Bull. Malays. Math. Sci. Soc. (2021) 44:3839–3859
https://doi.org/10.1007/s40840-021-01147-2

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Besov Estimates for Weak Solutions of the Parabolic *p*-Laplacian Equations

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Received: 20 July 2020 / Revised: 12 March 2021 / Accepted: 19 May 2021 / Published online: 4 June 2021

Abstract
In this paper, we obtain the local regularity estimates in Besov spaces of weak solutions for the following parabolic p-Laplacian equations:

$$u_t - \operatorname{div} a\,(Du, x, t) = \operatorname{div} \mathbf{F}$$

under some proper assumptions on the functions a and $\mathbf{F}$. Moreover, we would like to point out that our results improve the known results for such equations.

Keywords Besov spaces · Regularity · p-Laplacian · Parabolic · Weak solutions

Mathematics Subject Classification 35B65 · 35K55

1 Introduction

The aim of this paper is the study of the local regularity estimates in Besov spaces of weak solutions for the following quasilinear parabolic equations of p-Laplacian type

$$u_t - \operatorname{div} a\,(Du, x, t) = \operatorname{div} \mathbf{F} \quad \text{in } \Omega_T := \Omega \times (t_0, t_0 + T], \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $T > 0$, $n \geq 2$ and Ω is an open bounded domain in $\mathbb{R}^n$. Here, $\mathbf{F} = (f^1, \cdots, f^n)$ is a given vector-valued function, and $a\,(\xi, x, t)$ is a Carathéodory function satisfying the following conditions:

$$\gamma\,|\xi|^{p-2}\,|\eta|^2 \leq \langle D_\xi a\,(\xi, x, t)\,\eta, \eta\rangle \tag{1.2}$$

Communicated by Rosihan M. Ali.

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and

$$|a(\xi,x,t)|+|\xi|\left|D_\xi a(\xi,x,t)\right|\leq\Lambda|\xi|^{p-1} \tag{1.3}$$

for $p\geq 2$, every $\xi,\eta\in\mathbb{R}^n$, $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ and some constants γ, $\Lambda>0$. Moreover, the coefficients $a(\xi,x,t)$ are also assumed to satisfy some smoothness assumptions (see Assumption (A_1) and Assumption (A_2)). Additionally, we would like to point out that (1.2) and (1.3) can imply the following condition:

$$[a(\xi,x,t)-a(\eta,x,t)]\cdot(\xi-\eta)\geq\bar{\gamma}|\xi-\eta|^p, \tag{1.4}$$

where $\bar{\gamma}$ depends only on γ, Λ, n and p.

There has been a rapid scientific development in the theory of nonlinear parabolic equations of p-Laplacian type in divergence form. In [1], Acerbi and Mingione invent a new covering/iteration argument allowing them to prove the local Calderón–Zygmund estimates for the following parabolic system of p-Laplacian type:

$$u_t-\operatorname{div}\left(a(x,t)|Du|^{p-2}Du\right)=\operatorname{div}\left(|\mathbf{F}|^{\mathbf{p-2}}\mathbf{F}\right) \tag{1.5}$$

with coefficients of VMO/BMO type. Furthermore, Byun, Ok and Ryu [10] obtained the global Calderón–Zygmund estimates of the general case of p-Laplacian type

$$u_t-\operatorname{div}a(Du,x,t)=\operatorname{div}\left(|\mathbf{F}|^{\mathbf{p-2}}\mathbf{F}\right)\quad\text{in }\Omega_T. \tag{1.6}$$

Moreover, many authors [4–9,11,25] also studied the Calderón–Zygmund estimates for the parabolic equations of p-Laplacian type.

Recently, Baisón, Clop, Giova, Orobitg and Passarelli di Napoli [2] studied the local regularity estimates in Besov spaces for weak solutions of the following linear elliptic equation

$$\operatorname{div}\mathcal{A}(Du,x)=\operatorname{div}G\quad\text{in }\Omega, \tag{1.7}$$

where $\mathcal{A}$ is a Carathéodory function with linear growth satisfying

$$\begin{aligned}&<\mathcal{A}(\xi,x)-\mathcal{A}(\eta,x),\xi-\eta>\geq C|\xi-\eta|^2,\\&|\mathcal{A}(\xi,x)-\mathcal{A}(\eta,x)|\leq C|\xi-\eta|,\\&|\mathcal{A}(\xi,x)|\leq C\left(\mu^2+|\xi|^2\right)^{\frac{1}{2}}\end{aligned}$$

for any $\xi,\eta\in\mathbb{R}^n$, some $\mu\in[0,1]$ and a.e. $x\in\Omega$. Furthermore, Clop, Giova & Passarelli di Napoli [13] extended the results in the previous paper [2] to the more general case of (1.7), when $\mathcal{A}$ is a Carathéodory function with $p-1$ growth for $p\geq 2$ satisfying

$$<\mathcal{A}(\xi,x)-\mathcal{A}(\eta,x),\xi-\eta>\geq C\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|^2, \tag{1.8}$$

$$|\mathcal{A}(\xi,x)-\mathcal{A}(\eta,x)|\le C\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|, \tag{1.9}$$

$$|\mathcal{A}(\xi,x)|\le C\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}} \tag{1.10}$$

for any $\xi,\eta\in\mathbb{R}^n$, some $\mu\in[0,1]$ and a.e. $x\in\Omega$. Moreover, many authors [3,16,24] investigated the regularity theory in the context of Besov spaces for the elliptic p-Laplacian equation with $\mathcal{A}(Du,x)=|Du|^{p-2}Du$. Meanwhile, Eleuteri & Passarelli di Napoli [18] established the higher differentiability of the gradient of weak solutions to variational obstacle problems of the form:

$$\int_\Omega\langle\mathcal{A}(Du,x),D(\varphi-u)\rangle\,dx\ge 0,$$

where $\mathcal{A}(\xi,x)$ is a p-harmonic type operator satisfying (1.8)-(1.10) and the critical Besov spaces. Actually, many authors [12,14,15,17,20,22,26,28,30] also studied regularity estimates in Besov spaces for PDEs of various types. The aim of this paper is to study the corresponding regularity estimates in Besov spaces for the case of the parabolic p-Laplacian equation.

In the elliptic case

$$\operatorname{div}\mathcal{A}(Du,x)=0\quad\text{in }\Omega,$$

Giova & Passarelli di Napoli [19,31] obtained the higher differentiability from the following pointwise condition on the map $\mathcal{A}(\xi,x)$

$$|\mathcal{A}(\xi,x)-\mathcal{A}(\xi,y)|\le|x-y|\,(g(x)+g(y))\left(1+|\xi|^2\right)^{\frac{p-1}{2}}$$

for some $g(x)\in L^n_{loc}(\Omega)$, each $\xi\in\mathbb{R}^n$ and almost every $x,y\in\Omega$. Furthermore, the authors first studied the corresponding fractional higher differentiability for the linear (see [2]) and nonlinear (see [13]) elliptic equations with the following Triebel–Lizorkin coefficients

$$|\mathcal{A}(\xi,x)-\mathcal{A}(\xi,y)|\le|x-y|^\alpha\,(g(x)+g(y))\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}} \tag{1.11}$$

for some $g(x)\in L^{\frac{n}{\alpha}}_{loc}(\Omega)$, some $\alpha\in(0,1)$, some $\mu\in[0,1]$, each $\xi\in\mathbb{R}^n$ and almost every $x,y\in\Omega$. Actually, Kristensen & Mingione have used such assumption in the paper [29], where various higher integrability results are obtained. As in Sect. 1 in [2,13], we would like to mention that condition (1.11) for $0<\alpha<1$ says that $\mathcal{A}$ belongs to the Triebel–Lizorkin space $F^\alpha_{\frac{n}{\alpha},\infty}$ (see Remark 3.3 in [27] for details). In order to get the extra differentiability for the non-homogeneous equations and fit the Besov setting, the authors in [2,13] originally introduced the following condition

$$|\mathcal{A}(\xi,x)-\mathcal{A}(\xi,y)|\le|x-y|^\alpha\,(g_k(x)+g_k(y))\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}, \tag{1.12}$$

where $g_k(x) \in L_{loc}^{\frac{n}{\alpha}}(\Omega)$ is a sequence of measurable non-negative functions satisfying

$$\sum_k \|g_k(x)\|^q_{L_{loc}^{\frac{n}{\alpha}}(\Omega)} < \infty.$$

As mentioned in [2,13], if $\mathcal{A}(\xi, x) = A(x)|\xi|^{p-2}\xi$ and $\Omega = \mathbb{R}^n$, condition (1.12) says that $\mathcal{A}$ belongs to the Besov space $B^{\alpha}_{\frac{n}{\alpha},q}$ (see Theorem 1.2 in [27]).

