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Vanishing Theorems for Riemannian Manifolds with Nonnegative Scalar Curvature and Weighted *p*-Poincaré Inequality

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Abstract

In this paper, we give some vanishing theorems for harmonic p-forms on complete noncompact Riemannian manifolds satisfying a weighted p-Poincaré inequality with nonnegative scalar curvature and under pointwise curvature pinching conditions which are bounded from above by the weight function.

Keywords Harmonic p-form · Vanishing theorems · Weighted p-Poincaré inequality

Mathematics Subject Classification Primary 58J05; Secondary 58J35

1 Introduction

Let M be an n-dimensional complete noncompact orientable Riemannian manifold. Let d be the exterior differential operator, so its dual operator δ is defined by

$$\delta = (-1)^{n(p+1)+1} * d*,$$

where $*$ is the Hodge star operator acting on the space of smooth p-forms $\Lambda^p(M)$. Then, the Hogde–Laplace–Beltrami operator Δ acting on the space of smooth p-$\Lambda^p(M)$ is given by

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$$\Delta = -(\delta d + d\delta).$$

Recall that a p-form ω on a Riemannian manifold M is said to be harmonic if it satisfies $d\omega = 0$ and $\delta\omega = 0$. It is an interesting and important problem in geometry and topology to find sufficient conditions on a complete manifold M for the vanishing of harmonic forms by its applications in the study of the structure of complete manifolds. For instance, it is well known that the set of harmonic 1-forms has close relationship with the connectedness at infinity of the manifold, in particular, with nonparabolic end such as the topology at infinity of a complete Riemannian manifold or complete orientable δ-stable minimal hypersurface in $\mathbb{R}^{n+1}$ (see [3,13] and others).

We would like to emphasize that when M is compact, the space of harmonic p-forms is isomorphic to its p-th de Rham cohomology group. This is true when M is noncompact, but the theory of L^2 harmonic forms still has some interesting applications. (We refer the reader to [5,6] for details.) For this reason, it is naturally to study the L^2 Hodge theory. Li and Wang [14,15] proved vanishing-type theorems of L^2 harmonic 1-forms when Ricci curvature of the manifold is bounded from below in terms of the dimension and the first eigenvalue. After that, Lam [10] generalized results of Li and Wang to manifolds satisfying a weighted Poincaré inequality by assuming that the weight function is of sub-quadratic growth of the distance function. By using a weighted Poincaré inequality, Lin [17] established some vanishing theorems under various pointwise or integral curvature conditions. Besides, Chen and Sung [7], Dung and Sung [8] considered manifolds satisfying the following weighted p-Poincaré inequality

$$\int_M \rho(x)|\omega|^2 \le \int_M |d\omega|^2 + |\delta\omega|^2 \text{ for } \omega \in \Lambda_0^p(M),$$

in which the weight function is of exponential growth of the distance function and they obtained splitting and vanishing theorems for L^2 harmonic forms. Recently, Zhou [20] obtained some vanishing and splitting theorems which are established with a much weaker curvature condition and a lower bound of the first eigenvalue of the Laplacian.

It is worth to notice that the main tools to study the spaces of harmonic p-forms are the Bochner–Weitzenböck-type formulas and refined Kato-type inequalities under some conditions on the curvature operators of the manifolds such as conditions on nonnegative scalar curvature, Weyl curvature tensor, Ricci curvature and curvature tensor. We know that the major difficulty to compute the Bochner–Weitzenböck formula of harmonic p-forms of higher degrees is the nontriviality of the Weyl tensor. If the Weyl tensor vanishes, that is, M is locally conformally flat, there are many results for the vanishing of harmonic forms (see [9,16,18] and others).

When the norm of the Weyl conformal tensor is too large compared to the positive scalar curvature at each point, Lin [17] obtained a vanishing theorem which generalizes Bourguignon's result [1]. In addition, by assuming the norm of the Weyl tensor satisfies certain integral pinching conditions, he obtained several vanishing theorems for harmonic p-forms. His proof is based on a precise estimate of the curvature operator which appears in the Bochner–Weitzenböck formula on p-forms and together with Kato's inequality and the condition on weighted Poincaré inequality.

