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The Backward Problem for Nonlinear Fractional Diffusion Equation with Time-Dependent Order

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Abstract
This paper investigates the nonlinear backward fractional diffusion equations (BFDEs) having a variable order and variable diffusion coefficient. It is well-known that the BFDEs are ill-posed in the sense of Hadamard. Moreover, we investigate the problem taking into account the disturbance both variable diffusion coefficient and variable order. This may lead to some common regularization strategy to failure. Therefore, we propose a regularization method for this kind of problem. Under some appropriate regularity assumptions of the exact solution, a convergence estimate of Holder-type is proved.

Keywords Inverse problems · Regularization · Truncated method · Variable fractional orders

Mathematics Subject Classification 35R30 · 65F22 · 35R11

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1 Introduction

In recent years, the backward diffusion equations (BDEs) have attracted the attention of many researchers due to its application arose in a variety of professional fields. We can refer to two most important areas of application of the BDEs which are image deblurring and hydrologic inversion. In image deblurring, the BDEs can be applied for the deblurring process in image restoration (see e.g., [2,5,7]). In hydrologic inversion, the BDEs are also applied to identify sources of groundwater pollution by reconstructing the contaminant plume history (see e.g., [1]).

More recently, the orders of the fractional diffusion equations which may be a function of time (t), space (x) or both of them were studied (see e.g., [11,12]). These fractional orders help to describe the behavior of some heterogeneous diffusion processes better than the constant fractional orders. To the best of our knowledge, the fractional orders of the BDEs were only investigated as usual to be a constant. Hence, in this paper, we consider the problem with the time-dependent order as a benchmark case.

To state precisely the problem, we set up some notations. Let $T > 0$, and $\alpha, \kappa : [0, T] \to (0, \infty)$ be two continuous positive functions. Let H be a Hilbert space, and let $\mathbb{A} : D(\mathbb{A}) \subset H \to H$ be a positive, self-adjoint operator with compact inverse on H. Using the notations, we consider the problem of finding a function $u : [0, T] \to H$ satisfying

$$\begin{cases} u_t + \kappa(t)\mathbb{A}^{\alpha(t)}u = f(t, u(t)), & 0 < t < T, \\ u(T) = h. \end{cases} \tag{1}$$

The problem (1) is ill-posed; hence, a regularization method for this problem is in order. In the case that $\alpha(t) \equiv \alpha$ and $\kappa(t) \equiv \kappa$ are exactly known constants and $\mathbb{A} = (-\Delta)$, the regularization was studied in some recent papers. In [9], the authors introduced three methods, which are Tikhonov, Landweber iteration and iterative Lavrentiev to regularize the homogeneous problem; after that, they applied the results obtained in image deblurring. In the case of nonhomogeneous [13], the authors used the truncated method to regularize the fractional backward problem in an unbounded domain. In [8], the authors used the idea of the Tikhonov method to regularization the nonlinear backward problem in $\mathbb{R}^d$.

Although there are some works on this topic, the backward problem with perturbed variable diffusion coefficients and variable orders is still under investigation. In the real-world problems, the diffusion coefficients and fractional orders are established from statistical models as the Levy flight or from experiments; hence, the diffusion coefficients and orders are not exactly known. In fact, we can have a known approximations $\alpha_\epsilon, \kappa_\epsilon \in C[0, T]$ such that

$$\|\alpha_\epsilon - \alpha\|_{C[0,T]}, \ \ \|\kappa_\epsilon - \kappa\|_{C[0,T]} \le \epsilon,$$

where ϵ is a noise level. So, the stability of solution of these problems with respect to diffusion coefficients and fractional orders is worth considering. Besides, if the variable diffusion coefficients and variable orders are inexact, then some common regularization schemes could be disabled (see in sect. 4). Even so, papers devoted to the stability are quite rare. To the best of our knowledge, we can only list some papers dealt with this question in [4,10,14–17]. In this paper, we would like to fill a part of this gap.

We also emphasize that solving the problem (1) is rather interesting and challenging due to the disturbance both of the variable diffusion coefficient and variable order. Motivated by the above reasons, in this paper, we study the BFDEs with perturbed both variable diffusion coefficient and variable order, and a locally Lipschitz source.

This paper is structured as follows. In the second section, we introduce symbols and definitions, and we state an essential lemma which will be used to prove the main results of the present paper. The third section is devoted to investigating the formulation and uniqueness of the solution of the problem. The fourth section proposes a truncated method to regularize solution of the backward problem. The final section presents the conclusions of the paper and some problems can be expanded in future studies.

2 Preliminaries

In this section, we introduce some symbols, definitions, and lemmas which are used throughout the rest of the present paper.

