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Pairs of Positive Solutions for Nonhomogeneous Dirichlet Problems

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Abstract

We consider a nonlinear Dirichlet problem driven by a nonhomogeneous differential operator. The reaction has a parametric concave term and negative sublinear perturbation. In contrast to the case of a positive perturbation, we show that now for all big values of the parameter $\lambda > 0$, we have at least two positive solutions which do not vanish in the domain. In the process we prove a nonlinear maximum principle which is of independent interest.

Keywords Nonhomogeneous differential operator · Regularity theorem · Maximum principle · Positive solutions

Mathematics Subject Classification 35J20 · 35J60 · 35J92

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonhomogeneous parametric Dirichlet problem:

$$\begin{cases} -\mathrm{div} a(Du(z)) = \lambda u(z)^{q-1} - f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad \lambda > 0, \quad u > 0. \end{cases} \tag{p_λ}$$

In this paper the map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the differential operator, is continuous and strictly monotone, thus maximal monotone too. It exhibits balanced (p-1)-growth and $1 < q < p$. In the reaction (right-hand side) we have a parametric "concave" term $x \to \lambda x^{q-1}$ (since $q < p$) and there is a negative perturbation $-f(z, x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a.$z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ is $(q-1)$ sublinear as $x \to 0^+$ and as $x \to +\infty$. A typical case is when $f(x) = x^{\tau-1}$ for all $x \geq 0$ with $1 < \tau < q$. It is well known that if this perturbation enters in the reaction with a positive sign, then the problem has a unique positive solution. This was proved first by Brezis–Oswald [3] for problems driven by the Laplacian and was extended by Diaz–Saa [5] to equations driven by the Dirichlet p-Laplacian and by Fragnelli–Mugnal–Papageorgiou [7] for equations driven by a nonhomogeneous differential operator with Robin boundary condition. The case where the perturbation enters with a negative sign has not been studied. We show that in this case, uniqueness of the solution fails and for big values of the parameter $\lambda > 0$, we have at least two positive smooth solutions. However, these solutions do not belong in the interior of the positive cone of $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$, since the nonlinear Hopf's lemma cannot be used (see Pucci–Serrin [15], pp. 111, 120). Nevertheless, in Sect. 3, we prove a maximum principle which shows that our solutions are strictly positive in Ω. That result is of independent interest and can be useful in different contexts.

2 Mathematical Background Hypotheses

The analysis of problem (p_λ) will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega})$. By $\|\cdot\|$ we denote the norm of the Sobolev space. On account of the Poincaré inequality, we have $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$. The Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ is ordered with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by $\mathrm{int}C_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\}$ with $n(\cdot)$ being the outward unit norm on $\partial\Omega$.

If $v, u : \Omega \to \mathbb{R}$ are measurable functions such that $v(z) \leq u(z)$ for a.a $z \in \Omega$, then by $[v, u]$ we denote the order interval in $W_0^{1,p}(\Omega)$ defined by

$$[v, u] = \{h \in W_0^{1,p}(\Omega) : v(z) \leq h(z) \leq u(z) \quad \text{for a.a } z \in \Omega\}.$$

For $x \in \mathbb{R}$, let $x^{\pm} = \max\{\pm x, 0\}$. Then, given $u \in W_0^{1,p}(\Omega)$, we set $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that $u^{\pm} \in W_0^{1,p}(\Omega),\ u = u^+ - u^-$ and $|u| = u^+ + u^-$. By $|\cdot|_N$ we will denote the Lebesgue measure on $\mathbb{R}^N$. Also if X is a Banach space and $\varphi \in C^1(X)$, then $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Next, we will introduce the hypotheses on the map $a(\cdot)$. So, let $\theta \in C^1(0,\infty)$ be such that

$$0 < \widehat{c} \le \frac{\theta'(t)t}{\theta(t)} \le c_0 \quad \text{and } c_1 t^{p-1} \le \theta(t) \le c_2[t^{s-1} + t^{p-1}]$$
$$\text{for all } t > 0 \text{ and some } c_1, c_2 > 0,\ 1 < s < p. \tag{1}$$

