Bull. Malays. Math. Sci. Soc. (2021) 44:3725–3732
https://doi.org/10.1007/s40840-021-01142-7

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

The Maximum Number of Spanning Trees of a Graph with Given Matching Number

Muhuo Liu[1] · Guangliang Zhang[2] · Kinkar Chandra Das[3]

Received: 13 January 2021 / Revised: 28 April 2021 / Accepted: 13 May 2021 / Published online: 24 May 2021

Abstract
The number of spanning trees of a graph G is the total number of distinct spanning subgraphs of G that are trees. Feng et al. determined the maximum number of spanning trees in the class of connected graphs with n vertices and matching number β for $2 \le \beta \le n/3$ and $\beta = \lfloor n/2 \rfloor$. They also pointed out that it is still an open problem to the case of $n/3 < \beta \le \lfloor n/2 \rfloor - 1$. In this paper, we solve this problem completely.

Keywords Graph · Spanning tree · Matching number

Mathematics Subject Classification 05C50 · 15A18

1 Introduction

Throughout this paper, we consider simple graph $G = (V, E)$ with vertex set $V(G)$ ($|V(G)| = n$) and edge set $E(G)$. The Laplacian matrix of graph G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of G. Let $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge \mu_n = 0$ be the eigenvalues of $L(G)$. The number of spanning trees of G, denoted by $\kappa(G)$, is the total number of distinct spanning subgraphs of G that are trees. It is well known that

---

Communicated by Rosihan M. Ali.

✉ Kinkar Chandra Das
kinkardas2003@gmail.com

Muhuo Liu
liumuhuo@163.com

Guangliang Zhang
zhgl@gpnu.edu.cn

1 Department of Mathematics, South China Agricultural University, Guangzhou 510642, China

2 School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510631, P. R. China

3 Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea

$$\kappa(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i.$$

We consider the problem of determining the extremal graphs with the maximum number of spanning trees from some special classes of graphs. The application of this problem is (i) in the area of experimental design [2] and (ii) in the network theory [15]. For calculating $\kappa(G)$ of a graph G, we do not have any clear formula in the literature except for some special cases. However, in the design of reliable probabilistic networks, the value (or the bound) on the number of spanning trees is very important (see, [22]).

As usual, we denote by K_n and $K_{1,n-1}$, the complete graph and the star of order n, respectively. We now recall some definitions in graph theory. Two distinct edges in a graph G are independent if they do not have a common end vertex. A **matching** in G is a set of pairwise independent edges. A **maximum matching** in G is a matching of maximum cardinality. The **matching number** of the graph G, denoted by $\beta(G)$, is the number of edges in a maximum matching. Obviously, $\beta(G) = 0$ if and only if G is an empty graph. For a connected graph G with $n \geq 2$ vertices, $\beta(G) = 1$ if and only if $G \cong K_{1,n-1}$ or $G \cong K_3$. Thus, we consider graphs with matching number at least 2. The vertex disjoint union of the graphs G and H is denoted by $G \cup H$. The joint of two graphs G and H is $G \vee H = \overline{\overline{G} \cup \overline{H}}$, where $\overline{G}$ is the complement of graph G.

Problem 1.1 *Given a set $\mathcal{G}$ of graphs, find an upper bound for the number of spanning trees in this set and characterize the graphs at which the maximal number of spanning trees is attained.*

This problem attracts much attention in the literature. Li et al. [16] completely resolved this problem for given connectivity or chromatic number of graphs. For bicyclic and tricyclic graphs, this problem has been resolved in [21,22]. In [1], the authors obtained several sharp upper bounds for $\kappa(G)$ based on the so-called normalized Laplacian eigenvalues (for details on this subject, see [3,4]). Ma and Yao [18] studied the number of spanning trees of a class of self-similar fractal models and obtained an approximate numerical value of its spanning tree entropy. For some other excellent results on this topic including lower and upper bounds, please refer to [5,6,8–15,17,19,20].

Let $\mathcal{G}_{n,\beta}$ be the set of graphs on n vertices with matching number β, and denote by $\overline{G}$ the complement graph of G. Feng et al. presented the following result in the proof of Theorem 1.1 [7].

Proposition 1.2 [7] *Let $G \in \mathcal{G}_{n,\beta}$ and $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$\kappa(G) \leq \max\left\{ n^k (k+1)^{n-2\beta+k} (2\beta - k)^{2\beta-2k-2} : \ 0 \leq k \leq \beta - 1 \right\},$$

where equality holds if and only if $G \cong K_{k+1} \vee \left(K_{2\beta-2k-1} \cup \overline{K_{n+k-2\beta}}\right)$.