We denote the inner product in the Hilbert space H by $\langle .,. \rangle$ and the associated norm by $\|.\|$. From the assumptions, the operator $\mathbb{A}$ has the system (λ_k, ϕ_k) of eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{\phi_k\}$, respectively, with

$$0 < \lambda_1 < \lambda_2 < .. < \lambda_k < .., \ \lim_{k\to\infty} \lambda_k = \infty.$$

and the functions $\{\phi_k\}$ being an orthonormal basis of the space H. In present paper, we follow [18,19] to define the variable fractional order of the operator. For $\alpha \in C[0, T]$, $\alpha(t) > 0$ for all $t \in [0, T]$, we put

$$\mathbb{H}^\alpha := \left\{ w \in C([0,T]; H) : \ w = \sum_{k=1}^{+\infty} w_k(t)\phi_k \text{ and } \sup_{t\in[0,T]} \sum_{k=1}^{+\infty} \lambda_k^{\alpha(t)} |w_k(t)|^2 < +\infty \right\},$$

where $w_k(t) = \langle w(t), \phi_k \rangle$. We define the norm in $\mathbb{H}^\alpha$ by

$$\|w\|_\alpha = \sup_{t\in[0,T]} \left(\sum_{k=1}^{+\infty} \lambda_k^{\alpha(t)} |w_k(t)|^2 \right)^{1/2}.$$

Throughout this paper, for every function $\gamma \in C[0, T]$ arbitrary, we denote

$$\|\gamma\| = \max_{t\in[0,T]} |\gamma(t)|, \ \gamma_* = \min_{0\le t\le T} \gamma(t), \ \gamma^* = \max_{0\le t\le T} \gamma(t).$$

For any positive function α in $C[0, T]$, it is clear that $\mathbb{H}^{\alpha^*} \subset \mathbb{H}^\alpha \subset \mathbb{H}^{\alpha_*} \subset C([0, T]; H)$. We can easily verify that there exist three constants $C_1, \ C_2, \ C_3 > 0$

such that $\sup_{0\le t\le T}\|w(t)\| \le C_1\|w\|_{\alpha_*} \le C_2\|w\|_{\alpha} \le C_3\|w\|_{\alpha^*}$ for any $w \in \mathbb{H}^{\alpha^*}$. Using the above notations, we define $\mathbb{A}^{\alpha(t)}w$ by

$$\mathbb{A}^{\alpha(t)}w = \sum_{k=1}^{+\infty} \lambda_k^{\alpha(t)} w_k(t)\phi_k.$$

Suggested by ideas in [6], we also introduce the Gevrey class of functions of order $p > 0$ and index $\ell > 0$ as

$$\mathbb{G}_\ell^p = \left\{ v \in H : \sum_{k=1}^{\infty} e^{2\ell\lambda_k^p}|v_k|^2 < \infty \right\}.$$

We will justify this class in the next section.

In this paper, we deal with the source $f = f(t, z)$ in the class of functions which are locally Lipschitz with respect to the second variable, i.e., for every $z_1, z_2 \in H$ such that $\|z_1\|, \|z_2\| \le M$, there exists an $L(M) > 0$ such that

$$\|f(t, z_1) - f(t, z_2)\| \le L(M)\|z_1 - z_2\| \tag{2}$$

for any $t \in [0, T]$.

Next, we state an essential lemma which plays an important role in proving main results of this paper.

Lemma 1 *Let $T,\ a_1,\ a_2,\ b_1,\ b_2$ be positive numbers such that $a_2 \le a_1,\ b_2 \le b_1$. Let $\kappa_i, \alpha_i \in C[0, T]$ such that $a_2 \le \kappa_i(t) \le a_1$ and $b_2 \le \alpha_i(t) \le b_1$ for any $t \in [0, T]$, $(i = 1, 2)$. Fix $c > 1$, then for any $\lambda_0 \ge c$ and for every $\lambda \in (0, \lambda_0]$, we have*

$$\begin{aligned}&\left|\exp\left(\int_t^T \kappa_1(\tau)\lambda^{\alpha_1(\tau)}\,\mathrm{d}s\right) - \exp\left(\int_t^T \kappa_2(\tau)\lambda^{\alpha_2(\tau)}\,\mathrm{d}s\right)\right| \\ &\quad \le A\,|\!|\!|\alpha_1 - \alpha_2|\!|\!| + B\,|\!|\!|\kappa_1 - \kappa_2|\!|\!|,\end{aligned}$$

where $A = A_0 \exp\left(a_1(T-t)\lambda_0^{b_1}\right)\lambda_0^{b_1}\ln\lambda_0$, $B = B_0\exp\left(a_1(T-t)\lambda_0^{b_1}\right)\lambda_0^{b_1}$, and A_0, B_0 independent of $\alpha_1 - \alpha_2$, $\kappa_1 - \kappa_2$, λ, λ_0 and t.