Motivated by Lin's work as well as by Zhou's result, we will establish some results on vanishing of L^2 harmonic p-forms on complete Riemannian manifolds with nonnegative scalar curvature and satisfying a weighted p-Poincaré inequality.

Theorem 1.1 *Let M be an n-dimensional complete noncompact Riemannian manifold satisfying a weighted p-Poincaré inequality with weight function $\rho(x)$ and the scalar curvature $R \geq 0$. Assume that the Weyl conformal curvature tensor W and the traceless Ricci tensor E satisfy*

$$|W|(x) + a_p|E|(x) \leq a\rho(x)$$

for some constant $a > 0$. Then, $\mathcal{H}^p(L^2(M)) = \{0\}$ for all $2 \leq p \leq n-2$ but $p \neq \frac{n}{2}$.

Here, we denote by $\mathcal{H}^p(L^2(M))$ the space of L^2 harmonic p-forms on M and $a_p := \frac{2(n-1)|n-2p|}{(p-1)\sqrt{(n+1)(n-2)^3}}$.

In the case of $p = \frac{n}{2} = m$, we require the weight function $\rho \not\equiv 0$ or the scalar curvature $R \not\equiv 0$.

Theorem 1.2 *Let M be a $2m$-dimensional complete noncompact Riemannian manifold satisfying a weighted m-Poincaré inequality with weight function $\rho(x)$ and the scalar curvature $R \geq 0$. Assume that the Weyl conformal curvature tensor W satisfies*

$$|W|(x) \leq a\rho(x)$$

for some constant $a > 0$. Then, every L^2 harmonic m-form is parallel. In particular, if $\rho > 0$ or $R > 0$ at some point, then $\mathcal{H}^m(L^2(M)) = \{0\}$.

Similar to results of Vieira [19] and Zhou [20], in order to obtain the vanishing theorem with the weaker curvature assumption, we need the certain lower bound of the first eigenvalue of the Laplacian.

Theorem 1.3 *Let M be an n-dimensional complete noncompact Riemannian manifold satisfying a weighted p-Poincaré inequality with weight function $\rho(x)$ and the scalar curvature $R \geq 0$. Assume that the Weyl conformal curvature tensor W and the traceless Ricci tensor E satisfy*

$$|W|(x) + a_p|E|(x) \leq a\rho(x) + b$$

for two constants $a, b > 0$. Then, $\mathcal{H}^p(L^2(M)) = \{0\}$ for all $2 \leq p \leq n-2$ provided the first eigenvalue of the Laplacian satisfies $\lambda_1(M) > \frac{bp(p-1)}{2(1+k_p)}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}$, where $k_p := \frac{1}{\max\{p,n-p\}}$.

2 Preliminaries

Let M be a complete Riemannian manifold of dimension n and let R_{ijkl} and W_{ijkl} denote, respectively, the components of the Riemannian curvature tensor and the Weyl curvature tensor of M in local orthonormal frame fields. Then, we have the decomposition as follows

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) \\ - \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

where R_{ik} and R denote the component of the Ricci tensor and the scalar curvature, respectively. Let ω and θ be two p-forms on M; a pointwise inner product is defined by

$$\langle \omega, \theta \rangle = \sum_{i_1,\ldots,i_p=1}^{n} \omega(e_{i_1}, \ldots, e_{i_p})\theta(e_{i_1}, \ldots, e_{i_p}),$$

where we omit the normalizing factor $\frac{1}{p!}$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of TM with dual coframe $\{\theta^1, \ldots, \theta^n\}$. Take a representation of p-form ω in a local coordinate system as

$$\omega = \frac{1}{p!}\omega_{i_1\ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where repeated indices are contracted and summed and the indices $1 \leq i_i, i_2, \ldots, i_n \leq n$ are distinct with each other. Then, the Bochner–Weitzenböck formula gives

