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Strong Edge Geodetic Problem on Grids

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Abstract

Let $G = (V(G), E(G))$ be a simple graph. A set $S \subseteq V(G)$ is a strong edge geodetic set if there exists an assignment of exactly one shortest path between each pair of vertices from S, such that these shortest paths cover all the edges $E(G)$. The cardinality of a smallest strong edge geodetic set is the strong edge geodetic number $\mathrm{sg_e}(G)$ of G. In this paper, the strong edge geodetic problem is studied on the Cartesian product of two paths. The exact value of the strong edge geodetic number is computed for $P_n \,\square\, P_2$, $P_n \,\square\, P_3$ and $P_n \,\square\, P_4$. Some general upper bounds for $\mathrm{sg_e}(P_n \,\square\, P_m)$ are also proved.

Keywords Strong geodetic problem · Strong edge geodetic problem · Cartesian product of paths

Mathematics Subject Classification 05C12 · 05C70

1 Introduction

Different covering problems with shortest paths were studied in the literature. For example, the geodetic problem was introduced in 1993 in [3] and its edge version in 2007 in [11]. In 2016, the strong geodetic problem was introduced, the seminal paper [9] being published only recently. Since then, a lot of work was done on the strong geodetic problem.

The exact value of the strong geodetic number was computed for different families of graphs. For example, for complete bipartite graphs $K_{n,m}$ it was first computed for cases when $n = m$, and for $n \gg m$ in [4], and later in the general case in [1]. In [6], the strong geodetic number was computed for some balanced multipartite complete

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graphs and it was shown that computing the strong geodetic number of general complete multipartite graphs is NP-complete. The exact strong geodetic number was also computed for crown graphs S_0^n in [1], Hamming graphs $K_m \,\square\, K_n$ in [5], Cartesian products $K_{1,n} \,\square\, P_l$ in [5], thin ($n \gg m$) grids $P_n \,\square\, P_m$, and thin ($n \gg m$) cylinders $P_n \,\square\, C_m$ in [7], and i-level complete Apollonian networks $A(i)$ in [9].

Several general bounds on the strong geodetic number were given using different graphical invariants. In [4], bounds on $\mathrm{sg_e}(G)$ were given depending on the diameter $\mathrm{diam}(G)$ of G. In [12], an upper bound on the strong geodetic number was given using the connectivity number. In [9], lower and upper bounds were given using isometric path number. Gledel at al. [2] proved an upper bound on the strong geodetic number of Cartesian product graphs using the so-called strong geodetic core number. In [1], an upper bound on the strong edge geodetic number was given for hypercubes. A general upper bound was in [5] computed for the strong geodetic number of Cartesian product graphs, and an upper bound for the strong geodetic number of the Cartesian product of a path with an arbitrary graph was computed in [7].

Using reduction from NP-completeness of dominating set problem, Manuel et al. in [9] proved that the strong geodetic problem is NP-complete. On the positive side, Mezzini in [10] gave a polynomial algorithm for computing the strong geodetic number of outerplanar graphs. Some general properties about the strong geodetic number were also given. For example, in [12] the graphs with the strong geodetic number 2, $n(G)$, or $n(G) - 1$, were characterized. In [5], relation between the strong geodetic number of a graph and its induced, convex, or gated subgraphs was derived.

Like geodetic problem and many other problems in graph theory, there is an interesting edge version of the problem. In 2017, the edge version of the strong geodetic problem, called the strong edge geodetic problem, was introduced in [8], but it did not get as much attention as the vertex version, with the exception of the very recent paper [13]. This gap is in part filled in this paper. In the seminal paper [8], it was proved that the strong edge geodetic problem is NP-complete and some general upper and lower bounds were given using the isometric path number, the number of simplicial vertices in a graph, and the number of convex components in graph. In this article, we will show that even though the vertex and edge version of the strong geodetic problem seem similar at the first sign, they differ a lot. For example, in [7] it was proved that if $2 \le n \le m$, then $\mathrm{sg}(P_n \,\square\, P_m) \le \lceil 2\sqrt{n}\, \rceil$, while we will prove that this is not true for the strong edge geodetic number $\mathrm{sg_e}(P_n \,\square\, P_m)$ when $m = 3$ or 4. The main results of this article are the following three theorems.

Theorem 1.1 *If $n \ge 2$, then* $\mathrm{sg_e}(P_n \,\square\, P_2) = \lceil 2\sqrt{n}\, \rceil$.

Theorem 1.2 *If $n \ge 2$, then* $\mathrm{sg_e}(P_n \,\square\, P_3) = \lceil 2\sqrt{n+1}\, \rceil$.

Theorem 1.3 *If $n \ge 2$, then*

$$\mathrm{sg_e}(P_n \,\square\, P_4) = \begin{cases} 2k+1; & n = k^2 + h, 0 \le h \le k-1, \\ 2k+2; & n = k^2 + h, k \le h \le 2k-1, \\ 2k+3; & n = k^2 + 2k. \end{cases}$$

Theorem 1.3 implies that $\mathrm{sg_e}(P_n \,\square\, P_4) = \lceil 2\sqrt{n+2}\,\rceil$ for all $n \in \mathbb{N}$ except when $n = k^2 + k - 1$ for some $k \in \mathbb{N}$. This can also be interpreted as $\mathrm{sg_e}(P_n \,\square\, P_4) = \lceil 2\sqrt{n+1}\,\rceil$ for all $n \in \mathbb{N}$ except when $n = k^2 + 2k$ for some $k \in \mathbb{N}$. This shows that the pattern from Theorems 1.1 and 1.2 does not extend to $n \geq 4$.

In the next section, we formally define concepts needed in this paper and prepare several preliminary results. Then, in Sects. 3-5, we prove Theorems 1.1, 1.2, 1.3, respectively. In the last section, we give three general upper bounds on $\mathrm{sg_e}(P_n \,\square\, P_m)$.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple graph. A x, y-geodesic is a shortest path between vertices x and y. With $P(G; x, y)$, we denote the set of all shortest paths in G between vertices x and y. A set $S \subseteq V(G)$ is a *strong edge geodetic set* if there exists an assignment of shortest paths $P_{x,y} \in P(G; x, y)$ for every pair $x, y \in V(G)$, such that

$$\bigcup_{\{x,y\}\in\binom{S}{2}} E(P_{x,y}) = E(G),$$

where $E(P_{x,y})$ denotes the set of edges from the selected shortest path $P_{x,y}$. The set of these shortest paths is called the *strong edge geodetic covering*. The *strong edge geodetic number* of G, denoted by $\mathrm{sg_e}(G)$, is the cardinality of a smallest strong edge geodetic set of G.

The Cartesian product $G \,\square\, H$ of graphs G and H is the graph on the vertex set $V(G \,\square\, H) = V(G) \times V(H)$, where two vertices (g_1, h_1) and (g_2, h_2), $g_1, g_2 \in V(G)$, $h_1, h_2 \in V(H)$ are adjacent if $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or if $g_1 = g_2$ and $h_1 h_2 \in E(H)$. An edge $(g_1, h_1)(g_2, h_2) \in E(G \,\square\, H)$ is said to be *horizontal* if $h_1 = h_2$ and is said to be *vertical* if $g_1 = g_2$. A *grid* is the Cartesian product of two paths. The *i-th row*, $1 \leq i \leq m$, in $P_n \,\square\, P_m$ is the vertex set $\{(1, i), \ldots, (n, i)\}$ together with the horizontal edges between them. Similarly, the *j-th column*, $1 \leq j \leq n$, in $P_n \,\square\, P_m$ is the vertex set $\{(j, 1), \ldots, (j, m)\}$ together with the vertical edges between them. Because of the commutativity of the Cartesian product operation, $\mathrm{sg_e}(G \,\square\, H) = \mathrm{sg_e}(H \,\square\, G)$. A subgraph H of a graph G is *convex* if for every pair of vertices $\{x, y\}$ in H, every x, y-geodesic lies completely in H. A set of edges $F \subseteq E(G)$ is a *convex edge cut* if $G - F$ has precisely two convex components.

Lemma 2.1 *If $n \geq 2$ and $m \geq 2$, then* $\mathrm{sg_e}(P_n \,\square\, P_m) \geq \lceil 2\sqrt{n}\,\rceil$.

Proof In [8, Corollary 6.4], it is proved that if F is a convex edge cut of a graph G, then $\mathrm{sg_e}(G) \geq \lceil 2\sqrt{|F|}\rceil$. In our case, all vertical edges between the first and the second row in $P_n \,\square\, P_m$ represent a convex edge-cut set of $P_n \,\square\, P_m$ (see Fig. 1). Because there are exactly n vertical edges between the first and the second row in $P_n \,\square\, P_2$, the inequality holds.

