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Lelong–Demailly Numbers and Wedge Product of *S*-Plurisubahrmonic Currents

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Abstract
In this paper, we show the existence of the Lelong–Demailly numbers of positive S-plurisubharmonic currents. Moreover, a valid definition of $dd^c g \wedge T$ is obtained for plurisubharmonic currents T and unbounded plurisubharmonic functions g. The importance of this definition comes from the sharpness of our condition on the Hausdorff dimension of the locus points of g.

Keywords Lelong numbers · Plurisubharmonic functions · Plurisubharmonic currents

Mathematics Subject Classification 32U05 · 32U40

1 Introduction

Let Ω be an open subset of $\mathbb{C}^n$ and T be a positive current of bi-dimension (p, p). Recall that T is said to be closed if $dT = 0$, and is said to be S-plurisubharmonic (resp. S-plurisuperharmonic) if there exists a positive current S on Ω such that $dd^c T \geq -S$ (resp. $dd^c T \leq S$). Consider a non-negative plurisubharmonic function φ of class $\mathcal{C}^2$ on Ω such that $\log \varphi$ is plurisubharmonic and set the following notations for every $0 < r_1 < r_2$

$$B_\varphi(r_1) := \{z \in \Omega;\ \varphi(z) < r_1\},$$
$$B_\varphi(r_2, r_1) := B_\varphi(r_2) \backslash B_\varphi(r_1),$$
$$\beta_\varphi := dd^c \varphi,\ \alpha_\varphi = dd^c \log \varphi.$$

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Throughout this paper, we assume that φ is semi-exhaustive, which means that there exists $R_\varphi > 0$ so that $B_\varphi(R_\varphi)$ is relatively compact in Ω. The Lelong–Demailly number of T with respect to the weight φ is $\nu(T,\varphi) := \lim_{r\to 0^+} \nu(T,\varphi,r)$, where $\nu(T,\varphi,r) = \dfrac{1}{r^p}\displaystyle\int_{B_{\varphi(r)}} T\wedge\beta_\varphi^p$, $r\in(0,R_\varphi)$. By $Psh^-(\Omega)$ we denote the set of all negative plurisubharmonic functions on Ω. For a function $g\in Psh^-(\Omega)$, put L_g to be the set of all locus points of g which consists of the points $z\in\Omega$ where g is unbounded in every neighborhood of z.

The first part of this paper deals with Lelong–Demailly numbers. Actually, for S-plurisubharmonic currents T we give sufficient conditions on S that yield to the existence of $\nu(T,\varphi)$.

Theorem (Theorem 2.2) *Let T be a positive S-plurisubharmonic current of bi-dimension (p,p) on Ω. If the function $t\mapsto \dfrac{\nu(S,\varphi,t)}{t}$ is integrable in a neighborhood of 0, then the Lelong–Demailly number $\nu(T,\varphi)$ exists.*

The result generalizes essential results due to Pierre Lelong and Henri Skoda [11].

The second part treats the wedge product of currents. More precisely, we study existence of the current $dd^c g\wedge T$ in both cases when T is either plurisubharmonic or plurisuperharmonic.

Let T be a positive current of bi-dimension (p,p) on Ω and $g\in Psh^-(\Omega)\cap \mathcal{C}^1(\Omega\backslash L_g)$. Assume that $(g_j)_j$ is a decreasing sequence of smooth plurisubharmonic functions on Ω converging *to g in $\mathcal{C}^1(\Omega\backslash L_g)$. Then $dd^c g\wedge T$ is a well defined current as a limit of $dd^c g_j\wedge T$ on Ω in each of the following cases.*

(1) $dd^c T\geq 0$ and $\mathcal{H}_{2p-2}(L_g\cap \text{SuppT})$ is locally finite. (Theorem 3.8.)
(2) $dd^c T\leq 0$ and $\mathcal{H}_{2p-2}(L_g\cap \text{SuppT}) = 0$. (Theorem 3.5.)

