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Geometric Properties of the Triangular Ratio Metric and Related Metrics

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Abstract
We study the inclusion relation of the triangular ratio metric balls and the Cassinian metric balls in subdomains of $\mathbb{R}^n$. Moreover, we study distortion properties of Möbius transformations with respect to the triangular ratio metric in the punctured unit ball.

Keywords Triangular ratio metric · Metric ball · Ball inclusion · Möbius transformation

Mathematics Subject Classification 30F45 · 51M05

1 Introduction

In geometric function theory, various metrics relative to the boundary of domains in which families of functions are defined have been introduced and played important roles in the studies of geometric and analytic properties of these functions. In the planar case, the hyperbolic metric serves as an important example of such metrics [3,15]. The so-called hyperbolic-type metrics, defined as generalizations of the hyperbolic metric of the planar domains to subdomains of higher-dimensional Euclidean space, share some but not all properties of the hyperbolic metric [5,8]. Examples of well-known hyperbolic-type metrics include the quasihyperbolic metric, distance ratio metric, and Apollonian metric.

Most of the hyperbolic-type metrics belong to the family of relative metrics. A relative metric is a metric that is evaluated in a domain $D \subsetneq \mathbb{R}^n$ relative to its boundary.

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In 2002, Hästö [7] introduced the generalized relative metric named as the M−relative metric which is defined on a domain $D \subsetneq \mathbb{R}^n$ by the quantity

$$\rho_{M,D}(x, y) = \sup_{a\in\partial D} \frac{|x-y|}{M(|x-a|, |y-a|)},$$

where M is continuous in $(0, \infty)\times(0, \infty)$ and ∂D is the boundary of D. For $M(\alpha, \beta) = \alpha + \beta$, the corresponding relative metric is the so-called triangular ratio metric

$$s_D(x, y) = \sup_{a\in\partial D} \frac{|x-y|}{|x-a| + |y-a|}.$$

The triangular ratio metric has been recently investigated in [4,9–11,16]. Another example of generalized relative metric is the Cassinian metric defined by the choice $M(\alpha, \beta) = \alpha\beta$, i.e.,

$$c_D(x, y) = \sup_{a\in\partial D} \frac{|x-y|}{|x-a||y-a|}.$$

The geometric properties of the Cassinian metric have been studied in [13,14,17].

In this paper, we continue to study the geometric properties of the triangular ratio metric and Cassinian metric. In particular, we investigate the inclusion relation of the triangular ratio metric balls and the Cassinian metric balls in subdomains of $\mathbb{R}^n$. Also, we study distortion properties of Möbius transformations with respect to the triangular ratio metric in the punctured unit ball. By using the comparison between the triangular ratio metric and Ibragimov's metric, we show the quasiconformality of bilipschitz mappings in Ibragimov's metric.

2 Hyperbolic-Type Metrics

In this section, we collect the definitions and some basic properties of various hyperbolic-type metrics. We always denote by D the proper subdomain of the Euclidean space $\mathbb{R}^n$ and write $d(x) = d(x, \partial D)$ for the distance from x to the boundary of the domain D, and let $d_{xy} = \min\{d(x), d(y)\}$.

2.1 Hyperbolic Metric

The hyperbolic metrics $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and of the unit ball $\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$ are, respectively, defined as follows [2]: for $x, y \in \mathbb{H}^n$

$$\mathrm{ch}\rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n}, \tag{2.1}$$

and for $x, y \in \mathbb{B}^n$

$$\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}. \tag{2.2}$$

2.2 Distance Ratio Metric

For all $x, y \in D$, the distance ratio metric j_G is defined as

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{d_{xy}}\right).$$

This metric was introduced by Gehring and Palka [6] in a slightly different form and in the above form in [20]. It follows from [21, Lemma 2.41(2)] and [1, Lemma 7.56] that

$$j_D(x,y) \le \rho_D(x,y) \le 2 j_D(x,y)$$

for $D \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and all $x, y \in D$.

2.3 Quasihyperbolic Metric

For all $x,\ y \in D$, the quasihyperbolic metric k_D is defined as

$$k_D(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial D)} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in D [6]. It is well known that

$$j_D(x,y) \le k_D(x,y)$$

for all $x, y \in D$.

