rangle \leq 0$. So, $D_g(\overline{x}, z) + D_g(z, \overline{x}) \leq 0$ and hence $\overline{x} = z$. This completes the proof. □

In order to obtain the convergence of our method, we consider the following blanket assumptions imposed on the bifunction f.

(A1) f is pseudomonotone on C, i.e., for all $x, y \in C$,

$$
f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.
$$

(A2) f satisfies the Bregman–Lipschitz-type condition.

(A3) for any sequence $\{x_n\} \subset C$ and $x \in C$ such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} f(x_n, y) \geq 0$, for all $y \in C$, then $f(x, y) \geq 0$ [26].

(A4) $f(x, .)$ is convex, lower semicontinuous and subdifferentiable on C for every fixed $x \in C$. A bifunction f is called monotone on C if for all $x, y \in C$, $f(x, y) + f(y, x) \leq 0$. It is evident that any monotone bifunction is a pseudomonotone one, but not vice versa. A mapping $A : C \to X^*$ is pseudomonotone if and only if the bifunction $f(x, y) = \langle y - x, A(x)\rangle$ is pseudomonotone on C (see [47]).

Lemma 12 *[6] If the bifunction f satisfying conditions $A1 - A4$, then $EP(f)$ is closed and convex.*

Now, we present the following algorithm and we prove a weak convergence theorem (Theorem 5) and a strong convergence theorem (Theorem 6) for approximating a point of $EP(f)$.

Algorithm 1 (Extragradient algorithm for EP)

Initialization. Choose $x_0 \in C$, $\lambda_0 > 0$ and $\mu \in (0, 1)$.
Iterative steps: Assume that $x_n \in C$, λ_n $(n \geq 0)$ are known. Compute x_{n+1} and λ_{n+1} as follows:

$$y_n = prox^g_{\lambda_n f(x_n,.)}(x_n),$$
$$x_{n+1} = prox^g_{\lambda_n f(y_n,.)}(x_n),$$

and set

$$\lambda_{n+1} = \min\{\lambda_n, \frac{\mu(D_g(y_n, x_n) + D_g(x_{n+1}, y_n))}{[f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1})]_+}\}.$$

Stopping criterion: If $y_n = x_n$, then stop and x_n is a solution of $EP(f)$.

Remark 1 Under hypothesis $(A2)$, there exist some constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq& c_1 D_g(y_n, x_n) + c_2 D_g(x_{n+1}, y_n) \\ \leq& \max\{c_1, c_2\}\big(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)\big). \end{aligned}$$

Thus, from the definition of λ_n, we see that this sequence is bounded from below. Indeed, if $\lambda_0 \leq \frac{\mu}{\max\{c_1,c_2\}}$, then $\{\lambda_n\}$ is bounded from below by λ_0; otherwise, $\{\lambda_n\}$ is bounded from below by $\frac{\mu}{\max\{c_1,c_2\}}$. Moreover, the sequence $\{\lambda_n\}$ is nonincreasing. Therefore, there is a real $\lambda > 0$ such that $\lambda_n \to \lambda$, as $n \to \infty$.

Theorem 5 *Let $g : X \to \mathbb{R}$ be a Legendre and supercoercive function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X and let ∇g be weakly sequentially continuous. Under conditions $(A1) - (A4)$, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to $\bar{x} \in EP(f)$, where $\bar{x} = \lim_{n\to\infty} \overleftarrow{proj}^g_{EP(f)}(x_n)$.*

Proof It follows from Lemma 7 and the definition of x_{n+1} that

$$\langle x_{n+1} - y, \nabla g(x_n) - \nabla g(x_{n+1})\rangle \geq \lambda_n f(y_n, x_{n+1}) - \lambda_n f(y_n, y), \quad \forall y \in C. \tag{6}$$

From the definition of λ_{n+1}, we get

$$f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu\big(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)\big)}{\lambda_{n+1}},$$

which, by multiplying both sides of it by λ_n, implies that

$$\lambda_n f(y_n, x_{n+1}) \geq \lambda_n\big(f(x_n, x_{n+1}) - f(x_n, y_n)\big) - \frac{\mu\lambda_n\big(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)\big)}{\lambda_{n+1}}. \quad (7)$$

Combining relations (6) and (7), we obtain

$$\begin{aligned}\big\langle x_{n+1} - y, \nabla g(x_n) - \nabla g(x_{n+1})\big\rangle \geq &\lambda_n\big(f(x_n, x_{n+1}) - f(x_n, y_n)\big) - \lambda_n f(y_n, y) \\ &- \frac{\mu\lambda_n\big(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)\big)}{\lambda_{n+1}}. \end{aligned} \quad (8)$$

Similarly, from Lemma 7 and the definition of y_n we obtain

$$\big\langle y_n - x_{n+1}, \nabla g(y_n) - \nabla g(x_n)\big\rangle \leq \lambda_n\big(f(x_n, x_{n+1}) - f(x_n, y_n)\big). \quad (9)$$

Using (8) and (9), we have

$$\begin{aligned}\big\langle x_{n+1} - y, \nabla g(x_n) - \nabla g(x_{n+1})\big\rangle \geq &\big\langle y_n - x_{n+1}, \nabla g(y_n) - \nabla g(x_n)\big\rangle - \lambda_n f(y_n, y)) \\ &- \frac{\mu\lambda_n\big(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)\big)}{\lambda_{n+1}}. \end{aligned} \quad (10)$$