Hereafter, let

$$f(x) = n^x\ (x+1)^{n-2\beta+x}(2\beta-x)^{2\beta-2x-2}, \quad \text{and}$$
$$\varphi(x) = \ln\ f(x) = x\ln n + (n-2\beta+x)\ln(x+1) + (2\beta-2x-2)\ln(2\beta-x).$$

Then,

$$\varphi'(x) = \ln n + \ln(x+1) - 2\ln(2\beta-x) + \frac{n-2\beta-1}{x+1} + \frac{x+2}{2\beta-x}$$

and

$$\varphi'(0) = n-2\beta-1+\frac{1}{\beta}+\ln\left(\frac{n}{4\beta^2}\right), \quad \varphi'(\beta-1) = \frac{n-\beta-1}{\beta}+\ln\left(\frac{n\beta}{(\beta+1)^2}\right) > 0. \tag{1.1}$$

In [7], Feng et al. obtained a sharp upper bound on the number of spanning trees of graph G with given matching number β and the statement is as follows:

Theorem 1.3 [7] *Let $G \in \mathcal{G}_{n,\beta}$ and $2 \le \beta \le \lfloor \frac{n}{2} \rfloor$. The following statements hold:*

(1) For $\beta = \lfloor \frac{n}{2} \rfloor$, $\kappa(G) \le n^{n-2}$ with equality if and only if $G \cong K_n$;
(2) For $2 \le \beta \le \frac{n}{3}$, $\kappa(G) \le n^{\beta-1}\beta^{n-\beta-1}$ with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

For the case of $n = 2\beta+2$ and $\beta \ge 9$, Feng et al. pointed out that (See Remark 1 of [7]), in the class of $\mathcal{G}_{n,\beta}$, $K_1 \vee \left(K_{2\beta-1} \bigcup \overline{K_{n-2\beta}}\right)$ is the maximal graph of $\kappa(G)$. They also raised the problem to solve the case of $\frac{n}{3} < \beta < \lfloor \frac{n}{2} \rfloor - 1$. In this paper, we give a solution to this problem completely (as they have solved the situation of $\beta = \lfloor \frac{n}{2} \rfloor$ and $n = 2\beta+2$, we only consider the case of $n \ge 2\beta+3$ in the following).
For $n \ge 2.3\,\beta$ and $\beta \ge 9$, in this paper, we give an upper bound on the number of spanning trees of graph G in terms of n and β, and the statement is as follows.

Theorem 1.4 *Let $G \in \mathcal{G}_{n,\beta}$ with $n \ge 2.3\,\beta$ and $\beta \ge 9$. Then*

$$\kappa(G) \le n^{\beta-1}\beta^{n-\beta-1},$$

with equality if and only if $G \cong K_\beta \vee \overline{K_{n-\beta}}$.

For $2\beta+3 \le n < 2.3\beta$, in this paper, we present an upper bound on the number of spanning trees of graph G in terms of n and β, and the statement is as follows.

Theorem 1.5 *Let $G \in \mathcal{G}_{n,\beta}$ with $2\beta+3 \le n < 2.3\beta$.*

(1) If $n-2\beta-1+\frac{1}{\beta}+\ln\left(\frac{n}{4\beta^2}\right) < 0$, then

$$\kappa(G) \le \max\left\{(2\beta)^{2\beta-2},\ n^{\beta-1}\,\beta^{n-\beta-1}\right\}.$$

(2) If $n-2\beta-1+\frac{1}{\beta}+\ln\left(\frac{n}{4\beta^2}\right)>0$*, then* $\varphi'(x)$ *contains exactly two roots in the interval* $(0,\beta-1)$*. Let* α *be the smallest root of* $\varphi'(x)$ *in the interval* $(0,\beta-1)$*. Then,*

$$\kappa(G)\leq\max\left\{f(\beta-1),\ f(\lceil\alpha\rceil),\ f(\lfloor\alpha\rfloor)\right\}.$$

As K_n is a special case, by comparing the results of Theorem 1.3 (2) and Theorems 1.4–1.5, it is natural to ask the following problem:

Problem 1.6 *Let* $G\in\mathcal{G}_{n,\beta}$ *with* $2\beta+3\leq n<2.3\beta$*. Whether* $\kappa(G)\leq n^{\beta-1}\beta^{n-\beta-1}$ *is true or not, that is, whether* $n^{\beta-1}\beta^{n-\beta-1}$ *also uniquely maximizes* $\kappa(G)$ *in Theorem 1.5?*

However, the answer to Problem 1.6 is negative, as we have the following example.

