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Generalized Lie (Jordan) Triple Derivations on Arbitrary Triangular Algebras

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Abstract

In this paper, we give a description of Lie (Jordan) triple derivations and generalized Lie (Jordan) triple derivations of an arbitrary triangular algebra $\mathfrak{A}$ through a triangular algebra $\mathfrak{A}^0$, where $\mathfrak{A}^0$ is a triangular algebra constructed from the given triangular algebra $\mathfrak{A}$ using the notion of maximal left (right) ring of quotients such that $\mathfrak{A}$ is the subalgebra of $\mathfrak{A}^0$ having the same unity.

Keywords Lie triple derivation · Generalized Lie triple derivation · Jordan triple derivation · Generalized Jordan triple derivation · Maximal left (right) ring of quotients · Triangular algebra

Mathematics Subject Classification 16W25 · 16R60

1 Introduction

Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. An $\mathcal{R}$-linear mapping $\Delta : \mathcal{A} \to \mathcal{A}$ is said to be a *derivation* if $\Delta(xy) = \Delta(x)y + x\Delta(y)$ holds for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is said to be a *Lie derivation* (resp. *Lie triple derivation*) if $\mathcal{L}([x, y]) = [\mathcal{L}(x), y] + [x, \mathcal{L}(y)]$ (resp. $\mathcal{L}([[x, y], z]) = [[\mathcal{L}(x), y], z] + [[x, \mathcal{L}(y)], z] + [[x, y], \mathcal{L}(z)]$) holds for all $x, y, z \in \mathcal{A}$, where $[x, y] = xy - yx$ is the usual Lie product. An $\mathcal{R}$-linear mapping

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$\mathcal{D}_\Delta : \mathcal{A} \to \mathcal{A}$ is said to be a *generalized derivation* with associated derivation Δ on $\mathcal{A}$ if $\mathcal{D}_\Delta(xy) = \mathcal{D}_\Delta(x)y + x\Delta(y)$ for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $\mathcal{G}_\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is said to be a *generalized Lie derivation* (resp. *generalized Lie triple derivation*) with associated *Lie derivation* (resp. *Lie triple derivation*) $\mathcal{L}$ on $\mathcal{A}$ if $\mathcal{G}_\mathcal{L}([x, y]) = [\mathcal{G}_\mathcal{L}(x), y] + [x, \mathcal{L}(y)]$ (resp. $\mathcal{G}_\mathcal{L}([[x, y], z]) = [[\mathcal{G}_\mathcal{L}(x), y], z] + [[x, \mathcal{L}(y)], z] + [[x, y], \mathcal{L}(z)]$) holds for all $x, y, z \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $J : \mathcal{A} \to \mathcal{A}$ is said to be a *Jordan derivation* (resp. *Jordan triple derivation*) if $J(x^2) = J(x)x + xJ(x)$ (resp. $J(xyx) = J(x)yx + xJ(y)x + xyJ(x)$) holds for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $\mathcal{G}_J : \mathcal{A} \to \mathcal{A}$ is said to be a *generalized Jordan derivation* (resp. *generalized Jordan triple derivation*) with associated *Jordan derivation* (resp. *Jordan triple derivation*) J on $\mathcal{A}$ if $\mathcal{G}_J(x^2) = \mathcal{G}_J(x)x + xJ(x)$ (resp. $\mathcal{G}_J(xyx) = \mathcal{G}_J(x)yx + xJ(y)x + xyJ(x)$) holds for all $x, y \in \mathcal{A}$. It can be easily seen that every derivation is a Lie derivation as well as a Jordan derivation and every Lie derivation is a Lie triple derivation. Also, every Lie triple derivation is a generalized Lie triple derivation. However, the converse need not be true in general.