Proof First, we have for $0 < a \le b$ and $y, z \in [a, b]$, then

$$|e^y - e^z| \le e^b|y - z|, \tag{3}$$

$$|\lambda^y - \lambda^z| \le \lambda^b \ln\lambda|y - z|, \quad \forall\lambda \ge 1, \tag{4}$$

$$|\lambda^y - \lambda^z| \le \lambda^a|\ln\lambda||y - z|, \quad \forall\, 0 < \lambda < 1. \tag{5}$$

The proofs of the three latter inequalities are elementary and omitted. Now, using these inequalities, we will prove the result of the lemma. Indeed, if $\lambda \ge 1$, using (3)–(4), we obtain by direct computations

$$\begin{aligned}
&\left|\exp\left(\int_t^T \kappa_1(\tau)\lambda^{\alpha_1(\tau)}\,\mathrm{d}\tau\right) - \exp\left(\int_t^T \kappa_2(\tau)\lambda^{\alpha_2(\tau)}\,\mathrm{d}\tau\right)\right| \\
&\quad\le \exp\left(a_1\lambda^{b_1}(T-t)\right)\left|\int_t^T \left(\kappa_1(\tau)\lambda^{\alpha_1(\tau)} - \kappa_2(\tau)\lambda^{\alpha_2(\tau)}\right)\,\mathrm{d}\tau\right| \\
&\quad\le A_1 \,|||\alpha_1-\alpha_2||| + B_1\,|||\kappa_1-\kappa_2|||\,,
\end{aligned} \tag{6}$$

where $A_1 = Ta_1 \exp\left(a_1(T-t)\lambda_0^{b_1}\right)\lambda_0^{b_1}\ln\lambda_0$, $B_1 = T\exp\left(a_1(T-t)\lambda_0^{b_1}\right)\lambda_0^{b_1}$.

If $0<\lambda<1$, using (3) and (5), by the same method in estimating (6), we obtain the desired result. □

3 Formulation and Uniqueness of the Solution of the Problem

In this section, we give a uniqueness results for the problem (1). Firstly, we establish the formula solution of our problem. In fact, for each final data $h \in H$, a function $u \in \mathbb{H}^\alpha$ is called a weak solution of the problem (1) if u satisfies the weak form

$$\begin{cases} \langle u_t, w\rangle + \kappa(t)\langle \mathbb{A}^{\alpha(t)/2}u, \mathbb{A}^{\alpha(t)/2}w\rangle = \langle f(t,u), w\rangle, \ \forall\, w\in\mathbb{H}^\alpha, \\ \langle u(T), w\rangle = \langle h, w\rangle, \ \forall\, w\in\mathbb{H}^\alpha. \end{cases} \tag{7}$$

It is easy to see that the system (7) can be transformed to the integral equation

$$\begin{aligned}
u(t) = &\sum_{k=1}^{\infty}\exp\left(\int_t^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right)h_k\phi_k. \\
&-\sum_{k=1}^{\infty}\int_t^T \exp\left(\int_t^s \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right) f_k(u)(s)\,\mathrm{d}s\phi_k
\end{aligned} \tag{8}$$

We note that if $f\equiv 0$ and $\alpha(t)\equiv\alpha_0>0, \kappa(t)=\kappa_0>0$ the latter formula can be rewritten as

$$u(t) = \sum_{k=1}^{\infty}\exp\left(\kappa_0\lambda_k^{\alpha_0}(T-t)\right)h_k\phi_k.$$

Hence, in this special case, the condition $u(0)\in H$ is equivalent to $h\in\mathbb{G}_\ell^r$ with $p=\alpha_0,\ \ell=\kappa_0 T/2$. This is the justification of the Gevrey class defined in Sect. 2.

Theorem 1 *Assume that the condition* (2) *holds. Put*

$$\mathbb{K}^\alpha = \{w\in C([0,T],H) : \|w\|_{\mathbb{K}^\alpha} < \infty\}$$

where

$$\|w\|^2_{\mathbb{K}^\alpha} = \sup_{0\le t\le T} \sum_{k=1}^{\infty} \exp\left(2\int_t^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right) |\langle w(t), \phi_k\rangle|^2 < \infty.$$

If $u_1, u_2 \in \mathbb{K}^\alpha$ *satisfy* (7) *the* $u_1(t) = u_2(t)$ *for all* $t \in [0, T]$.

Proof Denote $w = u_2 - u_1$ and $M = \max\{\|u_1\|_\alpha, \|u_2\|_\alpha\}$. We obtain in view of (8)

$$w(t) = \sum_{k=1}^{\infty}\left(-\int_t^T \exp\left(\int_t^s \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right)(f_k(w+u_1)(s) - f_k(u_1)(s))\,\mathrm{d}s\right)\phi_k.$$

Denoting $w_k(t) = \langle w(., t), \phi_k\rangle$, we have

$$w(t) = \sum_{k=1}^{N}\left(-\int_t^T \exp\left(\int_t^s \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right)(f_k(w+u_1)(s) - f_k(u_1)(s))\,\mathrm{d}s\right)\phi_k + \sum_{k=N+1}^{\infty} w_k(t)\phi_k.$$

Hence

$$\|w(t)\|^2 \le \sum_{k=1}^{N} T\int_t^T \exp\left(2\int_t^s \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|f_k(w+u_1)(s) - f_k(u_1)(s)\|^2\,\mathrm{d}s + \sum_{k=N+1}^{\infty} |w_k(t)|^2.$$