Applying the three-point identity of Bregman distance in (10) and a simple calculation, we get

$$\begin{aligned}D_g(y, x_{n+1}) \leq &D_g(y, x_n) - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})D_g(y_n, x_n) - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})D_g(x_{n+1}, y_n)) \\ &+ \lambda_n f(y_n, y). \end{aligned} \quad (11)$$

Let $x^* \in EP(f)$. Since f is pseudomonotone, $f(y_n, x^*) \leq 0$. Hence, substituting $y = x^*$ into (11), we obtain

$$D_g(x^*, x_{n+1}) \leq D_g(x^*, x_n) - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})D_g(y_n, x_n) - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})D_g(x_{n+1}, y_n). \quad (12)$$

Let $\epsilon \in (0, 1 - \mu)$ be some fixed number. Applying Remark 1, $\lambda_n \to \lambda > 0$ and hence

$$\lim_{n\to\infty}(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}) = 1 - \mu > \epsilon > 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > \epsilon > 0, \quad \forall n \geq n_0. \quad (13)$$

From (12) and (13), we obtain

$$D_g(x^*, x_{n+1}) \leq D_g(x^*, x_n) - \epsilon(D_g(y_n, x_n) + D_g(x_{n+1}, y_n)), \tag{14}$$

or

$$a_{n+1} \leq a_n - b_n,$$

where $a_n = D_g(x^*, x_n)$ and $b_n = \epsilon(D_g(y_n, x_n) + D_g(x_{n+1}, y_n))$. Thus, there exists the limit of $\{a_n\}$ and $\lim_{n\to\infty} b_n = 0$. From Lemma 2, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{15}$$

and hence $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Therefore, from Lemma 1 we have

$$\lim_{n\to\infty} \|\nabla g(x_{n+1}) - \nabla g(x_n)\| = 0. \tag{16}$$

From Theorems 2 and 4, g^* is bounded on bounded subsets of X^* and hence ∇g^* is also bounded on bounded subsets of X^*. From this, (14) and Lemma 4, the sequence $\{x_n\}$ is bounded. Now, we prove that each weak cluster point of $\{x_n\}$ is in $EP(f)$. Suppose that $\overline{x}$ is a weak cluster point of $\{x_n\}$. That is, there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \overline{x}$ as $k \to \infty$. Since $\|x_n - y_n\| \to 0$ as $n \to \infty$, we also have that $y_{n_k} \rightharpoonup \overline{x}$ as $k \to \infty$. Passing to the limit in (11) as $n = n_k$, we obtain

$$\begin{aligned}
\limsup_{k\to\infty} f(y_{n_k}, y) &\geq \frac{1}{\lambda} \limsup_{k\to\infty}(D_g(y, x_{n_k+1}) - D_g(y, x_{n_k})) \\
&\geq \frac{1}{\lambda} \limsup_{k\to\infty}(D_g(y, x_{n_k+1}) - D_g(y, x_{n_k}) - D_g(x_{n_k}, x_{n_k+1})) \\
&= \frac{1}{\lambda} \lim_{k\to\infty} \langle x_{n_k} - y, \nabla g(x_{n_k+1}) - \nabla g(x_{n_k})\rangle.
\end{aligned}$$

From above inequality, (16), boundedness of $\{x_n\}$ and $(A3)$, we obtain $f(\overline{x}, y) \geq 0$ for all $y \in C$. Hence, $\overline{x} \in EP(f)$. Using (12) and (13), we get that $\{x_n\}_{n\geq n_0}$ is Bregman monotone with respect to $EP(f)$. Thus, applying Lemmas 10, 11 and 12, we get the desired result. □

4 Strong Convergence

In this section, we analyze the strong convergence of the sequence generated by Algorithm 1 to an element of $EP(f)$ with some various extra assumptions on the problem. In the following theorem, we also assume that the bifunction f satisfies the following condition:

(A5) for all bounded sequences $\{x_n\}$ and $\{y_n\}$ in C,

$$\|x_n - y_n\| \to 0 \Longrightarrow f(x_n, y_n) \to 0.$$

Definition 2 A bifunction f is called strongly pseudomonotone on C, if there exists $\beta > 0$ such that whenever $f(x, y) \geq 0$, then $f(y, x) \leq -\beta\|x - y\|^2$ for all $x, y \in C$.

Definition 3 A function $h : X \to (-\infty, +\infty]$ is called uniformly convex with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if ψ is increasing, vanishes only at 0, and for each pair $x, y \in \text{dom}\, h$ and each $\alpha \in (0, 1)$,

$$h(\alpha x + (1-\alpha)y) \leq \alpha h(x) + (1-\alpha)h(y) - \alpha(1-\alpha)\psi(\|x - y\|). \tag{17}$$

If (17) holds with $\psi = \frac{\sigma}{2}\|.\|^2$ for some $\sigma > 0$, then h is strongly convex with constant σ. We say that h is strongly concave whenever $-h$ is strongly convex.

Motivated by Theorem 4.3 of [26], we prove the following theorem.

