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On the A_α-Spectra of Some Join Graphs

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Abstract
Let G be a simple, connected graph and let $A(G)$ be the adjacency matrix of G. If $D(G)$ is the diagonal matrix of the vertex degrees of G, then for every real $\alpha \in [0, 1]$, the matrix $A_\alpha(G)$ is defined as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

The eigenvalues of the matrix $A_\alpha(G)$ form the A_α-spectrum of G. Let $G_1 \dot{\vee} G_2$, $G_1 \underline{\vee} G_2$, $G_1 \langle \text{v} \rangle G_2$ and $G_1 \langle \text{e} \rangle G_2$ denote the subdivision-vertex join, subdivision-edge join, R-vertex join and R-edge join of two graphs G_1 and G_2, respectively. In this paper, we compute the A_α-spectra of $G_1 \dot{\vee} G_2$, $G_1 \underline{\vee} G_2$, $G_1 \langle \text{v} \rangle G_2$ and $G_1 \langle \text{e} \rangle G_2$ for a regular graph G_1 and an arbitrary graph G_2 in terms of their A_α-eigenvalues. As an application of these results, we construct infinitely many pairs of A_α-cospectral graphs.

Keywords α-Adjacency matrix · A_α-Spectra · Subdivision-vertex join · Subdivision-edge join · R-vertex join · R-edge join

Mathematics Subject Classification 05C50 · 05C05

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1 Introduction

All graphs considered in this article are simple, undirected and connected. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The adjacency matrix of G on n vertices, denoted by $A(G)$, is an $n \times n$ symmetric matrix whose rows and columns are indexed by $V(G)$. The (i, j)-th entry of $A(G)$ is 1, if the vertices i and j are adjacent in G, and 0 otherwise. We denote the degree of the vertex v in G by $d_G(v)$, and define $D(G)$ to be the $n \times n$ diagonal matrix, whose diagonal entries are the degrees of the vertices of G. The Laplacian matrix of G, denoted by $L(G)$, is defined as $L(G) = D(G) - A(G)$. The signless Laplacian matrix of G, denoted by $Q(G)$, is defined as $Q(G) = D(G) + A(G)$. In [19], the author introduced a family of matrices $A_\alpha(G)$ as follows:

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \qquad \text{for any } \alpha \in [0, 1]. \tag{1}$$

It is clear that $A_\alpha(G)$ is equal to the adjacency matrix of G if $\alpha = 0$, and is equal to $\frac{1}{2}Q(G)$ if $\alpha = \frac{1}{2}$.

Given an $n \times n$ matrix M, let M^T, $\det(M)$ and $\operatorname{adj}(M)$ denote the transpose, the determinant and the adjugate of M, respectively. The characteristic polynomial of M is denoted by $\psi_M(x)$, which is defined as

$$\psi_M(x) = \det(xI_n - M),$$

where I_n is the identity matrix of order n. In particular, for a graph G on n vertices, $\psi_{A(G)}(x)$ and $\psi_{A_\alpha(G)}(x)$ denote the characteristic polynomial of $A(G)$ and $A_\alpha(G)$, respectively. The roots of the characteristic polynomial of M are called the M-eigenvalues. Let $\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \cdots \geq \lambda_n(A(G))$ and $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$ be the A-eigenvalues and A_α-eigenvalues of G, respectively. The set of all eigenvalues of $A(G)$ and $A_\alpha(G)$ together with their multiplicities is called the A-spectrum and the A_α-spectrum of G, respectively. If $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ are the distinct A_α-eigenvalues of G, then the A_α-spectrum of G can be written as

$$\sigma(A_\alpha(G)) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_k]^{m_k}\},$$

where m_i is the algebraic multiplicity of λ_i, for $1 \leq i \leq k$. Two graphs are said to be A-cospectral (respectively, A_α-cospectral) if they have the same A-spectrum (respectively, A_α-spectrum).

In spectral graph theory, computing the spectra and the characteristic polynomials of various classes of matrices associated with the graphs are interesting problems considered in the literature. Various graph operations such as the disjoint union, the Cartesian product, the Kronecker product, the corona, the edge corona, the neighborhood corona, the subdivision-edge neighborhood corona, the join, the subdivision-vertex join, the subdivision-edge join, the R-vertex join, the R-edge join etc., have been introduced and their adjacency, Laplacian and signless Laplacian spectra are computed in [1–6,8,9,11,14,15,17,18,21]. Recently, the A_α-spectra of some graph operations have been studied in [12,13,20]. Motivated by these works, in this article, we determine the A_α-spectra of subdivision-vertex join, subdivision-edge join, R-vertex join and R-edge join of two graphs G_1 and G_2, where G_1 is a regular graph and G_2 is an arbitrary graph. As applications of these results on the A_α-spectra, we construct infinitely many pairs of A_α-cospectral graphs. The results obtained in this paper extends the results presented in [6,15] for A_α-spectra.

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the disjoint union of G_1 and G_2 together with all possible edges connecting all the vertices of G_1 with all the vertices of G_2 [7]. The subdivision graph of a graph G, denoted by $S(G)$, is the graph obtained by inserting a new vertex in every edge of G, that is by replacing each edge of G by P_3, the path on 3 vertices [4]. Based on this subdivision graph, two new graph operations, namely the subdivision-vertex join and the subdivision-edge join are introduced in [10]. The subdivision-vertex join of two graphs G_1 and G_2, denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$. The subdivision-edge join of G_1 and G_2, denoted by $G_1 \underline{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$, where $I(G_1)$ is the set of inserted vertices of $S(G_1)$. The R-graph of a graph G, denoted by $\mathcal{R}(G)$, is the graph obtained from G by introducing a new vertex u_e for each edge $e \in E(G)$, and making u_e adjacent to both the end vertices of e [5]. In [16], the authors defined two new graph operations based on R-graph, namely the R-vertex join and the R-edge join. The R-vertex join of two graphs G_1 and G_2, denoted by $G_1 \langle \mathrm{v} \rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$. The R-edge join of G_1 and G_2, denoted by $G_1 \langle \mathrm{e} \rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$, where $I(G_1)$ is the set of inserted vertices of $\mathcal{R}(G_1)$.

This paper is organized as follows: in Sect. 2, we collect some preliminary results and define some useful notations. In Sects. 3, 4, 5 and 6, we obtain the characteristic polynomials of A_α-matrices for the graphs $G_1 \dot{\vee} G_2$, $G_1 \underline{\vee} G_2$, $G_1 \langle \mathrm{v} \rangle G_2$ and $G_1 \langle \mathrm{e} \rangle G_2$ respectively, where G_1 is an r_1-regular graph and G_2 is an arbitrary graph. In each of these four sections, we include some results on the eigenvalues of the said matrices taking G_2 as some particular graphs, like regular and complete bipartite. Also, as an application of these results, we construct infinitely many pairs of graphs having the same A_α-spectrum.

2 Preliminaries

Let G be a graph on n vertices and m edges. The *incidence matrix* $R(G)$ of the graph G is the (0, 1)-matrix, whose rows and columns are indexed by the vertex set and the edge set of G, respectively. The (i, j)-th entry of $R(G)$ is 1, if the vertex i is incident to the edge j, and 0 otherwise. The *line graph* of G, denoted by $\mathcal{L}(G)$, is the graph with vertices are the edges of G. Two vertices in $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges have a common end-vertex in G. It is well known [4] that

$$R(G)^T R(G) = A(\mathcal{L}(G)) + 2I_m. \tag{2}$$

If G is an r-regular graph, then

$$R(G)R(G)^T = A(G) + rI_n. \tag{3}$$

We will use the symbols 0_n and 1_n ($0_{m\times n}$ and $J_{m\times n}$) for the column vectors ($m \times n$ matrices) consisting all 0's and all 1's, respectively. The M-coronal $\Gamma_M(x)$ of an $n \times n$ square matrix M is defined [3,18] by

$$\Gamma_M(x) = 1_n^T(xI_n - M)^{-1}1_n. \tag{4}$$

Here the matrix $(xI_n - M)$ is considered as matrix over the field of rational functions $\mathbb{C}(x)$, and the inverse $(xI_n - M)^{-1}$ is considered in $\mathbb{C}(x)$. We will use this convention throughout this manuscript. It is known that [3, Proposition 2], if each row sum of an $n \times n$ matrix M is constant, say t, then

$$\Gamma_M(x) = \frac{n}{x-t}. \tag{5}$$

Now we state some lemmas which will be useful to prove our main results.

Lemma 2.1 (Schur complement formula)[22] *Let M_1, M_2, M_3 and M_4 be matrices of size $r \times r$, $r \times s$, $s \times r$ and $s \times s$, respectively. Then*

$$\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4)\cdot\det\left(M_1 - M_2M_4^{-1}M_3\right), \text{ when } M_4 \text{ is invertible}.$$
$$= \det(M_1)\cdot\det\left(M_4 - M_3M_1^{-1}M_2\right), \text{ when } M_1 \text{ is invertible}.$$

The matrices $M_1 - M_2M_4^{-1}M_3$ and $M_4 - M_3M_1^{-1}M_2$ are called the Schur complements of M_4 and M_1, respectively.

Lemma 2.2 [15, Proposition 2.2] *Let A be an $n \times n$ real matrix. Then*

$$\det(A + cJ_{n\times n}) = \det(A) + c1_n^T\operatorname{adj}(A)1_n,$$

where c is a real number.

Lemma 2.3 [15, Corollary 2.3] *Let A be an $n \times n$ real matrix and c be a real number. Then*

$$\det(xI_n - A - cJ_{n\times n}) = (1 - c\Gamma_A(x))\det(xI_n - A).$$

Lemma 2.4 [4] *Let G be an r-regular graph on n vertices, and let $\mathcal{L}(G)$ be the line graph of G. If the characteristic polynomials of the matrices $A(G)$ and $A(\mathcal{L}(G))$ are $\psi_{A(G)}(x)$ and $\psi_{A(\mathcal{L}(G))}(x)$, respectively, then*

$$\psi_{A(\mathcal{L}(G))}(x) = (x+2)^{m-n}\psi_{A(G)}(x - r + 2).$$

For two positive integers p, q, let K_p and $K_{p,q}$ denote the complete graph on p vertices, and complete bipartite graph on $p+q$ vertices, respectively.

Lemma 2.5 [19, Proposition 37] *The spectrum of* $A_\alpha(K_{p,q})$ *is* $\sigma(A_\alpha(K_{p,q})) = \left\{\frac{\alpha(p+q)+\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}, [\alpha p]^{q-1}, [\alpha q]^{p-1}, \frac{\alpha(p+q)-\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}\right\}$.