Now we shall study the extra fractional higher differentiability of weak solutions of the quasilinear parabolic equations of p-Laplacian type (1.1). In this work, we shall need the following assumptions (A_1) and (A_2) on $a(\xi, x, t)$ in the setting of the parabolic case.

Assumption (A_1). Given $0 < \alpha < 1$, we assume that there exists a measurable non-negative function $g(x, t) \in L_{loc}^{\frac{n+2}{\alpha}}(\Omega_T)$ such that

$$|a(\xi, x, t) - a(\xi, y, s)| \le \left[|x-y|^2 + |t-s|\right]^{\frac{\alpha}{2}} (g(x,t) + g(y,s))\, |\xi|^{p-1}$$

for each $\xi \in \mathbb{R}^n$ and almost every $(x, t), (y, s) \in \Omega_T$.

Assumption (A_2). Given $0 < \alpha < 1$ and $1 < q < \infty$, we assume that there exists a sequence of measurable non-negative functions $g_k(x, t) \in L_{loc}^{\frac{n+2}{\alpha}}(\Omega_T)$ such that

$$\sum_k \|g_k(x,t)\|^q_{L_{loc}^{\frac{n+2}{\alpha}}(\Omega_T)} < \infty \tag{1.13}$$

and

$$|a(\xi, x, t) - a(\xi, y, s)| \le \left[|x-y|^2 + |t-s|\right]^{\frac{\alpha}{2}} (g_k(x,t) + g_k(y,s))\, |\xi|^{p-1}$$

for each $\xi \in \mathbb{R}^n$ and almost every $(x, t), (y, s) \in \Omega_T$ such that $2^{-k}\operatorname{diam}(\Omega_T) \le (|x-y|^2 + |t-s|)^{\frac{1}{2}} \le 2^{-k+1}\operatorname{diam}(\Omega_T)$ for $k \in \mathbb{N}$.

As usual, the solutions of (1.1) are taken in a weak sense. More precisely, we have the following definition of weak solutions.

Definition 1.1 Assume that $\mathbf{F} \in L_{loc}^{\frac{p}{p-1}}(\Omega_T)$. A function $u \in C\big((t_0, t_0+T]; L^2(\Omega)\big) \cap L^p_{loc}\big((t_0, t_0+T]; W^{1,p}_{loc}(\Omega)\big)$ is a local weak solution of (1.1) in Ω_T if for any compact set $\mathcal{K}$ of Ω and any subinterval $[t_1, t_2]$ of $(t_0, t_0+T]$ we have

$$\int_{\mathcal{K}} u\varphi \mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} \big\{-u\varphi_t + a(Du, x, t)\cdot D\varphi\big\}\mathrm{d}x\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\mathcal{K}} \mathbf{F}\cdot D\varphi \mathrm{d}x\mathrm{d}t$$

for any $\varphi \in W^{1,2}_{loc}\big((t_0, t_0+T]; L^2(\Omega)\big) \cap L^p_{loc}\Big((t_0, t_0+T]; W^{1,p}_0(\Omega)\Big)$.

Let $a(\xi, x, t)$ be a Carathéodory function satisfying (1.2) and (1.3). Just like §2.2 in [13], we need a control on the oscillations, which is called "locally uniformly in VMO". Next, we shall give the exact definition of the parabolic version of the one used in this paper.

Definition 1.2 We say that $a\,(\xi, x, t)$ is locally uniformly in VMO if

$$\lim_{R\to 0} \sup_{0<\rho\le R} \sup_{Q_\rho(z)\subset\Omega_T} \fint_{Q_\rho(z)} V\big(x, t, Q_\rho\,(z)\big)\mathrm{d}x\mathrm{d}t = 0, \tag{1.14}$$

where $z = (y, s) \in \mathbb{R}^{n+1}$, $Q_\rho\,(z) = B_\rho\,(y) \times (s - \rho^2, s + \rho^2)$,

$$V\big(x, t, Q_\rho\,(z)\big) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\left|a\,(\xi, x, t) - \bar{a}_{Q_\rho(z)}\,(\xi)\right|}{|\xi|^{p-1}} \tag{1.15}$$

and

$$\bar{a}_{Q_\rho(z)}\,(\xi) := \fint_{Q_\rho(z)} a\,(\xi, x, t)\,\mathrm{d}x\mathrm{d}t = \frac{1}{\left|Q_\rho\,(z)\right|} \int_{Q_\rho(z)} a\,(\xi, x, t)\,\mathrm{d}x\mathrm{d}t.$$

We first recall the following definition of the Besov space $B^\alpha_{p,q}(\Omega)$.

Definition 1.3 *(see [32], Section 2.5.12)* Let $0 < \alpha < 1$ and $1 < p, q < \infty$. Then, the Besov space $B^\alpha_{p,q}(\Omega)$ is the set of all measurable functions v satisfying $v \in L^p(\Omega)$ and

$$\|v\|_{B^\alpha_{p,q}(\Omega)} := \|v\|_{L^p(\Omega)} + [v]_{\dot{\mathrm{B}}^\alpha_{p,q}(\Omega)} < \infty,$$

where

$$[v]_{\dot{\mathrm{B}}^\alpha_{p,q}(\Omega)} := \left(\int_{\mathbb{R}^n} \left(\int_\Omega \frac{|\Delta_h v(x)|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{q}{p}} \frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}},$$
$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v\,(x)\,|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

and $\Delta_h v(x) = v(x + h) - v(x)$. Moreover, we define $\Delta_h v(x)$ is zero if $x + h$ leaves Ω.

It is easy to check that $v \in L^p(\Omega)$ and $\frac{\Delta_h v}{|h|^\alpha} \in L^q\big(\frac{\mathrm{d}h}{|h|^n}; L^p(\Omega)\big)$ if $v \in B^\alpha_{p,q}(\Omega)$. Actually, we can only integrate over B_δ for a fixed $\delta > 0$ and then obtain an equivalent norm since

$$\left(\int_{\{|h|\ge\delta\}} \left(\int_\Omega \frac{|\Delta_h v(x)|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{q}{p}} \frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}} \le c(n, \alpha, p, q, \delta)\|v\|_{L^p(\Omega)}.$$

Meanwhile, we say that $v \in B^{\alpha}_{p,\infty}(\Omega)$ if $v \in L^p(\Omega)$ and

$$[v]_{\dot{\mathrm{B}}^{\alpha}_{p,\infty}(\Omega)} := \sup_{h\in\mathbb{R}^n} \left(\int_{\Omega} \frac{|\Delta_h v(x)|^p}{|h|^{\alpha p}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Similarly, we can only take the supremum over $|h| \le \delta$ and then obtain an equivalent norm.