The definition of $dd^c g\wedge T$ was considered before in the particular case when T is pluriharmonic. In fact, the case when Ω is a compact Kähler manifold and g is continuous on Ω is due to Dinh and Sibony [6]. One year later, Alessandrini and Bassanelli [2] defined $dd^c g\wedge T$ in the case where L_g is a proper analytic set with $dim(L_g) < p$. In [1] the author generalized the latter work to the case of closed obstacle L_g when $\mathcal{H}_{2p-2}(L_g\cap \text{SuppT})$ is locally finite. We sholud point out to the inspiring works of Demailly [5] and Fornæss-Sibony [7] where they established fabulous techniques to tackle the case of closed currents.

In order to grasp the above definitions, new versions of Chern–Levine–Nirenberg inequalities induced to the considered objects T and g. Namely, we prove what follows.

Lemma (Lemma 3.3) *Let T be a positive plurisubharmonic current of bi-dimension (p,p) on Ω. Let K and L be compact sets of Ω with $L\subset \overset{\circ}{K}$. Then there exist positive constant $C_{K,L}$, and a neighborhood V of $K\cap L_g$ such that for all $g\in Psh^-(\Omega)\cap\mathcal{C}^1(\Omega\backslash L_g)$ so that $\mathcal{H}_{2p-1}(L_g\cap \text{Supp } T) = 0$ we have*

$$\|dd^c g\wedge T\|_{L\backslash L_g} \leq C_{K,L}\|g\|_{\mathcal{C}^\infty(K\backslash V)}\|T\|_{K\backslash V}.$$

We also deduce a similar estimation when T and $dd^c T$ alternate in the sign as soon as $\mathcal{H}_{2p-2}(L_g\cap \text{Supp}T) = 0$.

2 Lelong–Demailly Numbers

We start here with a very famous version of Lelong–Jensen formula.

Lemma 2.1 *Let T be a positive or negative plurisubharmonic current of bi-dimension (p,p) on Ω. Then for all $0<r_1<r_2<R_\varphi$ we have*

$$\begin{aligned}\nu(T,\varphi,r_2)-\nu(T,\varphi,r_1)&=\int_{B_\varphi(r_2,r_1)}T\wedge\alpha_\varphi^p\\&+\int_{r_1}^{r_2}\left(\frac{1}{t^p}-\frac{1}{r_2^p}\right)\int_{B_\varphi(t)}dd^cT\wedge\beta_\varphi^{p-1}dt\\&+\left(\frac{1}{r_1^p}-\frac{1}{r_2^p}\right)\int_0^{r_1}\int_{B_\varphi(t)}dd^cT\wedge\beta_\varphi^{p-1}dt\end{aligned}\tag{2.1}$$

Theorem 2.2 *Let T be a positive S-plurisubharmonic current of bi-dimension (p,p) on Ω. If the function*

$$t\mapsto\frac{\nu(S,\varphi,t)}{t}\tag{2.2}$$

is integrable in a neighborhood of 0, then the Lelong–Demailly number $\nu(T,\varphi)$ exists.

Proof We follow a similar technique as in [8]. For all $0<r<R_\varphi$, let us define

$$\Gamma(r)=\nu(T,\varphi,r)+\int_0^r\left(1-\frac{t^p}{r^p}\right)\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\,dt.\tag{2.3}$$

By the properties of S, the function Γ is well-defined and non-negative. Now, for $0<r_1<r_2<R_\varphi$, we have

$$\begin{aligned}\Gamma(r_2)-\Gamma(r_1)&=\nu(T,\varphi,r_2)-\nu(T,\varphi,r_1)+\int_0^{r_2}\left(1-\frac{t^p}{r_2^p}\right)\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\,dt\\&-\int_0^{r_1}\left(1-\frac{t^p}{r_1^p}\right)\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\,dt\\&=\nu(T,\varphi,r_2)-\nu(T,\varphi,r_1)+\int_{r_1}^{r_2}\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}dt\\&-\int_0^{r_2}\frac{1}{r_2^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}dt+\int_0^{r_1}\frac{1}{r_1^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}dt.\end{aligned}\tag{2.4}$$