Throughout the rest of this paper, we will use Algorithm 1 to prove upper bounds on the strong edge geodetic number. The input of this algorithm is an integer n and

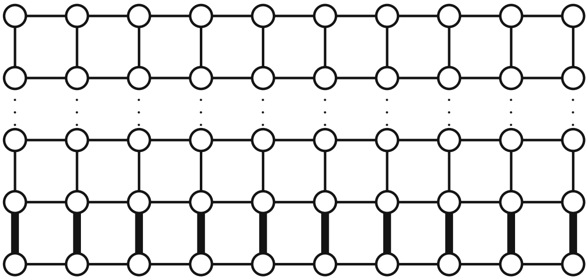

Fig. 1 Edge cut in $P_n \square P_m$

an integer m. Integer n can be uniquely written as the sum $n = k^2 + h$, where k and h are integers and $0 \leq h \leq 2k$. Algorithm defines a set of vertices S and a set of shortest paths, where for each pair of vertices in S it uses at most one shortest path between them, such that the union of this shortest paths covers all the vertical edges in $P_n \square P_m$.

Algorithm 1 first takes two vertices, $a_1 = (1, 1)$ and $b_1 = (1, m)$ from the first column, and covers the vertical edges in the first column by the unique a_1, b_1-geodesic. In next step, if $k^2 \geq 4$, the algorithm takes vertices $a_2 = (2^2, 1)$ and $b_2 = (2^2, m)$ and covers the second column of edges by the a_1,b_2-geodesic, the third column of edges by the a_2, b_1-geodesic and the fourth column of edges by the a_2, b_2-geodesic. After the $(i-1)$-th, $(i-1)^2 \leq n$, step, all the first $(i-1)^2$ columns are already covered. In the i-th step, if $i^2 \leq n$, we add vertices $a_i = (i^2, 1)$ and $b_i = (i^2, m)$. We cover the next $i - 1$ columns by a_1, b_i-, …, a_{i-1}, b_i-geodesic, respectively, and the way every geodesic covers the leftmost not yet covered column of edges. Similarly, we cover $i-1$ edges from the $(i-1)^2 + (i-1) + 1$-th to the $(i-1)^2 + 2(i-1)$-th column by a_i, b_1-, …, a_i, b_{i-1}-geodesic, respectively, covering the leftmost not yet covered column of edges. The algorithm covers the i-th column of edges by the unique a_i, b_i-geodesic. This way we cover all the vertical edges from the first k^2 columns.

If $1 \leq h \leq k$, we add the vertex $b_{k+1} = (n, m)$ and then cover the remaining columns of edges by a_1, b_{k+1}-, …, a_h, b_{k+1}-geodesic, respectively, covering the leftmost not yet covered column of edges. If $k + 1 \leq h \leq 2k$, we add vertices $a_{k+1} = (n, 1)$ and $b_{k+1} = (n, m)$. We cover the next k columns by a_1, b_{k+1}-, …, a_k, b_{k+1}-geodesic, respectively, and the way every geodesic covers the leftmost not yet covered column of edges. Similarly, we cover the remaining columns of edges by a_{k+1}, b_1-, …, a_{k+1}, b_{h-k}-geodesic, respectively, covering the leftmost not yet covered column of edges.

We conclude the preliminaries with the following technical lemma to be used in our proofs.

Lemma 2.2

$$\lceil 2\sqrt{n} \rceil = \begin{cases} 2k; & n = k^2,\ k \in \mathbb{N}, \\ 2k+1; & n = k^2 + h,\ k \in \mathbb{N}, 1 \leq h \leq k, \\ 2k+2; & n = k^2 + h,\ k \in \mathbb{N}, k+1 \leq h \leq 2k. \end{cases}$$

Algorithm 1: Covering vertical edges in grids

Input: integer $n = k^2 + h$, where $0 \leq h \leq 2k$, and integer m

Result: the set of vertices V in $P_n \,\square\, P_m$, $m \geq 2$, and the set of shortest paths between pairs of vertices from V that cover all vertical edges

for $i = 1, \dots, k$ **do**
  $a_i = \left(i^2, 1\right)$;
  $b_i = \left(i^2, m\right)$;
  for $j = 1, \dots, i-1$ **do**
    connect vertices a_i and b_j with a geodesic that covers all vertical edges in $((i-1)^2 + j)$-th column of $P_n \,\square\, P_m$
  end
  for $j = 1, \dots, i-1$ **do**
    connect vertices a_j and b_i with a geodesic that covers all vertical edges in $(i(i-1) + j)$-th column of $P_n \,\square\, P_m$
  end
  connect vertices a_i and b_i with the unique geodesic between them
end
if $1 \leq h \leq k$ **then**
  $b_{k+1} = (n, m)$;
  for $j = 1, \dots, h$ **do**
    connect vertices a_j and b_{k+1} with a geodesic that covers all vertical edges in $(k^2 + j)$-th column of $P_n \,\square\, P_m$
  end
end
else if $k + 1 \leq h \leq 2k$ **then**
  $a_{k+1} = (n, 1)$;
  $b_{k+1} = (n, m)$;
  for $j = 1, \dots, k$ **do**
    connect vertices a_j and b_{k+1} with a geodesic that covers all vertical edges in $(k^2 + j)$-th column of $P_n \,\square\, P_m$
  end
  for $j = 1, \dots, h - k$ **do**
    connect vertices a_{k+1} and b_j with a geodesic that covers all vertical edges in $(k(k+1)+j)$-th column of $P_n \,\square\, P_m$
  end
end

Proof If $n = k^2$ for some $k \in \mathbb{N}$, then $\lceil 2\sqrt{n} \rceil = \left\lceil 2\sqrt{k^2} \right\rceil = 2k$.

Suppose $n = k^2 + h$ for some $k, h \in \mathbb{N}$ where $1 \leq h \leq k$. Then, because $h > 0$ and $\left\lceil 2\sqrt{k^2} \right\rceil$ is an integer, it holds

$$\lceil 2\sqrt{n} \rceil = \left\lceil 2\sqrt{k^2 + h} \right\rceil > \left\lceil 2\sqrt{k^2} \right\rceil = 2k.$$

Also, because $h \leq k$ and $\left\lceil 2\sqrt{k^2} \right\rceil$ is an integer, it holds

$$\lceil 2\sqrt{n} \rceil \leq \left\lceil 2\sqrt{k^2+k} \right\rceil = \left\lceil 2\sqrt{(k^2+1/2)^2 - 1/4} \right\rceil \leq \left\lceil 2\sqrt{(k+1/2)^2} \right\rceil = 2k+1.$$

Because $\lceil 2\sqrt{n}\, \rceil$ is an integer greater than $2k$ and is also less or equal to $2k+1$, we conclude that $\lceil 2\sqrt{n}\, \rceil = 2k+1$ when $n = k^2+h$, $1 \leq h \leq k$.

Now, suppose that $n = k^2 + h$ for some $k, h \in \mathbb{N}$ where $k+1 \leq h \leq 2k$. Then, because $h \geq k+1$ and $\left\lceil 2\sqrt{(k+1/2)^2} \right\rceil$ is an integer, it holds

$$\lceil 2\sqrt{n} \rceil \geq \left\lceil 2\sqrt{k^2+k+1} \right\rceil = \left\lceil 2\sqrt{(k+1/2)^2 + 3/4} \right\rceil > \left\lceil 2\sqrt{(k+1/2)^2} \right\rceil = 2k+1.$$

Also, because $h \leq 2k$ and $\left\lceil 2\sqrt{k^2} \right\rceil$ is an integer, it holds

$$\lceil 2\sqrt{n} \rceil \leq \left\lceil 2\sqrt{k^2+2k} \right\rceil \leq \left\lceil 2\sqrt{k^2+2k+1} \right\rceil = 2k+2.$$

Because $\lceil 2\sqrt{n}\, \rceil$ is an integer greater than $2k+1$ and is also less or equal to $2k+2$, we get $\lceil 2\sqrt{n}\, \rceil = 2k+2$ when $n = k^2+h$, $k+1 \leq h \leq 2k$. □

3 Proof of Theorem 1.1

The lower bound $\mathrm{sg_e}(P_n \,\square\, P_2) \geq \lceil 2\sqrt{n}\, \rceil$ follows from Lemma 2.1. It remains to prove the upper bound $\mathrm{sg_e}(P_n \,\square\, P_2) \leq \lceil 2\sqrt{n}\, \rceil$.