Lemma 2.6 [20, Theorem 3] *For the graph* $K_{p,q}$*, the* A_α*-coronal is given by,*

$$\Gamma_{A_\alpha(K_{p,q})}(x) = \frac{(p+q)x - \alpha(p+q)^2 + 2pq}{x^2 - \alpha(p+q)x + (2\alpha-1)pq}.$$

In [6, Lemma 12], the authors have obtained an expression for the inverse of the matrix $(cI_n - dJ_{n\times n})$, for $c, d > 0$. We modify the conditions on c and d, and restate the result with proof.

Lemma 2.7 *Let* c *and* d *be two real numbers such that the matrix* $(cI_n - dJ_{n\times n})$ *is invertible. Then*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}. \tag{6}$$

Proof The eigenvalues of $J_{n\times n}$ are n and 0 with multiplicity $n-1$. Therefore, we have

$$\det(cI_n - dJ_{n\times n}) = c^{n-1}(c-nd).$$

Since the matrix $(cI_n - dJ_{n\times n})$ is invertible, we have $\det(cI_n - dJ_{n\times n}) = c^{n-1}(c-nd) \neq 0$. Thus $c \neq 0$ and $c - nd \neq 0$. So the expression on the right hand side of (6) is valid. Now,

$$\begin{aligned}
&(cI_n - dJ_{n\times n}) \cdot \left(\frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}\right)\\
&= I_n - \frac{d}{c}J_{n\times n} + \frac{d}{c-nd}J_{n\times n} - \frac{nd^2}{c(c-nd)}J_{n\times n}\\
&= I_n.
\end{aligned}$$

Hence the inverse of the matrix $(cI_n - dJ_{n\times n})$ is $\frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}$. □

Remark 2.1 If G_1 is a graph on n_1 vertices and m_1 edges, and G_2 is a graph on n_2 vertices, each of the graphs $G_1\dot{\vee}G_2$, $G_1\underline{\vee}G_2$, $G_1\langle v\rangle G_2$ and $G_1\langle e\rangle G_2$ has $(n_1+m_1+n_2)$ vertices. We consider the following partition of the vertex set of above graphs: $V(G_1)\cup I(G_1)\cup V(G_2)$, where $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ are vertex sets of G_1 and G_2, respectively, and $I(G_1) = \{v'_1, v'_2, \ldots, v'_{m_1}\}$ is the set of inserted vertices to construct the graphs $S(G_1)$ and $R(G_1)$ from G_1. In the following figure, we illustrate the labeling process with a particular example.

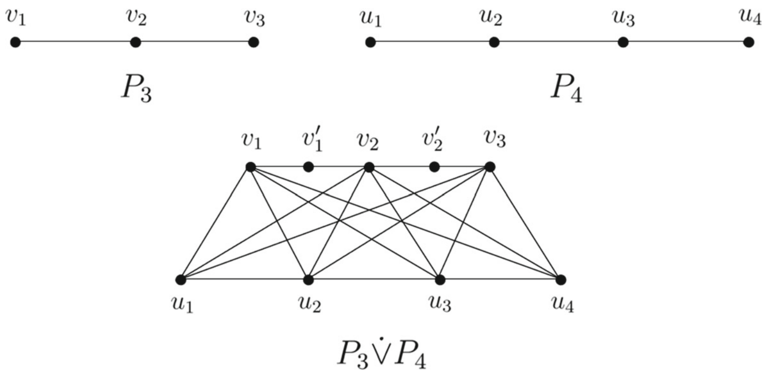

Fig. 1 The subdivision-vertex join of P_3 and P_4

3 A_α-spectrum of $G_1\dot{\vee}G_2$

In this section, we discuss results related to the computation of A_α-spectrum of $G_1\dot{\vee}G_2$, the subdivision-vertex join of the graphs G_1 and G_2. To begin with we obtain an expression for the A_α-characteristic polynomial of $G_1\dot{\vee}G_2$, where G_1 is an r_1-regular graph and G_2 is an arbitrary graph.

Theorem 3.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be a graph on n_2 vertices. Let $\Gamma_{A_\alpha(G_2)}(x)$ be the $A_\alpha(G_2)$-coronal of G_2. Then, for each $\alpha \in [0, 1]$, the A_α-characteristic polynomial of $G_1\dot{\vee}G_2$ is*

$$\begin{aligned}\psi_{A_\alpha(G_1\dot{\vee}G_2)}(x) = {} & (x-2\alpha)^{m_1-n_1}\cdot\psi_{A_\alpha(G_2)}(x-\alpha n_1)\\ & \cdot\prod_{i=2}^{n_1}\Bigg(x^2-\alpha(2+r_1+n_2)x+\alpha(\alpha r_1+r_1+2\alpha n_2)\\ & -(1-\alpha)\Big((1-\alpha)r_1+\lambda_i\big(A_\alpha(G_1)\big)\Big)\Bigg)\\ & \cdot\Bigg(x^2-\alpha(2+r_1+n_2)x-2(r_1-2\alpha r_1-\alpha^2 n_2)\\ & -n_1(1-\alpha)^2(x-2\alpha)\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Bigg).\end{aligned} \tag{7}$$

Proof With respect to the labeling of vertices considered in Remark 2.1, the adjacency matrix of $G_1\dot{\vee}G_2$ is

$$A(G_1\dot{\vee}G_2)=\begin{bmatrix}0_{n_1\times n_1} & R & J_{n_1\times n_2}\\ R^T & 0_{m_1\times m_1} & 0_{m_1\times n_2}\\ J_{n_2\times n_1} & 0_{n_2\times m_1} & A(G_2)\end{bmatrix}, \tag{8}$$

where R is the (0, 1)-incidence matrix of G_1.
The degrees of the vertices of the graph $G_1\dot{\vee}G_2$ are:

$$d_{G_1\dot{\vee}G_2}(v_i) = r_1 + n_2,\ \text{for } i = 1, 2, \dots, n_1;$$
$$d_{G_1\dot{\vee}G_2}(v'_j) = 2,\ \text{for } j = 1, 2, \dots, m_1;$$
$$d_{G_1\dot{\vee}G_2}(u_k) = d_{G_2}(u_k) + n_1,\ \text{for } k = 1, 2, \dots, n_2.$$

So the diagonal matrix of order $(n_1 + m_1 + n_2) \times (n_1 + m_1 + n_2)$, whose diagonal entries are the degrees of the vertices of the graph $G_1\dot{\vee}G_2$ is

$$D(G_1\dot{\vee}G_2) = \begin{bmatrix} (r_1+n_2)I_{n_1} & 0_{n_1\times m_1} & 0_{n_1\times n_2} \\ 0_{m_1\times n_1} & 2I_{m_1} & 0_{m_1\times n_2} \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & D(G_2)+n_1 I_{n_2} \end{bmatrix}. \tag{9}$$

Using (8) and (9), we have

$$A_\alpha(G_1\dot{\vee}G_2) = \begin{bmatrix} \alpha(r_1+n_2)I_{n_1} & (1-\alpha)R & (1-\alpha)J_{n_1\times n_2} \\ (1-\alpha)R^T & 2\alpha I_{m_1} & 0_{m_1\times n_2} \\ (1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} & A_\alpha(G_2)+\alpha n_1 I_{n_2} \end{bmatrix}.$$

Therefore, the characteristic polynomial of $A_\alpha(G_1\dot{\vee}G_2)$ is

$$\begin{aligned} &\psi_{A_\alpha(G_1\dot{\vee}G_2)}(x) \\ &= \det\left(xI_{n_1+m_1+n_2} - A_\alpha(G_1\dot{\vee}G_2)\right) \\ &= \det\begin{bmatrix} \left(x-\alpha(r_1+n_2)\right)I_{n_1} & -(1-\alpha)R & -(1-\alpha)J_{n_1\times n_2} \\ -(1-\alpha)R^T & \left(x-2\alpha\right)I_{m_1} & 0_{m_1\times n_2} \\ -(1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} & (x-\alpha n_1)I_{n_2} - A_\alpha(G_2) \end{bmatrix} \\ &= \det\left((x-\alpha n_1)I_{n_2} - A_\alpha(G_2)\right)\cdot \det S \quad \text{(by Lemma 2.1)}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} S = &\begin{bmatrix} \left(x-\alpha(r_1+n_2)\right)I_{n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & \left(x-2\alpha\right)I_{m_1} \end{bmatrix} \\ &- \begin{bmatrix} -(1-\alpha)J_{n_1\times n_2} \\ 0_{m_1\times n_2} \end{bmatrix} \left((x-\alpha n_1)I_{n_2} - A_\alpha(G_2)\right)^{-1} \begin{bmatrix} -(1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \text{Now, } \det S = \det\Bigg(&\begin{bmatrix} \left(x-\alpha(r_1+n_2)\right)I_{n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & \left(x-2\alpha\right)I_{m_1} \end{bmatrix} \\ &-(1-\alpha)^2 \begin{bmatrix} \Gamma_{A_\alpha(G_2)}(x-\alpha n_1)J_{n_1\times n_1} & 0_{n_1\times m_1} \\ 0_{m_1\times n_1} & 0_{m_1\times m_1} \end{bmatrix}\Bigg) \\ = \det &\begin{bmatrix} \left(x-\alpha(r_1+n_2)\right)I_{n_1} - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)J_{n_1\times n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & \left(x-2\alpha\right)I_{m_1} \end{bmatrix}. \end{aligned}$$

By Lemma 2.1, we have

$$\det S = \det\Big((x-2\alpha)I_{m_1}\Big)$$
$$\cdot \det\Big(\Big(x-\alpha(r_1+n_2)\Big)I_{n_1} - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)J_{n_1\times n_1}$$
$$-\Big((1-\alpha)R\Big)\Big((x-2\alpha)I_{m_1}\Big)^{-1}\Big((1-\alpha)R^T\Big)\Big)$$
$$= (x-2\alpha)^{m_1}\cdot \det\Big(\Big(x-\alpha r_1-\alpha n_2\Big)I_{n_1}$$
$$-(1-\alpha)^2\Big(\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big)J_{n_1\times n_1} - \frac{(1-\alpha)^2}{(x-2\alpha)}RR^T\Big)$$

Now, by Lemma 2.3, we have

$$\det S = (x-2\alpha)^{m_1}\cdot \det\Big(\Big(x-\alpha r_1-\alpha n_2\Big)I_{n_1} - \frac{(1-\alpha)^2}{(x-2\alpha)}RR^T\Big)$$
$$\cdot\Big(1-(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Gamma_{\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T}(x-\alpha r_1-\alpha n_2)\Big). \quad (11)$$

It is clear from Eq. (3) that the eigenvalues of the matrix RR^T are $r_1+\lambda_i(A(G_1))$ for $i=1,2,\ldots,n_1$. Again each row sum of the matrix $\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T$ is $\frac{2r_1(1-\alpha)^2}{(x-2\alpha)}$. Therefore from (5), we have

$$\Gamma_{\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T}(x) = \frac{n_1}{x-\frac{2r_1(1-\alpha)^2}{(x-2\alpha)}}.$$