Using Lemma 2.1, one has

$$\Gamma(r_2)-\Gamma(r_1)=\int_{r_1}^{r_2}\frac{1}{t^p}\int_{B_\varphi(t)}dd^cT\wedge\beta_\varphi^{p-1}dt-\int_0^{r_2}\frac{1}{r_2^p}\int_{B_\varphi(t)}dd^cT\wedge\beta_\varphi^{p-1}dt$$

$$
\begin{aligned}
&+\int_0^{r_1}\frac{1}{r_1^p}\int_{B_\varphi(t)}\mathrm{dd}^cT\wedge\beta_\varphi^{p-1}\mathrm{d}t+\int_{r_1}^{r_2}\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\mathrm{d}t\\
&-\int_0^{r_2}\frac{1}{r_2^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\mathrm{d}t+\int_0^{r_1}\frac{1}{r_1^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\mathrm{d}t\\
&+\int_{B_\varphi(r_2,r_1)}T\wedge\alpha_\varphi^p\\
=&\int_{B_\varphi(r_2,r_1)}T\wedge\alpha_\varphi^p+\int_{r_1}^{r_2}\left(\frac{1}{t^p}-\frac{1}{r_2^p}\right)\int_{B_\varphi(t)}(\mathrm{dd}^cT+S)\wedge\beta_\varphi^{p-1}\mathrm{d}t\\
&+\int_0^{r_1}\left(\frac{1}{r_1^p}-\frac{1}{r_2^p}\right)\int_{B_\varphi(t)}(\mathrm{dd}^cT+S)\wedge\beta_\varphi^{p-1}\mathrm{d}t\geq 0. \qquad (2.5)
\end{aligned}
$$

This shows that the function Γ is increasing, and therefore the limit $\lambda=\lim_{r\to 0^+}\Gamma(r)$ exists. Now, the integrability of $\dfrac{\nu(S,\varphi,t)}{t}$ together with uniform boundedness of $\left(1-\dfrac{t^p}{r^p}\right)$ imply that

$$
\lim_{r\to 0^+}\int_0^r\left(1-\frac{t^p}{r^p}\right)\frac{1}{t^p}\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\mathrm{d}t=0. \qquad (2.6)
$$

Hence, $\nu(T,\varphi)=\lim_{r\to 0^+}\nu(T,\varphi,r)=\lim_{r\to 0^+}\Gamma(r)=\lambda.$ □

We should point out to the work of Ghiloufi [8] where he made a nice contribution in the case when $\mathrm{dd}^cT\leq 0$.

Application For positive closed current S, if the Lelong–Demailly number $\nu(U,\varphi)$ of the potential U of S is obtained, then $\nu(T,\varphi)$ exists. Indeed, the fulfillment of the condition (2.2) can be checked by applying Lemma 2.1 to U and letting $r_1\to 0^+$. More precisely, we have

$$
\begin{aligned}
0\leq\int_0^{r_2}\left(\frac{1}{t^p}-\frac{1}{r_2^p}\right)\int_{B_\varphi(t)}S\wedge\beta_\varphi^{p-1}\mathrm{d}t&\leq\nu(U,\varphi,r_2)-\nu(U,\varphi)\\
&-\int_{B_\varphi(r_2)}U\wedge\alpha_\varphi^p\\
&<\infty
\end{aligned}
\qquad (2.7)
$$

Another useful application of Theorem 2.2 occurs when we consider a positive function $h\in\mathcal{C}^2(\Omega)$ alongside a positive closed current T. In such a situation, $\nu(hT,\varphi)$ exists regardless the fact that $\mathrm{dd}^c(hT)$ may not be positive nor negative. Actually, one can, locally, find a positive constant M such that

$$
\mathrm{dd}^c(hT)\geq -MT\wedge\beta.
$$

Now we achieve the required Lelong–Demailly number by taking $S=MT\wedge\beta$.

3 Wedge Product of Positive Currents

In this section we continue the work in [1]. From now on, in this section, we assume that $g \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \backslash L_g)$. Here we start by recalling a classical result due to Fornæss and Sibony [7].