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \langle \sum_{j,k=1}^{n} \theta^k\, i_{e_j} R(e_j, e_k)\omega, \omega\rangle \\ = |\nabla\omega|^2 - \langle (d\delta + \delta d)\omega, \omega\rangle + pF_p(\omega), \tag{2.1}$$

where

$$F_p(\omega) = R_{ij}\omega^{ii_2\ldots i_p}\omega^{j}_{i_2\ldots i_p} - \frac{p-1}{2}R_{ijkl}\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p}.$$

Recall that $W : \Lambda^2 TM \to \Lambda^2 TM$ can be interpreted as a trace-free symmetric endomorphism defined by

$$\mathcal{W}(e_i \wedge e_j) = \frac{1}{2}W_{ijkl}e_k \wedge e_l,$$

and the norm of $\mathcal{W}$ is given by $|\mathcal{W}|^2 = \sum_{i<j,k<l} \mathcal{W}^2_{ijkl} = \frac{1}{4}|W|^2$.

Then, Lin [17] had an estimate for F_p as follows

$$F_p(\omega) \geq -\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}|W||\omega|^2 - \frac{p|n-2p|}{n-2}\sqrt{\frac{n-1}{n}}|E||\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2,$$

where E is the traceless Ricci tensor, that is, $E_{ij} = R_{ij} - \frac{R}{n}\delta_{ij}$. For $a_p = \frac{2(n-1)|n-2p|}{(p-1)\sqrt{(n+1)(n-2)^3}}$, the above inequality is rewritten as follows

$$F_p(\omega) \geq -\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}(|W| + a_p|E|)|\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2. \tag{2.2}$$

For each harmonic p-form ω, from [2] we have the refined Kato's inequality

$$|\nabla\omega|^2 - |\nabla|\omega||^2 \geq k_p|\nabla|\omega||^2 \tag{2.3}$$

for all $\omega \in \mathcal{H}^p(L^2(M))$, where $k_p = \frac{1}{\max\{p, n-p\}}$.

Let now $\rho(x)$ be a nontrivial nonnegative function on M. We say that M satisfies a weighted Poincaré inequality if

$$\int_M \rho(x)\psi^2(x) \leq \int_M |\nabla\psi|^2(x)$$

is valid for all $\psi \in C_0^\infty(M)$. This notion is a natural generalization of the positivity of the bottom spectrum of the Laplacian. By variational principle, the first spectrum $\lambda_1(M) > 0$ implies the following Poincaré inequality, i.e.,

$$\lambda_1(M)\int_M \psi^2 \leq \int_M |\nabla\psi|^2 \text{ for } \psi \in C_0^\infty(M).$$

Here, the first eigenvalue of the Laplacian $\lambda_1(M)$ is given by

$$\lambda_1(M) = \inf_{\varphi\in C_0^\infty} \frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2}.$$

3 Proof of Theorem 1.1

Let ω be any harmonic p-form on M with finite L^2 norm. Then, we have $d\omega = 0$ and $\delta\omega = 0$. It follows from (2.1) that

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + pF_p(\omega) - \langle(\delta d + d\delta)\omega, \omega\rangle$$

$$= |\nabla\omega|^2 + pF_p(\omega). \tag{3.1}$$

By the assumption $R \geq 0$, from (2.2) we have

$$F_p(\omega) \geq -\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}(|W| + a_p|E|)|\omega|^2. \tag{3.2}$$

Combining (3.1–3.2) with equality

$$\frac{1}{2}\Delta|\omega|^2 = |\omega|\Delta|\omega| + |\nabla|\omega||^2,$$

and applying Kato's inequality (2.3), we get

$$\begin{aligned} |\omega|\Delta|\omega| &\geq |\nabla\omega|^2 - |\nabla|\omega||^2 + pF_p(\omega) \\ &\geq k_p|\nabla|\omega||^2 - \frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}(|W| + a_p|E|)|\omega|^2. \end{aligned} \tag{3.3}$$

Let φ be a smooth function with compact support on M. Multiplying both sides of inequality (3.3) by φ^2 and integrating over M gives

$$\begin{aligned} \int_M \varphi^2|\omega|\Delta|\omega| \geq k_p \int_M \varphi^2|\nabla|\omega||^2 \\ - \frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}\int_M (|W| + a_p|E|)\varphi^2|\omega|^2. \end{aligned} \tag{3.4}$$

We now give the estimates for each term in (3.4).