We note that

$$\sum_{k=1}^{N}\|f_k(w+u_1)(s) - f_k(u_1)(s)\|^2 \le \|f(w+u_1)(s) - f(u_1)(s)\|^2 \le L^2(M)\|w(s)\|^2.$$

Hence if we put

$$\psi(t) = \exp\left(2\int_0^t \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|w(t)\|^2,$$
$$R_N(t) = \exp\left(2\int_0^t \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\sum_{k=N+1}^{\infty}|w_k(t)|^2,$$

then

$$\psi(t) \le TL^2(M)\int_t^T \psi(s)\,\mathrm{d}s + R_N(t).$$

Using the Gronwall inequality, one gets

$$0 \leq \psi(t) \leq R_N(t) + TL^2(M) \int_t^T R_N(s) e^{TL^2(M)(s-t)} ds.$$

We note that $R_N(s) \leq \|w\|^2_{\mathbb{K}^\alpha}$. Hence the latter inequality yields

$$\|w(t)\| \leq \exp\left(-2\int_0^t \kappa(\tau)\lambda_N^{\alpha(\tau)}\, \mathrm{d}\tau\right) \|w\|_{\mathbb{K}^\alpha} e^{T^2L^2(M)}.$$

Letting $N \to \infty$, we obtain $w(t) = 0$ for all $0 < t \leq T$. By the continuity of w, we also have $w(0) = 0$. Therefore, we have $u_1(t) = u_2(t)$ for all $t \in [0, T]$. □

4 Regularization of the Problem

In this section, we investigate a truncated method to regularization the problem (1) in the case of inexact variable diffusion coefficient and variable order.

Firstly, we analyze the instability of the final value problem. For convenience, let us consider the homogeneous problem

$$\begin{cases} u_t + \kappa(t)\mathbb{A}^{\alpha(t)}u = 0, \quad 0 < t < T, \\ u(T) = h, \end{cases} \tag{9}$$

where $\mathbb{A} = \Delta$ and $u = u(x,t)$ with $x \in \mathbb{R}^d$. By Weyl's law, we known that $\lambda_k = O\left(k^{2/d}\right)$ and $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^d} < +\infty$. From (8), the problem (9) can be transformed to the following integral equation

$$u(t) = \sum_{k=1}^{\infty} \exp\left(\int_t^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\, \mathrm{d}\tau\right) h_k.$$

This gives

$$u(0) = \sum_{k=1}^{\infty} \exp\left(\int_0^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\, \mathrm{d}\tau\right) h_k.$$

We choose

$$h = \sum_{k=1}^{\infty} \frac{\phi_k}{\lambda_k^{d/2} \exp\left(\int_0^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\, \mathrm{d}\tau\right)}.$$

We have

$$\|u(0)\|^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^d} < +\infty.$$

Now we consider the truncated method as follows

$$R_N(h)(t) = \sum_{k=1}^{N} \exp\left(\int_t^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right) h_k.$$

Direct computations yield

$$\|u(0) - R_N(h)(0)\|^2 = \sum_{k=N+1}^{\infty} \frac{1}{\lambda_k^d} \to 0 \ \text{ as } \ N \to \infty.$$

This shows that the truncated regularization is convergent. Now we perturbed the order fractional $\alpha(t)$ by $\alpha_N(t) = \alpha(t) + r_N$ with $r_N = \frac{\ln\left(1+\frac{d\ln\lambda_N}{\kappa_* T\lambda_N^{\alpha_*}}\right)}{\ln\lambda_N}$ for some N large enough such that $\lambda_N > 1$. It is easy to see that $r_N \to 0$ as $N \to \infty$ and $\exp\left(\kappa_* T(\lambda_N^{r_N} - 1)\lambda_N^{\alpha_*}\right) = \lambda_N^d$. In this case, the truncated method become

$$R_{N,inexact}(h)(t) = \sum_{k=1}^{N} \exp\left(\int_t^T \kappa(\tau)\lambda_k^{\alpha_N(\tau)}\,\mathrm{d}\tau\right) h_k\phi_k,$$

where $h_k = \frac{1}{\lambda_k^{d/2}\exp\left(\int_0^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right)}$. By direct computation, we have

$$\begin{aligned}
&\|R_{N,inexact}(h)(0) - u(0)\|^2 \\
&\quad\geq \sum_{k=1}^{N}\left(\exp\left(\int_0^T \kappa(\tau)\lambda_k^{\alpha_N(\tau)}\,\mathrm{d}\tau\right) - \exp\left(\int_0^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\,\mathrm{d}\tau\right)\right)^2 h_k^2 \\
&\quad\geq \left(\exp\left(\int_0^T \kappa(\tau)\lambda_N^{\alpha_N(\tau)}\,\mathrm{d}\tau\right) - \exp\left(\int_0^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\right)^2 h_N^2 \\
&\quad= \frac{1}{\lambda_N^d}\left(\exp\left(\left(\lambda_N^{r_N} - 1\right)\int_0^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right) - 1\right)^2 \\
&\quad\geq \frac{1}{\lambda_N^d}\left(\exp\left(\kappa_* T\left(\lambda_N^{r_N} - 1\right)\lambda_N^{\alpha_*}\right) - 1\right)^2 \\
&\quad\geq \frac{\left(\lambda_N^d - 1\right)^2}{\lambda_N^d} \to +\infty,\ \ (N \to \infty).
\end{aligned}$$

The latter inequality shows that the solution of the final value problem is instability with respect to the fractional-order and leads to the common regularization truncated strategy to failure. In other words, we need a significant improvement for the current regularization strategy when the diffusion coefficient and fractional are inexact.