Lemma 3.1 *Let T be a positive closed current on Ω and $u \in Psh^-(\Omega)$. If $\mathcal{H}_{2p}(L_u \cap$ SuppT$) = 0$, then the currents uT together with $dd^c u \wedge T$ are well defined.*

The result was first considered by Demailly in [5] where he studied the case when $\mathcal{H}_{2p-1}(L_u \cap \mathrm{Supp}T) = 0$.

We proceed by studying the case of compact obstacles. More precisely, for positive plurisubharmonic currents T one can obtain the current $T \wedge dd^c g$.

Theorem 3.2 *Let T be a positive plurisubharmonic current of bi-dimension (p, p) on Ω. If L_g is compact, then*

(1) $g dd^c T$ is a well defined current on Ω, and the trivial extension $\widetilde{dd^c g \wedge T}$ exists.
(2) $dd^c g \wedge T$ is a well defined current as soon as L_g, in addition, is complete pluripolar and $p \geq 2$.
(3) $dd^c g \wedge T$ is a well defined current when L_g is considered to be a single point.

Proof Assume that $(g_j)_j$ is a decreasing sequence of smooth plurisubharmonic functions on Ω converging to g in $\mathcal{C}^1(\Omega \backslash L_g)$. Let W and W' be neighborhoods of L_g such that $W \Subset W'$, and take a positive function $f \in \mathcal{C}_0^\infty(W')$ so that $f = 1$ on a neighborhood of W. Then we have

$$\int_{W'} dd^c(f g_j) \wedge T \wedge \beta^{p-1} = \int_{W'} f g_j \wedge dd^c T \wedge \beta^{p-1} \leq 0. \tag{3.1}$$

This implies that

$$\begin{aligned} 0 &\leq \int_{W'} f dd^c g_j \wedge T \wedge \beta^{p-1} - \int_{W'} f g_j \, dd^c T \wedge \beta^{p-1} \\ &\leq \left| \int_{W'} dg_j \wedge d^c f \wedge T \wedge \beta^{p-1} \right| + \left| \int_{W'} df \wedge d^c g_j \wedge T \wedge \beta^{p-1} \right| \\ &\quad + \left| \int_{W'} g_j dd^c f \wedge T \wedge \beta^{p-1} \right|. \end{aligned} \tag{3.2}$$

Thanks to the properties of f, each term of the first line integrals of (3.2) is uniformly bounded. Therefore, one can infer the existence of both extensions $\widetilde{g dd^c T}$ and $\widetilde{dd^c g \wedge T}$. Notice that, the current $g dd^c T$ is well defined by the monotone convergence, and by Banach-Alaoglu, the sequence $(dd^c g_j \wedge T)$ has a subsequence $(dd^c g_{j_s} \wedge T)$ converges weakly* to a current denoted by $dd^c g \wedge T$. To show (2), we first note that $dd^c g \wedge dd^c T$ is well defined as well, thanks to the continuity of the operators d and d^c. Hence by [4] the residual current $R = dd^c \widetilde{g \wedge dd^c} T - dd^c(\widetilde{dd^c g} \wedge T)$ is positive and supported in L_g. Now, if we set $F := dd^c g \wedge T - \widetilde{dd^c g \wedge T}$ we find clearly that F is a positive current where

$$dd^c F = dd^c g \wedge dd^c T - dd^c(\widetilde{dd^c g \wedge T}) \geq dd^c g \wedge dd^c T - \widetilde{dd^c g \wedge dd^c T} \geq 0.$$

As F is a compactly supported current, one can infer that $F \equiv 0$ using [9]. The third statement comes immediately from the fact that the distribution $\mu := (dd^c g \wedge T - \widetilde{dd^c g \wedge T}) \wedge \beta^{p-1}$ is positive and supported in L_g. Indeed, L_g can be assumed to be the origin, and hence there exists a positive constant c such that $\mu = c\delta_0$ where δ_0 is the Dirac measure. Clearly, the constant c is independent from the choice of j_s since

$$\begin{aligned} c = \mu(f) &= \lim_{s\to\infty} \int_{W'} f dd^c g_{j_s} \wedge T \wedge \beta^{p-1} - \int_{W'} f \widetilde{dd^c g \wedge T} \wedge \beta^{p-1} \\ &= \int_{W'} (g dd^c f + 2dg \wedge d^c f) \wedge T \wedge \beta^{p-1} \\ &\quad + \int_{W'} f g dd^c T \wedge \beta^{p-1} - \int_{W'} f \widetilde{dd^c g \wedge T} \wedge \beta^{p-1}. \end{aligned}$$

