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The Mostar Index of Fibonacci and Lucas Cubes

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Abstract
The Mostar index of a graph was defined by Došlić, Martinjak, Škrekovski, Tipurić Spužević and Zubac in the context of the study of the properties of chemical graphs. It measures how far a given graph is from being distance-balanced. In this paper, we determine the Mostar index of two well-known families of graphs: Fibonacci cubes and Lucas cubes.

Keywords Fibonacci cube · Lucas cube · Mostar index

Mathematics Subject Classification 05C09 · 05C12 · 05A15

1 Introduction

We consider what is termed the *Mostar index* of Fibonacci and Lucas cubes. These two families of graphs are special subgraphs of hypercube graphs. They were introduced as alternative interconnection networks to hypercubes and have been studied extensively because of their interesting graph theoretic properties. The Mostar index of a graph was introduced in [4].

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Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $uv \in E(G)$, let $n_{u,v}(G)$ denote the number of vertices in $V(G)$ that are closer (w.r.t. the standard shortest path metric) to u than to v, and let $n_{v,u}(G)$ denote the number of vertices in $V(G)$ that are closer to v than to u. The *Mostar index* of G is defined in [4] as

$$\mathrm{Mo}(G) = \sum_{uv \in E(G)} |n_{u,v}(G) - n_{v,u}(G)| \, .$$

When G and uv is clear from the context, we will write $n_u = n_{u,v}(G)$ and $n_v = n_{v,u}(G)$.

Distance-related properties of graphs such as the Wiener index, irregularity and Mostar index have been studied for various families of graphs in the literature.

The Wiener index $W(G)$ of a connected graph G is defined as the sum of distances over all unordered pairs of vertices of G. It is determined for Fibonacci cubes and Lucas cubes in [9]. The irregularity of a graph is another distance invariant measuring how much the graph differs from a regular graph, and Albertson index (irregularity) is defined as the sum of $|deg(u) - deg(v)|$ over all edges uv in the graph [1]. The irregularity of Fibonacci cubes and Lucas cubes is studied in [2,5]. The relation between the Mostar index and the irregularity of graphs and their difference is investigated in [6]. Recently, the Mostar index of trees and product graphs has been investigated in [3].

In this work, we determine the Mostar index of Fibonacci cubes and Lucas cubes. As a consequence, we derive a relation between the Mostar and the Wiener indices for Fibonacci cubes, giving an alternate expression to the closed formula for $W(\Gamma_n)$ calculated in [9].

2 Preliminaries

We use the notation $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{Z}^+$. Let $B = \{0, 1\}$ and

$$B_n = \{b_1 b_2 \ldots b_n \mid \forall i \in [n] \;\; b_i \in B\}$$

denote the set of all binary strings of length n. Special subsets of B_n defined as

$$\mathcal{F}_n = \{b_1 b_2 \ldots b_n \mid \forall i \in [n-1] \;\; b_i \cdot b_{i+1} = 0\}$$

and

$$\mathcal{L}_n = \{b_1 b_2 \ldots b_n \mid \forall i \in [n-1] \;\; b_i \cdot b_{i+1} = 0 \text{ and } b_1 \cdot b_n = 0\}$$

are the set of all Fibonacci strings and Lucas strings of length n, respectively.

The n-dimensional hypercube Q_n has vertex set B_n. Two vertices are adjacent if and only if they differ in exactly one coordinate in their string representation. For $n \geq 1$, the Fibonacci cube Γ_n and the Lucas cube Λ_n are defined as the subgraphs of

Q_n induced by the Fibonacci strings $\mathcal{F}_n$ and Lucas strings $\mathcal{L}_n$ of length n [7,10]. For convenience, we take $\Gamma_0 = K_1$ whose only vertex is represented by the empty string.

One can classify the binary strings defining the vertices of Γ_n by the value of b_1. In this way, Γ_n decomposes into a subgraph Γ_{n-1} whose vertices start with 0 and a subgraph Γ_{n-2} whose vertices start with 10 in Γ_n. This decomposition can be denoted by

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} \; .$$

Furthermore, $0\Gamma_{n-1}$ in turn has a subgraph $00\Gamma_{n-2}$ and there is a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, whose edges are called *link edges*. This decomposition is the *fundamental decomposition* of Γ_n. In a similar way, we can also decompose Γ_n as

$$\Gamma_n = \Gamma_{n-1}0 + \Gamma_{n-2}01 \; .$$

We refer to [8] for further details on Γ_n .