Thus (11) reduces to

$$\det S = (x-2\alpha)^{m_1}\cdot\prod_{i=1}^{n_1}\Big(x-\alpha r_1-\alpha n_2-\frac{(1-\alpha)^2}{(x-2\alpha)}\Big(r_1+\lambda_i(A(G_1))\Big)\Big)$$
$$\cdot\Big(1-(1-\alpha)^2\frac{n_1(x-2\alpha)}{(x-2\alpha)(x-\alpha r_1-\alpha n_2)-2r_1(1-\alpha)^2}\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big)$$
$$= (x-2\alpha)^{m_1-n_1}\cdot\prod_{i=1}^{n_1}\Big(\Big(x-2\alpha\Big)\Big(x-\alpha r_1-\alpha n_2\Big)-(1-\alpha)^2\Big(r_1+\lambda_i(A(G_1))\Big)\Big)$$
$$\cdot\Big(\frac{(x-2\alpha)(x-\alpha r_1-\alpha n_2)-2r_1(1-\alpha)^2-(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\cdot n_1(x-2\alpha)}{(x-2\alpha)(x-\alpha r_1-\alpha n_2)-2r_1(1-\alpha)^2}\Big)$$
$$= (x-2\alpha)^{m_1-n_1}\cdot\prod_{i=2}^{n_1}\Big(\Big(x-2\alpha\Big)\Big(x-\alpha r_1-\alpha n_2\Big)-(1-\alpha)^2\Big(r_1+\lambda_i(A(G_1))\Big)\Big)$$
$$\cdot\Big((x-2\alpha)(x-\alpha r_1-\alpha n_2)-2r_1(1-\alpha)^2-n_1(1-\alpha)^2(x-2\alpha)$$
$$\cdot\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big).$$

Since $A_\alpha(G_1) = \alpha r_1 I_{n_1} + (1-\alpha)A(G_1)$, the eigenvalues of $A_\alpha(G_1)$ are $\lambda_i(A_\alpha(G_1)) = \alpha r_1 + (1-\alpha)\lambda_i(A(G_1))$ for $i = 1, 2, \ldots, n_1$. Thus

$$\begin{aligned}\det S = &(x-2\alpha)^{m_1-n_1}\\ &\cdot\prod_{i=2}^{n_1}\Big((x-2\alpha)\big(x-\alpha r_1-\alpha n_2\big)-(1-\alpha)^2 r_1-(1-\alpha)\lambda_i\big(A_\alpha(G_1)\big)+\alpha(1-\alpha)r_1\Big)\\ &\cdot\Big((x-2\alpha)(x-\alpha r_1-\alpha n_2)-2r_1(1-\alpha)^2-n_1(1-\alpha)^2(x-2\alpha)\\ &\cdot\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big).\end{aligned}$$

Simplifying this, we get the required result from (10). □

Now, in the following corollary, we obtain the A_α-eigenvalues of $G_1\dot{\vee}G_2$, where G_2 is an r_2-regular graph.

Corollary 3.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1\dot{\vee}G_2$ consists precisely of:*
   (i) $\alpha(1+n_2)$;
   (ii) $2\alpha + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$ *and*
   (iii) *three roots of the equation $F(x) = 0$, where*

$$F(x) = (x-2\alpha-r_2)\big(x^2-\alpha(3+n_2)x-2(1-2\alpha-\alpha^2 n_2)\big)-2n_2(1-\alpha)^2(x-2\alpha).$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1\dot{\vee}G_2$ consists precisely of:*
   (i) 2α, *repeated $m_1 - n_1$ times;*
   (ii) $\alpha n_1 + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$;
   (iii) *two roots of the equation $G_i(x) = 0$ for each $i = 2, 3, \ldots, n_1$, where*

$$\begin{aligned}G_i(x) = x^2 - \alpha(2+r_1+n_2)x + \alpha(\alpha r_1 + r_1 + 2\alpha n_2)\\ -(1-\alpha)\big(r_1(1-\alpha)+\lambda_i\big(A_\alpha(G_1)\big)\big)\end{aligned}$$

   and
   (iv) *three roots of the equation $F(x) = 0$, where*

$$\begin{aligned}F(x) = (x-\alpha n_1 - r_2)\big(x^2-\alpha(2+r_1+n_2)x-2(r_1-2\alpha r_1-\alpha^2 n_2)\big)\\ -n_1 n_2(1-\alpha)^2(x-2\alpha).\end{aligned}$$

Proof Since G_2 is an r_2-regular graph on n_2 vertices, each row sum of the matrix $A_\alpha(G_2)$ is r_2. Therefore from (5), we have

$$\Gamma_{A_\alpha(G_2)}(x-\alpha n_1) = \frac{n_2}{x-\alpha n_1 - r_2}. \quad (12)$$

Again r_2 is an eigenvalue of $A(G_2)$, therefore r_2 is an eigenvalue of $A_\alpha(G_2) = \alpha r_2 I_{n_2} + (1-\alpha)A(G_2)$. Using this and Eq. (12) in (7), we get

$$\begin{aligned}\psi_{A_\alpha(G_1\dot{\vee}G_2)}(x) = (x-2\alpha)^{m_1-n_1}\prod_{i=2}^{n_2}\Big(x-\alpha n_1-\lambda_i\big(A_\alpha(G_2)\big)\Big)\\ \cdot\prod_{i=2}^{n_1}\Big(x^2-\alpha\big(2+r_1+n_2\big)x+\alpha\big(\alpha r_1+r_1+2\alpha n_2\big)\\ -\big(1-\alpha\big)\big(r_1(1-\alpha)+\lambda_i(A_\alpha(G_1))\big)\Big)\\ \cdot\Big(\big(x-\alpha n_1-r_2\big)\big(x^2-\alpha(2+r_1+n_2)x\\ -2(r_1-2\alpha r_1-\alpha^2 n_2)\big)-n_1 n_2(1-\alpha)^2(x-2\alpha)\Big). \quad (13)\end{aligned}$$

1. If $r_1 = 1$, the only possibility for G_1 is P_2, the path on two vertices. In this case, $n_1 = 2$ and $m_1 = 1$. Using these particular values of r_1, n_1 and m_1 in (13), we get the desired result.
2. If $r_1 \geq 2$, the graph G_1 can not have any pendant vertices, so it is not a tree. Thus $m_1 \geq n_1$, and the result follows from (13).

□

Taking G_2 as $K_{p,q}$, we obtain the A_α-eigenvalues of $G_1\dot{\vee}G_2$ in the next corollary.

Corollary 3.2 *Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges. Let $p, q \geq 1$ be integers and $G_2 = K_{p,q}$.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1\dot{\vee}G_2$ consists precisely of:*
   (i) $\alpha(1+p+q)$;
   (ii) $\alpha(p+2)$, *repeated* $q-1$ *times;*
   (iii) $\alpha(q+2)$, *repeated* $p-1$ *times and*
   (iv) *four roots of the equation* $F(x) = 0$, *where*

$$\begin{aligned}F(x) = \big(x^2-\alpha(3+p+q)x-2(1-2\alpha-\alpha^2 p-\alpha^2 q)\big)\\ \cdot\big(x^2-\alpha(4+p+q)x+(4\alpha^2+2\alpha^2 p+2\alpha^2 q+2\alpha pq-pq)\big)\\ -2(1-\alpha)^2(x-2\alpha)\big((x-2\alpha)(p+q)-\alpha(p+q)^2+2pq\big).\end{aligned}$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1\dot{\vee}G_2$ consists precisely of:*
   (i) 2α, *repeated* m_1-n_1 *times;*
   (ii) $\alpha(n_1+p)$, *repeated* $q-1$ *times;*
   (iii) $\alpha(n_1+q)$, *repeated* $p-1$ *times;*
   (iv) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$G_i(x) = x^2-\alpha(2+r_1+p+q)x+\alpha(\alpha r_1+r_1+2\alpha p+2\alpha q)$$

$$-(1-\alpha)\big((1-\alpha)r_1+\lambda_i\big(A_\alpha(G_1)\big)\big)$$

and

(v) *four roots of the equation* $F(x)=0$, *where*

$$\begin{aligned}F(x)=&\big(x^2-\alpha(2+r_1+p+q)x-2(r_1-2\alpha r_1-\alpha^2 p-\alpha^2 q)\big)\\&\cdot\big(x^2-\alpha(2n_1+p+q)x+(\alpha^2 n_1^2+\alpha^2 n_1 p+\alpha^2 n_1 q+2\alpha pq-pq)\big)\\&-n_1(1-\alpha)^2(x-2\alpha)\big((x-\alpha n_1)(p+q)-\alpha(p+q)^2+2pq\big).\end{aligned}$$

Proof From Lemmas 2.5 and 2.6, we have $\sigma(A_\alpha(G_2))=\left\{\frac{\alpha(p+q)+\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2},[\alpha p]^{q-1},[\alpha q]^{p-1},\frac{\alpha(p+q)-\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}\right\}$ and

$$\Gamma_{A_\alpha(G_2)}(x)=\frac{(p+q)x-\alpha(p+q)^2+2pq}{x^2-\alpha(p+q)x+(2\alpha-1)pq}.$$

Using this in (7), we have

$$\begin{aligned}&\psi_{A_\alpha(G_1\dot{\vee}G_2)}(x)\\&=\big(x-2\alpha\big)^{m_1-n_1}\\&\quad\cdot\left(x-\alpha n_1-\frac{\alpha(p+q)+\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}\right)\\&\quad\big(x-\alpha n_1-\alpha p\big)^{q-1}\\&\quad\cdot\big(x-\alpha n_1-\alpha q\big)^{p-1}\left(x-\alpha n_1-\frac{\alpha(p+q)-\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}\right)\\&\quad\cdot\prod_{i=2}^{n_1}\Big(x^2-\alpha\big(2+r_1+p+q\big)x+\alpha\big(\alpha r_1+r_1+2\alpha(p+q)\big)\\&\quad-(1-\alpha)\big((1-\alpha)r_1+\lambda_i\big(A_\alpha(G_1)\big)\big)\Big)\\&\quad\cdot\Big(x^2-\alpha\big(2+r_1+p+q\big)x-2\big(r_1-2\alpha r_1-\alpha^2(p+q)\big)\\&\quad-n_1\big(1-\alpha\big)^2\big(x-2\alpha\big)\cdot\frac{(p+q)(x-\alpha n_1)-\alpha(p+q)^2+2pq}{(x-\alpha n_1)^2-\alpha(p+q)(x-\alpha n_1)+(2\alpha-1)pq}\Big).\end{aligned}\tag{14}$$