In other words, $dd^c g \wedge T$ is well defined. □

Lemma 3.3 *Let T be a positive plurisubharmonic current of bi-dimension (p, p) on Ω. Let K and L be compact sets of Ω with $L \subset \overset{\circ}{K}$. Then there exist positive constant $C_{K,L}$, and a neighborhood V of $K \cap L_g$ such that for all $g \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \backslash L_g)$ so that $\mathcal{H}_{2p-1}(L_g \cap Supp\, T) = 0$ we have*

$$\|dd^c g \wedge T\|_{L\backslash L_g} \leq C_{K,L} \|g\|_{\mathcal{L}^\infty(K\backslash V)} \|T\|_{K\backslash V} \tag{3.3}$$

Proof Assume that g is a smooth negative function and $0 \in \mathrm{Supp}\, T \cap L_g$. Since $\mathcal{H}_{2p-1}(L_g \cap \mathrm{Supp}\, T) = 0$, by Bishop [3] and Shiffman [10], there exist a system of coordinates $(z', z'') \in \mathbb{C}^s \times \mathbb{C}^{n-s}$, $s = p-1$ and a polydisk $\triangle^n = \triangle' \times \triangle''$ such that $\overline{\triangle'} \times \partial \triangle'' \cap (\mathrm{Supp}\, T \cap L_g) = \emptyset$. Now, take $0 < t < 1$ so that $\triangle' \times \{z'', t < |z''| < 1\} \cap (\mathrm{Supp}\, T \cap L_g) = \emptyset$. As $\overline{\triangle^n} \cap L_g$ is a compact set, one can find a neighborhood ω of $\overline{\triangle^n} \cap L_g$ such that $\omega \cap (\triangle' \times \{z'', t < |z''| < 1\}) = \emptyset$. Let $a \in (t, 1)$ and choose $\rho(z') \in \mathcal{C}_0^\infty(a\triangle')$ such that $0 \leq \rho \leq 1$ and $\rho = 1$ on $\frac{1}{2}a\triangle'$. Take $\chi \in \mathcal{C}_0^\infty(\omega)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on a neighborhood ω_0 of $\overline{\triangle^n} \cap L_g$. Obviously, the function $\chi(z)\rho(z')$ is positive smooth and compactly supported in $a\triangle^n$. For convenience, we set $\beta' = dd^c|z'|^2$, $\beta'' = dd^c|z''|^2$ and $\alpha(z') = \rho(z')\beta'^s$. By using Stokes' formula

$$\int_{\triangle'\times\triangle''} dd^c(\chi g) \wedge T \wedge \alpha(z') = \int_{\triangle'\times\triangle''} \chi g dd^c T \wedge \alpha(z'). \tag{3.4}$$

Then (3.4) implies that

$$\begin{aligned}0 &\leq \int_{\triangle'\times\triangle''} \chi \mathrm{dd}^c g \wedge T \wedge \alpha(z') - \int_{\triangle'\times\triangle''} \chi g \mathrm{dd}^c T \wedge \alpha(z') \\ &\leq \left|\int_{\triangle'\times\triangle''} g \mathrm{dd}^c \chi \wedge T \wedge \alpha(z')\right| \\ &\quad + \left|\int_{\triangle'\times\triangle''} dg \wedge d^c \chi \wedge T \wedge \alpha(z')\right| \\ &\quad + \left|\int_{\triangle'\times\triangle''} d\chi \wedge d^c g \wedge T \wedge \alpha(z')\right|.\end{aligned} \tag{3.5}$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\left|\int_{\triangle'\times\triangle''} \mathrm{d}g \wedge \mathrm{d}^c \chi \wedge T \wedge \alpha(z')\right| \\ &\leq \left(\int_{\triangle'\times\triangle''} \mathrm{d}g \wedge \mathrm{d}^c g \wedge T \wedge \alpha(z')\right)^{\frac{1}{2}} \left(\int_{\triangle'\times\triangle''} \mathrm{d}\chi \wedge \mathrm{d}^c \chi \wedge T \wedge \alpha(z')\right)^{\frac{1}{2}}.\end{aligned}$$