Integrating by parts and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \int_M \varphi^2|\omega|\Delta|\omega| &= -\int_M \left\langle \nabla(\varphi^2|\omega|), \nabla(|\omega|)\right\rangle \\ &= -\int_M \left\langle \varphi^2\nabla|\omega| + 2\varphi|\omega|\nabla\varphi, \nabla|\omega|\right\rangle \\ &= -\int_M \varphi^2|\nabla|\omega||^2 - 2\int_M \varphi|\omega| \left\langle \nabla\varphi, \nabla|\omega|\right\rangle \\ &\leq -\int_M \varphi^2|\nabla|\omega||^2 + 2\int_M \varphi|\omega||\nabla\varphi| \cdot |\nabla|\omega|| \\ &\leq -\int_M \varphi^2|\nabla|\omega||^2 + \left(\varepsilon\int_M \varphi^2|\nabla|\omega||^2 + \frac{1}{\varepsilon}\int_M |\omega|^2|\nabla\varphi|^2\right) \end{aligned}$$

$$= (\varepsilon - 1)\int\limits_M \varphi^2|\nabla|\omega||^2 + \frac{1}{\varepsilon}\int\limits_M |\omega|^2|\nabla\varphi|^2 \tag{3.5}$$

for all $\epsilon > 0$. Note that

$$\begin{aligned}|d(\varphi\omega)|^2 &= |d\varphi \wedge \omega|^2 = |d\varphi|^2|\omega|^2 - \langle d\varphi, \omega\rangle^2, \\ |\delta(\varphi\omega)|^2 &= |-i_{e_j}\nabla_{e_j}(\varphi\omega)|^2 = |-\langle d\varphi, \omega\rangle + \varphi\delta\omega|^2 = |\langle d\varphi, \omega\rangle|^2.\end{aligned} \tag{3.6}$$

Then, by the assumption, we have

$$\begin{aligned}\int\limits_M (|W| + a_p|E|)\varphi^2|\omega|^2 &\le a\int\limits_M \rho\varphi^2|\omega|^2 \\ &\le a\int\limits_M (|d(\varphi\omega)|^2 + |\delta(\varphi\omega)|^2) \\ &= a\int\limits_M |\nabla\varphi|^2|\omega|^2.\end{aligned} \tag{3.7}$$

Combining (3.4) with (3.5) and (3.7), we have

$$\left(1 - \varepsilon + k_p\right)\int\limits_M \varphi^2|\nabla|\omega||^2 \le \left(\frac{1}{\varepsilon} + a\cdot\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}\right)\int\limits_M |\omega|^2|\nabla\varphi|^2.$$

Put $B_\epsilon = 1 - \varepsilon + k_p$, $D_\epsilon = \frac{1}{\varepsilon} + a\cdot\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}$. The above inequality is rewritten in the following form

$$B_\epsilon\int\limits_M \varphi^2|\nabla|\omega||^2 \le D_\epsilon\int\limits_M |\omega|^2|\nabla\varphi|^2. \tag{3.8}$$

Fix a point $x_0 \in M$ and let $\zeta(x)$ be the geodesic distance on M from x_0 to x. Let us choose a nonnegative smooth function ψ such that

$$\psi = \begin{cases} 1 & \text{if } \zeta(x) \le r \\ 0 & \text{if } 2r \le \zeta(x) \end{cases}$$

and $|\nabla\psi| \le \frac{2}{r}$. Then, inequality (3.8) implies

$$B_\epsilon\int_{B_{x_0}(r)} |\nabla|\omega||^2 \le \frac{4D_\epsilon}{r^2}\int_M |\omega|^2.$$

Letting $r \to \infty$ and then letting $\epsilon \to 0$, we deduce that

$$\int_M |\nabla|\omega||^2 \leq 0, \text{ i.e., } \nabla|\omega| = 0. \tag{3.9}$$

It follows that $|\omega|$ is constant on M.