The zeros of the denominator of $\frac{(p+q)(x-\alpha n_1)-\alpha(p+q)^2+2pq}{(x-\alpha n_1)^2-\alpha(p+q)(x-\alpha n_1)+(2\alpha-1)pq}$ are

$$\alpha n_1+\frac{\alpha(p+q)+\sqrt{\alpha^2(p+q)^2+4pq(1-2\alpha)}}{2}\text{ and}$$

$$\alpha n_1 + \frac{\alpha(p+q) - \sqrt{\alpha^2(p+q)^2 + 4pq(1-2\alpha)}}{2}.$$

Using this in (14), we get

$$\begin{aligned}
&\psi_{A_\alpha(G_1\dot{\vee}G_2)}(x) \\
&\quad= \big(x-2\alpha\big)^{m_1-n_1}\big(x-\alpha n_1-\alpha p\big)^{q-1}\big(x-\alpha n_1-\alpha q\big)^{p-1} \\
&\quad\cdot\prod_{i=2}^{n_1}\Big(x^2-\alpha\big(2+r_1+p+q\big)x+\alpha\big(\alpha r_1+r_1+2\alpha(p+q)\big) \\
&\quad-\big(1-\alpha\big)\Big((1-\alpha)r_1+\lambda_i\big(A_\alpha(G_1)\big)\Big)\Big) \\
&\quad\cdot\Big(\Big(x^2-\alpha\big(2+r_1+p+q\big)x-2\big(r_1-2\alpha r_1-\alpha^2(p+q)\big)\Big) \\
&\cdot\Big(x^2-(2\alpha n_1+\alpha p+\alpha q)x+(\alpha^2 {n_1}^2+\alpha^2 n_1 p+\alpha^2 n_1 q+2\alpha pq-pq)\Big) \\
&\quad-n_1\big(1-\alpha\big)^2\big(x-2\alpha\big)\Big((p+q)(x-\alpha n_1)-\alpha(p+q)^2+2pq\Big)\Big).
\end{aligned} \tag{15}$$

1. If $r_1 = 1$, then $G_1 = P_2$. Thus $n_1 = 2$ and $m_1 = 1$. Using these particular values of r_1, n_1 and m_1 in (15), we get the desired result.
2. If $r_1 \geq 2$, the graph G_1 is not a tree. Then we have $m_1 \geq n_1$, and the result follows from (15) .

□

Finally, to conclude this section, we provide a construction of new pairs of A_α-cospectral graphs from a given pair of A_α-cospectral graphs in the following corollary.

Corollary 3.3

1. *Let G_1 and G_2 be two A_α-cospectral regular graphs for $\alpha \in [0, 1]$, and let H be an arbitrary graph. Then the graphs $G_1\dot{\vee}H$ and $G_2\dot{\vee}H$ are A_α-cospectral.*
2. *Let H_1 and H_2 be two A_α-cospectral graphs with $\Gamma_{A_\alpha(H_1)}(x) = \Gamma_{A_\alpha(H_2)}(x)$ for $\alpha \in [0, 1]$. If G is a regular graph, then the graphs $G\dot{\vee}H_1$ and $G\dot{\vee}H_2$ are A_α-cospectral.*

Proof If two regular graphs are A_α-cospectral, then they have same regularity with same number of vertices and same number of edges. By applying Theorem 3.1 on the concerned graphs and comparing their A_α-characteristic polynomials, we get the required results. □

4 A_α-spectrum of $G_1\underline{\vee}G_2$

This section is about the A_α-spectrum of $G_1\underline{\vee}G_2$, the subdivision-edge join of the graphs G_1 and G_2. We start by obtaining an expression for the A_α-characteristic polynomial of $G_1\underline{\vee}G_2$ for an r_1-regular graph G_1 and an arbitrary graph G_2.

Theorem 4.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be a graph on n_2 vertices. Let $\Gamma_{A_\alpha(G_2)}(x)$ be the $A_\alpha(G_2)$-coronal of G_2.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-characteristic polynomial of $G_1 \underline{\vee} G_2$ is given by,*

$$\begin{aligned}\psi_{A_\alpha(G_1\underline{\vee}G_2)}(x) = (x-\alpha)\cdot\psi_{A_\alpha(G_2)}(x-\alpha)\\ \cdot\Big(x^2-\alpha(3+n_2)x+(\alpha^2 n_2+4\alpha-2)\\ -(1-\alpha)^2(x-\alpha)\Gamma_{A_\alpha(G_2)}(x-\alpha)\Big).\end{aligned}$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-characteristic polynomial of $G_1 \underline{\vee} G_2$ is given by,*

$$\begin{aligned}&\psi_{A_\alpha(G_1\underline{\vee}G_2)}(x)\\ &= \big(x-2\alpha-\alpha n_2\big)^{m_1-n_1}\cdot\psi_{A_\alpha(G_2)}(x-\alpha m_1)\\ &\cdot\Big(x^2-\alpha(2+r_1+n_2)x+r_1(\alpha^2 n_2+4\alpha-2)\\ &-m_1(1-\alpha)^2(x-\alpha r_1)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\cdot\prod_{i=2}^{n_1}\Big(x^2-\alpha(2+r_1+n_2)x+r_1(\alpha^2 n_2+3\alpha-1)-(1-\alpha)\lambda_i\big(A_\alpha(G_1)\big)\Big).\end{aligned}$$

Proof Consider the labeling of the graph $G_1 \underline{\vee} G_2$ given in Remark 2.1. Then, the adjacency matrix of $G_1 \underline{\vee} G_2$ is

$$A(G_1\underline{\vee}G_2) = \begin{bmatrix} 0_{n_1\times n_1} & R & 0_{n_1\times n_2}\\ R^T & 0_{m_1\times m_1} & J_{m_1\times n_2}\\ 0_{n_2\times n_1} & J_{n_2\times m_1} & A(G_2)\end{bmatrix}, \tag{16}$$

where R is the (0, 1)-incidence matrix of G_1.

The degrees of the vertices of the graph $G_1 \underline{\vee} G_2$ are:

$$\begin{aligned}d_{G_1\underline{\vee}G_2}(v_i) &= r_1, \text{ for } i = 1, 2, \ldots, n_1;\\ d_{G_1\underline{\vee}G_2}(v'_j) &= 2+n_2, \text{ for } j = 1, 2, \ldots, m_1;\\ d_{G_1\underline{\vee}G_2}(u_k) &= d_{G_2}(u_k)+m_1, \text{ for } k = 1, 2, \ldots, n_2.\end{aligned}$$

So the diagonal matrix with diagonal entries are the degrees of the vertices of the graph $G_1 \underline{\vee} G_2$ is

$$D(G_1\underline{\vee}G_2) = \begin{bmatrix} r_1 I_{n_1} & 0_{n_1\times m_1} & 0_{n_1\times n_2}\\ 0_{m_\times n_1} & (2+n_2)I_{m_1} & 0_{m_1\times n_2}\\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & D(G_2)+m_1 I_{n_2}\end{bmatrix}. \tag{17}$$

Using (16) and (17), we get the A_α-matrix of $G_1 \underline{\vee} G_2$ as

$$A_\alpha(G_1 \underline{\vee} G_2) = \begin{bmatrix} \alpha r_1 I_{n_1} & (1-\alpha)R & 0_{n_1 \times n_2} \\ (1-\alpha)R^T & (2+n_2)\alpha I_{m_1} & (1-\alpha)J_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & (1-\alpha)J_{n_2 \times m_1} & A_\alpha(G_2) + \alpha m_1 I_{n_2} \end{bmatrix}.$$

Therefore, the characteristic polynomial of $A_\alpha(G_1 \underline{\vee} G_2)$ is

$$\begin{aligned} &\psi_{A_\alpha(G_1 \underline{\vee} G_2)}(x) \\ &\quad = \det\Big(x I_{n_1+m_1+n_2} - A_\alpha(G_1 \underline{\vee} G_2)\Big) \\ &\quad = \det \begin{bmatrix} \Big(x - \alpha r_1\Big) I_{n_1} & -(1-\alpha)R & 0_{n_1 \times n_2} \\ -(1-\alpha)R^T & \Big(x - 2\alpha - \alpha n_2\Big) I_{m_1} & -(1-\alpha)J_{m_1 \times n_2} \\ 0_{n_2 \times n_1} & -(1-\alpha)J_{n_2 \times m_1} & (x - \alpha m_1)I_{n_2} - A_\alpha(G_2) \end{bmatrix} \qquad (18) \\ &\quad = \det\Big((x - \alpha m_1)I_{n_2} - A_\alpha(G_2)\Big) \cdot \det S \quad \text{(by Lemma 2.1)}, \end{aligned}$$

where

$$\begin{aligned} S = &\begin{bmatrix} (x-\alpha r_1)I_{n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & (x - 2\alpha - \alpha n_2)I_{m_1} \end{bmatrix} \\ &- \begin{bmatrix} 0_{n_1 \times n_2} \\ -(1-\alpha)J_{m_1 \times n_2} \end{bmatrix} \Big((x - \alpha m_1)I_{n_2} - A_\alpha(G_2)\Big)^{-1} \big[0_{n_2 \times n_1} \; -(1-\alpha)J_{n_2 \times m_1}\big] \\ = &\begin{bmatrix} (x-\alpha r_1)I_{n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & (x - 2\alpha - \alpha n_2)I_{m_1} - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}\big(x - \alpha m_1\big) J_{m_1 \times m_1} \end{bmatrix}. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} \det S = &\det\Big((x - \alpha r_1)I_{n_1}\Big) \\ &\cdot \det\Big((x - 2\alpha - \alpha n_2)I_{m_1} - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x - \alpha m_1)J_{m_1 \times m_1} - \frac{(1-\alpha)^2}{(x - \alpha r_1)} R^T R\Big) \\ = &(x - \alpha r_1)^{n_1} \cdot \det\Big((x - 2\alpha - \alpha n_2)I_{m_1} - \frac{(1-\alpha)^2}{(x-\alpha r_1)} R^T R\Big) \\ &\cdot \Big(1 - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x - \alpha m_1) \Gamma_{\frac{(1-\alpha)^2}{(x-\alpha r_1)} R^T R}(x - 2\alpha - \alpha n_2)\Big), \\ &\text{(using Lemma 2.3)}. \qquad (19) \end{aligned}$$

The line graph of an r-regular graph is a $(2r-2)$-regular graph. Using this in (2) for the graph G_1, we get each row sum of $R^T R$ is $2r_1$. Therefore using (5), we get

$$\Gamma_{\frac{(1-\alpha)^2}{(x-\alpha r_1)} R^T R}(x - 2\alpha - \alpha n_2) = \frac{m_1}{x - 2\alpha - \alpha n_2 - \frac{2r_1(1-\alpha)^2}{(x-\alpha r_1)}}.$$

Using these information, (19) becomes

$$\det S = (x-\alpha r_1)^{n_1} \cdot \prod_{i=1}^{m_1} \Big(x - 2\alpha - \alpha n_2 - \frac{(1-\alpha)^2}{(x-\alpha r_1)} \lambda_i\big(A\big(\mathcal{L}(G_1)\big)\big) - 2\frac{(1-\alpha)^2}{(x-\alpha r_1)}\Big) \cdot \left(1 - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha m_1) \cdot \frac{m_1(x-\alpha r_1)}{(x-\alpha r_1)(x-2\alpha-\alpha n_2) - 2r_1(1-\alpha)^2}\right), \tag{20}$$

where $\mathcal{L}(G_1)$ is the line graph of G_1.