Several authors studied Lie derivations, Lie triple derivations, as well as other Lie mappings on different algebras for many years. A central problem is whether a Lie derivation on $\mathcal{A}$ can be decomposed into the sum of a derivation and a linear mapping from $\mathcal{A}$ to its center $Z(\mathcal{A})$ vanishing on each commutator. Lie derivations of the above form are called proper. Martindale [18] was the first one who showed that every Lie derivation on a certain primitive ring is proper. In the year 1993, Brešar [7] gave a characterization of Lie derivations on prime rings. In the year 2000, Cheung [9] initiated the study of linear mappings on triangular algebras. He described Lie derivations, commuting mappings and automorphisms of triangular algebras. In 2003, Cheung [10] studied Lie derivations on triangular algebras and proved that every Lie derivation on triangular algebras under certain conditions is proper. Further, Xiao and Wei [24] extended this result to Lie triple derivations on triangular algebras. Qi and Hou [19] studied additive generalized Lie derivations on nest algebras and proved that an additive mapping is an additive generalized Lie derivation if and only if it is the sum of an additive generalized derivation and an additive mapping from the algebra into its center vanishing on all commutators. Benkovič [6] obtained that under certain conditions each generalized Lie derivation of a triangular algebra is the sum of a generalized derivation and a central mapping which vanishes on all commutators of triangular algebra. In the year 2017, Ashraf and Jabeen [2] characterized nonlinear generalized Lie triple derivations of triangular algebras. In addition, the characterization of Lie derivations, Lie triple derivations and generalized Lie triple derivations on various algebras are considered in [1,3,5,13–17], etc.

In the year 1956, Utumi [20] introduced the concept of the maximal left ring of quotients and he proved that every unital ring has a maximal left (right) ring of quotients. For a detailed study of maximal left (right) ring of quotients, the reader is referred to [4,8]. Eremita [11] initiated the study of functional identities of degree 2 in certain triangular rings. Furthermore, Eremita [12] explored functional identities of degree 2 for a more general class of triangular rings using the notion of maximal left ring of quotient. Wang [21] considered functional identities of degree 2 in arbitrary triangular rings. Recently, Wang [22] constructed a triangular algebra from a given triangular algebra using the notion of maximal left (right) ring of quotients and characterized Lie (Jordan) derivations through the constructed triangular algebra.

Motivated by the above work, the objective of this article is to generalize the result of Wang [22] in the setting of Lie (Jordan) triple derivation and generalized Lie (Jordan) triple derivation. In fact, it is shown that under certain appropriate conditions every generalized Lie triple derivation on an arbitrary triangular algebra is proper.

2 Preliminaries

Throughout this paper, we shall use the following notations: For any ring (algebra) $\mathcal{A}$, the maximal left (resp. right) ring of quotients is denoted by $Q_\ell(\mathcal{A})$ (resp. $Q_r(\mathcal{A})$). The center of $Q_\ell(\mathcal{A})$ (resp. $Q_r(\mathcal{A})$), called left (resp. right) extended centroid of $\mathcal{A}$, is denoted by $C_\ell(\mathcal{A})$ (resp. $C_r(\mathcal{A})$).

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over $\mathcal{R}$ and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module, that is, for $a \in \mathcal{A}$, $a\mathcal{M} = 0$ implies $a = 0$ and also as a right $\mathcal{B}$-module, that is, for $b \in \mathcal{B}$, $\mathcal{M}b = 0$ implies $b = 0$. The $\mathcal{R}$-algebra

$$\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| \; a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. The center of $\mathfrak{A}$ is given by

$$Z(\mathfrak{A}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; a \in \mathcal{A}, b \in \mathcal{B}, am = mb \text{ for all } m \in \mathcal{M} \right\}.$$

Define two natural projections $\pi_\mathcal{A} : \mathfrak{A} \to \mathcal{A}$ and $\pi_\mathcal{B} : \mathfrak{A} \to \mathcal{B}$ by $\pi_\mathcal{A}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a$ and $\pi_\mathcal{B}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b$. Moreover, $\pi_\mathcal{A}(Z(\mathfrak{A})) \subseteq Z(\mathcal{A})$ and $\pi_\mathcal{B}(Z(\mathfrak{A})) \subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\eta : \pi_\mathcal{A}(Z(\mathfrak{A})) \to \pi_\mathcal{B}(Z(\mathfrak{A}))$ such that $am = m\eta(a)$ for all $m \in \mathcal{M}$.

Let $1_\mathcal{A}$ (resp.$1_\mathcal{B}$) be the identity of the algebra $\mathcal{A}$ (resp.$\mathcal{B}$) and let $I = \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 1_\mathcal{B} \end{pmatrix}$ be the identity of triangular algebra $\mathfrak{A}$. Set $e = \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ and $f = I - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_\mathcal{B} \end{pmatrix}$. Then, $\mathfrak{A}$ can be written as $\mathfrak{A} = e\mathfrak{A}e \oplus e\mathfrak{A}f \oplus f\mathfrak{A}f$, where $e\mathfrak{A}e$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{A}$, $f\mathfrak{A}f$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{B}$ and $e\mathfrak{A}f$ is a $(e\mathfrak{A}e, f\mathfrak{A}f)$-bimodule isomorphic to the bimodule $\mathcal{M}$.