Example 1.7 *Let* $G\in\mathcal{G}_{n,\beta}$*, where* $2\beta+3\leq n<2.3\beta$*.*

(i) Let $\beta=10000$ and $n=20004$. Then, $\varphi'(0)<4+\ln\left(\frac{5001}{10000^2}\right)<-5.9$, and thus, the condition of Theorem 1.5 (1) is satisfied. Using **MATLAB**, we have

$$\begin{aligned}&2(\beta-1)\ln 2\beta-(\beta-1)\ln n-(n-\beta-1)\ln\beta\\&\quad=2\times 9999\times\ln(2\times 10000)-9999\times\ln 20004-10003\times\ln 10000>6891,\end{aligned}$$

and thus,

$$2(\beta-1)\ln 2\beta>(\beta-1)\ln n+(n-\beta-1)\ \ln\ \beta,$$

which implies that $(2\beta)^{2\beta-2}>n^{\beta-1}\,\beta^{n-\beta-1}$.
(ii) Let $\beta=10000$ and $n=21000$. Then, using **Matlab** it follows that

$$\begin{aligned}&\varphi'(0)>999+\ln\left(\frac{5250}{10000^2}\right)>989,\ \varphi'(226)>0.005,\ \varphi'(227)<-0.009,\\&\varphi'(5598)<-3.6\times 10^{-4}\ \text{ and }\ \varphi'(5599)>1.79\times 10^{-5},\end{aligned}$$

and thus, Theorem 1.5 (2) implies that the smallest root of $\varphi'(x)$ is in $(226, 227)$. Using **Matlab**, one can easily see that

$$f(226)=\max\left\{f(226),\ f(227),\ f(9999)\right\}.$$

2 Proofs of Theorems 1.4 and 1.5

This section is dedicated to the proofs of Theorems 1.4 and 1.5 . We firstly give the proof to Theorem 1.4.

Proof of Theorem 1.4 Let $g(x) = 2.3\beta^{2.3} - (2\beta - x)^2(x+1)^{0.3}$, where $0 \le x \le \beta - 1$. Then,

$$g'(x) = 0.1(x+1)^{-0.7}(23x + 20 - 6\beta)(2\beta - x). \tag{2.1}$$

Since $2\beta - x > 0$ always holds for $0 \le x \le \beta - 1$ and $\beta \ge 9$, by (2.1) we have

$$\begin{aligned} g(x) &\ge g\left(\frac{6\beta - 20}{23}\right) \\ &= 2.3\beta^{2.3} - \frac{400}{12167} \times 3^{0.3} \times 23^{0.7}(2\beta + 1)^{2.3} \\ &= \frac{1}{121670}\beta^{2.3}\left(279841 - 4000 \times 3^{0.3} \times 23^{0.7}\left(2 + \frac{1}{\beta}\right)^{2.3}\right) \\ &\ge \frac{1}{121670}\beta^{2.3}\left(279841 - 4000 \times 3^{0.3} \times 23^{0.7}\left(2 + \frac{1}{9}\right)^{2.3}\right) > 0. \end{aligned} \tag{2.2}$$

When $n \ge 2.3\,\beta$ and $0 \le k \le \beta - 2$, by (2.2), we obtain

$$\begin{aligned} &\left(\frac{n(k+1)}{(2\beta - k)^2}\right)^{\beta-k-1}\left(\frac{\beta}{k+1}\right)^{n-\beta-1} \\ &= \left(\frac{n\beta^{1.3}(k+1)}{(2\beta - k)^2(k+1)^{1.3}}\right)^{\beta-k-1}\left(\frac{\beta}{k+1}\right)^{n-2.3\beta+1.3k+0.3} \\ &\ge \left(\frac{n\beta^{1.3}}{(2\beta - k)^2(k+1)^{0.3}}\right)^{\beta-k-1} \\ &\ge \left(\frac{2.3\beta^{2.3}}{(2\beta - k)^2(k+1)^{0.3}}\right)^{\beta-k-1} > 1. \end{aligned}$$