Then, the hypotheses on the map $a(\cdot)$ are the following:
H_0: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \to a_0(t)t$ is strictly increasing, $a_0(t)t \to 0^+$ as $t \to 0^+$ and if $l(t) = a_0(t)t$, then $l'(t)t \ge c^* l(t)$ for some $c^* > 0$ all $t > 0$;
(ii) $|\nabla a(y)| \le c_3 \frac{\theta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \backslash \{0\}$, some $c_3 > 0$;
(iii) $\frac{\theta(|y|)}{|y|}|\xi|^2 \le (\nabla \alpha(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \backslash \{0\}$ and all $\xi \in \mathbb{R}^N$.

Remark 1 These hypotheses on $a(\cdot)$ are dictated by the nonlinear regularity theory of Lieberman [11]. Also, they lead to the nonlinear maximum principle which we prove in the next section. The hypotheses are not restrictive and include many differential operators of interest (see the Examples below).

From these hypotheses, we see that the primitive function $t \to G_0(t)$ is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then, the function $G(\cdot)$ is convex, differentiable and $G(0) = 0$. Moreover, using the chain rule, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \backslash \{0\},\ \nabla G(0) = 0.$$

Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$. Since $G(\cdot)$ is convex and $G(0) = 0$, from the properties of convex functions we have

$$G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2}$$

From (1) and hypotheses H_0, we infer the following properties for the map $a(\cdot)$ (see Papageorgiou–Rădulescu [12]).

Lemma 1 *If hypotheses H_0 hold, then*

(a) $y \to a(y)$ *is continuous and strictly monotone (thus maximal monotone too);*
(b) $|a(y)| \le c_4[|y|^{s-1} + |y|^{p-1}]$ *for some* $c_4 > 0$, *all* $y \in \mathbb{R}^N$;
(c) $\frac{c_1}{p-1}|y|^p \le (a(y), y)_{\mathbb{R}^N}$ *for all* $y \in \mathbb{R}^N$.

This lemma and (2) lead to the following growth restrictions for the primitive $G(\cdot)$.

Corollary 2 *If hypotheses $H_0(i)$, (ii), (iii) hold, then* $\frac{c_1}{p(p-1)}|y|^p \le G(y) \le c_5[|y|^{s-1} + |y|^{p-1}]$ *for some* $c_5 > 0$, *all* $y \in \mathbb{R}^N$.

Hypotheses H_0 provide a broad framework in which we can fit many differential operators of interest.

Examples:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the (p, q)-Laplacian defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W_0^{1,p}(\Omega).$$

Such operators arise in many mathematical models of physical processes. We mention the works of Benci–D'Avenia–Fortunato–Pisani [2] (quantum physics), Cherfils–Ilyasov [4] (reaction–diffusion systems) and Bahrauni–Rădulescu–Repovš [1] (transonic flow problems). Some recent results in this direction can be found in the works of Goodrich–Ragusa [8],Goodrich–Ragusa–Scapellato [9], Papageorgiou–Scapellato [13] and Papageorgiou–Zhang [14].

(c) $a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y$ with $1 < p < \infty$.

This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}(1 + |Du|^2)^{\frac{p-2}{2}}Du \text{ for all } u \in W_0^{1,p}(\Omega).$$

(d) $a(y) = [1 + \frac{|y|^2}{(1+|y|^{2p})^{1/2}}]|y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the following differential operator which arises in problems of plasticity theory

$$\operatorname{div}\left(\left(1 + \frac{|Du|^2}{(1 + |Du|^{2p})^{1/2}}\right)|Du|^{p-2}Du\right) \text{ for all } u \in W_0^{1,p}(\Omega).$$

Let $A : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N} \mathrm{d}z \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

This operator is continuous and strictly monotone, thus maximal monotone too. Moreover, if we consider the integral functional $j : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$j(u) = \int_\Omega G(Du)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega),$$

then $j \in C^1(W_0^{1,p}(\Omega))$ and $j'(u) = A(u)$ for all $u \in W_0^{1,p}(\Omega)$.