The next result is the current version of Sobolev embedding theorem (see Lemma 2.2 in [18]), whose proof can be found in Proposition 7.12 of [23].

Lemma 1.4 *Assume that* $0 < \alpha < 1$. *If* $1 < p < \frac{n}{\alpha}$ *and* $1 < q \le p^*_\alpha := \frac{np}{n-\alpha p}$, *then there is a continuous embedding* $B^{\alpha}_{p,q}(\Omega) \subset L^{p^*_\alpha}(\Omega)$.

In this work we define the Besov space $B^{\alpha}_{p,q}(\Omega_T)$ for $0 < \alpha < 1$ and $1 < p, q < \infty$ as

$$B^{\alpha}_{p,q}(\Omega_T) := \left\{ v \in L^p(\Omega_T) : \|v\|_{B^{\alpha}_{p,q}(\Omega_T)} := \|v\|_{L^p(\Omega_T)} + [v]_{\dot{\mathrm{B}}^{\alpha}_{p,q}(\Omega_T)} < \infty \right\},$$

where

$$[v]_{\dot{\mathrm{B}}^{\alpha}_{p,q}(\Omega_T)} := \left(\int_{\mathbb{R}^n} \left(\int_{\Omega_T} \frac{|\Delta_h v(x,t)|^p}{|h|^{\alpha p}} \mathrm{d}x\mathrm{d}t \right)^{\frac{q}{p}} \frac{\mathrm{d}h}{|h|^n} \right)^{\frac{1}{q}},$$

$$|v\|_{L^p(\Omega_T)} := \left(\int_{\Omega_T} |v\,(x,t)\,|^p \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p}}$$

and $\Delta_h v(x,t) = v(x+h,t) - v(x,t)$ with $\Delta_h v(x,t) = 0$ if $x+h$ leaves Ω. Meanwhile, we say that $v \in B^{\alpha}_{p,\infty}(\Omega_T)$ if $v \in L^p(\Omega_T)$ and

$$\sup_{h\in\mathbb{R}^n} \left(\int_{\Omega_T} \frac{|\Delta_h v(x,t)|^p}{|h|^{\alpha p}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Similar to the cases of $[\cdot]_{\dot{\mathrm{B}}^{\alpha}_{p,q}(\Omega)}$ and $[\cdot]_{\dot{\mathrm{B}}^{\alpha}_{p,\infty}(\Omega)}$, for $|h| \le \delta$ we can obtain the equivalent norms.

Now we state the main results of this work. Our first result concerns the homogeneous case $\mathbf{F} = 0$.

Theorem 1.5 *Assume that* $0 < \alpha < 1$ *and* $a(\xi, x, t)$ *satisfies (1.2), (1.3) and* (A_1). *If* u *is a local weak solution of*

$$u_t - div\, a\,(Du, x, t) = 0 \quad in\ \Omega_T,$$

then we have $Du(x,t) \in B^{\frac{\alpha}{p-1}}_{p,\infty}(\Omega_T)$ *locally.*

Under the assumption (A_2), we are able to deal with non-homogeneous equations and obtain the higher fractional differentiability. More precisely, we establish the following result.

Theorem 1.6 *Assume that* $0 < \alpha < 1$ *and* $a(\xi, x, t)$ *satisfies* (1.2), (1.3) *and* (A_2). *If* u *is a local weak solution of*

$$u_t - diva\,(Du, x, t) = div\,\mathbf{F} \quad in \quad \Omega_T,$$

then we have

$$\begin{aligned}&\mathbf{F}(x,t) \in L^{\frac{np}{n(p-1)-p\beta}}\left((t_0, t_0+T];\, B^{\beta}_{\frac{p}{p-1},q}(\Omega)\right) \ \textit{locally}\\ &\Rightarrow \quad Du(x,t) \in B^{\frac{\alpha}{p-1}}_{p,q}(\Omega_T) \ \textit{locally}\end{aligned}$$

for any $1 \le \frac{q(p-1)}{p} \le \min\left\{\frac{n(p-1)}{n(p-1)-p\beta}, \frac{(n+2)(p-1)}{p\beta}\right\}$ *and* $\alpha < \beta < 1$.

2 Proofs of the Main Results

This section is devoted to the proofs of the main results stated in Theorem 1.5 and Theorem 1.6. Now we will give some lemmas which are essential for the proofs of our main results. First of all we recall the following result (see [13,18,21]).

Lemma 2.1 *Assume that* F *and* G *are two functions such that* $F, G \in W^{1,p}(\Omega)$ *for* $p > 1$ *and*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then, we have

(1) $\Delta_h F \in W^{1,p}(\Omega_{|h|})$ *and* $D_i(\Delta_h F) = \Delta_h(D_i F)$.
(2) If at least one of the functions F *or* G *has support contained in* $\Omega_{|h|}$, *then*

$$\int_\Omega F\Delta_h G \mathrm{d}x = \int_\Omega G\Delta_{-h} F \mathrm{d}x.$$

(3) $\Delta_h\,(FG)\,(x) = F\,(x+h)\,\Delta_h G\,(x) + G\,(x)\,\Delta_h F\,(x)$.
(4)

$$\begin{aligned}\int_{B_\rho} |\Delta_h F|^p \,\mathrm{d}x &\le C\,(n,p)\,|h|^p \int_{B_R} |DF|^p \,\mathrm{d}x \quad \textit{and}\\ \int_{B_\rho} |F(x+h)|^p \,\mathrm{d}x &\le \int_{B_R} |F(x)|^p \,\mathrm{d}x\end{aligned}$$

for $0 < \rho < R$ *and* $|h| < \frac{R-\rho}{2}$ *with* $B_R \subset \Omega$.

The following result is a modified version of Theorem 2.6 in [10], which is suitable for our purpose.

Lemma 2.2 *Assume that* $|\mathbf{F}|^{\frac{p}{p-1}} \in L^q(\Omega_T)$ *for any* $q \in (1, \infty)$ *and* $a(\xi, x, t)$ *satisfies (1.2),(1.3) and (1.14). If* $u \in W^{1,p}(\Omega_T)$ *is the weak solution of (1.1) with* $u = 0$ *on* $\partial_p \Omega_T$*, then* $|Du|^p \in L^q(\Omega_T)$ *with the following estimate:*

$$\int_{\Omega_T} |Du|^{pq} \, \mathrm{d}x\mathrm{d}t \le C \left(\int_{\Omega_T} |\mathbf{F}|^{\frac{pq}{p-1}} \, \mathrm{d}x\mathrm{d}t + 1 \right)^{p/2},$$

where $C = C(\gamma, \Lambda, n, p, q, |\Omega_T|)$.

Actually, from [10] we can also obtain the local L^q estimates

$$\int_{Q_\rho} |Du|^{pq} \, \mathrm{d}x\mathrm{d}t \le C \left[\left(\int_{Q_R} |Du|^p \, \mathrm{d}x\mathrm{d}t \right)^q + \int_{Q_R} |\mathbf{F}|^{\frac{pq}{p-1}} \, \mathrm{d}x\mathrm{d}t + 1 \right]^{p/2} \tag{2.1}$$

for weak solutions of (1.1) and $\rho < R$ with $Q_R \subset \Omega_T$.

Next, we can prove the following result, which is just the parabolic version of Lemma 17 in [2] or Lemma 3.1 in [13]. We report it here in order to describe integrality.