1. If $r_1 = 1$, then $G_1 = P_2$. Therefore, $\mathcal{L}(G_1)$ is just a vertex and 0 is the only eigenvalue of $A(\mathcal{L}(G_1))$. Thus (20) becomes

$$\begin{aligned}\det S &= (x-\alpha)^2 \cdot \Big(x - 2\alpha - \alpha n_2 - 2\frac{(1-\alpha)^2}{(x-\alpha)}\Big) \cdot \left(1 - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha) \cdot \frac{(x-\alpha)}{(x-\alpha)(x-2\alpha-\alpha n_2) - 2(1-\alpha)^2}\right) \\ &= (x-\alpha)\Big((x-\alpha)(x-2\alpha-\alpha n_2) - 2(1-\alpha)^2 - (1-\alpha)^2(x-\alpha)\Gamma_{A_\alpha(G_2)}(x-\alpha)\Big).\end{aligned}$$

After simplifying this, we get the required result from (18).

2. If $r_1 \geq 2$, then $m_1 \geq n_1$. By Lemma 2.4 on G_1, the eigenvalues of $A(\mathcal{L}(G_1))$ are $\lambda_i(A(G_1)) + r_1 - 2$ for $i = 1, 2, \ldots, n_1$ and -2, repeated for $(m_1 - n_1)$ times. Using these information, (20) becomes

$$\begin{aligned}\det S &= (x-\alpha r_1)^{n_1} \cdot \frac{1}{(x-\alpha r_1)^{m_1}} \cdot \prod_{i=1}^{n_1} \Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) - (1-\alpha)^2\big(\lambda_i(A(G_1)) + r_1 - 2\big)\big) - 2(1-\alpha)^2\Big) \\ &\cdot \prod_{n_1+1}^{m_1} \Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) - (1-\alpha)^2(-2) - 2(1-\alpha)^2\Big) \\ &\cdot \frac{(x-\alpha r_1)(x-2\alpha-\alpha n_2) - 2r_1(1-\alpha)^2 - m_1(1-\alpha)^2(x-\alpha r_1)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-\alpha r_1)(x-2\alpha-\alpha n_2) - 2r_1(1-\alpha)^2} \\ &= (x-2\alpha-\alpha n_2)^{m_1-n_1} \cdot \prod_{i=2}^{n_1} \Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) - (1-\alpha)^2\big(\lambda_i(A(G_1)) + r_1\big)\Big) \\ &\cdot \Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) - 2r_1(1-\alpha)^2 - m_1(1-\alpha)^2(x-\alpha r_1)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big).\end{aligned} \tag{21}$$

Since G_1 is an r_1-regular graph, therefore

$$A_\alpha(G_1) = \alpha r_1 I_{n_1} + (1-\alpha)A(G_1).$$

Thus

$$\lambda_i\big(A_\alpha(G_1)\big) = \alpha r_1 + (1-\alpha)\lambda_i\big(A(G_1)\big) \text{ for } i = 1, 2, \ldots, n_1,$$

and hence,

$$\lambda_i\big(A(G_1)\big) = \frac{1}{(1-\alpha)}\Big(\lambda_i\big(A_\alpha(G_1)\big) - \alpha r_1\Big) \text{ for } i = 1, 2, \ldots, n_1. \tag{22}$$

Now replace $\lambda_i\big(A(G_1)\big)$ in (21) using (22). After simplifying that, we get the desired result from (18).

□

Now, in the following corollary, we obtain the A_α-eigenvalues of $G_1 \underline{\vee} G_2$ taking G_2 as an r_2-regular graph.

Corollary 4.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \underline{\vee} G_2$ consists precisely of:*
   - (i) α;
   - (ii) $\alpha + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$ *and*
   - (iii) *three roots of the equation* $F(x) = 0$, *where*

$$F(x) = (x-\alpha-r_2)\big(x^2-\alpha(3+n_2)x+(\alpha^2 n_2+4\alpha-2)\big)-n_2(1-\alpha)^2(x-\alpha).$$

(2) *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \underline{\vee} G_2$ consists precisely of:*
   - (i) $\alpha(2+n_2)$, *repeated* $m_1 - n_1$ *times;*
   - (ii) $\alpha m_1 + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$;
   - (iii) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$G_i(x) = x^2 - \alpha(2+r_1+n_2)x + (\alpha^2 r_1 n_2 + 3\alpha r_1 - r_1) - (1-\alpha)\lambda_i\big(A_\alpha(G_1)\big)$$

   and
   - (iv) *three roots of the equation* $F(x) = 0$, *where*

$$F(x) = (x-\alpha m_1 - r_2)\big(x^2 - \alpha(2+r_1+n_2)x + r_1(\alpha^2 n_2 + 4\alpha - 2)\big) - m_1 n_2(1-\alpha)^2(x-\alpha r_1).$$

Proof Proof is similar to that of Corollary 3.1. □

Taking G_2 as $K_{p,q}$, we obtain the A_α-eigenvalues of $G_1 \underline{\vee} G_2$ in the next corollary.

Corollary 4.2 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges. Let $p, q \geq 1$ be integers and $G_2 = K_{p,q}$.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \underline{\vee} G_2$ consists precisely of:*

   (i) α;
   (ii) $\alpha(1 + p)$, *repeated* $q - 1$ *times;*
   (iii) $\alpha(1 + q)$, *repeated* $p - 1$ *times and*
   (iv) *four roots of the equation* $F(x) = 0$, *where*

$$\begin{aligned} F(x) = &\big(x^2 - \alpha(3 + p + q)x + (\alpha^2 p + \alpha^2 q + 4\alpha - 2)\big) \\ &\cdot\big(x^2 - \alpha(2 + p + q)x + (\alpha^2 + \alpha^2 p + \alpha^2 q + 2\alpha pq - pq)\big) \\ &-(1 - \alpha)^2 (x - \alpha)\big((x - \alpha)(p + q) - \alpha(p + q)^2 + 2pq\big). \end{aligned}$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \underline{\vee} G_2$ consists precisely of:*

   (i) $\alpha(2 + p + q)$, *repeated* $m_1 - n_1$ *times;*
   (ii) $\alpha(m_1 + p)$, *repeated* $q - 1$ *times;*
   (iii) $\alpha(m_1 + q)$, *repeated* $p - 1$ *times;*
   (iv) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$\begin{aligned} G_i(x) = &\, x^2 - \alpha(2 + r_1 + p + q)x + (\alpha^2 r_1 p + \alpha^2 r_1 q + 3\alpha r_1 - r_1) \\ &-(1 - \alpha)\lambda_i\big(A_\alpha(G_1)\big) \end{aligned}$$

   and
   (v) *four roots of the equation* $F(x) = 0$, *where*

$$\begin{aligned} F(x) = &\big(x^2 - \alpha(2 + r_1 + p + q)x + r_1(\alpha^2 p + \alpha^2 q + 4\alpha - 2)\big) \\ &\cdot\big(x^2 - \alpha(2m_1 + p + q)x + (\alpha^2 m_1^2 \\ &+\alpha^2 m_1 p + \alpha^2 m_1 q + 2\alpha pq - pq)\big) \\ &-m_1(1 - \alpha)^2 (x - \alpha r_1)\big((x - \alpha m_1)(p + q) - \alpha(p + q)^2 + 2pq\big). \end{aligned}$$

Proof Proof is similar to that of Corollary 3.2. □

Finally, to conclude this section, we provide a construction of new pairs of A_α-cospectral graphs from a given pair of A_α-cospectral graphs in the following corollary.

Corollary 4.3

1. *Let G_1 and G_2 be two A_α-cospectral regular graphs for $\alpha \in [0, 1]$, and let H be an arbitrary graph. Then the graphs $G_1 \underline{\vee} H$ and $G_2 \underline{\vee} H$ are A_α-cospectral.*
2. *Let H_1 and H_2 be two A_α-cospectral graphs with $\Gamma_{A_\alpha(H_1)}(x) = \Gamma_{A_\alpha(H_2)}(x)$ for $\alpha \in [0, 1]$. If G is a regular graph, then the graphs $G \underline{\vee} H_1$ and $G \underline{\vee} H_2$ are A_α-cospectral.*

Proof Proof is similar to that of Corollary 3.3. □

5 A_α-spectrum of $G_1\langle \mathrm{v}\rangle G_2$

In this section, we study about the A_α-spectrum of $G_1\langle \mathrm{v}\rangle G_2$, the R-vertex join of the graphs G_1 and G_2. We start with obtaining the expression of the A_α-characteristic polynomial of $G_1\langle \mathrm{v}\rangle G_2$, for an r_1-regular graph G_1 and an arbitrary graph G_2, in the following theorem.

Theorem 5.1 *Let G_1 be an r_1-regular graph on n_1 and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Let $\Gamma_{A_\alpha(G_2)}(x)$ be the $A_\alpha(G_2)$-coronal of G_2. Then for each $\alpha \in [0, 1]$, the A_α-characteristic polynomial of $G_1\langle \mathrm{v}\rangle G_2$ is given by,*

$$\begin{aligned}\psi_{A_\alpha(G_1\langle \mathrm{v}\rangle G_2)}(x) = {} & \big(x-2\alpha\big)^{m_1-n_1}\cdot \psi_{A_\alpha(G_2)}(x-\alpha n_1)\\ & \cdot\prod_{i=2}^{n_1}\Big(x^2-\big(2\alpha+\alpha r_1+\alpha n_2+\lambda_i\big(A_\alpha(G_1)\big)\big)x\\ & +2\alpha^2 n_2+3\alpha r_1-r_1+(3\alpha-1)\lambda_i\big(A_\alpha(G_1)\big)\Big)\\ & \cdot\Big(x^2-(2\alpha+\alpha r_1+\alpha n_2+r_1)x+2\alpha^2 n_2+6\alpha r_1-2r_1\\ & -n_1(1-\alpha)^2(x-2\alpha)\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big).\end{aligned}$$

Proof The adjacency matrix of $G_1\langle \mathrm{v}\rangle G_2$ is

$$A(G_1\langle \mathrm{v}\rangle G_2) = \begin{bmatrix} A(G_1) & R & J_{n_1\times n_2}\\ R^T & 0_{m_1\times m_1} & 0_{m_1\times n_2}\\ J_{n_2\times n_1} & 0_{n_2\times m_1} & A(G_2)\end{bmatrix}, \tag{23}$$

where R is the (0, 1)-incidence matrix of G_1.