For $n \geq 2$, Λ_n is obtained from Γ_n by deleting the vertices that start and end with 1. This gives the fundamental decomposition of Λ_n as

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0 \; .$$

Here, $0\Gamma_{n-1}$ has a subgraph $00\Gamma_{n-3}0$ and there is a perfect matching between $00\Gamma_{n-3}0$ and $10\Gamma_{n-3}0$.

Fibonacci numbers f_n are defined by the recursion $f_n = f_{n-1}+f_{n-2}$ for $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. Similarly, the Lucas numbers L_n are defined by $L_n = L_{n-1}+L_{n-2}$ for $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$. It is well known that $|V(Q_n)| = |B_n| = 2^n$, $|V(\Gamma_n)| = |\mathcal{F}_n| = f_{n+2}$ and $|V(\Lambda_n)| = |\mathcal{L}_n| = L_n$.

For any binary string s, let $w_H(s)$ denote the Hamming weight of s, that is, the number of its nonzero coordinates. The *XOR* of two binary strings s_1 and s_2 of length n, denoted by $s_1 \oplus s_2$, is defined as the string of length n whose coordinates are the modulo 2 sum of the coordinates of s_1 and s_2. The distance $d(u, v)$ between two vertices u and v of the hypercube, the Fibonacci cube and the Lucas cube is equal to the Hamming distance between the string representations of u and v. In other words, $d(u, v) = d_H(s_1, s_2) = w_H(s_1 \oplus s_2)$ for any of these graphs, by assuming u and v have string representations s_1 and s_2, respectively.

3 The Mostar Index of Fibonacci Cubes

For any $uv \in E(\Gamma_n)$, let the string representations of u and v be $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$, respectively. By the structure of Γ_n, we know that $d(u, v) = 1$; that is, there is only one index k for which $u_k \neq v_k$.

Lemma 1 *For $n \geq 2$, assume that $uv \in E(\Gamma_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then, $n_{u,v}(\Gamma_n) = f_{k+1} f_{n-k+2}$ and $n_{v,u}(\Gamma_n) = f_k f_{n-k+1}$.*

Proof The result is clear for $n = 2$. Assume that $n \geq 3$, $1 < k < n$ and let $\alpha \in V(\Gamma_n)$ have string representation $b_1 b_2 \ldots b_n$. Since $uv \in E(\Gamma_n)$, u and v must be of the form $a_1 \ldots a_{k-1} 0 a_{k+1} \ldots a_n$ and $a_1 \ldots a_{k-1} 1 a_{k+1} \ldots a_n$, respectively. Since $v \in V(\Gamma_n)$, we must have $a_{k-1} = a_{k+1} = 0$. From these representations, we observe that the difference between $d(\alpha, u)$ and $d(\alpha, v)$ depends on the value of b_k only. If $b_k = 0$, we have $d(\alpha, u) = d(\alpha, v) - 1$, and if $b_k = 1$, we have $d(\alpha, u) = d(\alpha, v) + 1$. Therefore, the vertices whose kth coordinate is 0 are closer to u than v, and the vertices whose kth coordinate is 1 are closer to v than u. Hence, $n_{u,v}(\Gamma_n)$ is equal to the number of vertices in Γ_n whose kth coordinate is 0. These vertices have string representation of the form $\beta_1 0 \beta_2$ where β_1 is any Fibonacci string of length $k - 1$ and β_2 is any Fibonacci string of length $n - k$. Consequently, $n_{u,v}(\Gamma_n) = f_{k+1} f_{n-k+2}$. Similarly, $n_{v,u}(\Gamma_n)$ is number of vertices of the form $\beta_3 010 \beta_4$, and this is equal to $f_k f_{n-k+1}$.