3 Lie (Jordan) Triple Derivations on Arbitrary Triangular Algebras

Many authors studied Lie triple derivations on different rings and algebras. Recently, Wang [22] constructed a triangular algebra from a given triangular algebra using the notion of maximal left (right) ring of quotients and characterized Lie (Jordan) derivations on the newly constructed triangular algebra. In this section, we extend this result in the setting of Lie triple derivation. In order to obtain our result, we begin with the following known results:

Lemma 3.1 [22, Propositions 2.1 & 2.2] *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra. Then, the following assertions hold:*

(i) $Z(\mathfrak{A}) = \{c \in e\mathfrak{A}e + f\mathfrak{A}f \mid cexf = exfc \text{ for all } x \in \mathfrak{A}\}$,
(ii) $C_\ell(\mathfrak{A}) = \{q \in eQ_\ell(\mathfrak{A})e + fQ_\ell(\mathfrak{A})f \mid qexf = exfq \text{ for all } x \in \mathfrak{A}\}$,
(iii) $C_r(\mathfrak{A}) = \{q \in eQ_r(\mathfrak{A})e + fQ_r(\mathfrak{A})f \mid qexf = exfq \text{ for all } x \in \mathfrak{A}\}$,
(iv) $Z(e\mathfrak{A}e) \subseteq C_\ell(\mathfrak{A})e$,
(v) $Z(f\mathfrak{A}f) \subseteq C_r(\mathfrak{A})f$,
(vi) *there exists a unique algebra isomorphism $\tau_\ell : C_\ell(\mathfrak{A})e \to C_\ell(\mathfrak{A})f$ such that $\lambda e.exf = exf.\tau_\ell(\lambda e)$ for all $x \in \mathfrak{A}, \lambda \in C_\ell(\mathfrak{A})$. Moreover, $\tau_\ell(Z(\mathfrak{A})e) = Z(\mathfrak{A})f$,*
(vii) *there exists a unique algebra isomorphism $\tau_r : C_r(\mathfrak{A})e \to C_r(\mathfrak{A})f$ such that $\lambda e.exf = exf.\tau_r(\lambda e)$ for all $x \in \mathfrak{A}, \lambda \in C_r(\mathfrak{A})$. Moreover, $\tau_r(Z(\mathfrak{A})e) = Z(\mathfrak{A})f$.*

Lemma 3.2 [22, Theorem 2.1] *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra. Then, $\mathfrak{A}^\circ = \begin{pmatrix} \mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B})) & \mathcal{M} \\ 0 & \mathcal{B}{\tau_\ell}^{-1}(Z(\mathcal{A})) \end{pmatrix}$ is a triangular algebra such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^\circ$ having the same unity.*

Lemma 3.3 [22, Lemma 3.1] *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $\mathfrak{A}_1 = Tri(\mathcal{A}_1, \mathcal{M}, \mathcal{B}_1)$ be triangular algebras such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_1$ both having the same unity. Moreover, $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B} \subseteq \mathcal{B}_1$. A linear mapping $\Delta : \mathfrak{A} \to \mathfrak{A}_1$ is a derivation if and only if the following conditions are satisfied:*

$$\Delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & an - nb + f(m) \\ 0 & p_{\mathcal{B}}(b) \end{pmatrix},$$

where $n \in \mathcal{M}$ and

(1) *$p_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_1$ is a derivation, $f(am) = p_{\mathcal{A}}(a)m + af(m)$;*
(2) *$p_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}_1$ is a derivation, $f(mb) = mp_{\mathcal{B}}(b) + f(m)b$.*

The main result of this section states as follows:

Theorem 3.4 *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{L}$ be a Lie triple derivation on $\mathfrak{A}$. Then, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and $\mathcal{L}$ can be written as $\mathcal{L} = \Delta + \chi$, where $\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a derivation and $\chi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\chi([[x, y], z]) = 0$ for all $x, y, z \in \mathfrak{A}$.*

Proof Consider

$$\mathfrak{A}^0 = \begin{pmatrix} \mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B})) & \mathcal{M} \\ 0 & \mathcal{B}{\tau_\ell}^{-1}(Z(\mathcal{A})) \end{pmatrix}.$$