The degrees of the vertices of the graph $G_1\langle \mathrm{v}\rangle G_2$ are:

$$\begin{aligned} d_{G_1\langle \mathrm{v}\rangle G_2}(v_i) &= 2r_1+n_2,\ \text{for } i=1,2,\ldots,n_1;\\ d_{G_1\langle \mathrm{v}\rangle G_2}(v'_j) &= 2,\ \text{for } j=1,2,\ldots,m_1;\\ d_{G_1\langle \mathrm{v}\rangle G_2}(u_k) &= d_{G_2}(u_k)+n_1,\ \text{for } k=1,2,\ldots,n_2.\end{aligned}$$

Therefore,

$$D(G_1\langle \mathrm{v}\rangle G_2) = \begin{bmatrix} (2r_1+n_2)I_{n_1} & 0_{n_1\times m_1} & 0_{n_1\times n_2}\\ 0_{m_1\times n_1} & 2I_{m_1} & 0_{m_1\times n_2}\\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & D(G_2)+n_1 I_{n_2}\end{bmatrix}. \tag{24}$$

Using (23) and (24), we get the A_α-matrix of $G_1\langle \text{v}\rangle G_2$ as

$$A_\alpha(G_1\langle \text{v}\rangle G_2) = \begin{bmatrix} A_\alpha(G_1) + (\alpha r_1 + \alpha n_2)I_{n_1} & (1-\alpha)R & (1-\alpha)J_{n_1\times n_2} \\ (1-\alpha)R^T & 2\alpha I_{m_1} & 0_{m_1\times n_2} \\ (1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} & A_\alpha(G_2) + \alpha n_1 I_{n_2} \end{bmatrix}.$$

Therefore, the characteristic polynomial of $A_\alpha(G_1\langle \text{v}\rangle G_2)$ is

$$\begin{aligned} &\psi_{A_\alpha(G_1\langle \text{v}\rangle G_2)}(x) \\ &\quad = \det\Big(xI_{n_1+m_1+n_2} - A_\alpha(G_1\langle \text{v}\rangle G_2)\Big) \\ &\quad = \det\begin{bmatrix} (x - \alpha r_1 - \alpha n_2)I_{n_1} - A_\alpha(G_1) & -(1-\alpha)R & -(1-\alpha)J_{n_1\times n_2} \\ -(1-\alpha)R^T & (x-2\alpha)I_{m_1} & 0_{m_1\times n_2} \\ -(1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} & (x-\alpha n_1)I_{n_2} - A_\alpha(G_2) \end{bmatrix} \\ &\quad = \det\Big((x-\alpha n_1)I_{n_2} - A_\alpha(G_2)\Big)\cdot \det S \quad \text{(by Lemma 2.1)}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} S &= \begin{bmatrix} (x - \alpha r_1 - \alpha n_2)I_{n_1} - A_\alpha(G_1) & -(1-\alpha)R \\ -(1-\alpha)R^T & (x-2\alpha)I_{m_1} \end{bmatrix} \\ &\quad - \begin{bmatrix} -(1-\alpha)J_{n_1\times n_2} \\ 0_{m_1\times n_2} \end{bmatrix}\Big((x-\alpha n_1)I_{n_2} - A_\alpha(G_2)\Big)^{-1}\begin{bmatrix} -(1-\alpha)J_{n_2\times n_1} & 0_{n_2\times m_1} \end{bmatrix} \\ &= \begin{bmatrix} (x - \alpha r_1 - \alpha n_2)I_{n_1} - A_\alpha(G_1) - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)J_{n_1\times n_1} & -(1-\alpha)R \\ -(1-\alpha)R^T & (x-2\alpha)I_{m_1} \end{bmatrix}. \end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned} \det S &= \det\Big((x-2\alpha)I_{m_1}\Big) \\ &\qquad \cdot \det\Big((x - \alpha r_1 - \alpha n_2)I_{n_1} - A_\alpha(G_1) - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)J_{n_1\times n_1} \\ &\qquad - \frac{(1-\alpha)^2}{(x-2\alpha)}RR^T\Big) \\ &= (x-2\alpha)^{m_1}\cdot \det\Big((x - \alpha r_1 - \alpha n_2)I_{n_1} - A_\alpha(G_1) \\ &\qquad - \frac{(1-\alpha)^2}{(x-2\alpha)}RR^T\Big) \\ &\qquad \cdot\Big(1 - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Gamma_{A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T}(x-\alpha r_1 - \alpha n_2)\Big), \\ &\qquad\qquad \text{by Lemma 2.3.} \end{aligned} \tag{26}$$

Now using (1) in (3) for the graph G_1, we get

$$RR^T = \frac{1}{(1-\alpha)}\Big(A_\alpha(G_1) - \alpha r_1 I_{n_1}\Big) + r_1 I_{n_1}.$$

Thus

$$A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T=\frac{1}{(x-2\alpha)}\Big((x-3\alpha+1)A_\alpha(G_1)+(r_1-3\alpha r_1+2\alpha^2 r_1)I_{n_1}\Big). \tag{27}$$

Again, each row sum of the matrix $A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T$ is $r_1+\frac{(1-\alpha)^2}{(x-2\alpha)}2r_1$.
Therefore from (5), we have

$$\Gamma_{A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha)}RR^T}(x-\alpha r_1-\alpha n_2)=\frac{n_1}{x-\alpha r_1-\alpha n_2-\Big(r_1+\frac{(1-\alpha)^2}{(x-2\alpha)}2r_1\Big)}. \tag{28}$$

Applying (27) and (28) in (26) and simplifying, we get

$$\begin{aligned}\det S=&(x-2\alpha)^{m_1-n_1}\\ &\cdot\prod_{i=2}^{n_1}\Big(x^2-\big(2\alpha+\alpha r_1+\alpha n_2+\lambda_i\big(A_\alpha(G_1)\big)\big)x\\ &+2\alpha^2 n_2+3\alpha r_1-r_1+(3\alpha-1)\lambda_i\big(A_\alpha(G_1)\big)\Big)\\ &\cdot\Big(x^2-(2\alpha+\alpha r_1+\alpha n_2+r_1)x+2\alpha^2 n_2+6\alpha r_1\\ &-2r_1-n_1(1-\alpha)^2(x-2\alpha)\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\Big).\end{aligned} \tag{29}$$

Finally, using (29) in (25), we get the required result. □

Now, in the following corollary, we obtain the A_α-eigenvalues of $G_1\langle \mathrm{v}\rangle G_2$ taking G_2 as an r_2-regular graph.

Corollary 5.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices.*

1. *If $r_1=1$, then for each $\alpha\in[0,1]$, the A_α-spectrum of $G_1\langle \mathrm{v}\rangle G_2$ consists precisely of:*
   (i) $\alpha(3+n_2)-1$;
   (ii) $2\alpha+\lambda_i\big(A_\alpha(G_2)\big)$, $i=2,3,\ldots,n_2$ *and*
   (iii) *three roots of the equation $F(x)=0$, where*

$$\begin{aligned}F(x)=&(x-2\alpha-r_2)\big(x^2-(3\alpha+\alpha n_2+1)x\\ &+2\alpha^2 n_2+6\alpha-2\big)-2n_2(1-\alpha)^2(x-2\alpha).\end{aligned}$$

2. *If $r_1\geq 2$, then for each $\alpha\in[0,1]$, the A_α-spectrum of $G_1\langle \mathrm{v}\rangle G_2$ consists precisely of:*
   (i) 2α, *repeated m_1-n_1 times;*
   (ii) $\alpha n_1+\lambda_i\big(A_\alpha(G_2)\big)$, $i=2,3,\ldots,n_2$;

(iii) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$G_i(x) = x^2 - \Big(2\alpha + \alpha r_1 + \alpha n_2 + \lambda_i\big(A_\alpha(G_1)\big)\Big)x$$
$$+2\alpha^2 n_2 + 3\alpha r_1 - r_1 + (3\alpha - 1)\lambda_i\big(A_\alpha(G_1)\big)$$

and

(iv) *three roots of the equation* $F(x) = 0$, *where*

$$F(x) = (x - \alpha n_1 - r_2)\Big(x^2 - (2\alpha + \alpha r_1 + \alpha n_2$$
$$+r_1)x + (2\alpha^2 n_2 + 6\alpha r_1 - 2r_1)\Big)$$
$$-n_1 n_2(1-\alpha)^2(x - 2\alpha).$$

Taking G_2 as $K_{p,q}$, we obtain the A_α-eigenvalues of $G_1\langle \mathrm{v}\rangle G_2$ in the next corollary.

Corollary 5.2 *Let* G_1 *be an* r_1*-regular graph on* n_1 *vertices and* m_1 *edges. Let* $p, q \geq 1$ *be integers and* $G_2 = K_{p,q}$.

1. *If* $r_1 = 1$, *then for each* $\alpha \in [0, 1]$, *the* A_α*-spectrum of* $G_1\langle \mathrm{v}\rangle G_2$ *consists precisely of:*

   (i) $\alpha(3 + p + q) - 1$;
   (ii) $\alpha(2 + p)$, *repeated* $q - 1$ *times;*
   (iii) $\alpha(2 + q)$, *repeated* $p - 1$ *times and*
   (iv) *four roots of the equation* $F(x) = 0$, *where*

$$F(x) = \big(x^2 - (3\alpha + \alpha p + \alpha q + 1)x + 2\alpha^2 p + 2\alpha^2 q + 6\alpha - 2\big)$$
$$\cdot\big(x^2 - (4\alpha + \alpha p + \alpha q)x + 4\alpha^2 + 2\alpha^2 p + 2\alpha^2 q + 2\alpha pq - pq\big)$$
$$-2(1-\alpha)^2(x - 2\alpha)\big((x - 2\alpha)(p + q) - \alpha(p + q)^2 + 2pq\big).$$

2. *If* $r_1 \geq 2$, *then for each* $\alpha \in [0, 1]$, *the* A_α*-spectrum of* $G_1\langle \mathrm{v}\rangle G_2$ *consists precisely of:*

   (i) 2α, *repeated* $m_1 - n_1$ *times;*
   (ii) $\alpha(n_1 + p)$, *repeated* $q - 1$ *times;*
   (iii) $\alpha(n_1 + q)$, *repeated* $p - 1$ *times;*
   (iv) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$G_i(x) = x^2 - \Big(2\alpha + \alpha r_1 + \alpha p + \alpha q + \lambda_i\big(A_\alpha(G_1)\big)\Big)x$$
$$+2\alpha^2 p + 2\alpha^2 q + 3\alpha r_1 - r_1 + (3\alpha - 1)\lambda_i\big(A_\alpha(G_1)\big)$$

   and

   (v) *four roots of the equation* $F(x) = 0$, *where*

$$F(x) = \Big(x^2 - (2\alpha + \alpha r_1 + \alpha p + \alpha q + r_1)x + 2\alpha^2 p + 2\alpha^2 q + 6\alpha r_1 - 2r_1\Big)$$

$$\cdot\Big(x^2-(2\alpha n_1+\alpha p+\alpha q)x+\alpha^2n_1^2+\alpha^2n_1p+\alpha^2n_1q+2\alpha pq-pq\Big)$$
$$-n_1(1-\alpha)^2(x-2\alpha)\Big((x-\alpha n_1)(p+q)-\alpha(p+q)^2+2pq\Big).$$

Finally, to conclude this section, we provide a construction of new pair of A_α-cospectral graphs from a given pair of A_α-cospectral graphs in the following corollary.