Now we introduce our hypotheses on the perturbation $f(z,x)$:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a.$z \in \Omega$ and

(i) $0 \le f(z,x) \le a(z)[1 + x^{p-1}]$ for a.a.$z \in \Omega$, all $x \ge 0$, with $a \in L^\infty(\Omega)$;

(ii) $\lim_{x\to+\infty} \dfrac{f(z,x)}{x^{q-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to 0^+} \dfrac{f(z,x)}{x^{q-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iv) there exists $\mu \in (1,q)$ such that for all $\rho > 0$, we can find $\widehat{\xi}_\rho^\lambda > 0$ for which we have $\lambda x^{q-1} - f(z,x) + \widehat{\xi}_\rho^\lambda x^{\mu-1} \ge 0$ for a.a.$z \in \Omega$, all $x \in [0,\rho]$.

Remark 2 In hypothesis $H_1(iv)$ we need $\mu \in (1,q)$. This is a consequence of hypothesis $H_1(iii)$ and of the fact that the perturbation $f(z,x)$ enters in the reaction with a negative sign. However, this prohibits us from having a nonlinear Hopf's lemma (see Pucci–Serrin [15], p. 120), since hypothesis (1.1.5) in [15] is no longer true. Therefore, we see that the negative sign in the perturbation changes the geometry and is a source of difficulties. Nevertheless, in the next section we prove a maximum principle which shows that the positive solutions of problem (p_λ) do not vanish in Ω. This maximum principle extends Theorem 1.1 of Zhang [16].

3 A Maximum Principle

In this section we prove a nonlinear maximum principle. Our result was inspired by the work of Zhang [16] (Theorem 1.1) and we extend the result of [16]. The hypotheses of Zhang [16] on $a(\cdot)$ are more restrictive and do not cover the important case of the (p,q)-Laplacian (see (12) in [16]). The result is of independent interest.

Proposition 3 *If $u \in C_+\backslash\{0\}$, $\widehat{\xi} > 0$ and $\mu \in (1,q)$ satisfy*

$$-diva(Du) + \widehat{\xi} u^{\mu-1} \ge 0 \quad in\ \Omega,$$

then $u(z) > 0 \quad for\ all\ z \in \Omega$.

Proof We argue by contradiction. So, suppose we can find $z_1, z_2 \in \Omega$ and $\rho > 0$ such that $\overline{B}_{2\rho}(z_2) \subseteq \Omega$ ($B_{2\rho}(z_2) = \{z \in \Omega : |z - z_2| < 2\rho\}$), $z_1 \in \partial B_{2\rho}(z_1)$, $u(z_1) = 0$, $u|_{B_{2\rho}(z_2)} > 0$. By varying z_2 with z_1 fixed, we see that we can choose $\rho > 0$ small.

Since $u(z_1) = 0 = \min_{\overline{\Omega}} u$ and $z_1 \in \Omega$, we have

$$Du(z_1) = 0. \tag{3}$$

Let $m_\rho = \min[u(z) : z \in \partial B_\rho(z_2)] > 0$. As $\rho \to 0^+$, z_2 converges to z_1 (which we fixed) and so $m_\rho \to 0^+$ and $\frac{m_\rho}{\rho} \to 0^+$ (by L'Hopital's rule).

We introduce the annulus (ring) $R \subseteq \Omega$ defined by

$$R = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}.$$

We set

$$\eta = -\ln \frac{m_\rho}{\rho} + \frac{N-1}{\rho} > 0 \quad (\text{for } \rho > 0 \text{ small}). \tag{4}$$

We consider the function

$$v_\rho(t) = \frac{m_\rho[\rho^{\frac{\eta t}{c^*}} - 1]}{\rho^{\frac{\eta\rho}{c^*}} - 1} \quad \text{for all } 0 \le t \le \rho.$$

Since $m_\rho \eta \to 0^+$ as $\rho \to 0^+$, for $\rho \in (0, 1)$ small we have

$$0 \le v_\rho(t) < 1 \quad \text{and } 0 < v'_\rho(t) < 1 \quad \text{for all } t \in [0, \rho], \tag{5}$$

$$v''_\rho(t) = \frac{\eta}{c^*} v'_\rho(t) \quad \text{for all } t \in [0, \rho]. \tag{6}$$