We will use Algorithm 1 for $m = 2$ to cover all the vertical edges in $P_n \,\square\, P_2$. When $n = k^2$ for some $k \in \mathbb{N}$, the number of vertices used in Algorithm 1 is exactly $2k$, and these are the vertices $a_1, \dots, a_k, b_1, \dots, b_k$. When $n = k^2 + h$, $k \in \mathbb{N}$, $1 \leq h \leq k$, the number of vertices used in Algorithm 1 is exactly $2k+1$, and when $n = k^2 + h$, $k \in \mathbb{N}$, $k+1 \leq h \leq 2k$, exactly $2k+1$ vertices are used. By Lemma 2.2, we see that Algorithm 1 uses exactly $\lceil 2\sqrt{n}\, \rceil$ vertices.

It remains to cover the horizontal edges. Observe that Algorithm 1 uses only the a_{j_1},b_{j_2}-geodesics for some $j_1, j_2 \in \mathbb{N}$.

If $n = k^2$, all horizontal edges from the first row can be covered by the unique a_1,a_k-geodesic. Similarly, all horizontal edges from the second row can be covered by the unique b_1,b_k-geodesic.

If $n = k^2 + h$, where $1 \leq h \leq k$, we can cover the horizontal edges in the first row with the unique a_1,a_k-geodesic. The remaining horizontal edges in the first row are already covered with the shortest path from Algorithm 1 that covers vertical edges in the n-th column. We can cover the horizontal edges from the second row with the unique b_1,b_{k+1}-geodesic.

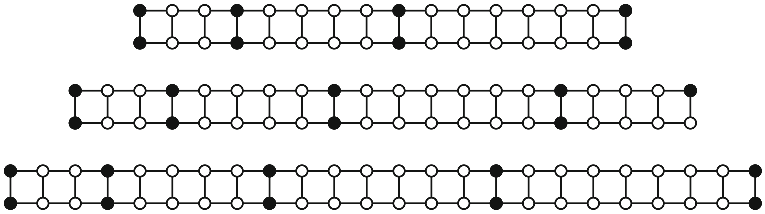

Fig. 2 Strong edge geodetic set for graphs $P_{16} \,\square\, P_2$, $P_{20} \,\square\, P_2$ and $P_{24} \,\square\, P_2$

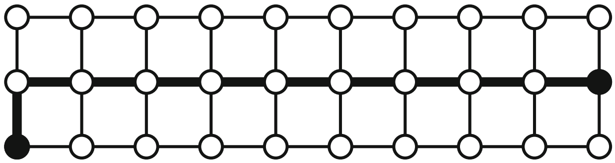

Fig. 3 Shortest path between $(1, 1)$ and $(n, 2)$

If $n = k^2 + h$, where $k + 1 \le h \le 2k$, we can cover the horizontal edges from the first row with the unique a_1,a_{k+1}-geodesic, and the horizontal edges from the second row with the unique b_1,b_{k+1}-geodesic.

This proves Theorem 1.1. See Fig. 2 for some typical optimal strong edge geodetic sets in $P_n \,\square\, P_2$.

4 Proof of Theorem 1.2

We first show:

Lemma 4.1 *If $n \ge 2$, then $\mathrm{sg_e}(P_n \,\square\, P_3) \le \lceil 2\sqrt{n}\, \rceil + 1$.*

Proof Use Algorithm 1 and then cover horizontal edges in the first row and in the last row in the same way as in the proof of Theorem 1.1. Add the vertex $(n, 2)$ to the existing vertex set from Algorithm 1, and connect it with $(1, 1)$ by a geodesic that covers all horizontal edges from the second row, see Fig. 3. □

By Lemmas 4.1 and 2.1 (for $m = 3$), we know that for every $n \ge 2$, the strong edge geodetic number $\mathrm{sg_e}(P_n \,\square\, P_3)$ is either $\lceil 2\sqrt{n}\, \rceil$ or $\lceil 2\sqrt{n}\, \rceil + 1$. Because by Lemma 2.2, it holds $\lceil 2\sqrt{n}\, \rceil = \lceil 2\sqrt{n+1}\, \rceil$, except when $n = k^2$ or $n = k^2 + k$ for some $k \in \mathbb{N}$, it is enough to prove that when $n = k^2$ or $n = k^2 + k$, there is no strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ and to find a strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ in the other cases.

First, let us show that there exists a strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ on $P_n \,\square\, P_3$ for $n = k^2 + h$, where $1 \le h \le k - 1$ or $k + 1 \le h \le 2k$.

Proposition 4.2 *If $n = k^2 + h$, $n \ge 3$, $k \ge 1$, where $1 \le h \le k - 1$ or $k + 1 \le h \le 2k$, then*

$$\mathrm{sg_e}(P_n \,\square\, P_3) \le \mathrm{sg_e}(P_n \,\square\, P_2).$$

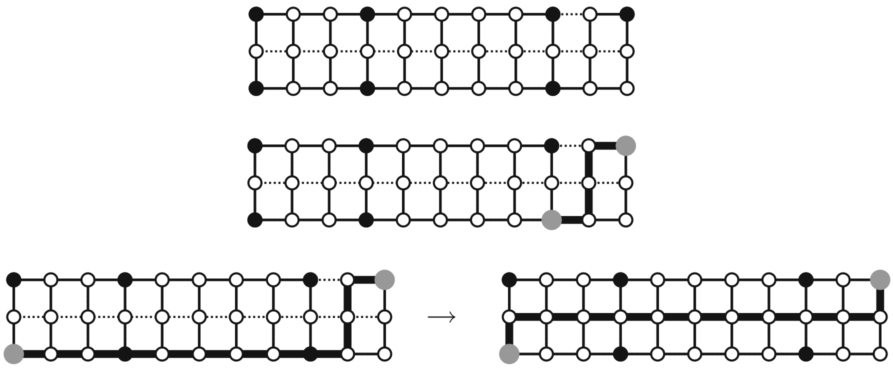

Fig. 4 Strong edge geodetic set in $P_{3^2+2} \,\square\, P_3$

Proof To cover all the vertical edges and all the horizontal edges in the second row, we will adjust Algorithm 1. We will divide this adjustment into two cases.

Case 1: $n = k^2 + h$, $1 \leq h \leq k - 1$ (see Fig. 4).
We can see that in this case, because $h \leq k - 1$, Algorithm 1 never uses a a_k,b_{k+1}-geodesic and that a a_1,b_{k+1}-geodesic covers all vertical edges in the (k^2+1)-th column. Because $a_k = (k^2, 1)$, we can add a a_k,b_{k+1}-geodesic that covers all the vertical edges in the $(k^2 + 1)$-th column and then replace the existing a_1,b_{k+1}-geodesic with the one that covers all the horizontal edges in the second row. This way we have covered all the vertical edges in $P_n \,\square\, P_3$ and also all the horizontal edges in the second row.

In the same way as in $P_n \,\square\, P_2$, we can cover the horizontal edges in the first row (using the a_1,a_k-geodesic and also the existing a_h,b_{k+1}-geodesic) and in the third row (using the b_1,b_{k+1}-geodesic) of $P_n \,\square\, P_3$.

Case 2: $n = k^2 + h$, $k + 1 \leq h \leq 2k$ (see Fig. 5).
In this case, Algorithm 1 does not use the a_{k+1},b_{k+1}-geodesic. This shortest path covers all the vertical edges in the n-th column. Because a a_{k+1},b_h-geodesic from Algorithm 1 also covers all the vertical edges from the n-th column, we can add a a_{k+1},b_{k+1}-geodesic and then replace an existing a_{k+1},b_h-geodesic with the one that covers all the vertical edges in the $(k^2 + 1)$-th column. This way we can then similar to the previous case replace the existing a_1,b_{k+1}-geodesic (which covers all the vertical edges in the $(k^2 + 1)$-th column in Algorithm 1) with the one that covers all the horizontal edges in the second row of $P_n \,\square\, P_3$.

In the same way as in $P_n \,\square\, P_2$, we now cover the horizontal edges in the first row (using the a_1,a_{k+1}-geodesic) and in the third row (using the b_1,b_{k+1}-geodesic) of $P_n \,\square\, P_3$.

In both cases, we have adjusted Algorithm 1 such that all the vertical edges together with all the horizontal edges are covered without changing the set of vertices used in the algorithm. It follows that $\mathrm{sg_e}(P_n \,\square\, P_3) \leq \lceil 2\sqrt{n}\, \rceil = \lceil 2\sqrt{n+1}\, \rceil$ for all $n \in \mathbb{N}$ when $n \neq k^2$ or $n \neq k^2 + k$ for some $k \in \mathbb{N}$. □

In the second part of the proof, we will prove that there is no strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ if $n = k^2$ or $n = k^2 + k$.