For the case $k = 1$, we have $u \in V(0\Gamma_{n-1})$ and $v \in V(10\Gamma_{n-2})$. Then, $n_{u,v}(\Gamma_n) = |V(0\Gamma_{n-1})| = f_{n+1}$ and $n_{v,u}(\Gamma_n) = |V(10\Gamma_{n-2})| = f_n$. Similarly, for $k = n$ we have $u \in V(\Gamma_{n-1}0)$ and $v \in V(\Gamma_{n-2}01)$. This gives again $n_{u,v}(\Gamma_n) = f_{n+1}$ and $n_{v,u}(\Gamma_n) = f_n$ for $k = n$. As $f_1 = f_2 = 1$, these are also of the form claimed. □

To find the Mostar index of Fibonacci cubes, we only need to find the number of edges uv in Γ_n for which $u_k = 0$ and $v_k = 1$ for a fixed $k \in [n]$ and add up these contributions over k.

Lemma 2 *For $n \geq 2$, assume that $uv \in E(\Gamma_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then, the number of such edges in Γ_n is equal to $f_k f_{n-k+1}$.*

Proof As in the proof of Lemma 1, the result is clear for $n = 2$. Assume that $n \geq 3$. For $1 < k < n$, we know that u and v are of the form $a_1 \ldots a_{k-2} 000 a_{k+2} \ldots a_n$ and $a_1 \ldots a_{k-2} 010 a_{k+2} \ldots a_n$. Then, the number of edges uv in Γ_n satisfying $u_k = 0$ and $v_k = 1$ is equal to the number of vertices of the form $a_1 \ldots a_{k-2} 000 a_{k+2} \ldots a_n$, which gives the desired result.

For the boundary cases $k = 1$ and $k = n$, we need to find the number of vertices of the form $00 a_3 \ldots a_n$ and $a_1 \ldots a_{n-2} 00$, respectively. Clearly, this number is equal to $|V(00\Gamma_{n-2})| = f_n$ and $f_1 = 1$. This completes the proof. □

Using Lemma 1 and Lemma 2, we obtain the following main result.

Theorem 1 *The Mostar index of Fibonacci cube Γ_n is given by*

$$\mathrm{Mo}(\Gamma_n) = \sum_{k=1}^{n} f_k f_{n-k+1} \left(f_{k+1} f_{n-k+2} - f_k f_{n-k+1} \right) . \tag{1}$$

Proof Let $uv \in E(\Gamma_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then, from Lemma 1 we know that

$$|n_u - n_v| = f_{k+1} f_{n-k+2} - f_k f_{n-k+1}$$

and therefore using Lemma 2, we have

$$\begin{aligned}\mathrm{Mo}(\Gamma_n) &= \sum_{uv\in E(\Gamma_n)} |n_u - n_v| \\ &= \sum_{k=1}^{n} f_k f_{n-k+1}\left(f_{k+1}f_{n-k+2} - f_k f_{n-k+1}\right) .\end{aligned}$$

□

Note that $f_{k+1}f_{n-k+2} - f_k f_{n-k+1} = f_k f_{n-k} + f_{k-1}f_{n-k+2}$ so that we can equivalently write

$$\mathrm{Mo}(\Gamma_n) = \sum_{k=1}^{n} f_k f_{n-k+1}\left(f_k f_{n-k} + f_{k-1}f_{n-k+2}\right) .$$

In Sect. 5, Theorem 3, we present a closed-form formula for $\mathrm{Mo}(\Gamma_n)$ obtained by using the theory of generating functions.

Next, we consider the Mostar index of Lucas cubes.

4 The Mostar Index of Lucas Cubes

We know that $\Lambda_2 = \Gamma_2$ and therefore $\mathrm{Mo}(\Gamma_2) = \mathrm{Mo}(\Lambda_2) = 2$.

For any $uv \in E(\Lambda_n)$, let the string representations of u and v be $u_1u_2\ldots u_n$ and $v_1v_2\ldots v_n$, respectively. We know that $d(u, v) = 1$ and there is only one index k for which $u_k \neq v_k$. Similar to Lemma 1 and Lemma 2, we have the following result.