By substituting the above cutoff function ψ into (3.7), we also have

$$\int_{B_{x_0}(r)} (|W| + a_p|E|)|\omega|^2 \leq \frac{4a}{r^2} \int_M |\omega|^2.$$

By letting $r \to \infty$, we get

$$(|W| + a_p|E|)|\omega|^2 \equiv 0. \tag{3.10}$$

Assume that $|\omega|$ is not identically zero on M. This implies that $\mathrm{vol}(M) < +\infty$ since $\omega \in \mathcal{H}^p(L^2(M))$. By the assumption $p \neq \frac{n}{2}$, we get $a_p > 0$. It follows from (3.10) that $|W| + a_p|E| \equiv 0$, and hence $|W| = |E| \equiv 0$ on M. Therefore, M is Einstein and locally conformally flat. According to the decomposition of the Riemannian curvature tensor, a locally conformally flat manifold has constant sectional curvature if and only if it is Einstein. Therefore, M is a space form. Note that M is complete noncompact with $R \geq 0$. So, by the Hopf classification theorem (see [11, Theorem 12.4]), M is a flat space form. Then, by [4] or [12], $\mathrm{vol}(B_{x_0}(r)) \geq Cr^n$ with some constant $C > 0$, which implies $\mathrm{vol}(M) = +\infty$. This is a contradiction. Therefore, we conclude $\mathcal{H}^p(L^2(M)) = \{0\}$. The proof of Theorem 1.1 is completed. □

4 Proof of Theorem 1.2

Let ω be any harmonic m-form on M with finite L^2 norm. By the same argument as in the proof of Theorem 1.1, we also have inequality (3.9), that is $\nabla|\omega| \equiv 0$, i.e., $|\omega|$ is constant on M. Since $p = \frac{n}{2} = m$, we have $a_p = 0$. Then, equality (3.10) implies $|W| \equiv 0$ on M.

By substituting $|W| \equiv 0$ into (2.2), we get $pF_p(\omega) \geq \frac{p(n-p)}{n(n-1)}R|\omega|^2$. Then, by (3.1), we obtain

$$\frac{1}{2}\Delta|\omega|^2 \geq |\nabla\omega|^2 + \frac{p(n-p)}{n(n-1)}R|\omega|^2.$$

This implies that $\nabla\omega \equiv 0$ and $R|\omega| \equiv 0$. Hence, we can conclude that ω is parallel.

If $R > 0$ at some point, immediately $\omega \equiv 0$.

We assume that $\rho > 0$ at some point. Since M satisfies the weighted p-Poincaré inequality and using (3.6), we get

$$\int_M \rho\varphi^2|\omega|^2 \leq \int_M (|d(\varphi\omega)|^2 + |\delta(\varphi\omega)|^2)$$

$$
\begin{aligned}
&= \int_M |\nabla\varphi|^2 |\omega|^2 \\
&\leq \frac{2}{r^2} \int_M |\omega|^2,
\end{aligned}
$$

where the cutoff function φ is given as in the proof of Theorem 1.1. By letting $r \to \infty$, we obtain

$$\int_M \rho|\omega|^2 \leq 0.$$

If $|\omega|$ is not identically zero, then

$$\int_M \rho \leq 0.$$

On the other hand, by the condition of the theorem, we have $\rho \geq 0$. It implies that $\rho \equiv 0$, which contradicts the assumption. Therefore, ω must be a zero constant on M. So we can conclude $\mathcal{H}^m(L^2(M)) = \{0\}$. The proof of Theorem 1.2 is completed. □