Now, we will construct a regularization for our problem. For $N \in \mathbb{N}$, $M > 0$, we approximate the problem (1) by the problem $\mathcal{P}(M, N, h, \alpha, \kappa)$ as below

$$\begin{cases} u_{N,t} + \kappa(t)\mathbb{A}^{\alpha(t)} u_N = \sum_{k=1}^{N} f_{M,k}(u_N)(t)\phi_k \\ u_N(T) = \sum_{k=1}^{N} h_k \phi_k, \end{cases} \tag{10}$$

where

$$f_M(t,u) = \begin{cases} f(t,u), & \text{if } \|u\| \le M, \\ f\left(t, M\frac{u}{\|u\|}\right), & \text{otherwise} \end{cases}$$

and $f_{M,k}(u_N)(t) = \langle f_M(t, u_N), \phi_k \rangle$. We can transform the problem (10) to the integral equation

$$\begin{aligned} u_N(t) = & \sum_{k=1}^{N} \exp\left(\int_t^T \kappa(\tau)\lambda_k^{\alpha(\tau)}\, \mathrm{d}\tau \right) h_k \phi_k. \\ & - \sum_{k=1}^{N} \left(\int_t^T \exp\left(\int_t^s \kappa(\tau)\lambda_k^{\alpha(\tau)}\, \mathrm{d}\tau \right) f_{M,k}(u_N)(s)\, \mathrm{d}s \right) \phi_k. \end{aligned} \tag{11}$$

From now on, we suppose further that N is larger enough such that $\lambda_N > 1$. Firstly, we show that, the approximate problem is well-posed.

Theorem 2 *Let f be a locally Lipschitz function as in (2), and such that $f(t, 0) \in L^2(0, T; H)$. Then*

(i) The problem $\mathcal{P}(M, N, h, \alpha, \kappa)$ has a unique solution in $\mathbb{H}^\alpha$, and we have the following estimate

$$\int_0^T \|f_N(t, u_N(t))\|^2\, \mathrm{d}t \le P_0 \exp\left(2\kappa T \lambda_N^{\alpha^*}\right), \tag{12}$$

where P_0 is a positive number which is independent of λ_N, t.

(ii) If v_N is the solution of the problem $\mathcal{P}(M, N, \widetilde{h}, \widetilde{\alpha}, \widetilde{\kappa})$ for every $\widetilde{h} \in H$ and $\widetilde{\kappa}$, $\widetilde{\alpha} \in C[0, T]$, we have

$$\|u_N(t) - v_N(t)\| \le Q_1\|h - \widetilde{h}\| + Q_2 \,|||\alpha - \widetilde{\alpha}||| + Q_3 \,|||\kappa - \widetilde{\kappa}|||, \tag{13}$$

where Q_1, Q_2, Q_3 are positive numbers independent of $h - \widetilde{h}$, $\alpha - \widetilde{\alpha}$ and $\kappa - \widetilde{\kappa}$.

Proof (i) We can easily verify that the function f_M satisfies the condition

$$\|f_M(t, z_1) - f_M(t, z_2)\| \le 2L(M)\|z_1 - z_2\| \tag{14}$$

for any $t \in [0, T]$ and $z_1, z_2 \in H$. On the other hand, the dimension of the approximate problem is finite; hence, we can use the Cauchy–Lipschitz–Picard

theorem (see [3], chapter 7, page 184) to deduce that the problem $\mathcal{P}(M, N, h, \alpha, \kappa)$ has a unique solution u_N in $\mathbb{H}^\alpha$.
First, we verify that

$$\|u_N(t)\|^2 \leq P_1 \exp\left(2\kappa T \lambda_N^{\alpha^*}\right),$$

where P_1 is a positive number which is independent of λ_N, t. Indeed, applying the Cauchy–Schwarz inequality for Eq. (11) yields

$$\begin{aligned}\|u_N(t)\|^2 &\leq 2\exp\left(2\int_t^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|h\|^2 \\ &\quad +2T\int_t^T \exp\left(2\int_t^s \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|f_M(s,u_N)\|^2\,\mathrm{d}s \\ &\leq 2\exp\left(2\int_t^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\left(\|h\|^2+2\sigma_T\right) \\ &\quad +16TL^2(M)\int_t^T \exp\left(2\int_t^s \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|u_N(s)\|^2\,\mathrm{d}s,\end{aligned}$$

where $\sigma_T = \int_0^T \|f(s,0)\|^2\,\mathrm{d}s$. The latter inequality deduces