Lemma 3 *For $n \geq 3$, assume that $uv \in E(\Lambda_n)$ with $u_k = 0$ and $v_k = 1$ for some $k \in [n]$. Then, $n_{u,v}(\Lambda_n) = f_{n+1}$ and $n_{v,u}(\Lambda_n) = f_{n-1}$.*

Proof Assume that $1 < k < n$ and let $\alpha \in V(\Lambda_n)$ having string representation $b_1b_2\ldots b_n$. Since $uv \in E(\Lambda_n)$, u must be of the form $a_1\ldots a_{k-2}000a_{k+2}\ldots a_n$ and v must be of the form $a_1\ldots a_{k-2}010a_{k+2}\ldots a_n$. Then, if $b_k = 0$, we have $d(\alpha, u) = d(\alpha, v) - 1$ and if $b_k = 1$, we have $d(\alpha, u) = d(\alpha, v) + 1$. Therefore, $n_{u,v}(\Lambda_n)$ and $n_{v,u}(\Lambda_n)$ are equal to the number of vertices in Λ_n whose kth coordinate is 0 and 1, respectively. Therefore, we need to count the number of Lucas strings of the form $\beta_1 0 \beta_2$ and $\beta_3 010 \beta_4$ which gives $n_{u,v}(\Lambda_n) = f_{n+1}$ and $n_{v,u}(\Lambda_n) = f_{n-1}$.

For the case $k = 1$, using the fundamental decomposition of Λ_n we have $u \in V(0\Gamma_{n-1})$ and $v \in V(10\Gamma_{n-3}0)$. Then, $n_{u,v}(\Lambda_n) = |V(0\Lambda_n)| = f_{n+1}$ and $n_{v,u}(\Lambda_n) = |V(10\Gamma_{n-3}0)| = f_{n-1}$. Similarly, for $k = n$ we have the same results $n_{u,v}(\Lambda_n) = f_{n+1}$ and $n_{v,u}(\Lambda_n) = f_{n-1}$. □

For any $uv \in E(\Lambda_n)$ using Lemma 3, we have

$$|n_{u,v}(\Lambda_n) - n_{v,u}(\Lambda_n)| = f_{n+1} - f_{n-1} = f_n .$$

Since the number of edges in Λ_n is nf_{n-1} [10], similar to Theorem 1 we have the following result.

Theorem 2 *The Mostar index of Lucas cube Λ_n is given by*

$$\mathrm{Mo}(\Lambda_n) = n f_n f_{n-1} \; .$$

Here, we remark that the vertices of Lucas cubes are represented by Lucas strings which are circular binary strings that avoid the pattern “11." Because of this symmetry, the derivation of a closed formula of Theorem 2 for the Mostar index of Lucas cube Λ_n is easier than the one for Γ_n, in which the first and the last coordinates behave differently from the others.

5 A Closed Formula for Mo(Γ_n)

By the fundamental decomposition of Γ_n, the set of edges $E(\Gamma_n)$ consists of three distinct types:

1. The edges in $0\Gamma_{n-1}$, which we denote by $E(0\Gamma_{n-1})$.
2. The link edges between $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$, denoted by C_n.
3. The edges in $10\Gamma_{n-2}$, which we denote by $E(10\Gamma_{n-2})$.

In other words, we have the partition

$$E(\Gamma_n) = E(0\Gamma_{n-1}) \cup C_n \cup E(10\Gamma_{n-2}) \; .$$

We keep track of the contribution of each part of this decomposition by setting for $n \geq 2$,

$$\begin{aligned} M_n(x, y, z) = & \sum_{uv \in E(0\Gamma_{n-1})} |n_u - n_v| x + \sum_{uv \in C_n} |n_u - n_v| y \\ & + \sum_{uv \in E(10\Gamma_{n-2})} |n_u - n_v| z \; . \end{aligned} \tag{2}$$

Clearly, $\mathrm{Mo}(\Gamma_n) = M_n(1, 1, 1)$. By direct inspection, we observe that

$$\begin{aligned} M_2 &= x + y \\ M_3 &= 4x + 2y + z \\ M_4 &= 16x + 6y + 6z \\ M_5 &= 54x + 15y + 23z \end{aligned}$$

which gives

$$\begin{aligned} \mathrm{Mo}(\Gamma_2) &= M_2(1, 1, 1) = 2 \\ \mathrm{Mo}(\Gamma_3) &= M_3(1, 1, 1) = 7 \\ \mathrm{Mo}(\Gamma_4) &= M_4(1, 1, 1) = 28 \\ \mathrm{Mo}(\Gamma_5) &= M_5(1, 1, 1) = 92 \; , \end{aligned}$$

consistent with the values that are calculated using Theorem 1.