5 Proof of Theorem 1.3

By the assumption and similar to inequality (3.7), we get

$$
\begin{aligned}
\int_M (|W| + a_p|E|)\varphi^2|\omega|^2 &\leq \int_M \left(a\rho\varphi^2|\omega|^2 + b\varphi^2|\omega|^2\right) \\
&\leq a\int_M |\nabla\varphi|^2|\omega|^2 + b\int_M \varphi^2|\omega|^2.
\end{aligned}
$$

Combining this with (3.4) and (3.5), we have

$$
\begin{aligned}
\left(1 - \varepsilon + k_p\right)\int_M \varphi^2|\nabla|\omega||^2 \leq& \left(\frac{1}{\varepsilon} + a \cdot \frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}\right)\int_M |\omega|^2|\nabla\varphi|^2 \\
&+ b \cdot \frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}\int_M \varphi^2|\omega|^2.
\end{aligned}
$$

Put $B_\epsilon = 1-\varepsilon+k_p$, $D_\epsilon = \frac{1}{\varepsilon}+a\cdot\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}$ and $E = b\cdot\frac{p(p-1)}{2}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}}$. Then, we have

$$B_\epsilon \int_M \varphi^2 |\nabla|\omega||^2 \leq D_\epsilon \int_M |\omega|^2 |\nabla\varphi|^2 + E \int_M \varphi^2 |\omega|^2. \tag{5.1}$$

By taking the cutoff function φ as in the proof of Theorem 1.1 and substituting it into inequality (5.1), we obtain

$$B_\epsilon \int_{B_{x_0}(r)} |\nabla|\omega||^2 \leq \frac{4D_\epsilon}{r^2} \int_M |\omega|^2 + E \int_M |\omega|^2.$$

By letting $r \to \infty$ and then letting $\epsilon \to 0$, we get

$$\int_M |\nabla|\omega||^2 \leq \frac{E}{B} \int_M |\omega|^2, \tag{5.2}$$

where $B := \lim_{\epsilon\to 0} B_\epsilon = 1 + k_p$.

By the assumption, we have $\lambda_1(M) > \frac{bp(p-1)}{2(1+k_p)}\sqrt{\frac{(n+1)(n-2)}{n(n-1)}} = \frac{E}{B} > 0$. By variational principle, we get

$$\lambda_1(M) \int_M \varphi^2 \leq \int_M |\nabla\varphi|^2 \quad \text{for any} \quad \varphi \in C_0^\infty(M).$$

Assume $|\omega| \not\equiv 0$ on M. Then, by taking the cutoff function ψ as in the proof of Theorem 1.1 again and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\lambda_1(M) \int_M \psi^2 |\omega|^2 &\leq \int_M |\nabla(\psi|\omega|)|^2 \\
&= \int_M \psi^2 |\nabla|\omega||^2 + \int_M |\omega|^2 |\nabla\psi|^2 + 2\int_M \langle \psi\nabla|\omega|, |\omega|\nabla\psi\rangle \\
&\leq (1+\epsilon) \int_M \psi^2 |\nabla|\omega||^2 + \big(1+\frac{1}{\epsilon}\big) \int_M |\omega|^2 |\nabla\psi|^2 \\
&\leq (1+\epsilon) \int_M \psi^2 |\nabla|\omega||^2 + \big(1+\frac{1}{\epsilon}\big)\frac{2}{r^2} \int_M |\omega|^2
\end{aligned}$$

for any $\epsilon > 0$. By letting $r \to \infty$ and then $\epsilon \to 0$, we obtain

$$\lambda_1(M) \int_M |\omega|^2 \leq \int_M |\nabla|\omega||^2. \tag{5.3}$$

Combining (5.2) with (5.3), we have $\lambda_1(M) \leq \frac{E}{B}$. This is a contradiction to the assumption. Hence $\omega \equiv 0$ on M, i.e., $\mathcal{H}^p(L^2(M)) = \{0\}$. The proof of Theorem 1.3 is completed. □

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