$$\begin{aligned}&\exp\left(2\int_0^t \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|u_N(t)\|^2 \\ &\quad\leq 2\exp\left(2\int_0^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\left(\|h\|^2+2\sigma_T\right) \\ &\quad\quad+16TL^2(M)\int_t^T \exp\left(2\int_0^s \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau\right)\|u_N(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Applying the Gronwall inequality and direct computation, we obtain

$$\|u_N(t)\|^2 \leq P_1 \exp\left(2\kappa^* T \lambda_N^{\alpha^*}\right) \tag{15}$$

due to $\int_0^T \kappa(\tau)\lambda_N^{\alpha(\tau)}\,\mathrm{d}\tau \leq \kappa^* T\lambda_N^{\alpha^*}$, where $P_1 = 2\left(\|h\|^2+2\sigma_T\right)\exp\left(16T^2L^2(M)\right)$.
Now we prove (12). It is easy to see that (14) implies

$$\|f_N(t,u_N(t))\|^2 \leq 8L^2(M)\|u_N(t)\|^2 + 2\|f(t,0)\|^2.$$

Combining the latter inequality with (15), we have

$$\int_0^T \|f_N(t,u_N(t))\|^2\,\mathrm{d}t \leq P_1\exp\left(2\kappa^*T\lambda_N^{\alpha^*}\right)+2\sigma_T \leq P_2\exp\left(2\kappa^*T\lambda_N^{\alpha^*}\right), \tag{16}$$

where $P_2 = P_1 + 2\sigma_T$ and P_1 defined in (15). This completes the proof of Part (i).

(ii) Firstly, we denote by v_N and w_N the solutions of the problem $\mathcal{P}(M, N, h, \widetilde{\alpha}, \widetilde{\kappa})$ and $\mathcal{P}(M, N, \widetilde{h}, \widetilde{\alpha}, \widetilde{\kappa})$ respectively. Put $\eta = \max\{\alpha^*, \widetilde{\alpha}^*\}$, $\mu = \max\{\kappa^*, \widetilde{\kappa}^*\}$. By the same way as in Part (i), we have

$$\|v_N(t) - w_N(t)\| \leq Q_1 \|h - \widetilde{h}\|, \tag{17}$$

where $Q_1 = \widetilde{Q_1} \exp\left(\mu T \lambda_N^\eta\right)$ and $\widetilde{Q_1} = \sqrt{2}\exp(4T^2 L^2(M))$.
Secondly, we will prove that

$$\|u_N(t) - v_N(t)\| \leq Q_2 \,\|\!|\alpha - \widetilde{\alpha}|\!\|\, + Q_3 \,\|\!|\kappa - \widetilde{\kappa}|\!\|\,, \tag{18}$$

where $Q_2 = Q_0 \exp\left(2\mu T \lambda_N^\eta\right) \lambda_N^\eta \ln \lambda_N$, $Q_3 = Q_0 \exp\left(2\mu T \lambda_N^\eta\right) \lambda_N^\eta$ and Q_0 independent of $\alpha - \widetilde{\alpha}$, $\kappa - \widetilde{\kappa}$, λ_N and t.
Indeed, using Lemma 1 and by direct computation, we have

$$\begin{aligned}
&\|u_N(t) - v_N(t)\|^2 \\
&\quad \leq (C_1 \,\|\!|\alpha - \widetilde{\alpha}|\!\| + C_2 \,\|\!|\kappa - \widetilde{\kappa}|\!\|)^2 \left(\|h\|^2 + \int_t^T \|f_N(s, u_N(s))\|^2 \,\mathrm{d}s\right) \\
&\qquad + 12 T L^2(M) \int_t^T \exp\left(2\mu(s-t)\lambda_N^\eta\right) \|u_N(s) - v_N(s)\|^2 \,\mathrm{d}s,
\end{aligned} \tag{19}$$

where $C_1 = C_{01} \exp\left(\mu(T-t)\lambda_N^\eta\right) \lambda_N^\eta \ln \lambda_N$, $C_2 = C_{02} \exp\left(\mu(T-t)\lambda_N^\eta\right) \lambda_N^\eta$ and C_{01}, C_{02} independent of $\alpha - \widetilde{\alpha}$, $\kappa - \widetilde{\kappa}$, λ_N and t. On the other hand, using (16), we have

$$\|h\|^2 + \int_t^T \|f_N(s, u_N(s))\|^2 \,\mathrm{d}s \leq \|h\|^2 + P_2 \exp\left(2\mu T \lambda_N^\eta\right) \leq P_3 \exp\left(2\mu T \lambda_N^\eta\right),$$

where P_3 independent of λ_N, t. Combining the latter inequality with (19), we have

$$\begin{aligned}
&\exp\left(2\mu\lambda_N^\eta t\right) \|u_N(t) - v_N(t)\|^2 \\
&\quad \leq D_0 (D_1 \,\|\!|\alpha - \widetilde{\alpha}|\!\| + D_2 \,\|\!|\kappa - \widetilde{\kappa}|\!\|)^2 \exp\left(2\mu T \lambda_N^\eta\right) \\
&\qquad + 12 T L^2(M) \int_t^T \exp\left(2\mu\lambda_N^\eta s\right) \|u_N(s) - v_N(s)\|^2 \,\mathrm{d}s.
\end{aligned}$$