2.4 Point Pair Function

We define for $x, y \in D \subsetneq \mathbb{R}^n$ the point pair function

$$p_D(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4\,d(x)\,d(y)}}.$$

This point pair function was introduced in [4] where it turned out to be a very useful function in the study of the triangular ratio metric. However, the function p_G is generally not a metric.

2.5 Ibragimov's Metric

For a domain $D \subsetneq \mathbb{R}^n$, Ibragimov's metric is defined as

$$u_D(x, y) = 2 \log \frac{|x-y| + \max\{d(x), d(y)\}}{\sqrt{d(x)\, d(y)}}, \qquad x, y \in D.$$

Several authors have studied comparison inequalities between Ibragimov's metric and the hyperbolic metric as well as some hyperbolic-type metrics [12,19,22,23].

3 Inclusion Properties

In this section, we study inclusion relation between triangular ratio metric balls and other hyperbolic-type metric balls. Let (D, d) be a metric space. A metric ball $B_d(x, r)$ is a set

$$B_d(x, r) = \{y \in D : d(x, y) < r\}.$$

Our first theorem shows the inclusion relation between the triangular ratio metric balls B_s and the Cassinian metric balls B_c.

Theorem 3.1 *For arbitrary $x \in D \subsetneq \mathbb{R}^n$ and $t \in (0, 1)$,*

$$B_c(x, r) \subset B_s(x, t) \subset B_c(x, R),$$

where $r = \frac{2t}{(1+2t)d(x)}$ and $R = \frac{2t}{(1-t)d(x)}$. Moreover, $R/r \to 1$ as $t \to 0$.

Proof For all $x, y \in D$, it is easy to see that

$$\inf_{z \in \partial D} |x-z||y-z| \le d_{xy}(d_{xy} + |x-y|).$$

By the definition of the Cassinian metric, we obtain

$$c_D(x, y) = \frac{|x-y|}{\inf_{z \in \partial D} |x-z||y-z|} \ge \frac{|x-y|}{d_{xy}(d_{xy} + |x-y|)},$$

and hence,

$$|x-y| \le \frac{c_D(x, y) d_{xy}^2}{1 - c_D(x, y) d_{xy}} < \frac{r d_{xy}^2}{1 - r d(x) \wedge d(y)} < 2t d_{xy}.$$

By the definition of the triangular ratio metric, we have

$$s_D(x, y) = \frac{|x-y|}{\inf_{z \in \partial D} |x-z| + |y-z|} \le \frac{|x-y|}{d(x) + d(y)} \le \frac{|x-y|}{2d_{xy}} < t.$$

Hence, we obtain $B_c(x, r) \subset B_s(x, t)$. As for the inclusion $B_s(x, t) \subset B_c(x, R)$, let $y \in B_s(x, t)$, then

$$\frac{|x-y|}{2d(y)+|x-y|} \leq s_D(x, y) < t,$$

which implies that $|x-y| < \frac{2td(y)}{1-t}$ and

$$c_D(x, y) \leq \frac{|x-y|}{d(x)d(y)} < \frac{2t}{(1-t)d(x)}.$$

Clearly,

$$\lim_{t\to 0} \frac{R}{r} = 1.$$

□

Theorem 3.2 shows the inclusion between the triangular ratio metric balls and distance ratio metric balls, which was conjectured in [11, Conjecture 7.7].

Theorem 3.2 *For arbitrary* $x \in D \subsetneq \mathbb{R}^n$ *and* $t \in (0, 1)$*,*

$$B_j(x, r) \subset B_s(x, t) \subset B_j(x, R),$$

where $r = \log(1+2t)$ *and* $R = \log(1+\frac{2t}{1-t})$*. Moreover,* $R/r \to 1$ *as* $t \to 0$*.*