Thus, for $0 \le k \le \beta - 2$ and $\beta \ge 9$,

$$n^k(k+1)^{n-2\beta+k}(2\beta - k)^{2\beta-2k-2} < n^{\beta-1}\beta^{n-\beta-1},$$

and hence, the result follows from Proposition 1.2. □

To present the proof of Theorem 1.5, we need more lemmas as follows:

Lemma 2.1 *If $2\beta + 3 \le n < 2.3\beta$, then there exists some real number p with $0 < p < \beta - 1$ such that $\varphi'(x)$ is a strictly decreasing function for $0 \le x \le p$, and $\varphi'(x)$ is a strictly increasing function for $p \le x \le \beta - 1$.*

Proof For $0 \le x \le \beta - 1$, we define $\psi(x)$ as follows:

$$\begin{aligned}\psi(x) &= (x+1)^2\,(2\beta - x)^2\,\varphi''(x)\\ &= -x^3 + (4\beta - n)x^2 + 2\,(2n\beta\\ &\quad -2\beta^2 + 2\beta + 1\Big)\,x + 8\beta^3 + 8\beta^2 + 6\beta + 2 - 4n\beta^2.\end{aligned}$$

Claim 1 $\psi(x)$ **is a strictly increasing function on** $[0,\ \beta - 1]$.

Proof of Claim 1 Note that $\psi''(x) = 2\,(4\beta - n - 3x)$. Then $\psi''(x) < 0$ for $x > \frac{1}{3}\,(4\beta - n)$ and $\psi''(x) > 0$ for $x < \frac{1}{3}\,(4\beta - n)$, that is, $\psi'(x)$ is a strictly increasing function on $0 \le x \le \frac{1}{3}\,(4\beta - n)$ and a strictly decreasing function on $\frac{1}{3}\,(4\beta - n) \le x \le \beta - 1$. Thus,

$$\psi'(x) \ge \min\Big\{\psi'(0),\ \psi'(\beta - 1)\Big\}.$$

Recall that $0 \le x \le \beta - 1$ and $n \ge 2\beta$. Thus, $\psi'(\beta - 1) = \beta^2 + 2(\beta+1)(n+1) - 3 > 0$ and $\psi'(0) = 4n\beta - 4\beta^2 + 4\beta + 2 \ge 4\beta^2 + 4\beta + 2 > 0$. Therefore, $\psi'(x) > 0$, and hence, $\psi(x)$ is a strictly increasing function on $[0,\ \beta - 1]$. This completes the proof of **Claim** 1.

Since $0 \le x \le \beta - 1$ and by **Claim** 1, we have $\psi(0) \le \psi(x) \le \psi(\beta - 1)$. Again since $\beta \ge 11$ by $2\beta + 3 \le n < 2.3\beta$, we obtain

$$\begin{aligned}\psi(0) &= -4\beta^2 n + 8\beta^3 + 8\beta^2 + 6\beta + 2\\ &\le -4\beta^2(2\beta + 3) + 8\beta^3 + 8\beta^2 + 6\beta + 2 = -2(2\beta^2 - 3\beta - 1) < 0,\end{aligned}$$

and

$$\begin{aligned}\psi(\beta - 1) &= (\beta + 1)(4\beta - n(\beta + 1) + 7\,\beta^2 + 1)\\ &> (\beta + 1)\Big(4\,\beta - 2.3\,\beta\,(\beta + 1) + 7\,\beta^2 + 1\Big)\\ &= (\beta + 1)(1.7\,\beta + 4.7\,\beta^2 + 1) > 0.\end{aligned}$$

Combining with **Claim** 1 and $\psi(0) < 0 < \psi(\beta - 1)$, we can conclude that there is some real number p with $0 < p < \beta - 1$ such that $\psi(p) = 0$. Once again, **Claim** 1 implies that $\psi(x) < 0$ for $0 \le x < p$, $\psi(x) > 0$ for $p < x \le \beta - 1$ and $\psi(p) = 0$. Recall that $\psi(x) = (x+1)^2(2\beta - x)^2\varphi''(x)$ and $0 \le x \le \beta - 1$. Thus, $\varphi''(x) < 0$ for $0 \le x < p$, $\varphi''(x) > 0$ for $p < x \le \beta - 1$ and $\varphi''(p) = 0$. This completes the proof of this result. □

Lemma 2.2 *If* $n - 2\beta - 1 + \frac{1}{\beta} + \ln\left(\frac{n}{4\beta^2}\right) > 0$ *and* $2\beta + 3 \le n < 2.3\beta$, *then* $\varphi'(0.2\beta) < 0$ *and* $\varphi'(x)$ *contains exactly two roots in the interval* $(0,\ \beta - 1)$.