Theorem 6 *Let $g : X \to \mathbb{R}$ be a Legendre and supercoercive function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X and let hypotheses $(A1) - (A5)$ be satisfied. If each one of the following conditions is satisfied:*

(i) f is strongly pseudomonotone,
(ii) for all $x \in C$, $f(x, .)$ is uniformly convex with modulus ψ,
(iii) for all $y \in C$, $f(., y)$ is strongly concave with constant σ, then the sequence $\{x_n\}$ made by Algorithm 1 is strongly convergent to an element of $EP(f)$.

Proof Observe that in Theorem 5, we proved that all cluster points of $\{x_n\}$ belong to $EP(f)$. Now, if $\{x_{n_m}\}$ and $\{x_{k_m}\}$ are arbitrary subsequences of $\{x_n\}$ that converge strongly to p and q, respectively, then

$$\begin{aligned}&\langle p - q, \nabla g(x_{n_m}) - \nabla g(x_{k_m})\rangle \\ &\quad = D_g(p, x_{k_m}) + D_g(q, x_{n_m}) - D_g(p, x_{n_m}) - D_g(q, x_{k_m}).\end{aligned}$$

From (14), $\lim_{n\to\infty} D_g(p, x_n)$ and $\lim_{n\to\infty} D_g(q, x_n)$ exist. Using this, Lemma 1 and taking limit when $m \to \infty$, we get $p = q$. That is, $\{x_n\}$ converges strongly to a point of $EP(f)$.
Therefore, in each item, it remains to be proved that if $x_{n_k} \rightharpoonup x^*$, then $x_{n_k} \to x^*$. Suppose that $x_{n_k} \rightharpoonup x^*$. Consequently by (15), $y_{n_k} \rightharpoonup x^*$. Substituting $y = x^*$ into (6), we have

$$\begin{aligned}0 \leq& \lambda_{n_k} f(y_{n_k}, x^*) - \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \langle x_{n_k+1} - x^*, \nabla g(x_{n_k}) - \nabla g(x_{n_k+1})\rangle \\ \leq& \lambda_{n_k} f(y_{n_k}, x^*) - \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \|x_{n_k+1} - x^*\|\|\nabla g(x_{n_k}) - \nabla g(x_{n_k+1})\|.\end{aligned}$$

From (15), (16), Remark 1, condition A5 and boundedness of $\{x_n\}$, we get

$$\liminf_{k\to\infty} f(y_{n_k}, x^*) \geq 0. \tag{18}$$

(i) Since $f(x^*, y_{n_k}) \geq 0$, there is a $\beta > 0$ such that $f(y_{n_k}, x^*) \leq -\beta\|y_{n_k} - x^*\|^2$. This together with (18) implies that

$$0 \leq \liminf_{k\to\infty} f(y_{n_k}, x^*) \leq \liminf_{k\to\infty}(-\beta\|y_{n_k} - x^*\|^2) \leq -\beta \limsup_{k\to\infty} \|y_{n_k} - x^*\|^2.$$

Therefore, $y_{n_k} \to x^*$, and hence, $x_{n_k} \to x^*$.
(ii) Let $\alpha \in (0, 1)$ and set $w_{n_k} = \alpha y_{n_k} + (1-\alpha)x^*$, for all $n \in \mathbb{N}$. Replacing y with w_{n_k} in (6) and using uniform convexity of $f(y_{n_k}, w_{n_k})$, we get

$$\begin{aligned} 0 \leq& \lambda_{n_k} f(y_{n_k}, w_{n_k}) - \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \langle x_{n_k+1} - w_{n_k}, \nabla g(x_{n_k}) - \nabla g(x_{n_k+1})\rangle \\ \leq& \lambda_{n_k}\alpha f(y_{n_k}, y_{n_k}) + \lambda_{n_k}(1-\alpha) f(y_{n_k}, x^*) - \lambda_{n_k}\alpha(1-\alpha)\psi(\|y_{n_k} - x^*\|) \\ &- \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \|x_{n_k+1} - w_{n_k}\|\|\nabla g(x_{n_k}) - \nabla g(x_{n_k+1})\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \lambda_{n_k}\alpha(1-\alpha)\psi(\|y_{n_k} - x^*\|) \leq& -\lambda_{n_k} f(y_{n_k}, x_{n_k+1}) \\ &+ \|x_{n_k+1} - w_{n_k}\|\|\nabla g(x_{n_k}) - \nabla g(x_{n_k+1})\|. \end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n > 0$, boundedness of the sequences $\{x_n\}$, $\{w_n\}$, condition A5 and (16) imply that $\lim_{k\to\infty} \psi(\|y_{n_k} - x^*\|) = 0$. Applying a similar discussion to the one utilized in the proof of Lemma 11 , we deduce that $y_{n_k} \to x^*$ and hence $x_{n_k} \to x^*$.
(iii) Let $\alpha \in (0, 1)$ and set $w_{n_k} = \alpha y_{n_k} + (1-\alpha)x^*$, for all $n \in \mathbb{N}$. Then, since $f(w_{n_k}, x^*)$ is strongly concave, we have

$$\alpha f(y_{n_k}, x^*) + (1-\alpha) f(x^*, x^*) + \frac{1}{2}\alpha(1-\alpha)\sigma\|y_{n_k} - x^*\|^2 \leq f(w_{n_k}, x^*) \leq 0.$$

Therefore, we get $f(y_{n_k}, x^*) \leq -\frac{1}{2}\sigma(1-\alpha)\|y_{n_k} - x^*\|^2$. Next, similar to item (i), we get the desired result. □

Remark 2 It is valuable to mention that in Theorem 6, unlike Theorem 5, we do not need that ∇g to be weakly sequentially continuous.