Proof Suppose that $y \in B_j(x, r)$. Then,

$$\log(1+\frac{|x-y|}{d_{xy}}) = j_D(x, y) < r = \log(1+2t),$$

which implies that

$$|x-y| < 2td_{xy}.$$

Since

$$\inf_{z\in\partial D} |x-z| + |y-z| \geq 2d_{xy},$$

by the definition of the triangular ratio metric we have

$$s_D(x, y) = \frac{|x-y|}{\inf_{z\in\partial D} |x-z| + |y-z|} \leq \frac{|x-y|}{2d_{xy}} < t.$$

Hence, $y \in B_s(x,t)$. Now we prove the second inclusion. It follows from triangle inequality that $\inf_{z\in\partial D} |x-z|+|y-z| \le 2d_{xy}+|x-y|$, and

$$t > s_D(x,y) \ge \frac{|x-y|}{2d_{xy}+|x-y|},$$

which implies

$$\frac{|x-y|}{d_{xy}} < \frac{2t}{1-t}.$$

Hence, the second inclusion holds. It is easy to check that

$$\lim_{t\to 0} \frac{R}{r} = 1.$$

□

From the well-known inequalities [1, Theorem 7.56]

$$j_{\mathbb{B}^n}(x,y) \le \rho_{\mathbb{B}^n}(x,y) \le 2 j_{\mathbb{B}^n}(x,y),$$

it follows that

$$B_\rho(x,r) \subset B_j(x,r) \subset B_\rho(x,2r). \tag{3.1}$$

Theorem 3.3 *Let $x \in \mathbb{B}^n$ and $t \in (0,1)$. Then,*

$$B_\rho(x,r) \subset B_s(x,t) \subset B_\rho(x,R),$$

where $r = \log(1+2t)$ and $R = 2\log(1+\frac{2t}{1-t})$. Moreover, $R/r \to 2$ as $t \to 0$.

Proof By Theorem 3.2, we have $B_s(x,t) \subset B_j(x, \log(1+\frac{2t}{1-t}))$, which together with the right-hand side of (3.1) implies the second inclusion with $R = 2\log(1+\frac{2t}{1-t})$. Similarly, Theorem 3.2 together with the left-hand side of (3.1) implies

$$B_\rho(x,r) \subset B_j(x,r) \subset B_s(x, \frac{e^r-1}{2}).$$

That is, $B_\rho(x,r) \subset B_s(x,t)$ with $r = \log(1+2t)$. By l'Hôpital rule, it is easy to see that

$$\lim_{t\to 0} \frac{R}{r} = 2.$$

□

In a convex domain $D \subset \mathbb{R}^n$, we recall the following inequality [4, Lemma 3.14]

$$s_D(x, y) \le p_D(x, y) \le \sqrt{2} s_D(x, y), \quad \text{for} \quad x, y \in D.$$

It follows immediately that

$$B_p(x, r) \subset B_s(x, r) \subset B_p(x, \sqrt{2} r).$$

Similarly, in a convex domain $D \subset \mathbb{R}^n$, the inequality [9, Theorem 2.17]

$$s_D(x, y) \le v_D(x, y) \le \pi s_D(x, y)$$

implies the inclusion

$$B_v(x, r) \subset B_s(x, r) \subset B_v(x, \pi r).$$

Lemma 3.4 [18, Corollary 3.4] *For $x \in \mathbb{B}^n$ and $r > 0$,*

$$B_j(x, r) \subset B_k(x, t) \subset B_j(x, R),$$

where

$$r = \max\{\log(1 + (1 + |x|) \sinh \frac{t}{4}), \log(1 + (1 - |x|) \frac{e^{t/2} - 1}{2})\}$$

and

$$R = \log(1 + (1 + |x|) \frac{e^t - 1}{2}).$$

Theorem 3.5 *Let $x \in \mathbb{B}^n$ and $t \in (0, 1)$. Then, the following inclusion relation holds:*

$$B_k(x, r) \subset B_s(x, t) \subset B_k(x, R),$$

where $r = \log(1 + \frac{4t}{1+|x|})$ and $R = \max\{R_1, R_2\}$ with

$$R_1 = 4 \operatorname{arsh}(\frac{2t}{(1 + |x|)(1 - t)}), \quad R_2 = 2 \log(1 + \frac{4t}{(1 - |x|)(1 - t)}).$$

Proof By Lemma 3.4, it is easy to see that

$$B_k(x, r) \subset B_j(x, \log(1 + (1 + |x|) \frac{e^r - 1}{2})),$$

and by Theorem 3.2, $B_j(x, r) \subset B_s(x, \frac{e^r - 1}{2})$, then we have

$$B_k(x, r) \subset B_s(x, (1 + |x|) \frac{e^r - 1}{4}).$$

Similarly, $B_k(x,r) \subset B_s(x,t)$ with $r = \log(1+\frac{4t}{1+|x|})$.