Corollary 5.3

1. *Let G_1 and G_2 be two A_α-cospectral regular graphs for $\alpha \in [0,1]$, and let H be an arbitrary graph. Then the graphs $G_1\langle \mathrm{v}\rangle H$ and $G_2\langle \mathrm{v}\rangle H$ are A_α-cospectral.*
2. *Let H_1 and H_2 be two A_α-cospectral graphs with $\Gamma_{A_\alpha(H_1)}(x) = \Gamma_{A_\alpha(H_2)}(x)$ for $\alpha \in [0,1]$. If G is a regular graph, then the graphs $G\langle \mathrm{v}\rangle H_1$ and $G\langle \mathrm{v}\rangle H_2$ are A_α-cospectral.*

6 A_α-spectrum of $G_1\langle \mathrm{e}\rangle G_2$

In this section, we study the A_α-spectrum of $G_1\langle \mathrm{e}\rangle G_2$, the R-edge join of the graphs G_1 and G_2. We start with obtaining the expression of the A_α-characteristic polynomial of $G_1\langle \mathrm{e}\rangle G_2$, for an r_1-regular graph G_1 and an arbitrary graph G_2, in the following theorem.

Theorem 6.1 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an arbitrary graph on n_2 vertices. Let $\Gamma_{A_\alpha(G_2)}(x)$ be the $A_\alpha(G_2)$-coronal of G_2. Then for each $\alpha \in [0,1]$, the A_α-characteristic polynomial of $G_1\langle \mathrm{e}\rangle G_2$ is given by,*

$$\begin{aligned}\psi_{A_\alpha(G_1\langle \mathrm{e}\rangle G_2)}(x) = &\big(x-2\alpha-\alpha n_2\big)^{m_1-n_1}\cdot \psi_{A_\alpha(G_2)}(x-\alpha m_1)\\ &\cdot\prod_{i=2}^{n_1}\Big(x^2-\big(2\alpha+\alpha r_1+\alpha n_2+\lambda_i\big(A_\alpha(G_1)\big)\big)x\\ &+\alpha^2 r_1 n_2+3\alpha r_1-r_1-(1-3\alpha-\alpha n_2)\lambda_i\big(A_\alpha(G_1)\big)\Big)\\ &\cdot\Big(x^2-(2\alpha+\alpha r_1+\alpha n_2+r_1)x+\alpha^2 r_1 n_2+6\alpha r_1-2r_1+\alpha r_1 n_2\\ &-m_1(1-\alpha)^2(x-\alpha r_1-r_1)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big).\end{aligned}$$

Proof The adjacency matrix of $G_1\langle \mathrm{e}\rangle G_2$ is

$$A(G_1\langle \mathrm{e}\rangle G_2) = \begin{bmatrix} A(G_1) & R & 0_{n_1\times n_2}\\ R^T & 0_{m_1\times m_1} & J_{m_1\times n_2}\\ 0_{n_2\times n_1} & J_{n_2\times m_1} & A(G_2)\end{bmatrix}, \tag{30}$$

where R is the $(0,1)$-incidence matrix of G_1.

The degrees of the vertices of the graph $G_1\langle \mathrm{e}\rangle G_2$ are:

$$d_{G_1\langle \mathrm{e}\rangle G_2}(v_i) = 2r_1, \text{ for } i = 1, 2, \ldots, n_1;$$
$$d_{G_1\langle \mathrm{e}\rangle G_2}(v'_j) = 2 + n_2, \text{ for } j = 1, 2, \ldots, m_1;$$
$$d_{G_1\langle \mathrm{e}\rangle G_2}(u_k) = d_{G_2}(u_k) + m_1, \text{ for } k = 1, 2, \ldots, n_2.$$

Therefore, we have

$$D(G_1\langle \mathrm{e}\rangle G_2) = \begin{bmatrix} 2r_1 I_{n_1} & 0_{n_1\times m_1} & 0_{n_1\times n_2} \\ 0_{m_1\times n_1} & (2+n_2)I_{m_1} & 0_{m_1\times n_2} \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & D(G_2)+m_1 I_{n_2} \end{bmatrix}. \tag{31}$$

Using (30) and (31), we get the A_α-matrix of $G_1\langle \mathrm{e}\rangle G_2$ as

$$A_\alpha(G_1\langle \mathrm{e}\rangle G_2) = \begin{bmatrix} A_\alpha(G_1)+\alpha r_1 I_{n_1} & (1-\alpha)R & 0_{n_1\times n_2} \\ (1-\alpha)R^T & \alpha(2+n_2)I_{m_1} & (1-\alpha)J_{m_1\times n_2} \\ 0_{n_2\times n_1} & (1-\alpha)J_{n_2\times m_1} & A_\alpha(G_2)+\alpha m_1 I_{n_2} \end{bmatrix}.$$

Therefore, the characteristic polynomial of $A_\alpha(G_1\langle \mathrm{e}\rangle G_2)$ is

$$\begin{aligned}
&\psi_{A_\alpha(G_1\langle \mathrm{e}\rangle G_2)}(x) \\
&= \det\Big(xI_{n_1+m_1+n_2} - A_\alpha(G_1\langle \mathrm{e}\rangle G_2)\Big) \\
&= \det\begin{bmatrix} (x-\alpha r_1)I_{n_1} - A_\alpha(G_1) & -(1-\alpha)R & 0_{n_1\times n_2} \\ -(1-\alpha)R^T & (x-2\alpha-\alpha n_2)I_{m_1} & -(1-\alpha)J_{m_1\times n_2} \\ 0_{n_2\times n_1} & -(1-\alpha)J_{n_2\times m_1} & (x-\alpha m_1)I_{n_2} - A_\alpha(G_2) \end{bmatrix} \\
&= \det\Big((x-\alpha m_1)I_{n_2} - A_\alpha(G_2)\Big)\cdot \det S \quad \text{(by Lemma 2.1)},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
S &= \begin{bmatrix} (x-\alpha r_1)I_{n_1} - A_\alpha(G_1) & -(1-\alpha)R \\ -(1-\alpha)R^T & (x-2\alpha-\alpha n_2)I_{m_1} \end{bmatrix} \\
&\quad - \begin{bmatrix} 0_{n_1\times n_2} \\ -(1-\alpha)J_{m_1\times n_2} \end{bmatrix} \Big((x-\alpha m_1)I_{n_2} - A_\alpha(G_2)\Big)^{-1} \begin{bmatrix} 0_{n_2\times n_1} & -(1-\alpha)J_{n_2\times m_1} \end{bmatrix} \\
&= \begin{bmatrix} (x-\alpha r_1)I_{n_1} - A_\alpha(G_1) & -(1-\alpha)R \\ -(1-\alpha)R^T & (x-2\alpha-\alpha n_2)I_{m_1} - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)J_{m_1\times m_1} \end{bmatrix}.
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
\det S &= \det\Big((x-2\alpha-\alpha n_2)I_{m_1} - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)J_{m_1\times m_1}\Big) \\
&\quad \cdot \det\Bigg((x-\alpha r_1)I_{n_1} - A_\alpha(G_1) \\
&\quad -(1-\alpha)^2 R\Big((x-2\alpha-\alpha n_2)I_{m_1} - (1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)J_{m_1\times m_1}\Big)^{-1} R^T\Bigg).
\end{aligned} \tag{33}$$

By Lemma 2.7, we can write (33) as

$$\begin{aligned}\det S &= \big(x-2\alpha-\alpha n_2\big)^{m_1-1}\Big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\quad\cdot\det\bigg((x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-(1-\alpha)^2R\Big(\frac{1}{(x-2\alpha-\alpha n_2)}I_{m_1}\\ &\quad+\frac{(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-2\alpha-\alpha n_2)(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1))}J_{m_1\times m_1}\Big)R^T\bigg)\\ &= \big(x-2\alpha-\alpha n_2\big)^{m_1-1}\Big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\quad\cdot\det\bigg((x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T\\ &\quad-\frac{(1-\alpha)^4\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-2\alpha-\alpha n_2)\big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\big)}RJ_{m_1\times m_1}R^T\bigg)\\ &= \big(x-2\alpha-\alpha n_2\big)^{m_1-1}\Big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\quad\cdot\det\bigg((x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T\\ &\quad-\frac{{r_1}^2(1-\alpha)^4\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-2\alpha-\alpha n_2)\big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\big)}J_{n_1\times n_1}\bigg)\end{aligned}$$

$$\begin{aligned}&= \big(x-2\alpha-\alpha n_2\big)^{m_1-1}\Big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\cdot\det\bigg((x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T\bigg)\\ &\cdot\bigg(1-\frac{{r_1}^2(1-\alpha)^4\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-2\alpha-\alpha n_2)\big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\big)}\\ &\cdot\Gamma_{A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T}(x-\alpha r_1)\bigg),\end{aligned}\tag{34}$$

by Lemma 2.3.