5 Application

In this section, we study the specific equilibrium problem related to the function f defined for every $x, y \in C$ by $f(x, y) = \langle y - x, Ax\rangle$ with $A : C \to X^*$. Doing so, we achieve the conventional variational inequality:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad \langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in C. \tag{19}$$

The set of solutions of (19) is denoted by $VI(A, C)$.

Lemma 13 *[14] Let C be a nonempty closed convex subset of a reflexive Banach space X, $A : C \to X^*$ be a mapping and $g : X \to \mathbb{R}$ be a Legendre function. Then,*

$$\overleftarrow{proj}_C^g(\nabla g^*[\nabla g(x) - \lambda A(y)]) = argmin_{w \in C}\{\lambda\langle w - y, A(y)\rangle + D_g(w, x)\},$$

for all $x \in X$, $y \in C$ and $\lambda \in (0, +\infty)$.

Let X be a real Banach space and $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_X : [0, 2] \to [0, 1]$ is defined by:

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}.$$

X is called uniformly convex if $\delta_X(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, p-uniformly convex if there is a $c_p > 0$ so that $\delta_X(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness $\rho_X(\tau) : [0, \infty) \to [0, \infty)$ is defined by:

$$\rho_X(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

X is called uniformly smooth if $\lim_{\tau \to 0} \dfrac{\rho_X(\tau)}{\tau} = 0$.

For the p-uniformly convex space, the metric and Bregman distance have the following relation [44]:

$$\tau\|x - y\|^p \leq D_{\frac{1}{p}\|.\|^p}(x, y) \leq \langle x - y, J_X^p(x) - J_X^p(y)\rangle, \tag{20}$$

where $\tau > 0$ is fixed number and duality mapping $J_X^p : X \to 2^{X^*}$ is defined by:

$$J_X^p(x) = \{f \in X^*, \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in X$. It is well known that X is smooth if and only if J_X^p is single-valued mapping of X into X^*. We also know that X is reflexive if and only if J_X^p is surjective, and X is strictly convex if and only if J_X^p is one-to-one. Therefore, if X is smooth, strictly convex and reflexive Banach space, then J_X^p is a single-valued bijection and in this case, $J_X^p = (J_{X^*}^q)^{-1}$ where $J_{X^*}^q$ is the duality mapping of X^*.

For $p = 2$, the duality mapping J_X^p is called the normalized duality and is denoted by J. The function $\phi : X^2 \to \mathbb{R}$ is defined by:

$$\phi(y, x) = \| y \|^2 - 2\langle y, Jx\rangle + \| x \|^2,$$

for all $x, y \in X$. The generalized projection Π_C from X onto C is defined by:

$$\Pi_C(x) = argmin_{y \in C}\phi(y, x), \quad \forall x \in X,$$

where C is a nonempty closed and convex subset of X. Let X be a uniformly convex and uniformly smooth Banach space and $g(.) = \frac{1}{2}\|.\|^2$. So, $\nabla g = J$, $D_{\frac{1}{2}\|.\|^2}(x, y) = \frac{1}{2}\phi(x, y)$ and $\overleftarrow{Proj}_C^{\frac{1}{2}\|.\|^2} = \Pi_C$. For solving the variational inequality (19) in 2-uniformly convex and uniformly smooth Banach space X, we consider the following algorithm.

Algorithm 2 (Extragradient algorithm for VI)

Initialization. Choose $x_0 \in C$, $\lambda_0 > 0$ and $\mu \in (0, 1)$.
Iterative steps: Assume that $x_n \in C$, λ_n $(n \geq 0)$ are known. Compute x_{n+1} and λ_{n+1} as follows:

$$y_n = \Pi_C(J^{-1}(J(x_n) - \lambda_n Ax_n)),$$
$$x_{n+1} = \Pi_C(J^{-1}(J(x_n) - \lambda_n Ay_n)),$$

and set

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu(\phi(y_n, x_n) + \phi(x_{n+1}, y_n))}{[\langle x_{n+1} - y_n, Ax_n - Ay_n\rangle]_+}\right\}.$$

Stopping criterion: If $y_n = x_n$, then stop and x_n is a solution of $VI(A, C)$.