Again from Theorem 3.2 and Lemma 3.4, it follows that

$$B_s(x,t) \subset B_j(x, \log(1+\frac{2t}{1-t}))$$

and

$$B_j(x,t) \subset B_k(x, \max\{4\,\mathrm{arsh}\frac{e^t-1}{1+|x|}, 2\log(1+\frac{2(e^t-1)}{1-|x|})\}).$$

Hence, the second inclusion holds with $R = \max\{R_1, R_2\}$, where

$$R_1 = 4\,\mathrm{arsh}(\frac{2t}{(1+|x|)(1-t)}) \quad \text{and} \quad R_2 = 2\log(1+\frac{4t}{(1-|x|)(1-t)}).$$

□

Lemma 3.6 [17, Theorem 5.4] *For given $a \in \mathbb{R}^n$, let domain $D = \mathbb{R}^n \backslash \{a\}$, $x \in D$ and $0 < t < 1/(2|x-a|)$. Then, we have the following inclusion relation*

$$B_j(x,r) \subset B_c(x,t) \subset B_j(x,R),$$

where $r = \log(1+t|x-a|)$ and $R = \log(\frac{1-t|x-a|}{1-2t|x-a|})$. Moreover, $R/r \to 1$ as $t \to 0$.

The following improved inclusion relation between the Cassinian metric balls and the distance ratio metric balls was conjectured in [17, Conjecture 5.5].

Theorem 3.7 *Let $D \subsetneq \mathbb{R}^n$ be a domain and $x \in D$. For $0 < t < \frac{1}{d(x)}$, the following inclusion holds:*

$$B_j(x,r) \subset B_c(x,t) \subset B_j(x,R),$$

where $r = \log(1+td(x))$ and $R = \log\left(\frac{1}{1-td(x)}\right)$. Moreover, $R/r \to 1$ as $t \to 0$.

Proof Suppose that $y \in B_j(x,r)$. Then, $j_D(x,y) < r$, and

$$\log\left(1+\frac{|x-y|}{\min\{d(x),d(y)\}}\right) < \log\left(1+td(x)\right).$$

On simplification, we get

$$|x-y| < td(x)\min\{d(x),d(y)\} < td(x)d(y),$$

which together with the inequality

$$\inf_{p\in\partial D} |x-p||y-p| \geq d(x)d(y)$$

implies

$$c_D(x, y) = \frac{|x-y|}{\inf\limits_{p\in\partial D}|x-p||y-p|} < t.$$

Hence, $y \in B_c(x, t)$ and $B_j(x, r) \subset B_c(x, t)$.

Now we prove the second inclusion. Let $p_0 \in \partial D$ with $|x - p_0| = d(x)$. The triangle inequality yields $|y - p_0| \le d(x) + |x - y|$, and then,

$$\inf_{p\in\partial D}|x-p||y-p| \le |x-p_0||y-p_0| \le d(x)(d(x)+|x-y|).$$

Similarly,

$$\inf_{p\in\partial D}|x-p||y-p| \le d(y)(d(y)+|x-y|).$$

Combining the above two inequalities, we have

$$\inf_{p\in\partial D}|x-p||y-p| \le d_{xy}(d_{xy}+|x-y|),$$

and then, for $y \in B_c(x, t)$,

$$\frac{|x-y|}{d_{xy}(d_{xy}+|x-y|)} \le \frac{|x-y|}{\inf_{p\in\partial D}|x-p||y-p|} < t,$$

which implies

$$\frac{|x-y|}{d_{xy}} < \frac{td_{xy}}{1-td_{xy}}.$$

Therefore,

$$\begin{aligned} j_D(x, y) &= \log\left(1+\frac{|x-y|}{d_{xy}}\right) \\ &\le \log\frac{1}{1-td_{xy}} < R \end{aligned}$$

and $y \in B_j(x, R)$. Hence, the second inclusion holds. Clearly, one can see that $R/r \to 1$ as $t \to 0$. □