Lemma 2.3 *Assume that* $a(\xi, x, t)$ *satisfies* (1.2), (1.3) *and* (A_1). *Then,* $a(\xi, x, t)$ *is locally uniformly in VMO.*

Proof Let $Q_\rho(y, s) \subset \Omega_T$. From (1.15), (A_1) and Hölder's inequality, we deduce that

$$\begin{aligned}
&\fint_{Q_\rho(y,s)} V\big(x, t, Q_\rho(y, s)\big) \mathrm{d}x\mathrm{d}t \\
&= \fint_{Q_\rho(y,s)} \sup_{\xi \in \mathbb{R}^n/\{0\}} \frac{\left| a(\xi, x, t) - \bar{a}_{Q_\rho(y,s)}(\xi) \right|}{|\xi|^{p-1}} \mathrm{d}x\mathrm{d}t \\
&\le \fint_{Q_\rho(y,s)} \sup_{\xi \in \mathbb{R}^n/\{0\}} \frac{\fint_{Q_\rho(y,s)} \left| a(\xi, x, t) - a\left(\xi, x', t'\right) \right| \mathrm{d}x'\mathrm{d}t'}{|\xi|^{p-1}} \mathrm{d}x\mathrm{d}t \\
&\le \fint_{Q_\rho(y,s)} \fint_{Q_\rho(y,s)} \left[\left| x - x' \right|^2 + \left| t - t' \right| \right]^{\frac{\alpha}{2}} \left[g(x, t) + g\left(x', t'\right) \right] \mathrm{d}x'\mathrm{d}t'\mathrm{d}x\mathrm{d}t \\
&\le C\rho^\alpha \fint_{Q_\rho(y,s)} g(x, t) \, \mathrm{d}x\mathrm{d}t \\
&\le C\rho^\alpha \left[\fint_{Q_\rho(y,s)} g(x, t)^{\frac{n+2}{\alpha}} \, \mathrm{d}x\mathrm{d}t \right]^{\frac{\alpha}{n+2}} \\
&= C \left[\int_{Q_\rho(y,s)} g(x, t)^{\frac{n+2}{\alpha}} \, \mathrm{d}x\mathrm{d}t \right]^{\frac{\alpha}{n+2}},
\end{aligned}$$

which implies that (1.14) is true. Thus, we finish the proof. □

Now it is time to prove the first one of the main results, Theorem 1.5.

Proof of Theorem 1.5 Let us fix a parabolic cylinder Q_{2R} such that $Q_{2R} \subset \Omega_T$ and consider a cut-off function $\eta(x,t) \in \mathcal{C}_0^\infty(\Omega_T)$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } Q_{R/2}, \quad \eta \equiv 0 \text{ in } \Omega_T \backslash Q_R, \quad |\nabla \eta| \le \frac{C}{R} \text{ and } 0 < |\eta_t| \le \frac{C}{R^2}. \tag{2.2}$$

Selecting $\varphi = \Delta_{-h}(\eta^2 \Delta_h u)$ as a test function, where $h \in B_\delta$ for some small constant $\delta < R$, from Definition 1.1 we have

$$\int_{B_R} u\varphi \mathrm{d}x \Big|_{t=-R^2}^{t=R^2} + \int_{-R^2}^{R^2} \int_{B_R} \{-u\varphi_t + a(Du, x, t) \cdot D\varphi\} \mathrm{d}x\mathrm{d}t = 0.$$

After a direct calculation, the resulting expression is shown as follows:

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6,$$

where

$$\begin{aligned}
I_1 &:= \frac{1}{2} \int_{B_R} \eta^2 (\Delta_h u)^2 \,\mathrm{d}x \Big|_{-R^2}^{R^2} = 0, \\
I_2 &:= \int_{Q_R} [a(Du(x+h,t), x+h, t) \\
&\quad - a(Du(x,t), x+h, t)] \cdot \eta^2 \Delta_h Du \,\mathrm{d}x\mathrm{d}t, \\
I_3 &:= \int_{Q_R} \eta \eta_t (\Delta_h u)^2 \,\mathrm{d}x\mathrm{d}t, \\
I_4 &:= \int_{Q_R} [a(Du(x,t), x+h, t) \\
&\quad - a(Du(x+h,t), x+h, t)] \cdot 2\eta D\eta \Delta_h u \,\mathrm{d}x\mathrm{d}t, \\
I_5 &:= \int_{Q_R} [a(Du(x,t), x, t) - a(Du(x,t), x+h, t)] \cdot 2\eta D\eta \Delta_h u \,\mathrm{d}x\mathrm{d}t, \\
I_6 &:= \int_{Q_R} [a(Du(x,t), x, t) - a(Du(x,t), x+h, t)] \cdot \eta^2 \Delta_h Du \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Estimate of I_2. It follows from (1.4) that

$$I_2 \ge \bar{\gamma} \int_{Q_R} \eta^2 |\Delta_h Du|^p \,\mathrm{d}x\mathrm{d}t.$$

Estimate of I_3. By virtue of (2.2) and Lemma 2.1, we find that

$$|I_3| \le \int_{Q_R} \eta |\eta_t| |\Delta_h u|^2 \,\mathrm{d}x\mathrm{d}t \le C \int_{Q_R} |\Delta_h u|^2 \,\mathrm{d}x\mathrm{d}t \le C |h|^2 \int_{Q_{2R}} |Du|^2 \,\mathrm{d}x\mathrm{d}t.$$

Estimate of I_4. Using (1.3), Lagrange's mean value theorem and Young's inequality, we obtain

$$\begin{aligned}|I_4| &\le C\int_{Q_R} |a(Du(x,t),x+h,t)\\ &\quad -a(Du(x+h,t),x+h,t)|\cdot\eta\cdot|\Delta_h u|\,\mathrm{d}x\mathrm{d}t\\ &\le C\int_{Q_R}\left|D_\xi a(\xi,x+h,t)\right|\cdot|\Delta_h Du|\cdot\eta\cdot|\Delta_h u|\,\mathrm{d}x\mathrm{d}t\\ &\le C\int_{Q_R}(|Du(x,t)|+|Du(x+h,t)|)^{p-2}\cdot\eta^{\frac{2}{p}}\cdot|\Delta_h Du|\cdot|\Delta_h u|\,\mathrm{d}x\mathrm{d}t\\ &\le C(\epsilon)\int_{Q_R}(|Du(x,t)|+|Du(x+h,t)|)^{\frac{p(p-2)}{p-1}}\cdot|\Delta_h u|^{\frac{p}{p-1}}\,\mathrm{d}x\mathrm{d}t\\ &\quad +\epsilon\int_{Q_R}\eta^2|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where ξ is between $Du(x,t)$ and $Du(x+h,t)$. Therefore, by using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}|I_4| &\le C(\epsilon)\left(\int_{Q_R}(|Du(x,t)|+|Du(x+h,t)|)^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{p-2}{p-1}}\\ &\quad\left(\int_{Q_R}|\Delta_h u|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p-1}}\\ &\quad +\epsilon\int_{Q_R}\eta^2|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\ &\le C(\epsilon)|h|^{\frac{p}{p-1}}\left(\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{p-2}{p-1}}\left(\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p-1}}\\ &\quad +\epsilon\int_{Q_R}\eta^2|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\ &\le C(\epsilon)|h|^{\frac{p}{p-1}}\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t+\epsilon\int_{Q_R}\eta^2|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t. \qquad (2.3)\end{aligned}$$

Estimate of I_5. Also, by (A_1) and Young's inequality we obtain

$$\begin{aligned}|I_5| &\le C\int_{Q_R}|a(Du(x,t),x,t)-a(Du(x,t),x+h,t)|\cdot\eta\cdot|\Delta_h u|\,\mathrm{d}x\mathrm{d}t\\ &\le C|h|^\alpha\int_{Q_R}[g(x,t)+g(x+h,t)]\cdot|Du(x,t)|^{p-1}\cdot|\Delta_h u|\,\mathrm{d}x\mathrm{d}t\\ &\le C|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}[g(x,t)+g(x+h,t)]^{\frac{p}{p-1}}|Du(x,t)|^p\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$+C\int_{Q_R}|\Delta_h u|^p\,\mathrm{d}x\mathrm{d}t,$$

which implies that

$$\begin{aligned}|I_5| \le\; & C\,|h|^{\frac{p\alpha}{p-1}}\left[\int_{Q_R}\left(g\,(x,t)+g\,(x+h,t)\right)^{\frac{n}{\alpha}}\mathrm{d}x\mathrm{d}t\right]^{\frac{p\alpha}{n(p-1)}}\\ & \left[\int_{Q_R}|Du|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{n(p-1)-p\alpha}{n(p-1)}}\\ & +C|h|^p\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where we used Hölder's inequality and Lemma 2.1.