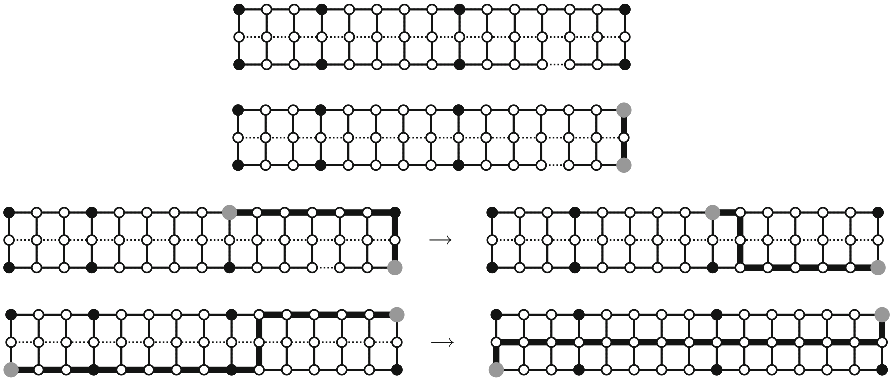

Fig. 5 Strong edge geodetic set in $P_{3^2+2\cdot 3} \,\square\, P_3$

Proposition 4.3 *If* $n = k^2$ *or* $n = k^2 + k$ *for some* $k \geq 2$, $k \in \mathbb{N}$, *then*

$$\mathrm{sg_e}(P_n \,\square\, P_3) > \mathrm{sg_e}(P_n \,\square\, P_2).$$

Proof Let us define the function

$$f_s(a, b, c) = ab + bc + 2ac$$

which gives an upper bound on how many different vertical edges can be covered with shortest paths between s vertices, where a vertices lie in the first row, b vertices in the second row, and c vertices in the third row in $P_n \,\square\, P_3$, and for each pair of these vertices, we use at most one shortest path. The extremes of f_s can be obtained by computer. For example, if $a \geq 0$, $b \geq 0$, and $c \geq 0$, $a, b, c \in \mathbb{R}$, then the maximum of f_s is equal to $s^2/2$ and if $a \geq 0$, $b \geq 1$, and $c \geq 0$, $a, b, c \in \mathbb{R}$, then the maximum of f_s is equal to $(s^2 - 1)/2$. We prove Proposition 4.3 by assuming the opposite in each case.

Case 1: $n = k^2$.
Suppose that there exists a strong edge geodetic set S in $P_n \,\square\, P_3$ with $2k$ elements. If this set contains a vertex from the second row (in which case $b \geq 1$), $f_s(a, b, c)$ is at most $(s^2 - 1)/2$, which is in our case (when $s = 2k$) equal to $(4k^2 - 1)/2$. This is less than the number $2k^2$ of all the vertical edges in $P_n \,\square\, P_2$, which in turn means that if there exists such a strong edge geodetic set, then all the vertices from it are in the first and the third row ($b = 0$).

Now, consider the edge $(1, 2)(2, 2)$. A shortest path that covers this edge has to have one endpoint at $(1, 1)$ or $(1, 3)$, which means that at least one of these vertices is in S. Without loss of generality, we can assume that it has one endpoint at $(1, 1)$ (Fig. 6a). This shortest path then also covers the edge $(1, 1)(1, 2)$. If $(1, 3) \in S$, then the edge $(1, 1)(1, 2)$ is also covered with the unique $(1, 1)$,$(1, 3)$-geodesic (Fig. 6b). Otherwise, the shortest path that covers the edge $(1, 2)(1, 3)$ also covers the edge $(1, 1)(1, 2)$ (Fig. 6c). In both ways, the edge $(1, 1)(1, 2)$ is covered at least twice,

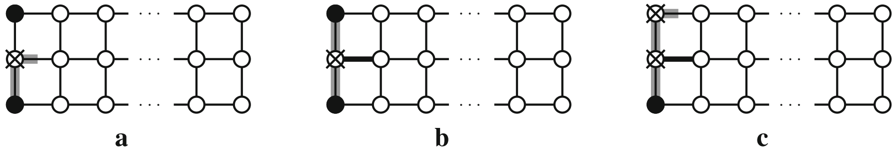

Fig. 6 Covering the edge $(1, 2)(2, 2)$ in $P_n \,\Box\, P_3$ when $n = k^2$

which means that with the set of $2k$ vertices, we can only find shortest paths between them that cover at most $\max(f_{2k} - 1) \leq 2k^2 - 1$ different vertical edges which is again less that the number $2k^2$ of all the vertical edges in $P_n \,\Box\, P_2$. This implies that such a strong edge geodetic set S cannot exist.

Case 2: $n = k^2 + k$.
Suppose that there exists a strong edge geodetic set S of $P_n \,\Box\, P_3$ with $2k+1$ elements. If this set does not include a vertex $(1, 2)$, we can similarly as in the previous case without loss of generality conclude that the edge $(1, 1)(1, 2)$ is covered at least twice, which implies that with a set of $2k+1$ vertices, we can only find shortest paths between them that cover at most $\max(f_{2k+1} - 1) \leq (2k+1)^2/2 - 1 = 2k^2 + 2k - 1/2$ different vertical edges which is less that the number $2(k^2+2)$ of all the vertical edges in $P_n \,\Box\, P_3$. This implies that if such a strong edge geodetic set in $P_n \,\Box\, P_3$ exists, it includes the vertex $(1, 2)$. By symmetry, it also includes the vertex $(n, 2)$. But then, because $b \geq 2$, the value of $f_{2k+1}(a, b, c)$ is less or equal to $((2k + 1)^2 - 4)/2$ which is again less than the number $2k^2 + 2k$ of different vertical edges in $P_n \,\Box\, P_3$.

Since in both cases we got a contradiction, we conclude that $\mathrm{sg_e}(P_n \,\Box\, P_3) > \mathrm{sg_e}(P_n \,\Box\, P_2)$. □

Propositions 4.2 and 4.3, together with Lemma 2.1 for $m = 3$, and Lemma 4.1 imply Theorem 1.2.

5 Proof of Theorem 1.3

Lemma 5.1 $\mathrm{sg_e}(P_n \,\Box\, P_4) \leq \lceil 2\sqrt{n}\, \rceil + 1$.

Proof Use Algorithm 1 and cover horizontal edges in the first row and in the last row in the same way as in the proof of Theorem 1.1. To the existing vertex set from Algorithm 1, add vertex $(n, 2)$ and connect it with $(1, 1)$ by a geodesic that covers all horizontal edges from the second row (Fig. 7a) and with $(1, 4)$ by a geodesic that covers all horizontal edges from the third row (Fig. 7b).

By Lemmas 5.1 and 2.1 for $m = 4$, we know that for every $n \geq 2$, the strong edge geodetic number $\mathrm{sg_e}(P_n \,\Box\, P_4)$ is either $\lceil 2\sqrt{n}\, \rceil$ or $\lceil 2\sqrt{n}\, \rceil + 1$. From Lemma 2.2, we see that Theorem 1.3 says that $\mathrm{sg_e}(P_n \,\Box\, P_4) = \lceil 2\sqrt{n}\, \rceil$ except when $n = k^2$, $n = k^2 + k$, or $n = k^2 + 2k$ for some $k \in \mathbb{N}$. Hence, it is enough to prove that when $n = k^2$, $n = k^2 + k$, or $n = k^2 + 2k$, there is no strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ and to find a strong edge geodetic set of size $\lceil 2\sqrt{n}\, \rceil$ in the other cases.

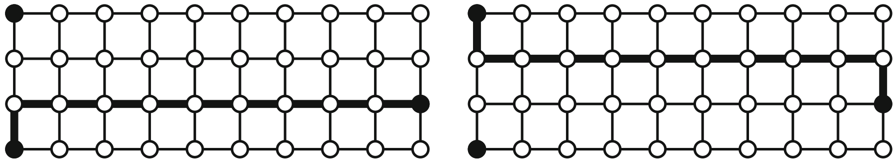

Fig. 7 $(1, 1)$,$(n, 2)$-geodesic and $(1, 4)$,$(n, 2)$-geodesic

First, let us show that there exists a strong edge geodetic set of size $\lceil 2\sqrt{n}\,\rceil$ for $P_n \,\square\, P_4$ when $n = k^2+h$ for some $k \in \mathbb{N}$, where $1 \le h \le k-1$, or $k+1 \le h \le 2k-1$.