As the forms $\mathrm{d}\chi$, $\mathrm{d}^c\chi$ and $\mathrm{dd}^c\chi$ vanish on some neighborhood V' of $\overline{\triangle^n} \cap L_g$, the limitation of the right hand side integrals of (3.5) is easily achieved. Therefore, by [1], there exists a constant $C \geq 0$ such that

$$\int_{\omega_0 \cap \frac{1}{2} a \triangle^n} \mathrm{dd}^c g \wedge T \wedge \beta^{p-1} \leq C \|g\|_{\mathcal{L}^\infty(\triangle^n \setminus V)} \|T\|_{\triangle^n \setminus V}. \tag{3.6}$$

Now, consider $g \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \setminus L_g)$. Set $\Gamma_m = \{z \in \overline{\triangle^n}, d(z, L_g) < \frac{1}{m}\}$ and put

$$a_m = \inf_{z \in \overline{\triangle^n} \setminus \Gamma_m} g(z) \tag{3.7}$$

Since g is continuous on $\Omega \setminus L_g$, then $a_m > -\infty$ for all m. For $\varepsilon_m > 0$ small enough, set

$$g_m = \max_{\varepsilon_m} \left(g, a_m - \frac{1}{2^m}\right) \tag{3.8}$$

Observe that, g_m is smooth plurisubharmonic function and $g_m = g$ on $\overline{\triangle^n} \setminus \Gamma_m$. Then by the previous part, for m sufficiently large we have

$$\begin{aligned}\int_{a\overline{\triangle^n} \setminus \Gamma_m} \mathrm{dd}^c g \wedge T \wedge \beta^{p-1} &\leq \int_{\overline{\triangle^n}} \mathrm{dd}^c g_m \wedge T \wedge \beta^{p-1} \\ &\leq C' \|g_m\|_{\mathcal{L}^\infty(\triangle^n \setminus V)} \|T\|_{\triangle^n \setminus V}.\end{aligned} \tag{3.9}$$

Clearly, the result follows by taking the limit over m as what we have shown is true for almost all choice of unitary coordinates (z', z''). □

The above version of Chern–Levine–Nirenberg inequality implies the existence of the trivial extension $\widetilde{dd^c g \wedge T}$ across L_g. Moreover, under the hypotheses of Theorem 3.3, the current $g dd^c T$ is well defined thanks to (3.5). Analogously, one can induce another version to plurisuperharmonic currents.

Corollary 3.4 *Let T be a positive plurisuperharmonic current of bi-dimension (p, p) on Ω. Let K and L be compact sets of Ω with $L \subset \overset{\circ}{K}$. Then there exist positive constant $C_{K,L}$, and a neighborhood V of $K \cap L_g$ such that for all $g \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \backslash L_g)$ so that $\mathcal{H}_{2p-2}(L_g \cap \text{Supp } T) = 0$ we have*

$$\|dd^c g \wedge T\|_{L \backslash L_g} \leq C_{K,L} \|g\|_{\mathcal{L}^\infty(K \backslash V)} \|T\|_{K \backslash V}. \tag{3.10}$$

Notice that the term $\|dd^c T\|$ is neglected in (3.10) as this term is dominated by $\|T\|$.