Estimate of I_6. Thanks to (A_1) and Young's inequality, we estimate I_6 as follows:

$$\begin{aligned}|I_6| &\le \int_{Q_R}|a\,(Du\,(x,t)\,,x,t)-a\,(Du\,(x,t)\,,x+h,t)|\cdot\eta^2\cdot|\Delta_h Du|\,\mathrm{d}x\mathrm{d}t\\ &\le C\,|h|^{\alpha}\int_{Q_R}[g\,(x,t)+g\,(x+h,t)]\cdot|Du\,(x,t)|^{p-1}\cdot\eta^{\frac{2}{p}}\cdot|\Delta_h Du|\,\mathrm{d}x\mathrm{d}t\\ &\le \epsilon\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t+C(\epsilon)\,|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}[g\,(x,t)\\ &\quad +g\,(x+h,t)]^{\frac{p}{p-1}}\,|Du\,(x,t)|^p\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Similar to the estimate of I_5, we have

$$\begin{aligned}|I_6| \le\; & C(\epsilon)\,|h|^{\frac{p\alpha}{p-1}}\left[\int_{Q_R}\left(g\,(x,t)+g\,(x+h,t)\right)^{\frac{n}{\alpha}}\mathrm{d}x\mathrm{d}t\right]^{\frac{p\alpha}{n(p-1)}}\\ & \left[\int_{Q_R}|Du|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{n(p-1)-p\alpha}{n(p-1)}}\\ & +\epsilon\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Combining all the estimates of I_i $(1\le i\le 6)$ and choosing ϵ small enough, we conclude that

$$\begin{aligned}&\int_{Q_{R/2}}|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\ &\quad\le \int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\ &\quad\le C\,|h|^{\frac{p\alpha}{p-1}}\left[\int_{Q_R}\left(g\,(x,t)+g\,(x+h,t)\right)^{\frac{n}{\alpha}}\mathrm{d}x\mathrm{d}t\right]^{\frac{p\alpha}{n(p-1)}}\end{aligned}$$

$$
\begin{aligned}
&\left[\int_{Q_R} |Du|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{n(p-1)-p\alpha}{n(p-1)}}\\
&+C|h|^p\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t + C\,|h|^{\frac{p}{p-1}}\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\\
&+C\,|h|^2\int_{Q_{2R}}|Du|^2\,\mathrm{d}x\mathrm{d}t\\
\le{}& C\,|h|^{\frac{p\alpha}{p-1}}\left[\int_{Q_R}(g\,(x,t)+g\,(x+h,t))^{\frac{n}{\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{p\alpha}{n(p-1)}}\\
&\left[\int_{Q_R} |Du|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{n(p-1)-p\alpha}{n(p-1)}}\\
&+C\,|h|^{\frac{p}{p-1}}\int_{Q_{2R}}|Du|^p+1\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

in view of the fact that $\frac{p}{p-1}\le 2\le p$. By (2.1), we know that $Du\in L^s_{loc}(\Omega_T)$ for any $s>p$ and so, in particular, $Du\in L_{loc}^{\frac{np(p-1)}{n(p-1)-p\alpha}}(\Omega_T)$. Also, we have $g\in L_{loc}^{\frac{n}{\alpha}}(\Omega_T)$ since $g\in L_{loc}^{\frac{n+2}{\alpha}}(\Omega_T)$. Furthermore, we divide both sides of the above inequality by $|h|^{\frac{p\alpha}{p-1}}$ and use Lemma 2.1 to obtain

$$
\begin{aligned}
&\int_{Q_{R/2}}\left|\frac{\Delta_h Du}{|h|^{\frac{\alpha}{p-1}}}\right|^p\mathrm{d}x\mathrm{d}t\\
&\le C\,|h|^{\frac{p(1-\alpha)}{p-1}}\int_{Q_{2R}}|Du|^p+1\mathrm{d}x\mathrm{d}t\\
&\quad +C\left[\int_{Q_R}(g\,(x,t)+g\,(x+h,t))^{\frac{n}{\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{p\alpha}{n(p-1)}}\\
&\quad\left[\int_{Q_R} |Du|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\,\mathrm{d}x\mathrm{d}t\right]^{\frac{n(p-1)-p\alpha}{n(p-1)}}\\
&\le C.
\end{aligned}
$$

Finally, we can take supremum over $h\in B_\delta$ for some $\delta<R$ and obtain

$$
\sup_{|h|<\delta}\int_{Q_{R/2}}\left|\frac{\Delta_h Du}{|h|^{\frac{\alpha}{p-1}}}\right|^p\mathrm{d}x\mathrm{d}t<+\infty,
$$

which implies that $Du\in B_{p,\infty}^{\frac{\alpha}{p-1}}(\Omega_T)$ locally. So, the proof of the lemma is completed. □

Moreover, we can prove the following result, whose proof is similar to the elliptic case (see Lemma 18 in [2] or Lemma 4.1 in [13]). We report it here just for the sake of completeness.

Lemma 2.4 *Assume that* $0 < \alpha < \beta < 1$, $1 < q \le \frac{n+2}{\beta}$ *and* $a(\xi, x, t)$ *satisfies* (1.2), (1.3) *and* (A_2). *Then,* $a(\xi, x, t)$ *is locally uniformly in VMO.*

Proof Let $(x, t), (y, s) \in \Omega_T$ and $A_k(x, t) := \{(y, s) \in \Omega_T : 2^{-k}\mathrm{diam}(\Omega_T) \le (|x-y|^2 + |t-s|)^{\frac{1}{2}} \le 2^{-k+1}\mathrm{diam}(\Omega_T)\}$. Then, by virtue of (1.15) and the assumption (A_2), we have