To simplify things, we may assume that $z_2 = 0$. Let $r = |z|$, $s = 2\rho - r$. For $s \in [0, \rho]$ and $r \in [\rho, 2\rho]$, we define

$$\begin{aligned} & y(r) = v_\rho(2\rho - r) = v_\rho(s), \\ \Rightarrow \ & y'(r) = -v'_\rho(s) \quad \text{and } y''(r) = v''_\rho(s). \end{aligned}$$

We set $y(z) = y(r)$ for all $z \in \Omega$ with $|z| = r$. Then, $y \in C^2(R)$ and using the function $l(\cdot)$ from hypothesis $H_0(i)$, we have

$$\begin{aligned} \operatorname{div} a(Dy) &= l'(v'_\rho(s))v''_\rho(s) - \frac{N-1}{r} l(v'_\rho(s)) \\ &= \frac{\eta}{c^*} l'(v'_\rho(s))v'_\rho(s) - \frac{N-1}{r} l(v'_\rho(s)) \quad (\text{see (6)}) \\ &\ge [\eta - \frac{N-1}{r}] l(v'_\rho(s)) \quad (\text{see hypothesis } H_0(i)) \\ &\ge (-\ln \frac{m_\rho}{\rho}) l(v'_\rho(s)) \quad (\text{see (4) and recall } r \ge \rho) \\ &\ge (-\ln \frac{m_\rho}{\rho}) \frac{c_1}{p-1} v'_\rho(s)^{p-1} \quad (\text{see Lemma 1}) \\ &\ge \widehat{\xi} v'(s)^{\mu-1} \quad \text{for } \rho \in (0,1) \text{ small} \end{aligned}$$

(note that $v'_\rho(0) > 0$ and $v'_\rho(\cdot)$ is increasing, see (6), (5)),

$$\Rightarrow \operatorname{div} a(Dy) + \widehat{\xi} y^{\mu-1} \le 0 \quad \text{in } R. \tag{7}$$

Note that $y \leq u$ on ∂R and by hypothesis

$$-\operatorname{diva}(Du) + \widehat{\xi} u^{\mu-1} \geq 0 \quad \text{in } R. \tag{8}$$

Then, from (7), (8) and Theorem 3.4.1, p. 61, of Pucci–Serrin [15] (the weak comparison principle), we have

$$y(z) \leq u(z) \quad \text{for all } z \in \overline{R}.$$

Then, we have

$$\begin{aligned}
&\lim_{\tau \to 0} \frac{y(z_1 + \tau(z_2 - z_1)) - y(z_1)}{\tau} \leq \lim_{\tau \to 0} \frac{u(z_1 + \tau(z_2 - z_1)) - u(z_1)}{\tau} \\
&\quad (\text{recall } u(z_1) = 0 \text{ and } y \geq 0) \\
&\quad \Rightarrow 0 < v'(0) \leq Du(z_1) = 0, \text{ a contradiction.}
\end{aligned}$$

So, we conclude that $u(z) > 0$ for all $z \in \Omega$. □

4 Positive Solutions

In this section we show that for $\lambda > 0$ big, problem (p_λ) admits a pair of positive solutions. We start by producing one positive solution.

Proposition 4 *If hypotheses H_0, H_1 hold, then for all $\lambda > 0$ big problem (p_λ) has a positive solution $u_\lambda \in C_+ \backslash \{0\}$, $0 < u_\lambda(z)$ for all $z \in \Omega$.*

Proof Let $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)\mathrm{d}z + \int_\Omega F(z, u^+)\mathrm{d}z - \frac{\lambda}{q}\|u^+\|_q^q \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $q < p$, using Corollary 2, we see that $\varphi_\lambda(\cdot)$ is coercive. Also, from the Sobolev embedding theorem, we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf[\varphi_\lambda(u) : u \in W_0^{1,p}(\Omega)]. \tag{9}$$