Corollary 1 *Let X be a 2-uniformly convex and uniformly smooth Banach space. Suppose that $A : C \to X^*$ is a strongly pseudomonotone, L-Lipschitz and weak to norm continuous mapping. Then, the sequence $\{x_n\}$ made by Algorithm 2 is strongly convergent to an element of $VI(A, C)$.*

Proof Let $f(x, y):=\langle y-x, A(x)\rangle$ for all $x, y \in C$. Since A is L-Lipschitz continuous, for all $x, y, z \in C$, we have

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &= \langle y - x, A(x)\rangle + \langle z - y, A(y)\rangle - \langle z - x, A(x)\rangle \\ &= -\langle y - z, A(y) - A(x)\rangle \\ &\geq -\|A(y) - A(x)\|\|y - z\| \\ &\geq -L\|y - x\|\|y - z\| \\ &\geq -\frac{L}{2}\|y - x\|^2 - \frac{L}{2}\|y - z\|^2 \\ &\geq -\frac{L}{4\tau}\phi(y, x) - \frac{L}{4\tau}\phi(z, y), \qquad by \quad (20). \end{aligned}$$

Therefore, f satisfies the Bregman–Lipschitz-type condition with respect to $g(.) = \frac{1}{2}\|.\|^2$ and $c_1 = c_2 = \frac{L}{2\tau}$. Moreover, the strong pseudomonotonicity of A certifies the strong pseudomonotonicity of f. Conditions $A3$ and $A4$ are satisfied automatically. Using Theorem 6 and Lemma 13, we get the desired result. □

6 Numerical Experiments

In this section, we will give two numerical examples to show that our algorithm is efficient and converges faster than Algorithm 1 of [22] . The optimization subproblems in these examples have been solved by FMINCON optimization toolbox in MATLAB software.

Theorem 7 *[5] Let $g : X \to \mathbb{R}$ be a Gâteaux differentiable function. The function g is strongly convex with constant $\sigma > 0$ if and only if*

$$g(x) - g(y) \geq \langle x - y, \nabla g(y)\rangle + \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in X.$$

Proposition 1 *[4] Let H be a Hilbert space and $g : H \to (-\infty, +\infty]$ be proper and let $\sigma > 0$. Then, g is strongly convex with constant $\sigma > 0$ if and only if $g - \frac{\sigma}{2}\|.\|^2$ is convex.*

Example 1 Let $X = \mathbb{R}$ and $C = [4, 10]$ with the usual norm $|.|$. Define the bifunction $f = C \times C \to \mathbb{R}$ as follows:

$$f(x, y) = (6x + 4y - 50)(y - x), \quad \forall x, y \in C.$$

Let $p \in (1, +\infty)$ and $g_p(.) = \frac{1}{p}\|.\|^p$. Using Proposition 1, we can easily see that the function g_p is strongly convex on C with constant

$$\sigma_p = \begin{cases} (p-1)10^{p-2}, & \text{if} \quad p < 2, \\ (p-1)4^{p-2}, & o.w. \end{cases} \tag{21}$$

Obviously, the bifunction f satisfies conditions $A1$, $A3$ and $A4$. Furthermore,

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &= 2(z - y)(y - x) \\ &\geq -2|z - y||y - x| \\ &\geq -|z - y|^2 - |y - x|^2 \\ &\geq \frac{-2}{\sigma_p} D_{g_p}(z, y) - \frac{2}{\sigma_p} D_{g_p}(y, x), \end{aligned}$$

which the last inequality follows from the strong convexity of g_p on C. Hence, f satisfies condition $A2$. Put $\lambda_0 = \frac{1}{10}$ and $\mu = \frac{3}{4}$. It can be seen that all the hypotheses of Theorem 5 are satisfied and $EP(f) = \{5\}$. Using Algorithm 1 with the initial point $x_0 = 4$, we have the numerical results in Figs. 1 and 2. In cases $p = 1.5$, 2 and 2.5, approximate solutions, respectively, are

$$x_9 = 4.9999990, \quad x_{21} = 4.9999994, \quad x_{53} = 4.9999990,$$

with the tolerance $\epsilon = 10^{-6}$. Therefore, the rate of convergence decreases with increasing p. Note that for $g(.) = \frac{1}{2}\|.\|^2$ our Algorithm 1 reduces to Algorithm 1 of [22] and we see that our Algorithm 1 with $g(.) = \frac{2}{3}\|.\|^{\frac{3}{2}}$ converges faster than Algorithm 1 of [22].

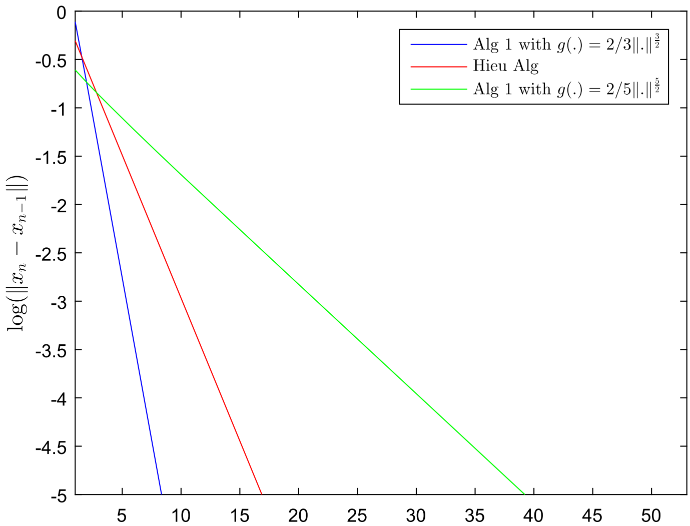

Fig. 1 Plotting of $\log \|x_n - x_{n-1}\|$ in Example 1

Theorem 8 *[5] The quadratic function $g(x) = x^T Ax + 2b^T x + c$ with $A = A^T \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ is strongly convex if and only if A is positive definite and in that case the strong convexity constant is $2\lambda_{\min}(A)$, where $\lambda_{\min}(A)$ is the minimum eigenvalue of A.*