Proof As $\mathcal{H}_{2p-2}(L_g \cap \text{Supp } T) = 0$, Lemma 3.1 shows that the current $g dd^c T$ is well defined. Now the statement follows by reenacting a similar technique as in the proof of Lemma 3.3 □

Theorem 3.5 *Let T be a positive plurisuperharmonic current of bi-dimension (p, p) on Ω and $g \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \backslash L_g)$. If $\mathcal{H}_{2p-2}(L_g \cap \text{SuppT}) = 0$ and $(g_j)_j$ converges to g in $\mathcal{C}^1(\Omega \backslash L_g)$, then $dd^c g \wedge T$ is a well defined current as a limit of $dd^c g_j \wedge T$ on Ω.*

Proof In virtue of Corollary 3.4 alongside Banach-Alaoglu, there exists a subsequence $dd^c g_{j_s} \wedge T$ converges to a current denoted by $dd^c g \wedge T$. Consider now the current $R := dd^c g \wedge T - \widetilde{dd^c g \wedge T}$. It is clear that R is a $\mathbb{C}$-flat current and supported in L_g. Therefore, $R \equiv 0$ by the support theorem. This means that $dd^c g \wedge T$ does not depend on the choice of j_s proving our statement. □

Remark 3.6 Theorem 3.5 is fulfilled for the case when T is defined on $\Omega \backslash L_g$. Indeed, by [4], the extension $\widetilde{T}$ exists and is positive plurisuperharmonic.

The condition on L_g in Theorem 3.5 is sharp and the thickness can not be reduced for anymore as illustrated in the following example.

Example 3.7 Let $T = \sum_{k=2}^{\infty} \frac{-1}{k \log^2 k} \log |z_1 - \frac{1}{k}|^2$ and $g(z) = \log |z_1|^2$, then T is a positive dd^c-negative current on $\frac{1}{2}\triangle^n$ of bi-dimension (n, n), and $g \in Psh(\triangle^n) \cap \mathcal{C}^\infty(\triangle^n \backslash \{z_1 = 0\})$. In spite that $\mathcal{H}_{2n-2}(\{z_1 = 0\})$ is locally finite, the mass of $dd^c g \wedge T$ explodes near $\{z_1 = 0\}$.

The cases studied previously can be employed to generalize the wedge products in [1,2] to the following assertion.

Theorem 3.8 *Let T be a positive plurisubharmonic current of bi-dimension (p,p) on Ω and $g \in Psh^{-}(\Omega)\cap\mathcal{C}^1(\Omega\backslash L_g)$. If $\mathcal{H}_{2p-2}(L_g \cap \mathrm{SuppT})$ is locally finite and $(g_j)_j$ is a decreasing sequence of smooth plurisubharmonic functions converging to g in $\mathcal{C}^1(\Omega\backslash L_g)$, then $\mathrm{dd}^c g\wedge T$ is a well defined current as a limit of $\mathrm{dd}^c g_j \wedge T$ on Ω.*

Proof We keep the notations of the proof of Lemma 3.3. For each z' we set $A_{z'} = (\mathrm{Supp}\, T\cap L_g)\cap(\{z'\}\times\triangle'')$. Since $\mathcal{H}_{2p-2}(\mathrm{Supp}\, T\cap L_g)$ is locally finite, then by Shiffman [10] the set $A_{z'}$ is a discrete subset for a.e. z'. Without loss of generality, we may assume that $A_{z'}$ is reduced to a single point $(z',0)$. On the other hand, T is $\mathbb{C}$-flat on Ω. Thus, the slice $\langle T,\pi,z'\rangle$ exists for a.e. z', and is a positive plurisubharmonic current of bidimension $(1,1)$ on Ω, supported in $\{z'\}\times\triangle^{n-p+1}$. Hence, by Theorem 3.2, the sequence $\langle \mathrm{dd}^c g_j\wedge T,\pi,z'\rangle$ is weakly* convergent since $A_{z'}$ is a single point. So by applying the slice formula we have

$$\begin{aligned}\int_{\triangle'\times\triangle''}\mathrm{dd}^c g\wedge T\wedge\pi^*\beta'^{p-1} &= \lim_{s\to\infty}\int_{\triangle'\times\triangle''}\mathrm{dd}^c g_{j_s}\wedge T\wedge\pi^*\beta'^{p-1}\\ &= \lim_{s\to\infty}\int_{z'}\langle \mathrm{dd}^c g_{j_s}\wedge T,\pi,z'\rangle\beta'^{p-1}\\ &= \lim_{j\to\infty}\int_{z'}\langle \mathrm{dd}^c g_{j}\wedge T,\pi,z'\rangle\beta'^{p-1}\\ &= \lim_{j\to\infty}\int_{\triangle'\times\triangle''}\mathrm{dd}^c g_{j}\wedge T\wedge\pi^*\beta'^{p-1}.\end{aligned}$$

This completes the proof. □

As a consequence of the precedent result, one can prolong Alessandirini-Bassanelli study [2] to plurisubharmonic currents.