In view of Lemma 3.2, we find that $\mathfrak{A}^0$ is a triangular algebra such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity. It follows from [16, Proposition 4.1] that

$$\mathcal{L}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_1(a) + g_1(b) & an - nb + h_2(m) \\ 0 & f_3(a) + g_3(b) \end{pmatrix},$$

where $a \in \mathcal{A}$; $n, m \in \mathcal{M}$; $b \in \mathcal{B}$ and $f_1 : \mathcal{A} \to \mathcal{A}, g_1 : \mathcal{B} \to [\mathcal{A}, \mathcal{A}]', h_2 : \mathcal{M} \to \mathcal{M}$, $f_3 : \mathcal{A} \to [\mathcal{B}, \mathcal{B}]'$, $g_3 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying

(1) f_1 is a Lie triple derivation on $\mathcal{A}$, $[[f_3(a), b_1], b_2] = 0$, $h_2(am) = f_1(a)m - mf_3(a) + ah_2(m)$ for all $a_1, a_2, a_3 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, $m \in \mathcal{M}$;
(2) g_3 is a Lie triple derivation on $\mathcal{B}$, $[[g_1(b), a_1], a_2] = 0$, $h_2(mb) = mg_3(b) - g_1(b)m + h_2(m)b$ for all $a_1, a_2 \in \mathcal{A}$, $b_1, b_2, b_3 \in \mathcal{B}$, $m \in \mathcal{M}$.

Using part (v) and (vii) of Lemma 3.1, we have

$$h_2(am) = (f_1(a) - {\tau_r}^{-1}(f_3(a)))m + ah_2(m) \tag{3.1}$$

for all $a \in \mathcal{A}$, $m \in \mathcal{M}$. Similarly, from part (iv) and (vi) of Lemma 3.1, we get

$$h_2(mb) = m(g_3(b) - \tau_\ell(g_1(b))) + h_2(m)b \tag{3.2}$$

for all $b \in \mathcal{B}$, $m \in \mathcal{M}$. Define $\Delta, \chi : \mathfrak{A} \to \mathfrak{A}^0$ by

$$\Delta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} f_1(a) - {\tau_r}^{-1}(f_3(a)) & an - nb + h_2(m) \\ 0 & g_3(b) - \tau_\ell(g_1(b)) \end{pmatrix},$$
$$\chi\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} {\tau_r}^{-1}(f_3(a)) + g_1(b) & 0 \\ 0 & f_3(a) + \tau_\ell(g_1(b)) \end{pmatrix}.$$

It is easy to see that Δ and χ are linear mappings and $\mathcal{L} = \Delta + \chi$.

Set $p_\mathcal{A} = f_1 - {\tau_r}^{-1} \circ f_3$ and $p_\mathcal{B} = g_3 - \tau_\ell \circ g_1$. It can be easily seen that $p_\mathcal{A} : \mathcal{A} \to \mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B}))$ and $p_\mathcal{B} : \mathcal{B} \to \mathcal{B}\tau_\ell(Z(\mathcal{A}))$ are linear mappings. We assert that $p_\mathcal{A}$ and $p_\mathcal{B}$ are derivations. From Eq. (3.1), we have

$$h_2(aa'm) = p_\mathcal{A}(aa')m + aa'h_2(m)$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. On the other hand, we get

$$\begin{aligned} h_2(aa'm) &= p_\mathcal{A}(a)a'm + ah_2(a'm) \\ &= p_\mathcal{A}(a)a'm + ap_\mathcal{A}(a')m + aa'h_2(m) \end{aligned}$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. Comparing the above two expressions, we arrive at

$$p_\mathcal{A}(aa')m = p_\mathcal{A}(a)a'm + ap_\mathcal{A}(a')m$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B}))$-module, we get

$$p_{\mathcal{A}}(aa') = p_{\mathcal{A}}(a)a' + ap_{\mathcal{A}}(a')$$

for all $a, a' \in \mathcal{A}$ and hence, $p_{\mathcal{A}}$ is a derivation. In a similar manner, from Eq. (3.2), we can prove that $p_{\mathcal{B}}$ is a derivation.