Remark 3 Let $g(x) = x^T Ax + 2b^T x + c$, where A is positive definite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, then

$$\nabla g(x) = 2(Ax + b), \quad g^*(y) = \frac{1}{4}(y - 2b)^T A^{-1}(y - 2b).$$

Example 2 Let $X = \mathbb{R}^3$ with the Euclidean norm and

$$C = \{x \in \mathbb{R}^3, \quad \|x - (4, 4, 4)\| \leq 2\}.$$

Define the bifunction $f = C \times C \to \mathbb{R}$ and the quadratic function $g : X \to \mathbb{R}$ as follows:

$$f(x, y) = \langle Px + Qy + q, y - x\rangle, \quad g(x) = x^T Ax + 2b^T x + c,$$

where

$$P = \begin{bmatrix} 8 & -3 & -1 \\ -3 & 10 & -7 \\ -1 & -7 & 13 \end{bmatrix}, \quad Q = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 8 & -6 \\ -1 & -6 & 11 \end{bmatrix}, \quad q = \begin{bmatrix} -21 \\ 0 \\ -27 \end{bmatrix},$$

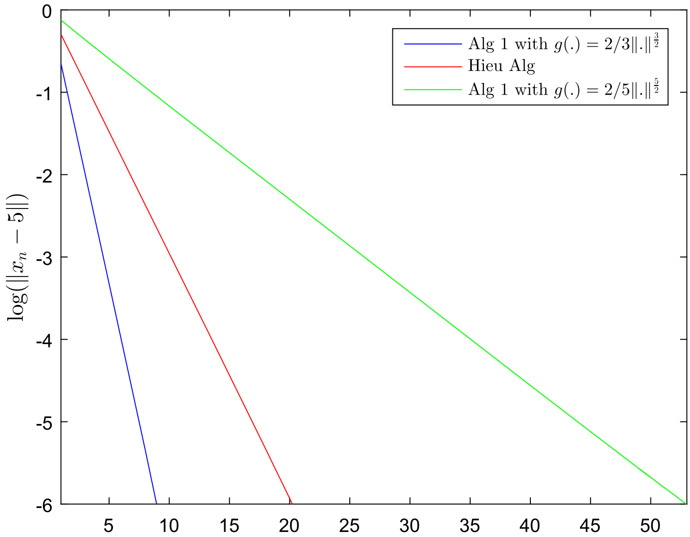

Fig. 2 Plotting of $\log \|x_n - 5\|$ in Example 1

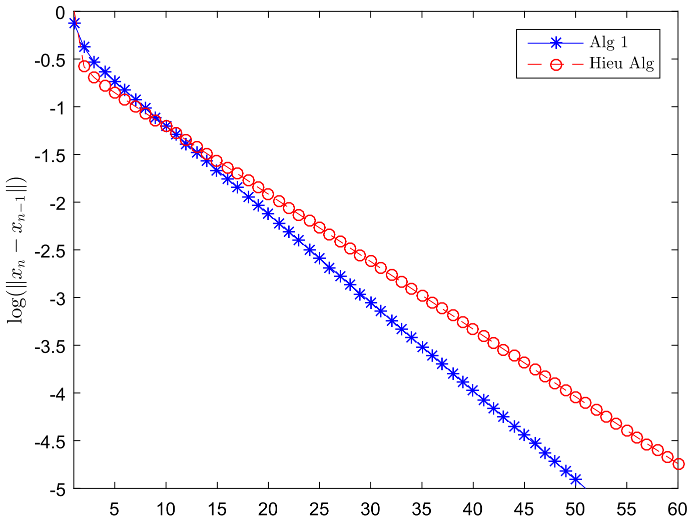

Fig. 3 Plotting of $\log \|x_n - x_{n-1}\|$ in Example 2

and

$$A = \begin{bmatrix} 11 & 5 & 1 \\ 5 & 5 & 6 \\ 1 & 6 & 13 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix},$$