Proof By (1.1), we have $\varphi'(0) > 0$ and $\varphi'(\beta - 1) > 0$. Combining this with Lemma 2.1, to complete the proof, it suffices to show that $\varphi'(0.2\beta) < 0$.

Since $2\beta + 3 \le n < 2.3\beta$, we have

$$\begin{aligned}\varphi'(0.2\beta) &= \ln(0.2\beta+1) - 2\ln(1.8\beta) + \ln n + \frac{n-2\beta-1}{0.2\beta+1} + \frac{0.2\beta+2}{1.8\beta} \\ &< \ln(0.2\beta+1) - 2\ln(1.8\beta) + \ln(2.3\beta) + \frac{0.3\beta-1}{0.2\beta+1} + \frac{0.2\beta+2}{1.8\beta} \\ &= \ln\left(\frac{2.3\beta\,(0.2\beta+1)}{(1.8\beta)^2}\right) + \frac{29\beta^2-60\beta+100}{18\beta(\beta+5)}. \qquad (2.3)\end{aligned}$$

Denote by

$$h(t) = \ln\left(\frac{2.3t\,(0.2t+1)}{(1.8t)^2}\right) + \frac{29t^2-60t+100}{18t(t+5)}.$$

Then, $h'(t) = \frac{5(23t^2-130t-100)}{18t^2(t+5)^2}$, which implies that $h'(t) > 0$ for $t > \frac{65+15\sqrt{29}}{23}$ and $h'(t) < 0$ for $11 \le t < \frac{65+15\sqrt{29}}{23}$. Furthermore, since $h(11) < -0.64$ and $h(+\infty) = \ln\left(\frac{23}{162}\right) + \frac{29}{18} < -0.34$, we can conclude that $h(t) < 0$ for $t \ge 11$. Combining this with $\beta \ge 11$ by $2\beta+3 \le n < 2.3\beta$, we have $\varphi'(0.2\beta) < 0$ by (2.3), as desired. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5 **(1)** $n - 2\beta - 1 + \frac{1}{\beta} + \ln\left(\frac{n}{4\beta^2}\right) < 0$. Then $\varphi'(0) < 0$ and $\varphi'(\beta-1) > 0$, by (1.1). Combining this with $\varphi'(x)$ being a strictly decreasing function on $[0, p]$ and a strictly increasing function on $[p, \beta-1]$ by Lemma 2.1, we can conclude the existence of some real number q with $0 < q < \beta - 1$ such that $\varphi'(x) < 0$ holds for $0 \le x < q$, $\varphi'(x) > 0$ holds for $q < x \le \beta - 1$ and $\varphi'(q) = 0$. Thus, $\varphi(x)$ is a strictly decreasing function on $[0, q]$ and a strictly increasing function on $[q, \beta-1]$. Since $\varphi(x) = \ln f(x)$ with Proposition 1.2, we obtain

$$\begin{aligned}\kappa(G) &\le \max\left\{n^k(k+1)^{n-2\beta+k}(2\beta-k)^{2\beta-2k-2} : 0 \le k \le \beta-1\right\} \\ &= \max\left\{(2\beta)^{2\beta-2},\ n^{\beta-1}\beta^{n-\beta-1}\right\}.\end{aligned}$$

(2) $n - 2\beta - 1 + \frac{1}{\beta} + \ln\left(\frac{n}{4\beta^2}\right) > 0$. By Lemma 2.2, $\varphi'(0.2\beta) < 0$ and $\varphi'(x)$ contains exactly two roots in the interval $(0, \beta-1)$. Let α be the smallest root of $\varphi'(x)$ in the interval $(0, \beta-1)$. Then $\varphi'(x) > 0$ holds for $0 < x < \alpha$, as $\varphi'(0) > 0$.

Recall that $\varphi'(x)$ is strictly decreasing on $[0, p]$ and strictly increasing on $[p, \beta-1]$ by Lemma 2.1. Combining this with $\varphi'(x) > 0 = \varphi'(\alpha)$ holds for $0 \le x < \alpha$, $\varphi'(\beta-1) > 0$ and $\varphi'(0.2\beta) < 0$ by Lemma 2.2, we can conclude that $\alpha < p < \beta-1$.