From Eqs. (3.1),(3.2) and Lemma 3.1, we conclude that $\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a derivation. We now prove that $\chi(\mathfrak{A}) \subseteq Z(\mathfrak{A}^\circ)$. Indeed, we have

$$\begin{aligned}({\tau_r}^{-1}(f_3(a)) + g_1(b))m &= {\tau_r}^{-1}(f_3(a))m + g_1(b)m\\ &= mf_3(a) + m\tau_\ell(g_1(b))\\ &= m(f_3(a) + \tau_\ell(g_1(b)))\end{aligned}$$

for all $m \in \mathcal{M}$. Hence, by Lemma 3.1(i), we get $\chi(\mathfrak{A}) \subseteq Z(\mathfrak{A}^\circ)$.

It remains to prove that $\chi[[x, y], z] = 0$ for all $x, y, z \in \mathfrak{A}$. Since $\mathcal{L} = \Delta + \chi$ is a Lie triple derivation, we have

$$\begin{aligned}\mathcal{L}([[x, y], z]) &= [[\mathcal{L}(x), y], z] + [[x, \mathcal{L}(y)], z] + [[x, y], \mathcal{L}(z)]\\ &= [[\Delta(x) + \chi(x), y], z] + [[x, \Delta(y) + \chi(y)], z]\\ &\quad + [[x, y], \Delta(z) + \chi(z)]\\ &= [[\Delta(x), y], z] + [[x, \Delta(y)], z] + [[x, y], \Delta(z)]\\ &= \Delta([[x, y], z])\\ &= \mathcal{L}([[x, y], z]) - \chi([[x, y], z])\end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. This implies that $\chi([[x, y], z]) = 0$ for all $x, y, z \in \mathfrak{A}$. □

It follows from [23, Theorem 4.4] that every Jordan triple derivation on a 2-torsion free triangular algebra is a Jordan derivation. Combining this fact with Theorem 3.4, we get the following corollary:

Corollary 3.5 *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and J be a Jordan triple derivation on $\mathfrak{A}$. Suppose that $\mathfrak{A}$ is 2-torsion free. Then, J is a derivation. Otherwise, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and J can be written as $J = \Delta + \chi$ where $\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a derivation and $\chi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\chi(xyx) = 0$ for all $x, y \in \mathfrak{A}$.*

Zhang and Yu [25, Theorem 2.1] proved that every Jordan derivation on a 2-torsion free triangular algebra is a derivation. In view of this result and Theorem 3.4, we obtain the following corollary:

Corollary 3.6 [22, Corollary 3.1] *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and J be a Jordan derivation on $\mathfrak{A}$. Suppose that $\mathfrak{A}$ is 2-torsion free. Then, J is a derivation. Otherwise, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and J can be written as $J = \Delta + \chi$ where $\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a derivation and $\chi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\chi(x^2) = 0$ for all $x \in \mathfrak{A}$.*

4 Generalized Lie (Jordan) Triple Derivations on Arbitrary Triangular Algebras

Motivated by the study of Qi and Hou [19], Benkovič [6] and Ashraf and Jabeen [2], in this section we extend Theorem 3.4 for generalized Lie triple derivations on arbitrary triangular algebras. In order to prove our main result of this section, the following Lemma is essential which can be easily obtained by using [2, Proposition 4.3].

Lemma 4.1 *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $\mathfrak{A}_1 = Tri(\mathcal{A}_1, \mathcal{M}, \mathcal{B}_1)$ be triangular algebras such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_1$ both having the same unity. Moreover, $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B} \subseteq \mathcal{B}_1$. A linear mapping $\mathcal{D}_\Delta : \mathfrak{A} \to \mathfrak{A}_1$ is a generalized derivation if and only if the following conditions are satisfied:*

$$\mathcal{D}_\Delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} P_{\mathcal{A}}(a) & an - nb + F(m) \\ 0 & P_{\mathcal{B}}(b) \end{pmatrix},$$

where $n \in \mathcal{M}$ and

(1) $P_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_1$ *is a generalized derivation,* $F(am) = P_{\mathcal{A}}(a)m + af(m)$;
(2) $P_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}_1$ *is a generalized derivation,* $F(mb) = F(m)b + mp_{\mathcal{B}}(b)$.