Before proving Theorem 3.8, we recall the following inequality [6, Lemma 2.1]:

$$j_D(x, y) \le k_D(x, y) \quad \text{for all} \quad x, y \in D. \tag{3.2}$$

Theorem 3.8 *Let $D \subsetneq \mathbb{R}^n$ be a domain and $x \in D$. For $t < \frac{1}{2d(x)}$, we have*

$$B_k(x, r) \subset B_c(x, t) \subset B_k(x, R),$$

where $r = \log(1 + td(x))$ and $R = \log(\frac{1-td(x)}{1-2td(x)})$. Moreover, $R/r \to 1$ as $t \to 0$.

Proof For arbitrary $y \in B_k(x, r)$, we have $k_D(x, y) < r$. Inequality (3.2) implies $j_D(x, y) < r$, and then $B_k(x, r) \subset B_j(x, r)$. Since $B_j(x, r) \subset B_c(x, t)$ by Theorem 3.7, the first inclusion follows.

Let $z \in \partial D$ such that $c_D(x, y) = c_{\mathbb{R}^n \setminus \{z\}}(x, y)$. Since $D \subset \mathbb{R}^n \setminus \{z\}$, it follows from the domain monotonicity of the distance ratio metric that

$$j_{\mathbb{R}^n \setminus \{z\}}(x, y) \le j_D(x, y).$$

Hence, we have $j_{\mathbb{R}^n \setminus \{z\}}(x, y) < r$. By Lemma 3.6, we obtain

$$c_{\mathbb{R}^n \setminus \{z\}}(x, y) < t$$

and $c_D(x, y) < t$, which implies $B_j(x, r) \subset B_c(x, t)$.

For the second inclusion relation, let $y \in B_c(x, t)$. It follows from Theorem 3.7 that $y \in B_j(x, \log(1/(1 - td(x))))$ and then

$$|x - y| < \frac{td^2(x)}{1 - td(x)}.$$

Since $t < 1/(2d(x))$, we have $|x - y| < d(x)$. By [21, Lemma 3.7],

$$\begin{aligned} k_D(x, y) &\le \log(1 + \frac{|x - y|}{d(x) - |x - y|}) \\ &< \log(1 + \frac{td(x)}{1 - 2td(x)}) \\ &= \log(\frac{1 - td(x)}{1 - 2td(x)}). \end{aligned}$$

It is easy to see that

$$\lim_{t \to 0} \frac{R}{r} = 1.$$

□

4 Distortion Property of Möbius Transformations

The distortion property of the triangular ratio metric under Möbius transformations of the unit ball has been studied in [4,10]. In this section, we study the similar property but under Möbius transformations of a punctured unit ball.

For $a \in \mathbb{R}^n \backslash \{0\}$, let $a^* = \frac{a}{|a|^2}$, $0^* = \infty$, and $\infty^* = 0$. Let

$$\sigma_a(x) = a^* + s^2(x - a^*)^*, \quad s^2 = |a^*|^2 - 1$$

be the inversion in the sphere $S^{n-1}(a*, s)$.

Let f be a Möbius transformation of the unit ball. Since the triangular ratio metric s_D is invariant under orthogonal transformations, it follows from [2, Theorem 3.5.1] that

$$s_{\mathbb{B}^n}(f(x), f(y)) = s_{\mathbb{B}^n}(\sigma_a(x), \sigma_a(y)) \quad \text{for} \quad x, y, a \in \mathbb{B}^n.$$