By using the fundamental decomposition of Γ_n, we obtain the following useful result.

Proposition 1 *For $n \geq 2$, the polynomial $M_n(x, y, z)$ satisfies*

$$M_n(x, y, z) = M_{n-1}(x+z, 0, x) + M_{n-2}(2x+z, x+z, x+z) \\ + f_{n-1}\,(f_n + f_{n-2})\,x + f_n f_{n-1} y$$

where $M_0(x, y, z) = M_1(x, y, z) = 0$.

Proof By the definition (2), there are three cases to consider:

1. Assume that $uv \in C_n$ such that $u \in V(0\Gamma_{n-1})$ and $v \in V(10\Gamma_{n-2})$:
   We know that $d(u, v) = 1$ and the string representations of u and v must be of the form $00b_3 \ldots b_n$ and $10b_3 \ldots b_n$, respectively. Then, using Lemma 1 with $k = 1$ we have $|n_u - n_v| = f_{n+1} - f_n = f_{n-1}$ for each edge uv in C_n. As $|C_n| = f_n$, all of these edges contribute $f_n f_{n-1} y$ to $M_n(x, y, z)$.
2. Assume that $uv \in E(10\Gamma_{n-2})$:
   Let the string representations of u and v be $10u_3 \ldots u_n$ and $10v_3 \ldots v_n$, respectively. Using the fundamental decomposition of Γ_n, there exist vertices of the form $u' = 0u_3 \ldots u_n$ and $v' = 0v_3 \ldots v_n$ in $V(\Gamma_{n-1})$; $u'' = u_3 \ldots u_n$ and $v'' = v_3 \ldots v_n$ in $V(\Gamma_{n-2})$. Then, n_u counts the number of vertices $0\alpha \in V(0\Gamma_{n-1})$ and $10\beta \in V(10\Gamma_{n-2})$ satisfying

   $$d(0\alpha, u) < d(0\alpha, v) \text{ and } d(10\beta, u) < d(10\beta, v)\ .$$

   For any $0\alpha \in V(0\Gamma_{n-1})$, we know that $d(0\alpha, u) = d(\alpha, u') + 1$ and $d(0\alpha, v) = d(\alpha, 0v') + 1$. Therefore, for a fixed $0\alpha \in V(0\Gamma_{n-1})$, $d(\alpha, u') < d(\alpha, v')$ if and only if $d(0\alpha, u) < d(0\alpha, v)$. Similarly, for any $10\beta \in V(10\Gamma_{n-2})$ we have $d(10\beta, u) = d(\beta, u'')$ and $d(\beta, v) = d(\beta, v'')$. Then, we can write

   $$\sum_{uv \in E(10\Gamma_{n-2})} \left|n_{u,v}(\Gamma_n) - n_{v,u}(\Gamma_n)\right| = \sum_{u'v' \in E(\Gamma_{n-1})} \left|n_{u',v'}(\Gamma_{n-1}) - n_{v',u'}(\Gamma_{n-1})\right| \\ + \sum_{u''v'' \in E(\Gamma_{n-2})} \left|n_{u'',v''}(\Gamma_{n-2}) - n_{v'',u''}(\Gamma_{n-2})\right|\ .$$

   Note that $\Gamma_{n-1} = 0\Gamma_{n-2} + 10\Gamma_{n-3}$ and the edge $u'v' \in E(\Gamma_{n-1})$ is an edge in the set $E(0\Gamma_{n-2})$. Furthermore, $u''v'' \in E(\Gamma_{n-2})$ is an arbitrary edge. Then, by the definition (2) of M_n we have

   $$\sum_{u'v' \in E(\Gamma_{n-1})} \left|n_{u',v'}(\Gamma_{n-1}) - n_{v',u'}(\Gamma_{n-1})\right| = M_{n-1}(1, 0, 0)$$

   and

   $$\sum_{u''v'' \in E(\Gamma_{n-2})} \left|n_{u'',v''}(\Gamma_{n-2}) - n_{v'',u''}(\Gamma_{n-2})\right| = M_{n-2}(1, 1, 1)\ .$$

Hence, all of these edges $uv \in E(10\Gamma_{n-2})$ contribute $\big(M_{n-1}(1,0,0) + M_{n-2}(1,1,1)\big)z$ to $M_n(x, y, z)$.