Corollary 3.9 *Let T be a positive plurisubharmonic current of bidimension (p,p) on an open subset Ω of $\mathbb{C}^n$ and A be an analytic subset of Ω, $dimA < p$. Let F be a positive closed current of bidimension $(n-1,n-1)$ on Ω and smooth on $\Omega\backslash A$. Then there exists a unique current on Ω denoted by $F\wedge T$ with the following property.*

If g is a solution of $\mathrm{dd}^c g = F$ on an open subset $U\subset\Omega$, and $(g_j)_j$ is a decreasing sequence of smooth plurisubharmonic functions on U converging pointwise to g on $U\backslash A$ such that g_j converges to g in $\mathcal{C}^1(\Omega\backslash A)$, then $\boldsymbol{F}\wedge T = \lim_{j\to\infty}\mathrm{dd}^c g_j\wedge T$ on U.

Another useful application appears when we take $g = \log|f|^2$, where f is a holomorphic function on Ω. In such a situation, the current $\mathrm{dd}^c g\wedge T$ is well defined as soon as the zero divisors Z_f of f are at most of dimension $p-1$. Notice that slice of T is related to the current $\mathrm{dd}^c g\wedge T$.

Remark 3.10 The previous proof can be implemented to relax the condition of Lemma 3.1 for a special case. In fact, the statement of that lemma remains true when $\mathcal{H}_{2p}$ is locally finite as soon as T has a locally bounded potential U. The proof comes as an application of Theorem 3.8 on the current U.

We finally, by induction, deduce some consequences for longer wedge product.

Theorem 3.11 *Let T be a positive plurisubharmonic (resp. plurisuperharmonic) current of bi-dimension (p, p) on an open subset Ω of $\mathbb{C}^n$ and let $A_1, \dots, A_q$ be closed subsets of Ω such that* $\mathcal{H}_{2(p-m)+1}(\text{Supp T} \cap A_{j_1} \cap \cdots \cap A_{j_m}) = 0$ *(resp.$\mathcal{H}_{2(p-m)-1}(SuppT \cap A_{j_1} \cap \cdots \cap A_{j_m}) = 0$) for all choices of $j_1 < \cdots < j_m$ in $\{1, \dots, q\}$. Let K and L be compact sets of Ω with $L \subset \overset{\circ}{K}$. Then there exist a positive constant $I_{K,L}$ and a neighborhoods V_j of $K \cap A_j$ such that for all $g_j \in Psh^-(\Omega) \cap \mathcal{C}^1(\Omega \backslash A_j)$, $L_{g_j} \subset A_j$ we have*

$$\left\| \bigwedge_{j=1}^{q} \mathrm{dd}^c g_j \wedge T \right\|_{L \backslash \bigcup_{j=1}^{q} A_j} \leq I_{K,L} \prod_{j=1}^{q} \|g_j\|_{\mathcal{L}^\infty(K \backslash V_j)} \|T\|_K. \tag{3.11}$$

Theorem 3.12 *Under the same hypotheses of Theorem 3.11, if T is positive plurisubharmonic and each A_j is analytic so that* $\text{Supp T} \cap A_{j_1} \cap \cdots \cap A_{j_m}$ *is at most of dimension $p - m$ for all choices of $j_1 < \cdots < j_m$ in $\{1, \dots, q\}$, then $\bigwedge_{j=1}^{q} \mathrm{dd}^c g_j \wedge T$ is well defined.*

These results generalize the case of closed current which was proved by Demailly [5].

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