where $D_1 = \exp\left(\mu T \lambda_N^\eta\right) \lambda_N^\eta \ln \lambda_N$, $D_2 = \exp\left(\mu T \lambda_N^\eta\right) \lambda_N^\eta$ and D_0 independent of $\alpha - \widetilde{\alpha}$, $\kappa - \widetilde{\kappa}$, λ_N and t. Applying the Gronwall inequality yields

$$\|u_N(t) - v_N(t)\| \leq Q_2 \,\|\!|\alpha - \widetilde{\alpha}|\!\|\, + Q_3 \,\|\!|\kappa - \widetilde{\kappa}|\!\|\,, \tag{20}$$

where $Q_2 = Q_0 \exp\left(2\mu T \lambda_N^\eta\right) \lambda_N^\eta \ln \lambda_N$, $Q_3 = Q_0 \exp\left(2\mu T \lambda_N^\eta\right) \lambda_N^\eta$ and $Q_0 = \sqrt{D_0} \exp(6T^2 L^2(M))$. The proof of the inequality (18) is completed.

Lastly, by using the triangle inequality, combining (17) and (18), we obtain the result of Part (ii). This completes the proof of the Theorem 2. □

In the next theorem, we give a regularization result for the problem (1) under some regularity assumptions of its exact solution.

Theorem 3 *Suppose that the assumptions in Theorem 2 hold. Suppose further that the problem (1) has a solution* $u \in C\left([0,T];\mathbb{G}_\theta^{\alpha^*}\right)$ *for* $\theta > \kappa^* T$, *and* $\sup_{t\in[0,T]}\|u(t)\| \le M$ *for some positive number* M. *Let* $\epsilon \in (0,1)$, $h_\epsilon \in H$ *and* $\kappa_\epsilon, \alpha_\epsilon \in C[0,T]$ *be the measurement data such that*

$$\|h - h_\epsilon\| + \|\!|\alpha - \alpha_\epsilon\|\!| + \|\!|\kappa - \kappa_\epsilon\|\!| \le \epsilon, \tag{21}$$

where $h = u(T)$. *Then there exist* M_0 *independent of* ϵ *such that*

$$\|u(t) - u_\epsilon(t)\| \le M_0 \epsilon^{\rho_0},$$

where u_ϵ *is the solution of problem* $\mathcal{P}(M, N, h_\epsilon, \alpha_\epsilon, \kappa_\epsilon)$, *and* $\rho_0 = (\theta - \kappa^* T) r_0/(1 + (\theta - \kappa^* T) r_0 + 2\kappa^* T)$ *with* $r_0 = e^{-1/e}$.

Proof For any $N \in \mathbb{N}$, assign the solution of the problem $\mathcal{P}(M, N, h, \alpha, \kappa)$ by u_N. Due to $\sup_{t\in[0,T}\|u(t)\| \le M$, this implies that $f_M(t, u(t)) = f(t, u(t))$, then by straightforward, we have

$$\|u(t) - u_N(t)\|^2 \le T L^2(M) \int_t^T \exp\left(2\kappa^* \lambda_N^{\alpha^*}(s - t)\right) \|u(s) - u_N(s)\|^2 \, ds + \sum_{k=N+1}^{\infty} |u_k(t)|^2. \tag{22}$$

Now, we estimate the second term in the right-hand side of the inequality (22). Since $u \in C\left([0,T];\mathbb{G}_\theta^{\alpha^*}\right)$, we have

$$\sum_{k=N+1}^{\infty} |u_k(t)|^2 \le \exp\left(-2\theta\lambda_N^{\alpha^*}\right) \sum_{k=N+1}^{\infty} \exp\left(2\theta\lambda_k^{\alpha^*}\right)|u_k(t)|^2 = \Lambda \exp\left(-2\theta\lambda_N^{\alpha^*}\right).$$

Substituting the latter inequality into (22) and by direct computation gives

$$\begin{aligned} &\exp\left(2\kappa^*\lambda_N^{\alpha^*} t\right) \|u(t) - u_N(t)\|^2 \\ &\quad \le 4T L^2(M) \int_t^T \exp\left(2\kappa^* \lambda_N^{\alpha^*} s\right) \|u(s) - u_N(s)\|^2 \, ds + \Lambda \exp\left(-2r\lambda_N^{\alpha^*}\right), \end{aligned}$$

where $r = \theta - \kappa^* T$. Applying the Gronwall inequality yields

$$\|u(t) - u_N(t)\| \le \Lambda_0 \exp\left(-r\lambda_N^{\alpha^*}\right), \tag{23}$$

where $\Lambda_0 = \sqrt{\Lambda}\exp(2T^2 L^2(M))$.