Proposition 5.2 *If $n = k^2 + h$, $n \ge 3$, $k \ge 1$, where $1 \le h \le k-1$ or $k+1 \le h \le 2k-1$, then*

$$\mathrm{sg_e}(P_n \,\square\, P_4) \le \mathrm{sg_e}(P_n \,\square\, P_2).$$

Proof To cover all the vertical edges and the horizontal edges in the second row, we will adjust Algorithm 1. We divide this adjustment into two cases.

Case 1: $n = k^2 + h$, $1 \le h \le k-1$ (see Fig. 8).
First, let us use Algorithm 1. In the next step, replace the existing vertex $b_{k+1} = (k^2, 1)$ with the vertex $c = (n, 3)$ and for $j = h, \dots, 1$, replace the existing a_j,b_{k+1}-geodesic with two shortest paths, the a_{j+1},c-geodesic (it exists because $h \le k-1$) that covers the edges $(k^2 + j, 1)(k^2 + j, 2)$ and $(k^2 + j, 2)(k^2 + j, 3)$, and the b_{j+1},c-geodesic that covers the edge $(k^2 + j, 3)(k^2 + j, 4)$. This replacement is well defined because we replaced all the shortest paths that had one endpoint at b_{k+1} and added some new shortest paths that have one endpoint at c, so the condition that for each pair of vertices from the set $\{a_1, \dots, a_k, b_1, \dots, b_k, c\}$ we only use at most one shortest path still holds.

In this way we have adjusted Algorithm 1 such that it covers all vertical edges in $P_n \,\square\, P_4$, while it uses neither the a_1,c-geodesics nor the b_1,c-geodesics. This means that we can add the a_1,c-geodesic that covers all the horizontal edges in the second row and the b_1,c-geodesic that covers all the horizontal edges in the third row. Some edges from the first row are already covered with the a_{h+1},c-geodesic, but the rest of them can be covered with the unique a_1,a_k-geodesic. By symmetry, some of the horizontal edges from the fourth row are covered with the b_{h+1},c-geodesic, and the other ones are covered by the b_1,b_k-geodesic.

Case 2: $n = k^2 + h$, $k+1 \le h \le 2k-1$ (see Fig. 9).
In this case, because $h \le 2k-1$, Algorithm 1 never uses a a_{k+1},b_k-geodesic. It also never uses the unique a_{k+1},b_{k+1}-geodesic. If we add this shortest path, we can replace the existing b_{h-k},a_{k+1}-geodesic (this shortest path covers the vertical edges in the n-th column) with the one that covers the vertical edges in the $(k(k+1)+1)$-th column (in Algorithm 1 covered by the a_{k+1},b_1-geodesic). Observe that this step does nothing if $h = 1$. This way we can replace the existing a_{k+1},b_1-geodesic with the one that covers all the horizontal edges in the third row.

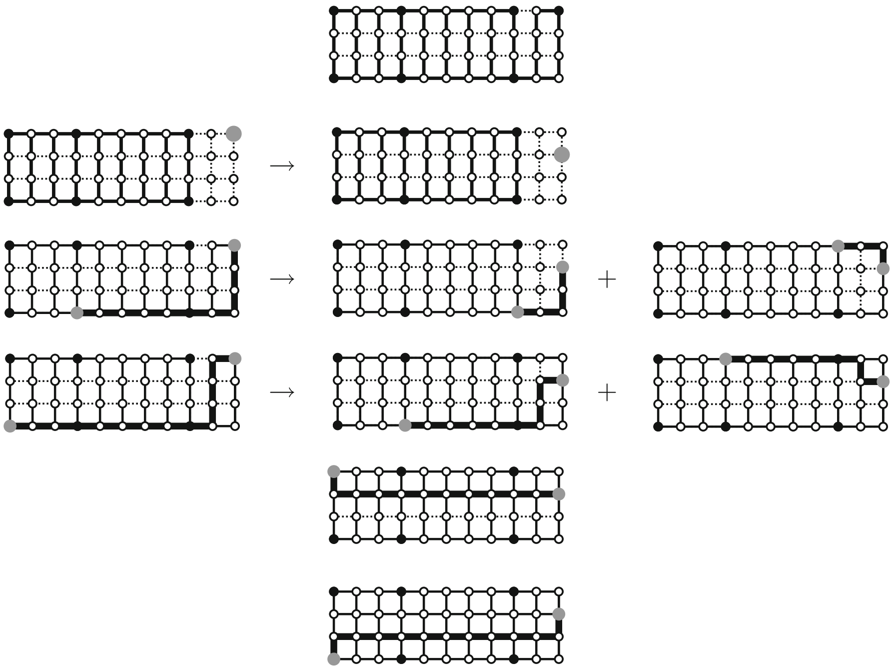

Fig. 8 Strong edge geodetic set in $P_{3^2+2} \,\square\, P_4$

Also, if we add the a_{k+1},b_k-geodesic that covers all the vertical edges in the (k^2+1)-th column, we can replace the existing a_1,b_{k+1}-geodesic with the one that covers all the horizontal edges in the second row.

In the same way as in $P_n \,\square\, P_2$, we now cover the horizontal edges in the first row (using the a_1,a_{k+1}-geodesic) and in the fourth row (using the b_1,b_{k+1}-geodesic) of $P_n \,\square\, P_4$.

Since in all three cases we found a strong edge geodetic set of size $\mathrm{sg_e}(P_n \,\square\, P_2)$, we can conclude that $\mathrm{sg_e}(P_n \,\square\, P_4) \leq \mathrm{sg_e}(P_n \,\square\, P_2)$ for all $n \in \mathbb{N}$ when $n \neq k^2$, $n \neq k^2 - 1$, and $n \neq k^2 + k$ for some $k \in \mathbb{N}$. □

We will now show that there is no strong edge geodetic set of size $\lceil 2\sqrt{n} \rceil$ for $P_n \,\square\, P_4$ when $n = k^2 - 1$, k^2, or $k^2 + k$ for some $k \in \mathbb{N}$.

Proposition 5.3 *If $n = k^2$, $n = k^2 - 1$, or $n = k^2 + k$ for some $k \geq 2$, $k \in \mathbb{N}$, then*

$$\mathrm{sg_e}(P_n \,\square\, P_4) > \mathrm{sg_e}(P_n \,\square\, P_2).$$

Proof Let us define the function

$$f_s(a, b, c, d) = ab + bc + cd + 2ac + 2bd + 3ad$$

which gives an upper bound on how many different vertical edges can be covered with shortest paths between s vertices, where a vertices are in the first row, b vertices in the

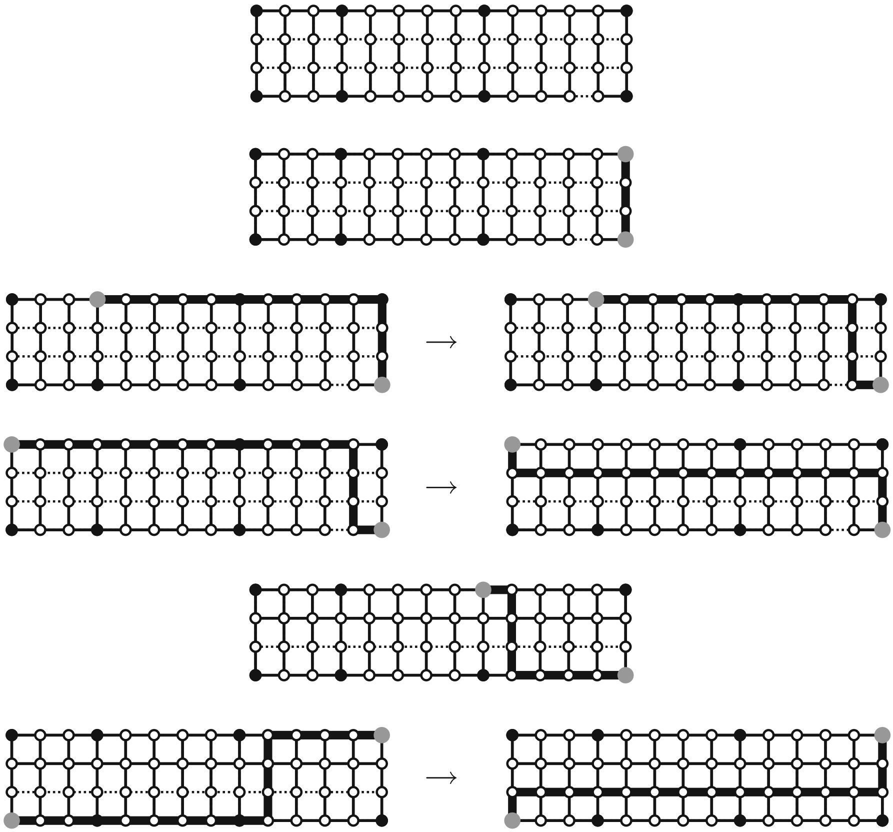

Fig. 9 Strong edge geodetic set in $P_{3^2+2\cdot3-1} \,\square\, P_4$

Table 1 Maximal values of function $f_s(a, b, c, d)$

Conditions	Upper bound
$a, b, c, d \geq 0$	$f_s(a, b, c, d) \leq \frac{1}{4}(3s^2)$
$a, c, d \geq 0, b \geq 1$	$f_s(a, b, c, d) \leq \frac{1}{12}(9s^2 - 8)$
$a, b, c, d \geq 0, b + c \geq 2$	$f_s(a, b, c, d) \leq \frac{1}{4}(3s^2 - 8)$
$a, b, c, d \geq 0, b + c \geq 3$	$f_s(a, b, c, d) \leq \frac{1}{4}(3s^2 - 18)$

second row, c vertices in the third row, and d vertices in the fourth row of $P_n \,\square\, P_4$. The maximum values of function f_s under some bounds are again computed by computer and are gathered in Table 1.