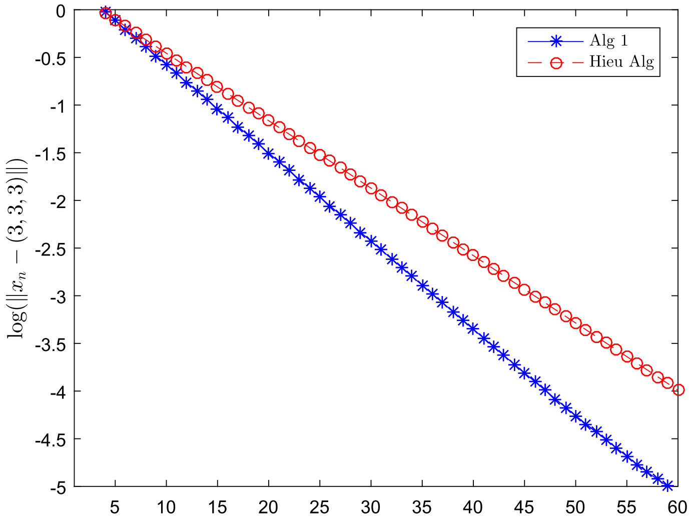

Fig. 4 Plotting of $\log \|x_n - (3, 3, 3)\|$ in Example 2

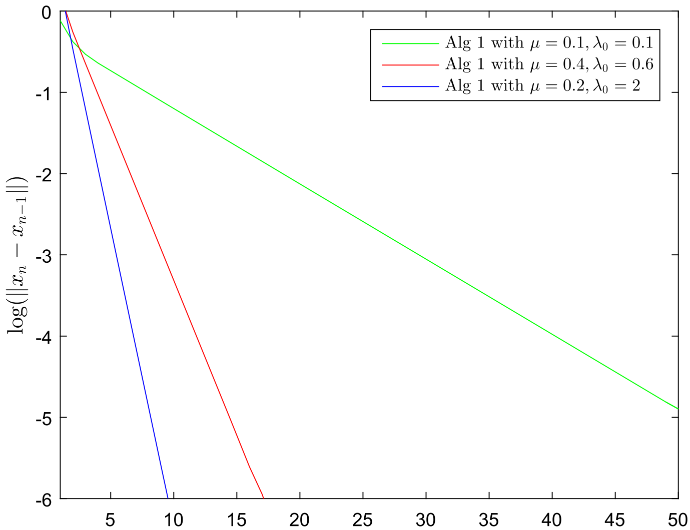

Fig. 5 Plotting of Example 2 with different choices of μ and λ_0

and $c \in \mathbb{R}$. Obviously, the bifunction f satisfies conditions $A1$, $A3$ and $A4$. Furthermore, using [36, Lemma 6.2], Theorems 7 and 8, we have

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &\geq -\frac{\|Q - P\|}{2}\|x - y\|^2 - \frac{\|Q - P\|}{2}\|y - z\|^2 \\ &\geq -\frac{\|Q - P\|}{2\lambda_{\min(A)}} D_g(y, x) - \frac{\|Q - P\|}{2\lambda_{\min(A)}} D_g(z, y), \end{aligned}$$

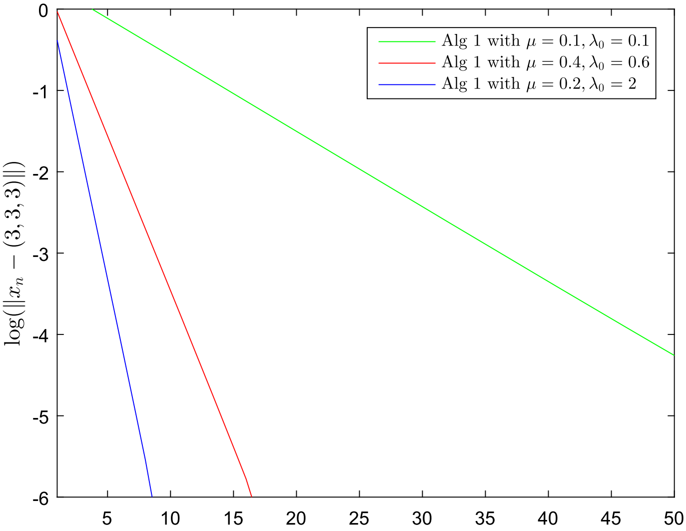

Fig. 6 Plotting of Example 2 with different choices of μ and λ_0

which shows the bifunction f satisfying $A2$. Since $q = (-Q-P)(3, 3, 3)^T$ and Q is a symmetric positive semidefinite matrix, $EP(f) = \{(3, 3, 3)^T\}$. Using Theorem 3 and Remark 3, we get that g is supercoercive function and total convexity of g on bounded subsets of X follows from the strong convexity of g. Put $\lambda_0 = \frac{1}{100}$ and $\mu = \frac{1}{10}$. It can be seen that all the hypotheses of Theorem 5 are satisfied. Applying Algorithm 1 with the initial point $x_0 = (4, 4, 5)$, we have the numerical results in Figs. 3 and 4 (See also Figs. 5, 6). In this example, as the first one, we see that our Algorithm 1 converges faster than Algorithm 1 of [22].

References

1. Antipin, A.S.: Extraproximal approach to calculating equilibriums in pure exchange models. Comput. Math. Math. Phys. **46**, 1687–1998 (2006)
2. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Bregman monotone optimization algorithms. SIAM J. Control Optim. **42**, 596–636 (2003)
3. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. **3**, 615–647 (2001)
4. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. Springer, New York (2017).. (**CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC**)
5. Beck, A.: Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB. SIAM, New Delhi (2014)
6. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibriume problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
7. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
8. Bonnans, J.F., Shapiro, A.: Perturbation analysis of optimization problems. Springer, New York (2000)