Let $\Omega_0 \subseteq \Omega$ be open subset such that $\overline{\Omega}_0 \subseteq \Omega$. Consider a function $y \in C_c^1(\Omega)$ such that

$$0 \leq y \leq 1 \text{ and } y|_{\overline{\Omega}_0} \equiv 1$$

(such a function is called "cut-off function" and is obtained by mollification, see, for example, Evans [6], p. 310). Hypotheses $H_1(i)(ii)$ imply that given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) > 0$ such that

$$0 \leq F(z,x) \leq \frac{\varepsilon}{q}x^q + c_6 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{10}$$

Then, we have

$$\begin{aligned}\varphi_\lambda(y) &\leq \int_\Omega G(Dy)\mathrm{d}z - \frac{\lambda-\varepsilon}{q}\|y\|_q^q + c_6|\Omega|_N \quad \text{(see (10))}\\ &\leq c_7 - \frac{\lambda-\varepsilon}{q}\int_{\Omega_0} y^q \mathrm{d}z \quad \text{for some } c_7 > 0 \quad \text{(see Corollary 2)}.\end{aligned}$$

Therefore, we can find $\lambda_* > \varepsilon$ such that

$$\begin{aligned}&c_7 < \frac{\lambda-\varepsilon}{q}\int_{\Omega_0} y^q \mathrm{d}z \quad \text{for all } \lambda > \lambda_*,\\ &\Rightarrow \varphi_\lambda(y) < 0,\\ &\Rightarrow \varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \quad \text{(see (9))},\\ &\Rightarrow u_\lambda \neq 0.\end{aligned}$$

From (9) we have

$$\begin{aligned}&\varphi_\lambda'(u_\lambda) = 0\\ &\Rightarrow \langle A(u_\lambda), h\rangle = \int_\Omega [\lambda u_\lambda^+ - f(z, u_\lambda^+)]h\mathrm{d}z \quad \text{for all } h \in W_0^{1,P}(\Omega).\end{aligned} \tag{11}$$

In (11) we choose $h = -u_\lambda^- \in W_0^{1,P}(\Omega)$, and using Lemma 1 we obtain

$$\frac{c_1}{p-1}\|Du_\lambda^-\|_p^p \leq 0, \quad \Rightarrow \quad u_\lambda \geq 0,\ u_\lambda \neq 0.$$

So, u_λ is a positive solution of (p_λ). Invoking Theorem 7.1, p. 286 of Ladyzhenskaya–Uraltseva [10], we have that $u_\lambda \in L^\infty(\Omega)$. Then, the nonlinear regularity theory of Lieberman [11] implies that $u_\lambda \in C_+\backslash\{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis $H_1(iv)$. We have

$$-\operatorname{diva}(Du_\lambda) + \widehat{\xi}_\rho^\lambda u_\lambda^{\mu-1} \geq 0 \text{ in } \Omega, \quad \Rightarrow \quad 0 < u_\lambda(z) \text{ for all } z \in \Omega \text{ (see Proposition 3)}.$$

□

Using this first solution, we can produce a second one.

Proposition 5 *If hypotheses H_0, H_1 hold and $\lambda > \lambda_*$, then problem (p_λ) has a second positive solution $\widehat{u}_\lambda \in C_+\backslash\{0\}$, $\widehat{u}_\lambda \neq u_\lambda$ and $0 < \widehat{u}_\lambda(z)$ for all $z \in \Omega$.*

Proof Let $k_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$k_\lambda(z, x) = \begin{cases} \lambda(x^+)^{q-1} - f(z, x^+) & \text{if } x \leq u_\lambda(z) \\ \lambda u_\lambda(z)^{q-1} - f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{12}$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \int_\Omega G(Du)\mathrm{d}z - \int_\Omega K_\lambda(z, u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: $K_{\widehat{\varphi}_\lambda} \subseteq [0, u] \cap C_+$.