Let us denote the solution of the problem $\mathcal{P}(M, N, h_\epsilon, \alpha_\epsilon, \kappa_\epsilon)$ by u_ϵ and $\nu = \max\{\alpha^*, \alpha_\epsilon^*\}$, $\upsilon = \max\{\kappa^*, \kappa_\epsilon^*\}$. Thank to (13), we have

$$\|u_N(t) - u_\epsilon(t)\| \le Q_1 \|h - h_\epsilon\| + Q_2 \|\|\alpha - \alpha_\epsilon\|\| + Q_3 \|\|\kappa - \kappa_\epsilon\|\|, \qquad (24)$$

where $Q_1 = \widetilde{Q_1} \exp\left(\upsilon T \lambda_N^\eta\right)$, $Q_2 = Q_0 \exp\left(2\upsilon T \lambda_N^\eta\right) \lambda_N^\eta \ln \lambda_N$, and $Q_3 = Q_0 \exp\left(2\upsilon T \lambda_N^\eta\right)\lambda_N^\eta$. Herein $\widetilde{Q_1}$ defined in (17) and Q_0 defined in (20).

Since $\exp\left(\upsilon T \lambda_N^\nu\right) \le \exp\left(2\upsilon T \lambda_N^\nu\right) \lambda_N^\nu \le \exp\left(2\upsilon T \lambda_N^\nu\right) \lambda_N^\nu \ln \lambda_N$ for N larger enough; hence, we can combine (21) with (24) to obtain

$$\|u_N(t) - u_\epsilon(t)\| \le Q_4 \epsilon \exp\left(2\upsilon T \lambda_N^\nu\right) \lambda_N^\nu \ln \lambda_N, \qquad (25)$$

where $Q_4 = \max\left\{\widetilde{Q_1}, Q_0\right\}$. Combining (23) with (25) and applying the triangle inequality, we obtain

$$\|u(t) - u_\epsilon(t)\| \le Q_4 \left(\epsilon \exp\left(2\upsilon T \lambda_N^\nu\right) \lambda_N^\nu \ln \lambda_N + \exp\left(-r \lambda_N^{\alpha^*}\right)\right), \qquad (26)$$

where $Q_4 = \max\{\Lambda_0, Q_3\}$. Since $\ln \lambda_N \le \lambda_N$ for any $N \ge 1$ and $\lambda^k e^{-\lambda} \le k^k e^{-k} \le k^k$ for any $k > 0$, we deduce

$$\lambda_N^\nu \ln \lambda_N \le \lambda_N^{\nu+1} = e^{\lambda_N^\nu} e^{-\lambda_N^\nu} \lambda_N^{\nu(1+1/\nu)} \le e^{\lambda_N^\nu} (1 + 1/\alpha^*)^{(1+1/\alpha^*)}.$$

From the latter inequality and (26), we obtain

$$\|u(t) - u_\epsilon(t)\| \le Q_5 \left(\epsilon \exp\left((2\upsilon T + 1)\lambda_N^\nu\right) + \exp\left(-r \lambda_N^{\alpha^*}\right)\right), \qquad (27)$$

where $Q_5 = Q_4(1 + 1/\alpha^*)^{(1+1/\alpha^*)}$. Now we can choose the parameter λ_N (or N) such that the right-hand side of (27) convergence to zero as ϵ to 0. For example, we put $\rho = (1 + r r_0 + 2\kappa^* T)^{-1}$, and choose $\lambda_N^\nu = \rho \ln(1/\epsilon)$. We have

$$\begin{aligned} \epsilon \exp\left((2\upsilon T + 1)\lambda_N^\nu\right) &= \epsilon \epsilon^{-(2\upsilon T+1)\rho} \\ &\le \epsilon^{1-\rho(1+2\kappa^* T)} \epsilon^{-2T(\upsilon - \kappa^*)\rho} \le Q_6 \epsilon^{1-\rho(1+2\kappa^* T)}, \end{aligned} \qquad (28)$$

where $Q_6 = \epsilon^{-2T\rho\epsilon} < +\infty$. On the other hand, for any $\epsilon \le e^{-1/\rho}$, we have $\lambda_N^{-\epsilon} = (\rho \ln(1/\epsilon))^{-\epsilon} \ge (\ln(1/\epsilon))^{-\epsilon} \ge \epsilon^\epsilon \ge e^{-1/e} := r_0$. This lead to

$$\exp\left(-r \lambda_N^{\alpha^*}\right) \le \exp\left(-r \lambda_N^\nu \lambda_N^{-\epsilon}\right) \le \epsilon^{r r_0 \rho}. \qquad (29)$$

Since $r r_0 \rho = 1 - \rho(1 + 2\kappa^* T)$, we can substitute (28) and (29) into (27) to obtain the desired result. □

5 Conclusions

In this work, we presented a truncated method to regularized solution of the nonlinear backward fractional diffusion problem with inexact variable diffusion coefficient and variable order. Under appropriate regularity assumptions of the exact solution, we obtained the order of convergence is $O(\epsilon^{\rho_0})$. It would be interesting to extend this work for problems with the diffusion coefficient and fractional order dependent on both x and t.

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