In this case, $\varphi'(p) < \varphi'(\alpha) = 0$. Since $\varphi'(x)$ is strictly increasing on $[p, \beta-1]$ by Lemma 2.1 and since $\varphi'(\beta-1) > 0$, there exists $\gamma \in (p, \beta-1)$ such that $\varphi'(\gamma) = 0$.

Once again, Lemma 2.2 implies that $\varphi'(x) < 0$ for $x \in (\alpha, \gamma)$ and $\varphi'(x) > 0$ for $x \in (0, \alpha) \bigcup (\gamma, \beta - 1)$. Thus, $\varphi(x)$ is strictly increasing for $x \in [0, \alpha]$, $\varphi(x)$ is strictly decreasing for $x \in [\alpha, \gamma]$ and $\varphi(x)$ is strictly increasing for $x \in [\gamma, \beta - 1]$. This completes the proof of (2). □

Acknowledgements The authors would like to thank two referees for their valuable comments which lead to an improvement of the original manuscript.

References

1. Bozkurt, S.B.: Upper bounds for the number of spanning trees of graphs. J. Inequal. Appl. **2012**, 269 (2012)
2. Cheng, C.-S.: Maximizing the total number of spanning trees in a graph: Two related problems in graph theory and optimum design theory. J. Combin. Theory Ser. B **31**, 240–248 (1981)
3. Chung, F.: Spectral Graph Theory. CBMS, USA (1997)
4. Chung, F.: Discrete isoperimetric inequalities. Surveys in Differential Geometry IX, International Press **9**, 53–82 (2004)
5. Das, K.C.: A sharp upper bounds for the number of spanning trees of a graph. Graphs Combin. **23**, 625–632 (2007)
6. Das, K.C., Çevik, A.S., Cangül, I.N.: The number of spanning trees of a graph. J. Inequal. Appl. **2013**, 395 (2013)
7. Feng, L., Xu, K., Das, K.C., Ilić, A., Yu, G.: The number of spanning trees of a graph with given matching number. Int. J. Comput. Math. **93**, 837–843 (2016)
8. Feng, L., Yu, G., Jiang, Z., Ren, L.: Sharp upper bounds for the number of spanning trees of a graph. Appl. Anal. Discrete Math. **2**, 255–259 (2008)
9. Grimmett, G.R.: An upper bound for the number of spanning trees of a graph. Discrete Math. **16**, 323–324 (1976)
10. Grone, R., Merris, R.: A bound for the complexity of a simple graph. Discrete Math. **69**, 97–99 (1988)
11. Gu, X.: Spectral conditions for edge connectivity and packing spanning trees in multigraphs. Linear Algebra Appl. **493**, 82–90 (2016)
12. Ha, P.H., Hanh, D.D.: Spanning trees of connected $K_{1,t}$-free graphs whose stems have a few leaves. Bull. Malays. Math. Sci. Soc. **43**, 2373–2383 (2020)
13. Haghighi, M.H.S., Bibak, K.H.: The number of spanning trees in some classes of graphs. Rocky Mountain J. Math. **42**, 1183–1195 (2012)
14. Kao, S., Pai, K., Hsieh, S., Wu, R., Chang, J.: Amortized efficiency of constructing multiple independent spanning trees on bubble-sort networks. J. Comb. Optim. **38**, 972–986 (2019)
15. Kelmans, A.K.: On graphs with the maximum number of spanning trees. Random Struct. Algorithms **9**, 177–192 (1996)
16. Li, J., Shiu, W.C., Chang, A.: The number of spanning trees of a graph. Appl. Math. Lett. **23**, 286–290 (2010)
17. Liu, R., Lai, H.-J., Tian, Y.: Spanning tree packing number and eigenvalues of graphs with given girth. Linear Algebra Appl. **578**, 411–424 (2019)
18. Ma, F., Yao, B.: The number of spanning trees of a class of self-similar fractal models. Inform. Process. Lett. **136**, 64–69 (2018)
19. Nikolopoulos, S.D., Palios, L., Papadopoulos, C.: Maximizing the number of spanning trees in K_n-complements of asteroidal graphs. Discrete Math. **309**, 3049–3060 (2009)
20. Teranishi, Y.: The number of spanning forests of a graph. Discrete Math. **290**, 259–267 (2005)
21. Tseng, S.S., Wang, L.R.: Maximizing the number of spanning trees of networks based on cycle basis representation. Int. J. Comput. Math. **28**, 47–56 (1989)
22. Wu, M.: Maximizing the number of spanning trees on the $(p,\ p+2)$ graphs. Int. J. Comput. Math. **32**, 27–38 (1990)