Let $u \in K_{\widehat{\varphi}_\lambda}$. We have

$$\begin{aligned} &\widehat{\varphi}_\lambda'(u) = 0, \\ &\Rightarrow \langle A(u), h\rangle = \int_\Omega K_\lambda(z, u)h\mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{13}$$

In (13) we use the test function $h = -u^- \in W_0^{1,p}(\Omega)$. Then, from (12) and Lemma 1, we have

$$\frac{c_1}{p-1}\|Du^-\|_p^p \leq 0, \quad \Rightarrow \quad u \geq 0.$$

Next, we test (13) with $h = [u - u_\lambda]^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} \langle A(u), (u - u_\lambda)^+\rangle &= \int_\Omega [\lambda u_\lambda^{q-1} - f(z, u_\lambda)](u - u_\lambda)^+\mathrm{d}z \quad \text{(see (12))} \\ &= \langle A(u_\lambda), (u - u_\lambda)^+\rangle \quad \text{(since } u_\lambda \text{ is a solution of } (p_\lambda)), \\ &\Rightarrow u \leq \widehat{u}_\lambda \quad \text{(from the monotonicity of } A). \end{aligned}$$

We have proved that $u \in [0, u_\lambda]$. Moreover, the nonlinear regularity theory of Lieberman [11] implies that $u \in C_+$. Therefore, we conclude that $K_{\widehat{\varphi}_\lambda} \subseteq [0, u] \cap C_+$. This proves Claim 1.

Claim 2: We can find $\rho_0 > 0$ such that

$$0 < m_0 \leq \widehat{\varphi}_\lambda^+(u) \quad \text{for all } u \in W_0^{1,p}(\Omega),\ \|u\| = \rho_0.$$

Hypotheses $H_1(i), (iii)$ imply that given $\eta > \lambda$, we can find $c_8 = c_8(\eta) > 0$ such that

$$f(z, x) \geq \eta x^{q-1} - c_8 x^{p-1} \quad \text{for a.a.} z \in \Omega,\ \text{all } x \geq 0. \tag{14}$$

It follows that

$$\lambda x^{q-1} - f(z, x) \leq c_8 x^{p-1} - (\eta - \lambda)x^{q-1} \quad \text{for a.a.} z \in \Omega,\ \text{all } x \geq 0.$$

Since $q < p$ and $\eta > \lambda$, we see that we can find $\delta \in (0, 1)$ small such that

$$\lambda x^{q-1} - f(z, x) \leq 0 \quad \text{for a.a.} z \in \Omega, \text{ all } 0 \leq x \leq \delta, \tag{15}$$

$$\Rightarrow \frac{\lambda}{q} x^q - F(z, x) \leq 0 \quad \text{for a.a.} z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{16}$$

Let $u \in W_0^{1,p}(\Omega)$ and introduce the set $\Omega_\delta^u = \{z \in \Omega : u(z) > \delta\}$. Using Corollary 2, we have

$$\widehat{\varphi}_\lambda(u) \geq \frac{c_1}{p(p-1)} \|Du\|_p^p - \int_\Omega K_\lambda(z, u)\mathrm{d}z. \tag{17}$$

We estimate the integral in the right-hand side of (17). We have

$$\int_\Omega K_\lambda(z, u)\mathrm{d}z = \int_{\Omega\setminus\Omega_\delta^u} K_\lambda(z, u)\mathrm{d}z + \int_{\Omega_\delta^u} K_\lambda(z, u)\mathrm{d}z. \tag{18}$$

We examine the first integral in the right-hand side of (18). Then,

$$\int_{\Omega\setminus\Omega_\delta^u} K_\lambda(z, u)\mathrm{d}z = \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u\leq u_\lambda\}} K_\lambda(z, u)\mathrm{d}z + \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} K_\lambda(z, u)\mathrm{d}z \tag{19}$$

Using (12), we see that

$$\int_{(\Omega\setminus\Omega_\delta^u)\cap\{u\leq u_\lambda\}} K_\lambda(z, u)\mathrm{d}z = \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u\leq u_\lambda\}} \left[\frac{\lambda}{q}(u^+)^q - F(z, u^+)\right]\mathrm{d}z \leq 0 \quad (\text{see } (16)). \tag{20}$$