$$\begin{aligned}
&\fint_{Q_\rho(y,s)} V\big(x, t, Q_\rho(y, s)\big)\mathrm{d}x\mathrm{d}t \\
&= \fint_{Q_\rho(y,s)} \sup_{\xi\in\mathbb{R}^n/\{0\}} \frac{\left|a(\xi, x, t) - \bar{a}_{Q_\rho(y,s)}(\xi)\right|}{|\xi|^{p-1}}\mathrm{d}x\mathrm{d}t \\
&\le \fint_{Q_\rho(y,s)} \sup_{\xi\in\mathbb{R}^n/\{0\}} \frac{\fint_{Q_\rho(y,s)} \left|a(\xi, x, t) - a\left(\xi, x', t'\right)\right| \mathrm{d}x'\mathrm{d}t'}{|\xi|^{p-1}}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{\left|Q_\rho(y, s)\right|^2} \int_{Q_\rho(y,s)} \sup_{\xi\in\mathbb{R}^n/\{0\}} \int_{Q_\rho(y,s)} \frac{\left|a(\xi, x, t) - a\left(\xi, x', t'\right)\right| \mathrm{d}x'\mathrm{d}t'}{|\xi|^{p-1}}\mathrm{d}x\mathrm{d}t \\
&\le \frac{1}{\left|Q_\rho(y, s)\right|^2} \int_{Q_\rho(y,s)} \sup_{\xi\in\mathbb{R}^n/\{0\}} \\
&\quad \sum_k \int_{Q_\rho(y,s)\cap A_k(x,t)} \frac{\left|a(\xi, x, t) - a\left(\xi, x', t'\right)\right| \mathrm{d}x'\mathrm{d}t'}{|\xi|^{p-1}}\mathrm{d}x\mathrm{d}t \\
&\le \frac{C\rho^\alpha}{\left|Q_\rho(y, s)\right|^2} \sum_k \int_{Q_\rho(y,s)} \int_{Q_\rho(y,s)\cap A_k(x,t)} \big(g_k(x, t) + g_k(x', t')\big)\, \mathrm{d}x'\mathrm{d}t'\mathrm{d}x\mathrm{d}t \\
&\le \frac{C\rho^\alpha}{\left|Q_\rho(y, s)\right|^2} \sum_k \left|Q_\rho(y, s) \cap A_k(x, t)\right| \int_{Q_\rho(y,s)} g_k(x, t)\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{C\rho^\alpha}{\left|Q_\rho(y, s)\right|} \sum_k \int_{Q_\rho(y,s)\cap A_k(x,t)} g_k(x', t')\mathrm{d}x'\mathrm{d}t' \\
&=: I_1 + I_2.
\end{aligned}$$

Estimate of I_1. Using Hölder's inequality, we have

$$\begin{aligned}
I_1 &\le \frac{C}{\left|Q_\rho(y, s)\right|} \sum_k \left|Q_\rho(y, s) \cap A_k(x, t)\right| \left[\int_{Q_\rho(y,s)} g_k^{\frac{n+2}{\alpha}}\, \mathrm{d}x\mathrm{d}t\right]^{\frac{\alpha}{n+2}} \\
&\le \frac{C}{\left|Q_\rho(y, s)\right|} \left[\sum_k \left(\int_{Q_\rho(y,s)} g_k^{\frac{n+2}{\alpha}}\, \mathrm{d}x\mathrm{d}t\right)^{\frac{\alpha q}{n+2}}\right]^{\frac{1}{q}} \\
&\quad \left[\sum_k \left|Q_\rho(y, s) \cap A_k(x, t)\right|^{\frac{q}{q-1}}\right]^{\frac{q-1}{q}}
\end{aligned}$$

$$\leq C\left[\sum_k \|g_k\|^q_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,s))}\right]^{\frac{1}{q}}.$$

Estimate of I_2. In view of the facts that $0<\alpha<\beta<1$ and $1<q\leq\frac{n+2}{\beta}$, we find that $\frac{(n+2-\alpha)q}{(n+2)(q-1)}>1$. Using Hölder's inequality again, we deduce that

$$\begin{aligned}
I_2 &\leq \frac{C\rho^\alpha}{|Q_\rho(y,s)|}\sum_k\left[\int_{Q_\rho(y,s)\cap A_k(x,t)} g_k^{\frac{n+2}{\alpha}}\mathrm{d}x'\mathrm{d}t'\right]^{\frac{\alpha}{n+2}}\left|Q_\rho(y,s)\cap A_k(x,t)\right|^{1-\frac{\alpha}{n+2}}\\
&\leq \frac{C\rho^\alpha}{|Q_\rho(y,s)|}\left[\sum_k\left(\int_{Q_\rho(y,s)} g_k^{\frac{n+2}{\alpha}}\mathrm{d}x\mathrm{d}t\right)^{\frac{\alpha q}{n+2}}\right]^{\frac{1}{q}}\\
&\qquad\left[\sum_k\left|Q_\rho(y,s)\cap A_k(x,t)\right|^{\frac{(n+2-\alpha)q}{(n+2)(q-1)}}\right]^{\frac{q-1}{q}}\\
&\leq C\left[\sum_k \|g_k\|^q_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,s))}\right]^{\frac{1}{q}}.
\end{aligned}$$

Combining the estimates of I_1 and I_2, we have

$$⨍_{Q_\rho(y,s)} V\big(x,t,Q_\rho(y,s)\big)\mathrm{d}x\mathrm{d}t\leq C\left[\sum_k \|g_k\|^q_{L^{\frac{n+2}{\alpha}}(Q_\rho(y,s))}\right]^{\frac{1}{q}}.$$

Thus, from (1.13) and the dominated convergence theorem, we can get the desired result (1.14). This finishes the proof. □

Finally, we shall finish the proof of Theorem 1.6.

Proof of Theorem 1.6 Let us fix a parabolic cylinder Q_{2R} such that $Q_{2R}\subset\Omega_T$ and select $\varphi=\Delta_{-h}(\eta^2\Delta_h u)$ as a test function, where $h\in B_\delta$ for some small constant $\delta<R$ and $\eta(x,t)\in C_0^\infty(\Omega_T)$ is a cut off function satisfying

$$0\leq\eta\leq 1,\quad \eta\equiv 1\ \text{ in } Q_{R/2},\quad \eta\equiv 0\ \text{ in } \Omega_T\backslash Q_R,\quad |\nabla\eta|\leq\frac{C}{R}\ \text{ and }\ 0<|\eta_t|\leq\frac{C}{R^2}. \tag{2.4}$$

From the definition of weak solution, we have

$$\begin{aligned}
&\int_{B_R} u\varphi\mathrm{d}x\Big|_{t=-R^2}^{t=R^2}+\int_{-R^2}^{R^2}\int_{B_R}\Big\{-u\varphi_t+a(Du,x,t)\cdot D\varphi\Big\}\mathrm{d}x\mathrm{d}t\\
&\quad=-\int_{Q_R}\mathbf{F}\cdot D\varphi\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

After simple computations, we can write the resulting expression as

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6 + I_7,$$

where

$$\begin{aligned}
I_1 &:= \frac{1}{2}\int_{B_R} \eta^2 (\Delta_h u)^2 \,\mathrm{d}x\Big|_{-R^2}^{R^2} = 0,\\
I_2 &:= \int_{Q_R} [a(Du(x+h,t), x+h, t)\\
&\quad - a(Du(x,t), x+h, t)] \cdot \eta^2 \Delta_h Du \,\mathrm{d}x\mathrm{d}t,\\
I_3 &:= \int_{Q_R} \eta\eta_t (\Delta_h u)^2 \,\mathrm{d}x\mathrm{d}t,\\
I_4 &:= \int_{Q_R} [a(Du(x,t), x+h, t)\\
&\quad - a(Du(x+h,t), x+h, t)] \cdot 2\eta D\eta \Delta_h u \,\mathrm{d}x\mathrm{d}t,\\
I_5 &:= \int_{Q_R} [a(Du(x,t), x, t) - a(Du(x,t), x+h, t)] \cdot 2\eta D\eta \Delta_h u \,\mathrm{d}x\mathrm{d}t,\\
I_6 &:= \int_{Q_R} [a(Du(x,t), x, t) - a(Du(x,t), x+h, t)] \cdot \eta^2 \Delta_h Du \,\mathrm{d}x\mathrm{d}t,\\
I_7 &:= -\int_{Q_R} \Delta_h \mathbf{F} \cdot 2\eta D\eta \Delta_h u + \Delta_h \mathbf{F} \cdot \eta^2 \Delta_h Du \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Estimates of I_2-I_4. Similar to the proof of Theorem 1.5, we deduce that