3. Assume that $uv \in E(0\Gamma_{n-1})$:
   Since $0\Gamma_{n-1} = 00\Gamma_{n-2} + 010\Gamma_{n-3}$, we have three subcases to consider here.

   (a) Assume that $uv \in C_{n-1}$ such that $u \in 00\Gamma_{n-2}$ and $v \in 010\Gamma_{n-3}$.
   Then, using Lemma 1 with $k = 2$ we have

$$|n_u - n_v| = f_3 f_n - f_2 f_{n-1} = 2f_n - f_{n-1} = f_n + f_{n-2}$$

   for each edge uv in C_n. As $|C_{n-1}| = f_{n-1}$, all of these edges contribute $f_{n-1}(f_n + f_{n-2})x$ to $M_n(x, y, z)$.

   (b) Assume that $uv \in E(010\Gamma_{n-3})$:
   Let the string representations of u and v are of the form $010u_4 \ldots u_n$ and $010v_4 \ldots v_n$, respectively. Using the fundamental decomposition of Γ_n, there exist vertices of the form $u' = 000u_4 \ldots u_n$ and $v' = 000v_4 \ldots v_n$ in $V(0\Gamma_{n-1})$; $u'' = 0u_4 \ldots u_n$ and $v'' = 0v_4 \ldots v_n$ in $V(\Gamma_{n-2})$. Then, for any $10\alpha \in V(10\Gamma_{n-2})$ we know that $d(10\alpha, u) = d(10\alpha, u') + 1 = d(\alpha, u'') + 2$ and we know that $d(10\alpha, v) = d(10\alpha, v') + 1 = d(\alpha, v'') + 2$. Therefore, for all $10\alpha \in V(10\Gamma_{n-2})$ we count their total contribution to M_n by $M_{n-2}(1,0,0)x$ in this case. Furthermore, as $uv \in E(010\Gamma_{n-3})$, we have $uv \in E(0\Gamma_{n-1})$, and for all $0\alpha \in V(0\Gamma_{n-1})$, we count their total contribution to M_n by $M_{n-1}(0,0,1)x$ by using the definition of M_{n-1}. Hence, the edges $uv \in E(010\Gamma_{n-3})$ contribute $\big(M_{n-1}(0,0,1) + M_{n-2}(1,0,0)\big)x$ to $M_n(x, y, z)$.

   (c) Assume that $uv \in E(00\Gamma_{n-2})$.
   These edges are the ones of $E(0\Gamma_{n-1})$ that are not in $E(010\Gamma_{n-3})$ and C_{n-1} (not created during the connection of $00\Gamma_{n-2}$ and $010\Gamma_{n-3}$). Then, similar to the Case 2 and using the definition (2) of M_n these edges contribute $\big(M_{n-1}(1,0,0) + M_{n-2}(1,1,1)\big)x$ to $M_n(x, y, z)$.

Combining all of the above cases and noting $M_{n-1}(0,0,1)x = M_{n-1}(0,0,x)$, $M_{n-2}(1,0,0)x = M_{n-2}(x,0,0)$, $M_{n-2}(1,1,1)x = M_{n-2}(x,x,x)$, we complete the proof. □

If we write $M_n(x, y, z) = a_n x + b_n y + c_n z$, then from the recursion in Proposition 1, we obtain for $n \geq 2$

$$\begin{aligned} a_n &= a_{n-1} + c_{n-1} + 2a_{n-2} + b_{n-2} + c_{n-2} + f_{n-1}(f_n + f_{n-2}) \\ b_n &= f_n f_{n-1} \\ c_n &= a_{n-1} + a_{n-2} + b_{n-2} + c_{n-2} \,. \end{aligned}$$

Eliminating b_n, this is equivalent to the system

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + c_{n-1} + c_{n-2} + f_{n-2}f_{n-3} + f_{n-1}f_{n-2} + f_n f_{n-1} \\ c_n &= a_{n-1} + a_{n-2} + c_{n-2} + f_{n-2}f_{n-3} \,. \end{aligned} \tag{3}$$