The main result of this section states as follows:

Theorem 4.2 *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{G}_{\mathcal{L}}$ be a generalized Lie triple derivation on $\mathfrak{A}$. Then, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and $\mathcal{G}_{\mathcal{L}}$ can be written as $\mathcal{G}_{\mathcal{L}} = \mathcal{D}_\Delta + \xi$, where $\mathcal{D}_\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a generalized derivation and $\xi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\xi([[x, y], z]) = 0$ for all $x, y, z \in \mathfrak{A}$.*

Proof Suppose that $\mathcal{L}$ is an associated Lie triple derivation of $\mathcal{G}_{\mathcal{L}}$. It view of Theorem 3.4, there exists a triangular algebra

$$\mathfrak{A}^0 = \begin{pmatrix} \mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B})) & \mathcal{M} \\ 0 & \mathcal{B}{\tau_\ell}^{-1}(Z(\mathcal{A})) \end{pmatrix},$$

such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and $\mathcal{L} = \Delta + \chi$, where $\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a derivation and $\chi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\chi([[x, y], z]) = 0$ for all $x, y, z \in \mathfrak{A}$.

It follows from [2, Proposition 4.17] that

$$\mathcal{G}_{\mathcal{L}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} F_1(a) + G_1(b) & an - nb + H_2(m) \\ 0 & F_3(a) + G_3(b) \end{pmatrix},$$

where $a \in \mathcal{A}, n, m \in \mathcal{M}, b \in \mathcal{B}$ and $F_1 : \mathcal{A} \to \mathcal{A}$, $G_1 : \mathcal{B} \to \mathcal{A}$, $H_2 : \mathcal{M} \to \mathcal{M}$, $F_3 : \mathcal{A} \to \mathcal{B}$, $G_3 : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying

(1) F_1 is a generalized Lie triple derivation on $\mathcal{A}$, $[[F_3(a), b_1], b_2] = 0$, $H_2(am) = F_1(a)m - mF_3(a) + ah_2(m)$ for all $a_1, a_2, a_3 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}, m \in \mathcal{M}$;

(2) G_3 is a generalized Lie triple derivation on $\mathcal{B}$, $[[G_1(b), a_1], a_2] = 0$, $H_2(mb) = H_2(m)b + mg_3(b) - g_1(b)m$ for all $a_1, a_2 \in \mathcal{A}$, $b_1, b_2, b_3 \in \mathcal{B}$, $m \in \mathcal{M}$.

By part (v) of Lemma 3.1, we get $Z(\mathcal{B}) \subseteq C_r(\mathfrak{A})f$. Using Lemma 3.1$(vii)$, we have

$$H_2(am) = (F_1(a) - {\tau_r}^{-1}(F_3(a)))m + ah_2(m) \tag{4.1}$$

for all $a \in \mathcal{A}$, $m \in \mathcal{M}$. From Lemma 3.1(iv), we get $Z(\mathcal{A}) \subseteq C_\ell(\mathfrak{A})e$. In view of Lemma 3.1$(vi)$, we find that

$$H_2(mb) = H_2(m)b + m(g_3(b) - \tau_\ell(g_1(b))) \tag{4.2}$$

for all $b \in \mathcal{B}$, $m \in \mathcal{M}$. Define $\mathcal{D}_\Delta, \xi : \mathfrak{A} \to \mathfrak{A}^0$ by

$$\mathcal{D}_\Delta \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} F_1(a) - {\tau_r}^{-1}(F_3(a)) & an - nb + H_2(m) \\ 0 & G_3(b) - \tau_\ell(G_1(b)) \end{pmatrix},$$
$$\xi \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} {\tau_r}^{-1}(F_3(a)) + G_1(b) & 0 \\ 0 & F_3(a) + \tau_\ell(G_1(b)) \end{pmatrix}.$$

It is easy to see that $\mathcal{D}_\Delta$ and ξ are linear mappings and $\mathcal{G}_\mathcal{L} = \mathcal{D}_\Delta + \xi$.

Suppose that $P_\mathcal{A} = F_1 - {\tau_r}^{-1} \circ F_3$ and $P_\mathcal{B} = G_3 - \tau_\ell \circ G_1$. It is clear that $P_\mathcal{A} : \mathcal{A} \to \mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B}))$ and $P_\mathcal{B} : \mathcal{B} \to \mathcal{B}\tau_\ell(Z(\mathcal{A}))$ are linear mappings. Now, we prove that $P_\mathcal{A}$ and $P_\mathcal{B}$ are generalized derivations with associated derivations $p_\mathcal{A}$ and $p_\mathcal{B}$, respectively.