Every strong edge geodetic set S has to include at least one vertex from the first column; otherwise, it would not be possible to cover the edge $(1, 1)(1, 2)$. Similarly, every strong edge geodetic set includes at least one vertex from the last column. Depending on a strong edge geodetic set S in $P_n \,\square\, P_4$, we will define a type of first or last column of vertices. All the types for a strong edge geodetic set with at

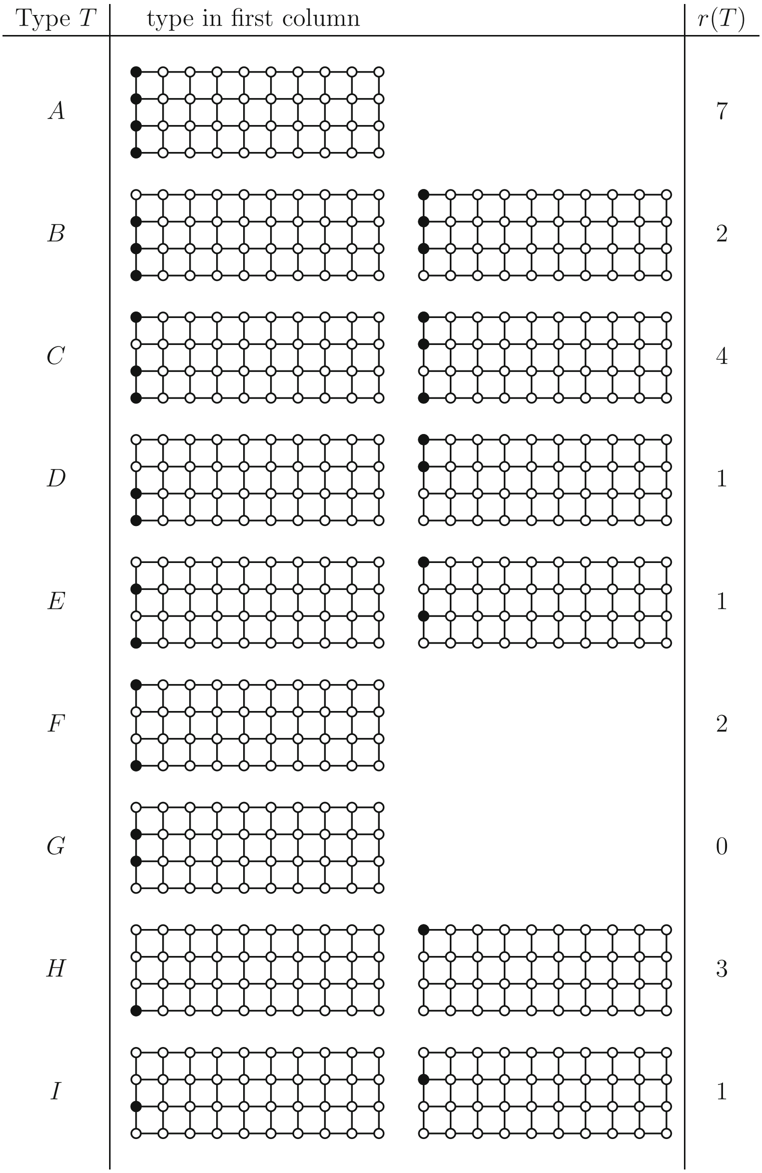

Fig. 10 Different types of a strong edge geodetic set in the first or the last column

least one vertex in the first and at least one vertex in the last column are gathered in Fig. 10. For example, if $(1,1), (1,2) \in S$ and $(1,3), (1,4) \notin S$, we say that the first column of vertices in $P_n \,\square\, P_4$ is of type D. Symmetrically, if $(1,1), (1,2) \notin S$ and $(1,3), (1,4) \in S$, we also say that the first column of vertices in $P_n \,\square\, P_4$ is of type D. We symmetrically define the type of the last column.

For a shortest path P in $P_n \,\square\, P_4$, let $E_i(P)$ denote the set of edges in i-th column, that is,

$$E_i(P) = E(P) \cap \{(i,1)(i,2), (i,2)(i,3), (i,3)(i,4)\}.$$

For a strong edge geodetic covering $C = \{P_{x,y}\}$ set

$$r_C^1 = \sum_{\{x,y\} \in \binom{S}{2}} |E_1(P_{x,y})| - 3.$$

Roughly speaking, r_C^1 measures redundancy of the strong edge geodetic covering in the first column of $P_n \,\square\, P_4$. Analogously, the redundancy r_C^n with respect to the last column is introduced.

For each type T of the first column determined by S, we can now compute the minimum number of redundant coverings in the first column of edges as $r_1(T) = \min_C\{r_C^1\}$, where C is a strong edge geodetic covering for a strong edge geodetic set S, where the first column in $P_n \,\square\, P_4$ is of type T. Similarly, we can define the minimum number of redundant coverings in the last column of edges, $r_n(T)$. By symmetry, these two numbers are the same, so we can denote $r(T) = r_1(T) = r_n(T)$. For each type T, the number $r(T)$ is listed in the last column of Fig. 10.

We will show how to compute $r(C)$ for type C, for other types it is similar. We can without loss of generality assume $\{(1,1), (1,2), (1,4)\} \subset S$. Between the pairs of these three vertices, there are three unique shortest paths P_1, P_2, and P_3. To cover the horizontal edge $(1,3)(2,3)$, we need another shortest path P_4 that has one end vertex in the vertex set $\{(1,1), (1,2), (1,4)\}$. Because $(1,3) \notin S$, this path includes at least one vertical edge from the first column, which implies

$$\sum_{\{x,y\} \in \binom{S}{2}} |E_1(P_{x,y})| - 3 \geq |E_1(P_1)| + |E_1(P_2)| + |E_1(P_3)| + |E_1(P_4)| - 3 \geq 4.$$

Suppose now that S is a strong edge geodetic set for $P_n \,\square\, P_4$ of cardinality $|S| = \mathrm{sg_e}(P_n \,\square\, P_2)$. We distinguish four different cases.

Case 1: S does not contain any vertex from the set $\{(1,2), (1,3), (n,2), (n,3)\}$. In this case, the first and the last column is either of type F or type G, which implies that $r_1 + r_n \geq 4$. Because $f_s(a,b,c,d) \leq \frac{3s^2}{4}$ when $a,b,c,d \geq 0$, it holds

$$f_s(a,b,c,d) - 4 \leq \frac{3 \cdot (2k)^2}{4} - 4 = 3k^2 - 4 < 3n \text{ for } n = k^2 - 1;$$
$$f_s(a,b,c,d) - 4 \leq \frac{3 \cdot (2k)^2}{4} - 4 = 3k^2 - 4 < 3n \text{ for } n = k^2;$$
$$f_s(a,b,c,d) - 4 \leq \frac{3 \cdot (2k+1)^2}{4} - 4 = 3k^2 + 3k - \frac{13}{4} < 3n \text{ for } n = k^2 + k,$$

for $s = \mathrm{sg_e}(P_n \,\square\, P_2)$.

Case 2: S contains at least three vertices from the second and the third row. Because $f_s(a,b,c,d) \leq \frac{1}{4}(3s^2-18)$ when $a,b,c,d \geq 0$, $b+c \geq 3$, it holds

$$f_s(a,b,c,d) \leq \frac{1}{4}(3(2k)^2-18) = 3k^2 - \frac{18}{4} < 3n \text{ for } n = k^2-1;$$
$$f_s(a,b,c,d) \leq \frac{1}{4}(3(2k)^2-18) = 3k^2 - \frac{18}{4} < 3n \text{ for } n = k^2;$$
$$f_s(a,b,c,d) \leq \frac{1}{4}(3(2k+1)^2-18) = 3k^2+3k - \frac{15}{4} < 3n \text{ for } n = k^2+k,$$

for $s = \mathrm{sg_e}(P_n \,\square\, P_2)$.