$$\begin{aligned}
I_2 &\ge \bar{\gamma} \int_{Q_R} \eta^2 |\Delta_h Du|^p \,\mathrm{d}x\mathrm{d}t,\\
|I_3| &\le C|h|^2 \int_{Q_{2R}} |Du|^2 \,\mathrm{d}x\mathrm{d}t,\\
|I_4| &\le C(\epsilon)|h|^{\frac{p}{p-1}} \int_{Q_{2R}} |Du|^p \,\mathrm{d}x\mathrm{d}t + \epsilon \int_{Q_R} \eta^2 |\Delta_h Du|^p \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Estimate of I_5. Without loss of generality, we may as well assume that $\operatorname{diam}(\Omega) = K_0 R$ for some constant $K_0 > 1$ and $2^{-k} K_0 R \le |h| \le 2^{-k+1} K_0 R \le R$ for $\mathbb{N} \ni k > k_0$, where $k_0 = [1 + \log_2 K_0] \in \mathbb{N}$. Therefore, from (A_2), (2.4), Young's inequality and Lemma 2.1 we have

$$\begin{aligned}
|I_5| &\le C \int_{Q_R} \big|a(Du(x,t), x, t) - a(Du(x,t), x+h, t)\big| \cdot |\Delta_h u| \,\mathrm{d}x\mathrm{d}t\\
&\le C|h|^\alpha \int_{Q_R} (g_k(x,t) + g_k(x+h,t)) \cdot |Du(x,t)|^{p-1} \cdot |\Delta_h u| \,\mathrm{d}x\mathrm{d}t\\
&\le C|h|^{\frac{p\alpha}{p-1}} \int_{Q_R} (g_k(x,t) + g_k(x+h,t))^{\frac{p}{p-1}} |Du(x,t)|^p \,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+C\int_{Q_R}|\Delta_h u|^p\,\mathrm{d}x\mathrm{d}t\\
&\le C\,|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}\left(g_k\,(x,t)+g_k\,(x+h,t)\right)^{\frac{p}{p-1}}|Du\,(x,t)|^p\,\mathrm{d}x\mathrm{d}t\\
&+C\,|h|^p\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Estimate of I_6-I_7. Thanks to (A_2), Young's inequality and Lemma 2.1, we estimate I_6-I_7 as follows:

$$
\begin{aligned}
|I_6| &\le \int_{Q_R}\left|a\,(Du\,(x,t)\,,x,t)-a\,(Du\,(x,t)\,,x+h,t)\right|\cdot\eta^2\cdot|\Delta_h Du|\,\mathrm{d}x\mathrm{d}t\\
&\le C\,|h|^\alpha\int_{Q_R}\left(g_k\,(x,t)+g_k\,(x+h,t)\right)\cdot|Du\,(x,t)|^{p-1}\cdot\eta^2\cdot|\Delta_h Du|\,\mathrm{d}x\mathrm{d}t\\
&\le \epsilon\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t+C(\epsilon)\,|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}\left(g_k\,(x,t)\right.\\
&\quad\left.+g_k\,(x+h,t)\right)^{\frac{p}{p-1}}|Du\,(x,t)|^p\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
|I_7| &\le C\int_{Q_R}|\Delta_h\mathbf{F}|\cdot|\Delta_h u|+|\Delta_h\mathbf{F}|\cdot\eta^2\cdot|\Delta_h Du|\,\mathrm{d}x\mathrm{d}t\\
&\le C(\epsilon)\int_{Q_R}|\Delta_h\mathbf{F}|^{\frac{p}{p-1}}\,\mathrm{d}x\mathrm{d}t+C\int_{Q_R}|\Delta_h u|^p\,\mathrm{d}x\mathrm{d}t\\
&\quad+\epsilon\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\
&\le C(\epsilon)\int_{Q_R}|\Delta_h\mathbf{F}|^{\frac{p}{p-1}}\,\mathrm{d}x\mathrm{d}t+C|h|^p\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\\
&\quad+\epsilon\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Combining the estimates of I_i $(1\le i\le 7)$ and choosing ϵ small enough, we conclude that

$$
\begin{aligned}
&\int_{Q_{R/2}}|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\
&\le\int_{Q_R}\eta^2\,|\Delta_h Du|^p\,\mathrm{d}x\mathrm{d}t\\
&\le C\,|h|^2\int_{Q_{2R}}|Du|^2\,\mathrm{d}x\mathrm{d}t+C\,|h|^p\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\\
&\quad+C\,|h|^{\frac{p}{p-1}}\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+C\,|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}\left(g_k\left(x,t\right)+g_k\left(x+h,t\right)\right)^{\frac{p}{p-1}}|Du\left(x,t\right)|^p\,\mathrm{d}x\mathrm{d}t\\
&+C\int_{Q_R}|\Delta_h\mathbf{F}|^{\frac{p}{p-1}}\,\mathrm{d}x\mathrm{d}t\\
\le{}& C\,|h|^{\frac{p}{p-1}}\int_{Q_{2R}}|Du|^p+1\mathrm{d}x\mathrm{d}t\\
&+C\,|h|^{\frac{p\alpha}{p-1}}\int_{Q_R}\left(g_k\left(x,t\right)+g_k\left(x+h,t\right)\right)^{\frac{p}{p-1}}|Du\left(x,t\right)|^p\,\mathrm{d}x\mathrm{d}t\\
&+C\int_{Q_R}|\Delta_h\mathbf{F}|^{\frac{p}{p-1}}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where we used the fact that $\frac{p}{p-1}\le 2\le p$. Furthermore, by dividing both sides by $|h|^{\frac{p\alpha}{p-1}}$ and taking the L^q norm with the measure $\frac{\mathrm{d}h}{|h|^n}$ restricted to the ball B_δ for some $\delta<R$, from Lemma 2.1 we conclude that

$$
\begin{aligned}
&\left(\int_{B_\delta}\left(\int_{Q_{R/2}}\left|\frac{\Delta_h Du}{|h|^{\frac{\alpha}{p-1}}}\right|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}\\
&\le C\left(\int_{B_\delta}|h|^{\frac{(1-\alpha)q}{p-1}}\left(\int_{Q_{2R}}|Du|^p+1\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}\\
&\quad+C\left(\int_{B_\delta}|h|^{\frac{(\beta-\alpha)q}{p-1}}\left(\int_{Q_R}\left|\frac{\Delta_h\mathbf{F}}{|h|^\beta}\right|^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}\\
&\quad+C\left(\int_{B_\delta}\left(\int_{Q_R}\left(g_k\left(x,t\right)+g_k\left(x+h,t\right)\right)^{\frac{p}{p-1}}|Du\left(x,t\right)|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}\\
&=: J_1+J_2+J_3.
\end{aligned}
\tag{2.5}
$$

Estimate of J_1. In view of the facts that $\delta<R$ and $p\ge 2$, we observe that

$$
\begin{aligned}
J_1&\le C\left(\int_{Q_{2R}}|Du|^p+1\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\delta\rho^{\frac{(1-\alpha)q}{p-1}-1}\mathrm{d}\rho\right)^{\frac{1}{q}}\\
&\le C\left(\int_{Q_{2R}}|Du|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}<+\infty,
\end{aligned}
$$

since $u\in L^p_{loc}\left((t_0,t_0+T];W^{1,p}_{loc}(\Omega)\right)$.