From Eq. (4.1), we have

$$H_2(aa'm) = P_\mathcal{A}(aa')m + aa'h_2(m)$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. On the other hand, by Eqs. (3.1) and (4.1), we get

$$\begin{aligned} H_2(aa'm) &= P_\mathcal{A}(a)a'm + ah_2(a'm) \\ &= P_\mathcal{A}(a)a'm + ap_\mathcal{A}(a')m + aa'h_2(m) \end{aligned}$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. Comparing the above two relations, we find that

$$P_\mathcal{A}(aa')m = P_\mathcal{A}(a)a'm + ap_\mathcal{A}(a')m$$

for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}{\tau_r}^{-1}(Z(\mathcal{B}))$-module, we get

$$P_\mathcal{A}(aa') = P_\mathcal{A}(a)a' + ap_\mathcal{A}(a')$$

for all $a, a' \in \mathcal{A}$. Hence, $P_\mathcal{A}$ is a generalized derivation with associated derivation $p_\mathcal{A}$. Similarly, using Eqs. (3.2) and (4.2), we can prove that $P_\mathcal{B}$ is a generalized derivation with associated derivation $p_\mathcal{B}$.

In view of Lemma 4.1, we get from Eqs. (4.1) and (4.2) that $\mathcal{D}_\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a generalized derivation associated with derivation Δ. Now, we claim that $\xi(\mathfrak{A}) \subseteq Z(\mathfrak{A}^\circ)$. In fact, we have

$$\begin{aligned}({\tau_r}^{-1}(F_3(a)) + G_1(b))m &= {\tau_r}^{-1}(F_3(a))m + G_1(b)m \\ &= mF_3(a) + m\tau_\ell(G_1(b)) \\ &= m(F_3(a) + \tau_\ell(G_1(b)))\end{aligned}$$

for all $m \in \mathcal{M}$. From part (i) of Lemma 3.1, we get $\xi(\mathfrak{A}) \subseteq Z(\mathfrak{A}^\circ)$.

Finally, we show that $\xi[[x, y], z] = 0$ for all $x, y, z \in \mathfrak{A}$. Since $\mathcal{G}_\mathcal{L} = \mathcal{D}_\Delta + \xi$ is a generalized Lie triple derivation, we have

$$\begin{aligned}\mathcal{G}_\mathcal{L}([[x, y], z]) &= [[\mathcal{G}_\mathcal{L}(x), y], z] + [[x, \mathcal{L}(y)], z] + [[x, y], \mathcal{L}(z)] \\ &= [[\mathcal{D}_\Delta(x) + \xi(x), y], z] + [[x, \Delta(y) + \chi(y)], z] \\ &\quad + [[x, y], \Delta(z) + \chi(z)] \\ &= [[\mathcal{D}_\Delta(x), y], z] + [[x, \Delta(y)], z] + [[x, y], \Delta(z)] \\ &= \mathcal{D}_\Delta([[x, y], z]) \\ &= \mathcal{G}_\mathcal{L}([[x, y], z]) - \xi([[x, y], z])\end{aligned}$$

for all $x, y, z \in \mathfrak{A}$. This implies that $\xi([[x, y], z]) = 0$ for all $x, y, z \in \mathfrak{A}$. □

Wei and Xiao [23, Theorem 4.7] proved that generalized Jordan triple derivation, generalized Jordan derivation and generalized derivation are equivalent on a 2-torsion free triangular algebra. In view of this fact and Theorem 4.2, we immediately obtain the following corollaries:

Corollary 4.3 *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{G}_J$ be a generalized Jordan triple derivation on $\mathfrak{A}$. Suppose that $\mathfrak{A}$ is 2-torsion free. Then, $\mathcal{G}_J$ is a generalized derivation. Otherwise, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and $\mathcal{G}_J$ can be written as $\mathcal{G}_J = \mathcal{D}_\Delta + \xi$ where $\mathcal{D}_\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a generalized derivation and $\xi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\xi(xyx) = 0$ for all $x, y \in \mathfrak{A}$.*

Corollary 4.4 *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{G}_J$ be a generalized Jordan derivation on $\mathfrak{A}$. Suppose that $\mathfrak{A}$ is 2-torsion free. Then, $\mathcal{G}_J$ is a generalized derivation. Otherwise, there exists a triangular algebra $\mathfrak{A}^0$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^0$ having the same unity and $\mathcal{G}_J$ can be written as $\mathcal{G}_J = \mathcal{D}_\Delta + \xi$ where $\mathcal{D}_\Delta : \mathfrak{A} \to \mathfrak{A}^0$ is a generalized derivation and $\xi : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A}^0)$ is a linear mapping such that $\xi(x^2) = 0$ for all $x \in \mathfrak{A}$.*

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