9. Bregman, L.M.: A relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200–217 (1967)
10. Butnariu, D., Iusem, A.N.: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization. Kluwer Academic Publishers, Dordrecht (2000)
11. Butnariu, D., Iusem, A.N., Zălinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach Spaces. J Convex Anal **10**, 35–61 (2003)
12. Butnariu, D., Kassay, G.: A proximal-projection method for finding zeroes of setvalued operators. SIAM J. Control Optim. **47**, 2096–2136 (2008)
13. Butnariu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **84919**, 1–39 (2006)
14. Eskandani, G.Z., Raeisi, M., Rassias, ThM: A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance. J. Fixed Point Theory Appl. **20**, 132 (2018)
15. Eskandani, G.Z., Azarmi, S. & Raeisi, M. On the generalized Bregman projection operator in reflexive Banach spaces. J. Fixed Point Theory Appl. **22**, 16 (2020). https://doi.org/10.1007/s11784-019-0749-0
16. Flam, S.D., Antipin, A.S.: Equilibrium programming and proximal-like algorithms. Math. Prog. **78**, 29–41 (1997)
17. Hieu, D.V., Cho, Y.J., Xiao, Y.B.: Modified extragradient algorithms for solving equilibrium problems. Optimization **67**(11), 2003–2029 (2018)
18. Hieu, D.V., Cho, Y.J., Xiao, Y.B.: Golden ratio algorithms with new stepsize rules for variational inequalities. Math. Meth. Appl. Sci. **42**, 6067–6082 (2019)
19. Hieu, D.V., Duong, H.N., Thai, B.H.: Convergence of relaxed inertial methods for equilibrium problems. J. Appl. Numer. Optim. **3**(1), 215–229 (2021)
20. Hieu, D.V., Je, C.Y., Xiao, Y.B., Kumam, P.: Modified extragradient method for pseudomonotone variational inequalities in infinite dimensional Hilbert spaces. Vietnam J. Math (2020). https://doi.org/10.1007/s10013-020-00447-7
21. Hieu, D.V., Muu, L.D., Quy, P.K., Duong, H.N.: New extragradient methods for solving equilibrium problems in Banach spaces. Banach J. Math. Anal. (2021). https://doi.org/10.1007/s43037-020-00096-5
22. Hieu, D.V., Quy, P.K., Vy, L.V.: Explicit iterative algorithms for solving equilibrium problems. Calcolo (2019). https://doi.org/10.1007/s10092-019-0308-5
23. Hieu, D.V., Strodiot, J.J.: Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces. J. Fixed Point Theory Appl. **20**, 131 (2018)
24. Iusem, A.N., Nasri, M.: Inexact proximal point methods for equilibrium problems in Banach spaces. Numer. Funct. Anal. Optim. **28**, 1279–1308 (2007)
25. Jouymandi, Z., Moradlou, F.: Extragradient methods for split feasibility problems and generalized equilibrium problems in Banach spaces. Math. Meth. Appl. Sci. **41**, 826–838 (2018)
26. Khatibzadeh, H., Mohebbi, V.: On the proximal point method for an infinite family of equilibrium problems in Banach spaces. Bull. Korean Math. Soc. **56**, 757–777 (2019)
27. Konnov, I.V.: Application of the proximal point method to nonmonotone equilibrium problems. J. Optim. Theory Appl. **119**, 317–333 (2003)
28. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekonomika i Matematicheskie Metody **12**, 747–756 (1976). ((**In Russian**))
29. Martín-Márquez, V., Reich, S., Sabach, S.: Bregman strongly nonexpansive operators in reflexive Banach spaces: Bregman strongly nonexpansive operators in reflexive Banach spaces. J. Math. Anal. Appl. **400**, 597–614 (2013)
30. Martín-Márquez, V., Reich, S., Sabach, S.: Iterative methods for approximating fixed points of Bregman nonexpansive operators. Discrete Contin. Dyn. Syst. **6**(4), 1043–1063 (2013)
31. Mastroeni, G.: Gap function for equilibrium problems. J. Glob. Optim. **27**, 411–426 (2003)
32. Moharami, R., Eskandani, G.Z.: An extragradient algorithm for solving equilibrium problem and zero point problem in Hadamard spaces. RACSAM **114**, 152 (2020). https://doi.org/10.1007/s13398-020-00885-5
33. Moudafi, A.: Proximal point algorithm extended to equilibrum problem. J. Nat. Geom. **15**, 91–100 (1999)
34. Nadezhkina, N., Takahashi, W.: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191–201 (2013)

35. Naraghirad, E., Yao, J.C.: Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. (2013). https://doi.org/10.1186/1687-1812-2013-141
36. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization **57**(6), 749–776 (2008)
37. Raeisi, M., Chalack, M., Zamani Eskandani G.: Gradient projection-type algorithms for solving ϕ-strongly pseudomonotone equilibrium problems in Banach spaces. Optimization (2021) https://doi.org/10.1080/02331934.2021.1884860
38. Raeisi, M., Eskandani, G.Z., Eslamian, M.A.: general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings. Optimization **68**, 309–327 (2018)
39. Reich, S., Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **10**, 471–485 (2009)
40. Reich, S., Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. **73**, 122–135 (2010)
41. Reich, S., Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**, 22–44 (2010)
42. Semenov, V.V.: A strongly convergent splitting method for systems of operator inclusions with monotone operators. J. Autom. Inf. Sci. **46**, 45–56 (2014)
43. Semenov, V.V.: Hybrid splitting methods for the system of operator inclusions with monotone operators. Cybern. Syst. Anal. **50**, 741–749 (2014)
44. Schöproofer, F., Schuster, T., Louis, A.K.: An iterative regularization method for the solution of the split feasibility problem in Banach spaces. Inverse Probl. **24**, (2008)
45. Takahashi, W., Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**, 417–428 (2003)
46. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**(2), 431–446 (2000)
47. Yao, J.C.: Variational inequalities with generalized monotone operators. Math. Oper. Res. **19**, 691–705 (1994)
48. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing, Singapore (2002)