Let $A(t)$, $B(t)$, $C(t)$ be the generating functions of the sequences a_n, b_n, c_n, $(n \geq 2)$, respectively. We already know that ([11, A001654])

$$B(t) = \sum_{n \geq 2} f_n f_{n-1} t^n = \frac{t^2}{(1+t)(1-3t+t^2)} . \tag{4}$$

From (3), we obtain

$$\begin{aligned} A(t) &= (t+2t^2)A(t) + (t+t^2)C(t) + (1+t+t^2)B(t) \\ C(t) &= (t+t^2)A(t) + t^2 C(t) + t^2 B(t) . \end{aligned} \tag{5}$$

Solving the system of equations (5) and using (4), we calculate

$$\begin{aligned} A(t) &= \frac{t^2}{(1+t)^2(1-3t+t^2)^2} , \\ C(t) &= \frac{t^3+2t^4-t^5}{(1+t)^2(1-3t+t^2)^2} . \end{aligned} \tag{6}$$

Since $\mathrm{Mo}(\Gamma_n) = M_n(1,1,1) = a_n + b_n + c_n$, adding the generating functions $A(t)$, $B(t)$, $C(t)$ we obtain

$$\sum_{n \geq 2} \mathrm{Mo}(\Gamma_n) t^n = \frac{(2-t)t^2}{(1+t)^2(1-3t+t^2)^2} . \tag{7}$$

Using partial fractions decomposition in (7) and the expansions

$$\frac{1}{1-3t+t^2} = \sum_{n \geq 0} f_{2n+2} t^n , \tag{8}$$

$$\frac{1}{(1-3t+t^2)^2} = \sum_{n \geq 0} \frac{1}{5}\big((4n+2) f_{2n+2} + (3n+3) f_{2n+1}\big) t^n , \tag{9}$$

we obtain

$$\mathrm{Mo}(\Gamma_n) = \frac{1}{25}\Big((3n+2)(-1)^n + (4n-5) f_{2n+2} + (3n+3) f_{2n+1} - (4n-3) f_{2n} - 3n f_{2n-1}\Big) ,$$

which can be simplified to the closed-form expression for $\mathrm{Mo}(\Gamma_n)$ in Theorem 3. This is another way of writing the sum given in Theorem 1.

Theorem 3 *The Mostar index of Fibonacci cube Γ_n is*

$$\mathrm{Mo}(\Gamma_n) = \frac{1}{25}\Big((3n-2) f_{2n+2} + n f_{2n+1} + (3n+2)(-1)^n\Big) .$$

6 The Wiener Index and Remarks

In [9], it is shown that

$$W(\Gamma_n) = \sum_{k=1}^{n} f_k f_{k+1} f_{n-k+1} f_{n-k+2} \tag{10}$$

and that this sum can be evaluated as

$$W(\Gamma_n) = \frac{1}{25}\big(4(n+1)f_n^2 + (9n+2)f_n f_{n+1} + 6nf_{n+1}^2\big)\,. \tag{11}$$

In view of our formula (1) of Theorem 1 and (10), this means that

$$W(\Gamma_n) = \mathrm{Mo}(\Gamma_n) + \sum_{k=1}^{n}(f_k f_{n-k+1})^2\,.$$

The sum above is the sequence [11, A136429] with generating function

$$\frac{t(1-t)^2}{(1+t)^2(1-3t+t^2)^2}\,.$$

Adding the generating function (7) to this, we get

$$\sum_{n\geq 1} W(\Gamma_n)t^n = \frac{t}{(1+t)^2(1-3t+t^2)^2}\,. \tag{12}$$

Using partial fractions and the expansions (8) and (9), $W(\Gamma_n)$ $(n \geq 2)$ is found to be

$$W(\Gamma_n) = \frac{1}{25}\big((3n+2)f_{2n+3} + (n-2)f_{2n+2} - (n+2)(-1)^n\big)$$

which is a somewhat simpler expression than (11).

It is also curious that in view of their generating functions (6) and (12) which differ only by factor of t, we have

$$a_n = M_n(1,0,0) = W(\Gamma_{n-1})\,.$$

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