Case 3: S contains exactly two vertices from the set $\{(1,2),(1,3),(n,2),(n,3)\}$ and no other vertex from the second and the third row.
If we look at all types of the first and the last column such that the condition holds, we see that $r_1 + r_n$ is always at least 2. (Observe that type G can only be combined with types F and H.) Because $f_s(a,b,c,d) \leq \frac{1}{4}(3s^2-8)$ when $b+c \geq 2$, and $a,b,c,d \geq 0$, it holds

$$f_s(a,b,c,d) - 2 \leq \frac{1}{4}(3(2k)^2-8) - 2 = 3k^2 - 4 < 3n \text{ for } n = k^2-1;$$
$$f_s(a,b,c,d) - 2 \leq \frac{1}{4}(3(2k)^2-8) - 2 = 3k^2 - 4 < 3n \text{ for } n = k^2;$$
$$f_s(a,b,c,d) - 2 \leq \frac{1}{4}(3(2k+1)^2-8) - 2 = 3k^2+3k - \frac{13}{4} < 3n \text{ for } n = k^2+k,$$

for $s = \mathrm{sg_e}(P_n \,\square\, P_2)$.

Case 4: S contains exactly one vertex from the set $\{(1,2),(1,3),(n,2),(n,3)\}$ and no other vertex from the second and the third row.
By symmetry, we can without loss of generality assume that $(1,3),(n,2),(n,3) \notin S$ and $(1,2) \in S$. This implies that the first column is of type C, D, E, or H and the last column is of type F or G. The sum of the number of redundant coverings for the first and the last column is than at least 3. Because $f_s(a,b,c,d) \leq \frac{1}{12}(9s^2-8)$ when $b+c \geq 2$ and $a,b,c,d \geq 0$, it holds

$$f_s(a,b,c,d) - 2 \leq \frac{1}{12}(9(2k)^2-8) - 3 = 3k^2 - \frac{11}{3} < 3n \text{ for } n = k^2-1;$$
$$f_s(a,b,c,d) - 2 \leq \frac{1}{12}(9(2k)^2-8) - 3 = (3k^2-3) - \frac{2}{3} < 3n \text{ for } n = k^2;$$
$$f_s(a,b,c,d) - 2 \leq \frac{1}{12}(9(2k+1)^2-8) - 3 = (3k^2+3k) - \frac{37}{12} < 3n \text{ for } n = k^2+k,$$

for $s = \mathrm{sg_e}(P_n \,\square\, P_2)$.

Since $3n$ is the number of all vertical edges in $P_n \square P_4$, it holds $|S| > \mathrm{sg_e}(P_n \square P_2)$ in every case above. This is a contradiction with the assumption that S is a strong edge geodetic set of $P_n \square P_4$ of cardinality $\mathrm{sg_e}(P_n \square P_2)$.

Because all four cases led us to a contradiction, we can conclude that for $P_n \square P_4$, where $n = k^2 - 1, k^2$, or $k^2 + k$ for some $k \in \mathbb{N}$, there is no strong edge geodetic set of size $\mathrm{sg_e}(P_n \square P_2)$. By Lemma 5.1, this means that $\mathrm{sg_e}(P_n \square P_4) = \mathrm{sg_e}(P_n \square P_2) + 1$ when $n = k^2 - 1, k^2$, or $k^2 + k$ for some $k \in \mathbb{N}$. In other words, $\mathrm{sg_e}(P_{k^2-1} \square P_4) = \mathrm{sg_e}(P_{k^2} \square P_4) = 2k + 1$ and $\mathrm{sg_e}(P_{k^2+k} \square P_4) = 2k + 2$. Also, because $k^2 - 1 = (k-1)^2 + 2(k-1)$, we have $\mathrm{sg_e}(P_{k^2+2k} \square P_4) = 2k + 3$. □

6 Upper Bounds

In this section, we give two upper bounds on $\mathrm{sg_e}(P_n \square P_m)$.

Proposition 6.1 *If $n \geq 2$ and $m \geq 2$, then*

$$\mathrm{sg_e}(P_n \square P_m) \leq \left\lceil 2\sqrt{n} \right\rceil + \left\lceil 2\sqrt{m-2} \right\rceil.$$

Proof First, we cover all the vertical edges in $P_n \square P_m$ by using Algorithm 1. This algorithm uses $\lceil 2\sqrt{n} \rceil$ vertices from the first and the last row. We also, similarly as in $P_n \square P_2$, use these vertices to cover the horizontal edges from the first and the last row (Fig. 11a).

In second step, we look at the subgraph H of $P_n \square P_m$ without the first and the last row (Fig. 11b). H is isomorphic to $P_n \square P_{m-2}$. To cover all the horizontal edges from H, we use Algorithm 1 on H rotated by 90 degrees, which is the same as using Algorithm 1 on $P_{m-2} \square P_n$ (Fig. 11c). This algorithm uses exactly $\lceil 2\sqrt{m-2} \rceil$ vertices and covers all the horizontal edges from H, which are exactly the edges that have not been covered in the first part of the proof. □

With a similar, but a bit more involved, idea as in the proof of Proposition 6.1, we are going to prove the following proposition.

Proposition 6.2 *If $n \geq 3$ and $m \geq 3$, then*

$$\mathrm{sg_e}(P_n \square P_m) \leq \left\lceil 2\sqrt{n+2} \right\rceil + \left\lceil 2\sqrt{m} \right\rceil - 4.$$

Proof First, we will adjust Algorithm 1 such that it will use all the corner vertices, that is $(1, 1), (1, m), (n, 1), (n, m)$. We call the new algorithm, Algorithm 1*. For $n = k^2 + h$, where $h = 0$ or $k + 1 \leq h \leq 2k$, Algorithm 1 already uses all the corner vertices. When $n = k^2 + h$, $1 \leq h \leq k$, redefine the vertex a_k as $a_k = (n, 1)$. The shortest paths between vertices a_k and b_i in Algorithm 1* are the ones that cover the same vertical edges as shortest paths between vertices a_k and b_i in Algorithm 1. An example output of this algorithm is shown in Fig. 12.

When $n = k^2$, $n = k^2 + k - 1$, $n = k^2 + k$, or $n = k^2 + 2k$ for some $k \in \mathbb{N}$, we will cover the vertical edges in $P_n \square P_m$ in the following way. First, we use Algorithm 1*

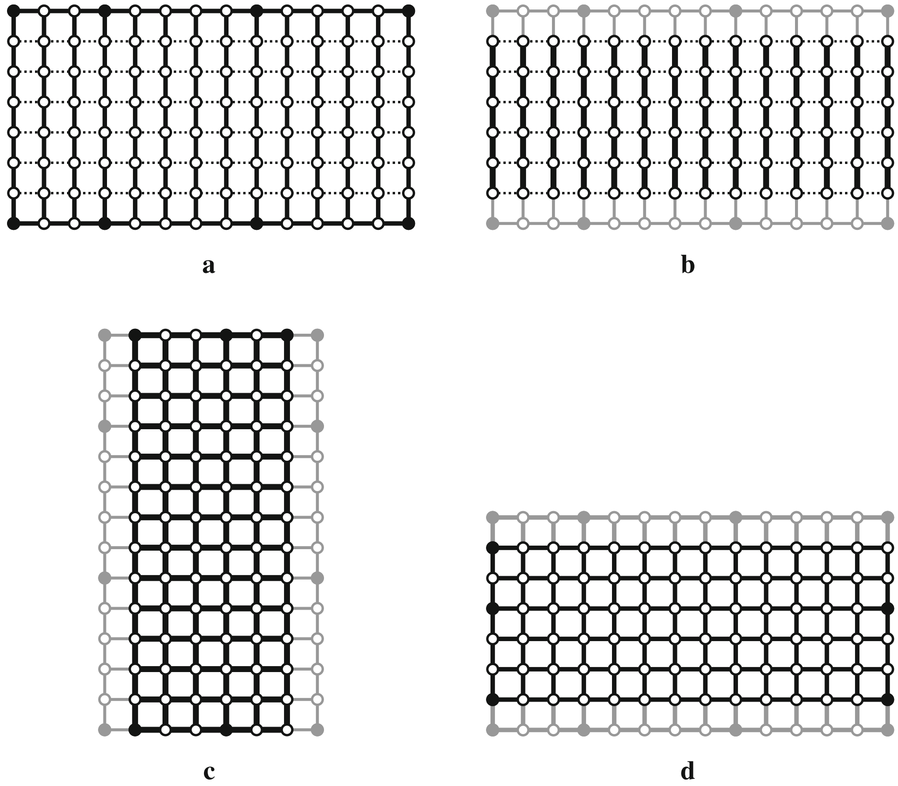

Fig. 11 Strong edge geodetic set for $P_n \,\square\, P_m$ with cardinality $\lceil 2\sqrt{n}\,\rceil + \lceil 2\sqrt{m-2}\,\rceil$ for $n = 14$ and $m = 8$

with $\lceil 2\sqrt{n}\,\rceil$ vertices V_1. To V_1 we add $c = ((k-1)^2 + k, m)$. We can then cover the vertical edges covered by the $(1, 1), (n, m)$-geodesic and the $(n, 1), (1, m)$-geodesic with $(1, 1), c$-geodesic and $(n, 1), c$-geodesic. In this way, we can remove shortest paths between corner vertices (Fig. 13).