Theorem 4.1 *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{B}^n \backslash \{a\}$ be a Möbius transformation with $f(0) = a$. Then, for $x, y \in \mathbb{B}^n \backslash \{0\}$, it holds*

$$\frac{1 - |a|}{1 + |a|} s_{\mathbb{B}^n \backslash \{0\}}(x, y) \leq s_{\mathbb{B}^n \backslash \{a\}}(f(x), f(y)) \leq \frac{1 + |a|}{1 - |a|} s_{\mathbb{B}^n \backslash \{0\}}(x, y).$$

Proof If $a = 0$, i.e., $f(0) = 0$, then f is a rotation and preserves the triangular ratio metric. Now we suppose that $a \neq 0$ and then $f(a) = 0$.

$$s_{\mathbb{B}^n \backslash \{0\}}(x, y) = \frac{|x - y|}{P},$$

where

$$P = \min\{|x| + |y|, \inf_{w \in \partial \mathbb{B}^n} |x - w| + |y - w|\}$$

and

$$s_{\mathbb{B}^n \backslash \{a\}}(\sigma_a(x), \sigma_a(y)) = \frac{|\sigma_a(x) - \sigma_a(y)|}{T},$$

with

$$T = \min\{|\sigma_a(x) - a| + |\sigma_a(y) - a|, \inf_{z \in \partial \mathbb{B}^n} |\sigma_a(x) - \sigma_a(z)| + |\sigma_a(z) - \sigma_a(y)|\}.$$

We first prove the right-hand side inequality.

If $T = \inf_{z \in \partial \mathbb{B}^n} |\sigma_a(x) - \sigma_a(z)| + |\sigma_a(z) - \sigma_a(y)|$, then the distortion of the triangular ratio metric under Möbius transformations of the punctured unit ball is the same as the case of the unit ball [4, Theorem 3.31].

Now we suppose that $T = |\sigma_a(x) - a| + |\sigma_a(y) - a|$. Then,

$$\begin{aligned} s_{\mathbb{B}^n\setminus\{a\}}(\sigma_a(x), \sigma_a(y)) &= \frac{|\sigma_a(x) - \sigma_a(y)|}{|\sigma_a(x) - a| + |\sigma_a(y) - a|} \\ &= \frac{\frac{s^2|x-y|}{|x-a^*||y-a^*|}}{\frac{s^2|x|}{|x-a^*||a^*|} + \frac{s^2|y|}{|y-a^*||a^*|}} \\ &= \frac{|x - y|}{\frac{|y-a^*|}{|a^*|}|x| + \frac{|x-a^*|}{|a^*|}|y|} \\ &= \frac{|x - y|}{\beta|x| + \gamma|y|}, \end{aligned}$$

where $\beta = \frac{|y-a^*|}{|a^*|}$ and $\gamma = \frac{|x-a^*|}{|a^*|}$. Clearly,

$$|a^*| - 1 \le |x - a^*|, |y - a^*| \le |a^*| + 1$$

which together with $\beta, \gamma \ge 1 - |a|$ implies

$$s_{\mathbb{B}^n\setminus\{a\}}(\sigma_a(x), \sigma_a(y)) \le \frac{1}{1 - |a|} \frac{|x - y|}{|x| + |y|} \le \frac{1 + |a|}{1 - |a|} s_{\mathbb{B}^n\setminus\{0\}}(x, y).$$

Next we prove the left-hand side of the inequality. If $P = \inf_{w\in\partial\mathbb{B}^n} |x-w|+|y-w|$, the distortion of the triangular ratio metric under Möbius transformations of the punctured unit ball is the same as the case of the unit ball [4, Theorem 3.31]. Now we assume $P = |x| + |y|$. Then,

$$\begin{aligned} s_{\mathbb{B}^n\setminus\{0\}}(x, y) &= \frac{|x - y|}{|x| + |y|} \\ &= \frac{\frac{|\sigma_a(x)-\sigma_a(y)||x-a^*||y-a^*|}{s^2}}{\frac{|\sigma_a(x)-a||x-a^*||a^*|}{s^2} + \frac{|\sigma_a(y)-a||y-a^*||a^*|}{s^2}} \\ &= \frac{|\sigma_a(x) - \sigma_a(y)|}{\frac{1}{\beta}|\sigma_a(x) - a| + \frac{1}{\gamma}|\sigma_a(y) - a|}, \end{aligned}$$