Similarly, using once again (12), we obtain

$$\begin{aligned}\int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} K_\lambda(z, u)\mathrm{d}z &= \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} \left[\lambda u_\lambda^{q-1} - f(z, u_\lambda)\right](u - u_\lambda)\mathrm{d}z \\ &\leq \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} \left[(\lambda - \eta)u_\lambda^{q-1} + c_8 u_\lambda^{p-1}\right](u - u_\lambda)\mathrm{d}z \quad (\text{see } (14)) \\ &\leq c_8 \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} u^p\mathrm{d}z \quad (\text{since } \eta > \lambda).\end{aligned} \tag{21}$$

Returning to (19) and using (20) and (21), we obtain

$$\int_{\Omega\setminus\Omega_\delta^u} K_\lambda(z, u)\mathrm{d}z \leq c_8 \int_{(\Omega\setminus\Omega_\delta^u)\cap\{u_\lambda<u\}} u^p\mathrm{d}z.$$

Since $u_\lambda \in C_+\setminus\{0\}$, from the absolute continuity of the Lebesgue integral, we see that given $\varepsilon > 0$, we can choose $\delta \in (0, 1)$ even smaller if necessary so that

$$\int_{\Omega\setminus\Omega_\delta^u} K_\lambda(z, u)\mathrm{d}z \leq \varepsilon. \tag{22}$$

Next, we estimate the second integral in the right-hand side of (18). Using (12), we have

$$\int_{\Omega_\delta^u} K_\lambda(z,u)\mathrm{d}z = \int_{\Omega_\delta^u\cap\{u\le u_\lambda\}} \left[\frac{\lambda}{q}(u^+)^q - F(z,u^+)\right]\mathrm{d}z + \int_{\Omega_\delta^u\cap\{u_\lambda<u\}} \left[\lambda u_\lambda^{q-1} - f(z,u_\lambda)\right](u-u_\lambda)\mathrm{d}z. \tag{23}$$

Since $F \ge 0$ (see hypothesis $H_1(i)$), we have

$$\int_{\Omega_\delta^u\cap\{u\le u_\lambda\}} \left[\frac{\lambda}{q}(u^+)^q - F(z,u^+)\right]\mathrm{d}z \le \frac{\lambda}{q}\int_{\Omega_\delta^u\cap\{u\le u_\lambda\}} (u^+)^q\mathrm{d}z. \tag{24}$$

Similarly, since $f \ge 0$, we have

$$\int_{\Omega_\delta^u\cap\{u_\lambda<u\}} \left[\lambda u_\lambda^{q-1} - f(z,u)\right](u-u_\lambda)\mathrm{d}z \le \lambda\int_{\Omega_\delta^u\cap\{u_\lambda<u\}} u^q\mathrm{d}z. \tag{25}$$

We return to (23) and use (24) and (25). We obtain

$$\begin{aligned}
\int_{\Omega_\delta^u} K_\lambda(z,u)\mathrm{d}z &\le \lambda\int_{\Omega_\delta^u} |u|^q\mathrm{d}z \\
&\le \lambda c_9\int_{\Omega_\delta^u} |u|^p\mathrm{d}z \\
&\quad \text{for some } c_9>0 \quad (\text{since } \delta>0 \text{ and } q<p) \\
&\le \lambda c_9|\Omega_\delta^u|_N^{1-\frac{p}{r}}\left[\int_{\Omega_\delta^u}|u|^r\mathrm{d}z\right]^{p/r} \\
&\quad \text{with } p<r<p^* \quad (\text{by Hölder's inequality }) \\
&= \lambda c_9|\Omega_\delta^u|_N^{1-\frac{p}{r}}\|u\|_r^p \\
&\le \lambda c_{10}|\Omega_\delta^u|_N^{1-\frac{p}{r}}\|u\|^p \\
&\quad \text{for some } c_{10}>0 \quad (\text{since } W_0^{1,p}(\Omega)\hookrightarrow L^r(\Omega)).
\end{aligned} \tag{26}$$