When $n = k^2 + h$; $k, h \in \mathbb{N}$; $1 \le h \le k-2$, we will adjust Algorithm 1* to cover all the vertical edges with the same vertex set and without using the shortest paths between vertices a_1 and b_{k+1} and between vertices a_k and b_1. First, we notice that Algorithm 1* does not use the unique a_k,b_{k+1}-geodesic. If we add it, we can replace the a_h,b_{k+1}-geodesic with the one that covers all the vertical edges in the (k^2+1)-th column. We can then remove the a_1,b_{k+1}-geodesic. (It also covers the vertical edges in the (k^2+1)-th column.) Algorithm 1* also does not use the a_{k-1},b_{k+1}-geodesic. If we add the a_{k-1},b_{k+1}-geodesic that covers all the vertical edges in the $((k-1)^2+k)$-th column, we can remove the a_k,b_{k+1}-geodesic.

When $n = k^2 + h$, $k, h \in \mathbb{N}$, $k+1 \le h \le 2k-1$, we can use a similar adjustment as in the proof of Theorem 1.3 for $n = k^2 + h$, $k+1 \le h \le 2k-1$, and remove the shortest paths between vertices a_1 and b_{k+1} and between vertices a_{k+1} and b_1 that cover the horizontal edges in the second and the third row.

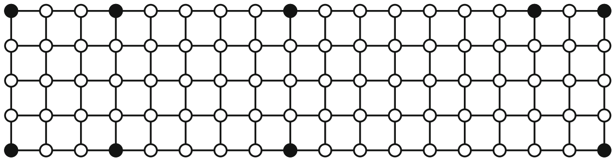

Fig. 12 Strong edge geodetic set of $P_{4^2+2} \square P_5$ from Algorithm 1*

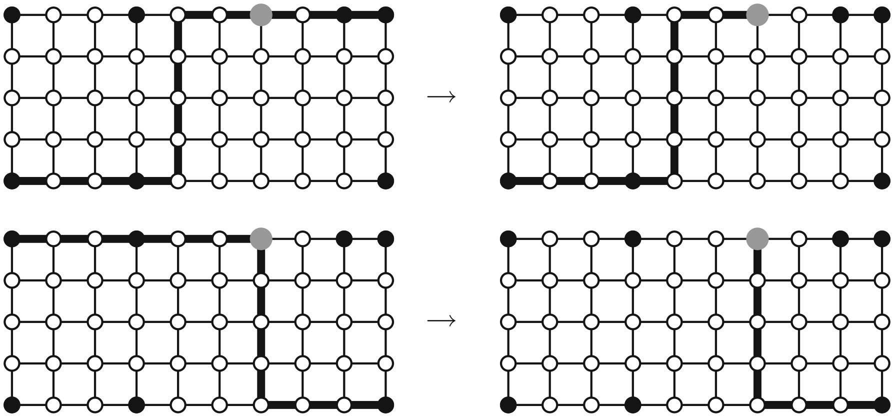

Fig. 13 Covering vertical edges in $P_n \square P_m$ without using shortest paths between vertices $(1, 1)$ and (n, m) and between vertices $(n, 1)$ and $(1, m)$

In the first part of the proof, we defined the algorithm that uses $\lceil 2\sqrt{n+2}\, \rceil$ vertices and covers all the vertical rows in $P_n \square P_m$, while it uses neither the $(1, 1)$,(n, m)-geodesic, nor the $(1, m)$,$(n, 1)$-geodesic.

To cover the horizontal edges, we use Algorithm 1* on rotated $P_n \square P_m$ by 90 degrees. This part will use $\lceil 2\sqrt{m}\, \rceil$ vertices, where four ($(1, 1)$, $(1, m)$, $(n, 1)$, (n, m)) of them are already used in the first part. Because the first part does not use shortest paths between these four vertices, the condition that between any two vertices we use at most one shortest path still holds.

In the first and in the second part, we have covered all the edges in $P_n \square P_m$, using exactly $\lceil 2\sqrt{n+2}\, \rceil + \lceil 2\sqrt{m}\, \rceil - 4$ vertices. □

If we combine all the results from this paper, we can see that the bound from Proposition 6.1 is sharp when $m = 2$ and is not sharp for $m = 3$ and $m = 4$. The bounds from Proposition 6.2 are sharp when $m = 3$ and $n = k^2$ or $k^2 + k$ for some integer k, as well as when $m = 4$ and $n = k^2$, $k^2 + k$ or $k^2 + 2k$ for some integer k.

7 Conclusion

We determined the strong edge geodetic number for Cartesian products $P_n \square P_2$, $P_n \square P_3$, and $P_n \square P_4$. To prove the corresponding upper bounds, we found strong edge geodetic sets by adjusting the algorithm that covers all the vertical edges in graph $P_n \,\square\, P_m$ by using at most one shortest path between every pair from the set of exactly $\lceil 2\sqrt{n}\, \rceil$ vertices. Using this algorithm, we also proved some general upper bounds for $\mathrm{sg_e}(P_n \,\square\, P_m)$ where $n, m \geq 3$.

The ultimate goal would of course be to determine the strong edge geodetic number for all graphs $P_n \,\square\, P_m$. If this task will be too demanding, one could at least try to find better general upper and lower bounds, in particular improving Lemma 2.1. It would also be of interest to investigate the strong edge geodetic problem on other products of graphs.

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References

1. Gledel, V., IrŠič, V.: Strong geodetic number of complete bipartite graphs, crown graphs and hypercubes. Bull. Malays. Math. Sci. Soc. **43**, 2757–2767 (2020)
2. Gledel, V., IrŠič, V., Klavžar, S.: Strong geodetic cores and Cartesian product graphs. Appl. Math. Comput. **363**, 124609 (2019)
3. Harary, F., Loukakis, E., Tsouros, C.: The geodetic number of a graph. Math. Comput. Model. **17**, 89–95 (1993)
4. IrŠič, V.: Strong geodetic number of complete bipartite graphs and of graphs with specified diameter. Graphs Combin. **43**, 443–456 (2018)
5. IrŠič, V., Klavžar, S.: Strong geodetic problem on Cartesian products of graphs. RAIRO Oper. Res. **52**, 205–216 (2018)
6. IrŠič, V., Konvalinka, M.: Strong geodetic problem on complete multipartite graphs. Ars Math. Contemp. **17**, 481–491 (2019)
7. Klavžar, S., Manuel, P.: Strong geodetic problem in grid-like architectures. Bull. Malays. Math. Sci. Soc. **41**, 1671–1680 (2018)
8. Manuel, P., Klavžar, S., Xavier, A., Arokiaraj, A., Thomas, E.: Strong edge geodetic problem in networks. Open Math. **15**, 1225–1235 (2017)
9. Manuel, P., Klavžar, S., Xavier, A., Arokiaraj, A., Thomas, E.: Strong geodetic problem in networks. Discuss. Math. Graph Theory **40**, 307–321 (2020)
10. Mezzini, M.: An $O(mn^2)$ algorithm for computing the strong geodetic number in outerplanar graphs. Discuss. Math. Graph Theory (2020). https://doi.org/10.7151/dmgt.2311
11. Santhakumaran, A.P., John, J.: Edge geodetic number of a graph. J. Discrete Math. Sci. Cryptogr. **10**, 415–432 (2007)
12. Wang, Z., Mao, Y., Ge, H., Magnant, C.: Strong geodetic number of graphs and connectivity. Bull. Malays. Math. Sci. Soc. **43**, 2443–2453 (2020)
13. Xavier, D.A., Mathew, D., Theresal, S., Varghese, E.V.: Some results on strong edge geodetic problem in graphs. Commun. Math. Appl. **11**, 403–413 (2020)