where $\beta, \gamma \le 1 + |a|$. Therefore,

$$\begin{aligned} s_{\mathbb{B}^n\setminus\{0\}}(x, y) &\le (1 + |a|)\frac{|\sigma_a(x) - \sigma_a(y)|}{|\sigma_a(x) - a| + |\sigma_a(y) - a|} \\ &\le (1 + |a|)s_{\mathbb{B}^n\setminus\{a\}}(\sigma_a(x), \sigma_a(y)) \\ &\le \frac{1 + |a|}{1 - |a|} s_{\mathbb{B}^n\setminus\{a\}}(\sigma_a(x), \sigma_a(y)). \end{aligned}$$

□

5 Quasiconformality of a Bilipschitz Mapping in Ibragimov's Metric

Bilipschitz mappings with respect to the triangular ratio metric have been studied in [10]. In this section, we use the comparison inequality between the triangular ratio metric and Ibragimov's metric to investigate the quasiconformality of bilipschitz mappings in Ibragimov's metric.

Theorem 5.1 [10, Theorem 4.4] *Let $G \subsetneq \mathbb{R}^n$ be a domain and let $f : G \to fG \subset \mathbb{R}^n$ be a sense-preserving homeomorphism, satisfying L-bilipschitz condition with respect to the triangular ratio metric, i.e.,*

$$s_G(x, y)/L \le s_{fG}(f(x), f(y)) \le L s_G(x, y),$$

holds for all $x, y \in G$. Then, f is quasiconformal with the linear dilatation $H(f) \le L^2$.

Lemma 5.2 [22, Theorem 3.10] *Let $D \subsetneq \mathbb{R}^n$. For $x, y \in D$,*

$$(2\log 3)s_D(x, y) \le u_D(x, y) \le 3\log\frac{1 + s_D(x, y)}{1 - s_D(x, y)},$$

and the inequalities are sharp.

Theorem 5.3 *Let $D \subsetneq \mathbb{R}^n$ be a domain and $f : D \to fD \subset \mathbb{R}^n$ is a sense-preserving homeomorphism satisfying the L-bilipschitz condition in Ibragimov's metric*

$$\frac{1}{L}u_D(x, y) \le u_{f(D)}(f(x), f(y)) \le L u_D(x, y). \tag{5.1}$$

Then, f is a quasiconformal mapping with the linear dilatation $H(f) \le \frac{9L^2}{\log^2 3}$.

Proof Since, by Lemma 5.2,

$$u_D(x, y) \le 3\log\frac{1 + s_D(x, y)}{1 - s_D(x, y)} = 3\log\left(1 + \frac{2s_D(x, y)}{1 - s_D(x, y)}\right),$$

for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in D$ satisfying $s_D(x, y) < \delta$, we have

$$(2\log 3)s_D(x, y) \le u_D(x, y) \le 6(1 + \varepsilon)s_D(x, y).$$

By Lemma 5.2, we have

$$\begin{aligned} s_{fD}(f(x), f(y)) &\leq \frac{1}{2\log 3} u_{fD}(f(x), f(y)) \\ &\leq \frac{L}{2\log 3} u_D(x, y) \\ &\leq \frac{3(1+\epsilon)L}{\log 3} s_D(x, y). \end{aligned}$$

Similarly,

$$\begin{aligned} s_{fD}((f(x), f(y)) &\geq \frac{1}{6(1+\epsilon)} u_{fD}(f(x), f(y)) \\ &\geq \frac{1}{6L(1+\epsilon)} u_D(x, y) \\ &\geq \frac{\log 3}{3(1+\epsilon)L} s_D(x, y). \end{aligned}$$

Therefore, an L-bilipschitz mapping under Ibragimov's metric is a $\frac{3L(1+\varepsilon)}{\log 3}$-bilipschitz mapping under the triangle ratio metric. It follows from Theorem 5.1 that f is a quasiconformal mapping with the linear dilatation $H(f) \leq (\frac{3L(1+\varepsilon)}{\log 3})^2$. Let $\varepsilon \to 0$, we obtain the desired linear dilatation. □

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