Again, each row sum of the matrix $A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T$ is $r_1+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}2r_1$. Therefore from (5), we have

$$\begin{aligned}&\Gamma_{A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T}(x-\alpha r_1)\\ &\quad=\frac{n_1}{x-\alpha r_1-\Big(r_1+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}2r_1\Big)}\\ &\quad=\frac{n_1(x-2\alpha-\alpha n_2)}{(x-\alpha r_1)(x-2\alpha-\alpha n_2)-r_1(x-2\alpha-\alpha n_2)-2r_1(1-\alpha)^2}\end{aligned}$$

and using this, the expression

$$\bigg(1-\frac{{r_1}^2(1-\alpha)^4\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{(x-2\alpha-\alpha n_2)\big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\big)}$$

$$\cdot\Gamma_{A_\alpha(G_1)+\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T}(x-\alpha r_1)\Big)$$

becomes

$$\frac{f(x)-{r_1}^2 n_1(1-\alpha)^4(x-2\alpha-\alpha n_2)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)}{f(x)}, \tag{35}$$

where

$$\begin{aligned} f(x)&=(x-2\alpha-\alpha n_2)\Big(x-2\alpha-\alpha n_2-m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big)\\ &\quad\cdot\Big((x-\alpha r_1)(x-2\alpha-\alpha n_2)-r_1(x-6\alpha+2\alpha^2-\alpha n_2+2)\Big). \end{aligned}$$

Again,

$$\begin{aligned} &(x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T\\ &\quad=(x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}\big(A(G_1)+D(G_1)\big)\\ &\quad=(x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}\left(\frac{A_\alpha(G_1)-\alpha D(G_1)}{(1-\alpha)}+D(G_1)\right)\\ &\quad=\left(x-\alpha r_1-\frac{r_1(1-3\alpha+2\alpha^2)}{(x-2\alpha-\alpha n_2)}\right)I_{n_1}\\ &\qquad-\frac{(x+1-3\alpha-\alpha n_2)}{(x-2\alpha-\alpha n_2)}A_\alpha(G_1),\ \text{as } D(G_1)=r_1I_{n_1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\det\left((x-\alpha r_1)I_{n_1}-A_\alpha(G_1)-\frac{(1-\alpha)^2}{(x-2\alpha-\alpha n_2)}RR^T\right)\\ &\quad=\prod_{i=1}^{n_1}\left(x-\alpha r_1-\frac{r_1(1-3\alpha+2\alpha^2)}{(x-2\alpha-\alpha n_2)}-\frac{(x+1-3\alpha-\alpha n_2)}{(x-2\alpha-\alpha n_2)}\lambda_i\big(A_\alpha(G_1)\big)\right)\\ &\quad=\frac{1}{(x-2\alpha-\alpha n_2)^{n_1}}\prod_{i=1}^{n_1}\Big((x-\alpha r_1)(x-2\alpha-\alpha n_2)-r_1(1-3\alpha+2\alpha^2)\\ &\qquad-(x+1-3\alpha-\alpha n_2)\lambda_i\big(A_\alpha(G_1)\big)\Big)\\ &\quad=\frac{1}{(x-2\alpha-\alpha n_2)^{n_1}}\Big((x-\alpha r_1)(x-2\alpha\\ &\qquad-\alpha n_2)-r_1(x-6\alpha+2\alpha^2-\alpha n_2+2)\Big)\\ &\qquad\cdot\prod_{i=2}^{n_1}\Big((x-\alpha r_1)(x-2\alpha \end{aligned}$$

$$-\alpha n_2) - r_1(1-3\alpha+2\alpha^2) - (x+1-3\alpha-\alpha n_2)\lambda_i\big(A_\alpha(G_1)\big)\Big). \tag{36}$$

Using (35) and (36) in (34), we get

$$\begin{aligned} \det S = {} & (x-2\alpha-\alpha n_2)^{m_1-n_1-1}\cdot F_1(x) \\ & \cdot \prod_{i=2}^{n_1}\Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) \\ & -r_1(1-3\alpha+2\alpha^2) - (x+1-3\alpha-\alpha n_2)\lambda_i\big(A_\alpha(G_1)\big)\Big), \end{aligned} \tag{37}$$

where

$$\begin{aligned} F_1(x) = {} & \Big(x-2\alpha-\alpha n_2 - m_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big) \\ & \cdot\Big((x-\alpha r_1)(x-2\alpha-\alpha n_2) \\ & -r_1(x-6\alpha+2\alpha^2-\alpha n_2+2)\Big) - {r_1}^2 n_1(1-\alpha)^4\Gamma_{A_\alpha(G_2)}(x-\alpha m_1). \end{aligned}$$

Since G_1 is an r_1-regular graph with n_1 vertices and m_1 edges, $r_1 n_1 = 2m_1$. Using this in the expression of $F_1(x)$ and then simplifying it, we get

$$\begin{aligned} F_1(x) = {} & (x-2\alpha-\alpha n_2)\Big(x^2-(2\alpha+\alpha n_2+\alpha r_1+r_1)x \\ & +\alpha^2 r_1 n_2 + 6\alpha r_1 + \alpha r_1 n_2 - 2r_1 \\ & -m_1(1-\alpha)^2(x-\alpha r_1-r_1)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1)\Big). \end{aligned}$$

We use this expression in (37). Using (32), we get the required result. □

Now, in the following corollary, we obtain the A_α-eigenvalues of $G_1\langle \mathrm{e}\rangle G_2$ taking G_2 as an r_2-regular graph.

Corollary 6.1 *Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1\langle \mathrm{e}\rangle G_2$ consists precisely of:*

   (i) $3\alpha - 1$;
   (ii) $\alpha + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$ *and*
   (iii) *three roots of the equation $F(x) = 0$, where*

$$\begin{aligned} F(x) = {} & (x-\alpha-r_2)\big(x^2-(3\alpha+\alpha n_2+1)x+\alpha^2 n_2+6\alpha+\alpha n_2-2\big) \\ & -n_2(1-\alpha)^2(x-\alpha-1). \end{aligned}$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \langle e \rangle G_2$ consists precisely of:*
   (i) $\alpha(2 + n_2)$, *repeated $m_1 - n_1$ times;*
   (ii) $\alpha m_1 + \lambda_i\big(A_\alpha(G_2)\big)$, $i = 2, 3, \ldots, n_2$;
   (iii) *two roots of the equation $G_i(x) = 0$ for each $i = 2, 3, \ldots, n_1$, where*

$$G_i(x) = x^2 - \Big(2\alpha + \alpha r_1 + \alpha n_2 + \lambda_i\big(A_\alpha(G_1)\big)\Big)x \\ +\alpha^2 r_1 n_2 + 3\alpha r_1 - r_1 + (3\alpha + \alpha n_2 - 1)\lambda_i\big(A_\alpha(G_1)\big)$$

   and
   (iv) *three roots of the equation $F(x) = 0$, where*

$$F(x) = (x - \alpha m_1 - r_2) \\ \cdot\Big(x^2 - (2\alpha + \alpha r_1 + \alpha n_2 + r_1)x + (\alpha^2 r_1 n_2 + 6\alpha r_1 + \alpha r_1 n_2 - 2r_1)\Big) \\ -m_1 n_2 (1-\alpha)^2 (x - \alpha r_1 - r_1).$$

Taking G_2 as $K_{p,q}$, we obtain the A_α-eigenvalues of $G_1 \langle e \rangle G_2$ in the next corollary.

Corollary 6.2 *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges. Let $p, q \geq 1$ be integers and $G_2 = K_{p,q}$.*

1. *If $r_1 = 1$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \langle e \rangle G_2$ consists precisely of:*
   (i) $3\alpha - 1$;
   (ii) $\alpha(1 + p)$, *repeated $q - 1$ times;*
   (iii) $\alpha(1 + q)$, *repeated $p - 1$ times and*
   (iv) *four roots of the equation $F(x) = 0$, where*

$$F(x) = \Big(x^2 - (2\alpha + \alpha p + \alpha q + \alpha + 1)x \\ +\alpha^2 p + \alpha^2 q + 6\alpha + \alpha p + \alpha q - 2\Big) \\ \cdot\Big(x^2 - (2\alpha + \alpha p + \alpha q)x + \alpha^2 + \alpha^2 p + \alpha^2 q + 2\alpha pq - pq\Big) \\ -(1-\alpha)^2 (x - \alpha - 1)\big(px + qx - \alpha p - \alpha q \\ -\alpha p^2 - \alpha q^2 - 2\alpha pq + 2pq\big).$$

2. *If $r_1 \geq 2$, then for each $\alpha \in [0, 1]$, the A_α-spectrum of $G_1 \langle e \rangle G_2$ consists precisely of:*
   (i) $\alpha(2 + p + q)$, *repeated $m_1 - n_1$ times;*
   (ii) $\alpha(m_1 + p)$, *repeated $q - 1$ times;*
   (iii) $\alpha(m_1 + q)$, *repeated $p - 1$ times;*

(iv) *two roots of the equation* $G_i(x) = 0$ *for each* $i = 2, 3, \ldots, n_1$, *where*

$$G_i(x) = x^2 - \Big(2\alpha + \alpha r_1 + \alpha p + \alpha q + \lambda_i\big(A_\alpha(G_1)\big)\Big)x + \alpha^2 r_1 p + \alpha^2 r_1 q + 3\alpha r_1 - r_1 + (3\alpha + \alpha p + \alpha q - 1)\lambda_i\big(A_\alpha(G_1)\big);$$

and

(v) *four roots of the equation* $F(x) = 0$, *where*

$$\begin{aligned} F(x) &= \Big(x^2 - (2\alpha + \alpha r_1 + \alpha p + \alpha q + r_1)x + \alpha^2 r_1 p + \alpha^2 r_1 q \\ &\quad +6\alpha r_1 + \alpha r_1 p + \alpha r_1 q - 2r_1\Big) \\ &\quad \cdot\Big(x^2 - (2\alpha m_1 + \alpha p + \alpha q)x + \alpha^2 m_1^2 + \alpha^2 m_1 p + \alpha^2 m_1 q + 2\alpha pq - pq\Big) \\ &\quad -m_1(1-\alpha)^2(x - \alpha r_1 - r_1)\big(px + qx - \alpha m_1 p - \alpha m_1 q \\ &\quad -\alpha p^2 - \alpha q^2 - 2\alpha pq + 2pq\big). \end{aligned}$$

Finally, to conclude this section, we provide a construction of new pair of A_α-cospectral graphs from a given pair of A_α-cospectral graphs in the following corollary.

Corollary 6.3 1. *Let* G_1 *and* G_2 *be two* A_α*-cospectral regular graphs for* $\alpha \in [0, 1]$, *and let* H *be an arbitrary graph. Then the graphs* $G_1\langle \mathrm{e}\rangle H$ *and* $G_2\langle \mathrm{e}\rangle H$ *are* A_α*-cospectral.*

2. *Let* H_1 *and* H_2 *be two* A_α*-cospectral graphs with* $\Gamma_{A_\alpha(H_1)}(x) = \Gamma_{A_\alpha(H_2)}(x)$ *for* $\alpha \in [0, 1]$. *If* G *is a regular graph, then the graphs* $G\langle \mathrm{e}\rangle H_1$ *and* $G\langle \mathrm{e}\rangle H_2$ *are* A_α*-cospectral.*

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