Estimate of J_2. Since the proof in the case that $p=2$ is trivial, we may as well assume that $p>2$. Applying Hölder's inequality and Fubini's theorem, we get

$$
\begin{aligned}
J_2 &= C\left(\int_{B_\delta}|h|^{\frac{-n}{p-1}}\left(\int_{Q_R}\left|\frac{\Delta_h\mathbf{F}}{|h|^\beta}\right|^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}|h|^{\frac{(\beta-\alpha)q-(p-2)n}{p-1}}\mathrm{d}h\right)^{\frac{1}{q}}\\
&\le C\left(\int_{B_\delta}\left(\int_{Q_R}\left|\frac{\Delta_h\mathbf{F}}{|h|^\beta}\right|^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{q(p-1)}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q(p-1)}}\\
&\quad\left(\int_{B_\delta}|h|^{\frac{(\beta-\alpha)q}{p-2}-n}\mathrm{d}h\right)^{\frac{p-2}{q(p-1)}}\\
&\le C\left(\int_{B_\delta}\int_{-R^2}^{R^2}\left(\int_{B_R}\left|\frac{\Delta_h\mathbf{F}}{|h|^\beta}\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{q(p-1)}{p}}\mathrm{d}t\,\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q(p-1)}}\\
&\quad\left(\int_0^\delta\rho^{\frac{(\beta-\alpha)q}{p-2}-1}\mathrm{d}\rho\right)^{\frac{p-2}{q(p-1)}}\\
&\le C\left(\int_{-R^2}^{R^2}\int_{B_\delta}\left(\int_{B_R}\left|\frac{\Delta_h\mathbf{F}}{|h|^\beta}\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{q(p-1)}{p}}\frac{\mathrm{d}h}{|h|^n}\mathrm{d}t\right)^{\frac{1}{q(p-1)}}\\
&\le C\,\|\mathbf{F}\|^{\frac{1}{p-1}}_{L^q\left((-R^2,R^2);B^\beta_{\frac{p}{p-1},q}(B_R)\right)}<+\infty,
\end{aligned}
$$

where we also used the facts that $\mathbf{F}(x,t)\in L^{\frac{np}{n(p-1)-p\beta}}\left((t_0,t_0+T];B^\beta_{\frac{p}{p-1},q}(\Omega)\right)$ locally and $q\le\frac{np}{n(p-1)-p\beta}$ for $1\le\frac{q(p-1)}{p}\le\min\left\{\frac{n(p-1)}{n(p-1)-p\beta},\frac{(n+2)(p-1)}{p\beta}\right\}$.

Estimate of J_3. Since $\mathbf{F}(\cdot,t)\in B^\beta_{\frac{p}{p-1},q}(\Omega)$ locally for any $t\in(t_0,t_0+T)$, $0<\beta<1$ and $q\le\left(\frac{p}{p-1}\right)^*_\beta=:\frac{np}{n(p-1)-p\beta}$, from Lemma 1.4 we find that $\mathbf{F}(\cdot,t)\in L_{loc}^{\frac{np}{n(p-1)-p\beta}}(\Omega)$ and then $\mathbf{F}(x,t)\in L_{loc}^{\frac{np}{n(p-1)-p\beta}}(\Omega_T)$ due to the fact that $\mathbf{F}(x,t)\in L^{\frac{np}{n(p-1)-p\beta}}\left((t_0,t_0+T];B^\beta_{\frac{p}{p-1},q}(\Omega)\right)$ locally. So, we conclude that $|\mathbf{F}(x,t)|^{\frac{p}{p-1}}\in L_{loc}^{\frac{n(p-1)}{n(p-1)-p\beta}}(\Omega_T)$ and then $|\mathbf{F}(x,t)|^{\frac{p}{p-1}}\in L_{loc}^{\frac{n(p-1)}{n(p-1)-p\alpha}}(\Omega_T)$ by the fact that $0<\alpha<\beta<1$. Therefore, from (2.1) we know that $Du\in L_{loc}^{\frac{np(p-1)}{n(p-1)-p\alpha}}(\Omega_T)$. Let $r_k=2^{-k}K_0R$. Furthermore, we use the assumption (A_2) and Hölder's inequality to obtain

$$\begin{aligned}
J_3 &\leq C\left(\int_{B_\delta}\left(\int_{Q_R}(g_k(x,t)+g_k(x+h,t))^{\frac{p}{p-1}}|Du(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}}\\
&\leq C\left(\int_0^{2^{-k_0}K_0R}\int_{\partial B_r}\left(\int_{Q_R}(g_k(x,t)\right.\right.\\
&\quad\left.\left.+g_k(x+h,t))^{\frac{p}{p-1}}|Du(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\mathrm{d}S(h)\,\mathrm{d}r\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_{k=k_0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\partial B_r}\left(\int_{Q_R}(g_k(x,t)\right.\right.\\
&\quad\left.\left.+g_k(x+h,t))^{\frac{p}{p-1}}|Du(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\mathrm{d}S(h)\,\mathrm{d}r\right)^{\frac{1}{q}}\\
&\leq C\|Du\|^q_{L^{\frac{np(p-1)}{n(p-1)-p\alpha}}(Q_R)}\left[\sum_{k=k_0}^{\infty}\int_{r_{k+1}}^{r_k}\int_{\partial B_r}\right.\\
&\quad\left.\left(\|g_k(x,t)\|^q_{L^{\frac{n+2}{\alpha}}(Q_{2R})}+1\right)\mathrm{d}S(h)\,\mathrm{d}r\right]^{\frac{1}{q}}\\
&\leq C\|Du\|^q_{L^{\frac{np(p-1)}{n(p-1)-p\alpha}}(Q_R)}\left[\|\{g_k\}_k\|_{l^q\left(L^{\frac{n+2}{\alpha}}(Q_{2R})\right)}+1\right]\\
&< +\infty,
\end{aligned}$$

where we have used (1.13) and

$$\begin{aligned}
&\left(\int_{Q_R}(g_k(x,t)+g_k(x+h,t))^{\frac{p}{p-1}}|Du(x,t)|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\\
&\quad\leq\left(\int_{Q_R}(g_k(x,t)+g_k(x+h,t))^{\frac{n}{\alpha}}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q\alpha}{n(p-1)}}\\
&\quad\left(\int_{Q_R}|Du|^{\frac{pn(p-1)}{n(p-1)-\alpha p}}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}\cdot\frac{n(p-1)-\alpha p}{n(p-1)}}\\
&\quad\leq C\|g_k(x,t)\|^{\frac{q}{p-1}}_{L^{\frac{n}{\alpha}}(Q_{2R})}\|Du\|^q_{L^{\frac{np(p-1)}{n(p-1)-p\alpha}}(Q_R)}\\
&\quad\leq C\|g_k(x,t)\|^{\frac{q}{p-1}}_{L^{\frac{n+2}{\alpha}}(Q_{2R})}\|Du\|^q_{L^{\frac{np(p-1)}{n(p-1)-p\alpha}}(Q_R)}\\
&\quad\leq C\left(\|g_k(x,t)\|^q_{L^{\frac{n+2}{\alpha}}(Q_{2R})}+1\right)\|Du\|^q_{L^{\frac{np(p-1)}{n(p-1)-p\alpha}}(Q_R)}
\end{aligned}$$

for $p \geq 2$. Finally, combining the estimates of J_1, J_2, J_3 and using (2.5), we obtain

$$\left(\int_{B_\delta}\left(\int_{Q_{R/2}}\left|\frac{\Delta_h Du}{|h|^{\frac{\alpha}{p-1}}}\right|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}} < +\infty.$$

Therefore, we obtain the desired result $Du \in B^{\frac{\alpha}{p-1}}_{p,q}(\Omega_T)$ locally for $0 < \alpha < \beta < 1$ and $1 \leq \frac{q(p-1)}{p} \leq \min\left\{\frac{n(p-1)}{n(p-1)-p\beta}, \frac{(n+2)(p-1)}{p\beta}\right\}$. This completes the proof of Theorem 1.6. □

Acknowledgements The authors wish to thank the anonymous reviewers for many valuable comments and suggestions to improve the expressions.

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