We return to (18) and use (22) and (26). Then,

$$\int_\Omega K_\lambda(z,u)\mathrm{d}z \le \varepsilon + \lambda c_{10}|\Omega_\delta^u|_N^{1-\frac{p}{r}}\|u\|^p. \tag{27}$$

From (17) and (27), we have

$$\widehat{\varphi}_\lambda(u) \ge \left[\frac{c_1}{p(p-1)} - \lambda c_{10}|\Omega_\delta^u|_N^{1-\frac{p}{r}}\right]\|u\|^p - \varepsilon.$$

If $\|u\| \to 0$, then $|u(z)| \to 0$ for a.a. $z \in \Omega$ and $|\Omega_\delta^u|_N \to 0$ uniformly for $\delta \in (0,1)$ small. So, we can find $\rho_0 \in (0, \|u_\lambda\|)$ small such that if $\|u\| = \rho_0$, then

$$\lambda c_{10} |\Omega_\delta^u|_N^{1-\frac{p}{r}} \leq \frac{c_1}{2p(p-1)}.$$

Hence, for $\|u\| = \rho_0$, we have

$$\widehat{\varphi}_\lambda(u) \geq \frac{c_1}{2p(p-1)} \rho_0^p - \varepsilon.$$

Recall that $\varepsilon > 0$ is arbitrary. So, we choose $\varepsilon \in (0,1)$ small so that

$$\widehat{\varphi}_\lambda(u) \geq m_0 > 0 \quad \text{for all } \|u\| = \rho_0. \tag{28}$$

This proves Claim 2.

Consider the set $\overline{B}_{\rho_0} = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho_0\}$. From the reflexivity of $W_0^{1,p}(\Omega)$ and the Eberlein–Smulian theorem, we have that $\overline{B}_{\rho_0}$ is sequentially weakly compact. Also $\widehat{\varphi}_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\widehat{u}_\lambda \in \overline{B}_{\rho_0}$ such that

$$\widehat{\varphi}_\lambda(\widehat{u}_\lambda) = \inf[\widehat{\varphi}_\lambda(u) : u \in \overline{B}_{\rho_0}].$$

From (12), for $\lambda > 0$ big, we have

$$\begin{aligned} &\widehat{\varphi}_\lambda(\widehat{u}_\lambda) < 0 = \widehat{\varphi}_\lambda(0), \\ &\quad \Rightarrow 0 < \|\widehat{u}_\lambda\| < \rho_0 \quad (\text{see } (28)) \\ &\quad \Rightarrow \widehat{u}_\lambda \neq u_\lambda \quad (\text{recall } \rho_0 < \|u_\lambda\|) \text{ and } \widehat{u}_\lambda \neq 0. \end{aligned}$$

Moreover, from Claim 1 and (12), we have that

$$\widehat{u}_\lambda \in C_+ \backslash \{0\} \quad \text{is a positive solution of } (p_\lambda)$$

Finally, as for u_λ, using Proposition 3, we have $0 < \widehat{u}_\lambda(z)$ for all $z \in \Omega$. ☐

So, summarizing the situation for problem (p_λ), we can state the following multiplicity theorem for problem (p_λ).

Theorem 6 *If hypotheses H_0, H_1 hold, then for all $\lambda > 0$ big problem (p_λ) has at least two positive solutions $u_\lambda, \widehat{u}_\lambda \in C_+ \backslash \{0\}$, $u_\lambda \neq \widehat{u}_\lambda$ and $0 < u_\lambda(z), \widehat{u}_\lambda(z)$ for all $z \in \Omega$.*

Remark 3 If for a.a. $z \in \Omega$, the quotient $x \to \frac{f(z,x)}{x^{p-1}}$ is strictly decreasing on $\overset{\circ}{\mathbb{R}}_+ = (0, \infty)$, then if the reaction is $\lambda x^{q-1} + f(z,x)$, the problem has a unique positive solution. However, if the reaction is $\lambda x^{q-1} - f(z,x)$ as in (p_λ), then we no longer have uniqueness of the positive solution and in fact for $\lambda > 0$ big enough we can guarantee the existence of at least two positive smooth solutions which do not vanish in Ω.

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