Bull. Malays. Math. Sci. Soc. (2021) 44:2447–2464
https://doi.org/10.1007/s40840-021-01077-z

BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Approximations to Weighted Sums of Random Variables

Amit N. Kumar[1]

Received: 9 July 2020 / Revised: 15 October 2020 / Accepted: 5 January 2021 /
Published online: 16 January 2021

Abstract
In this paper, we obtain error bound for pseudo-binomial and negative binomial approximations to weighted sums of locally dependent random variables, using Stein's method. We also discuss approximation results for weighted sums of independent random variables. We demonstrate our results through some applications in finance and runs in statistics.

Keywords Locally dependent random variables · Pseudo-binomial and negative binomial approximations · Error bounds · Stein's method

Mathematics Subject Classification Primary: 62E17 · 62E20; Secondary: 60F05 · 60E05

1 Introduction and Preliminaries

Sums of random variables (rvs) have always a special attraction as it raises in relevant theoretical challenges. Moreover, several linear statistics can be represented as weighted sums of rvs. Also, it is difficult to find the exact distribution of weighted sums of rvs, especially, if the underlying rvs are non-identical. So, it is of interest to study the behavior of such distributions. Many researchers studied the limiting behavior of weighted sums of rvs such as Chow and Lai [7], Olvera-Cravioto [18], and Zhengyan [28], among many others. But if weights are natural numbers, then it is also difficult to get asymptotic limits. Therefore, the study of the proximity of such distributions with a suitable distribution is of interest when the summation is taken over a finite set.

In this paper, we consider weighted sums of $\mathbb{Z}_+$-valued rvs, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, the set of nonnegative integers, and propose its approximation with pseudo-binomial and negative binomial distributions by matching the first two moments. Also, we assume weights are natural numbers. Let $X_1 \sim \mathrm{PB}(N, p)$, the

✉ Amit N. Kumar
amit.kumar2703@gmail.com

[1] Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

pseudo-binomial distribution (see Čekanavičius and Roos [24], p. 370), and $X_2 \sim \mathrm{NB}(r, \bar{p})$, the negative binomial distribution, then their probability mass functions are given by

$$\mathbb{P}(X_1 = k) = \frac{1}{\delta}\binom{N}{k} p^k q^{N-k}, \quad k = 0, 1, \ldots, \lfloor N \rfloor$$

and

$$\mathbb{P}(X_2 = k) = \binom{r+k-1}{k} \bar{p}^r \bar{q}^k, \quad k = 0, 1, \ldots,$$

respectively, where $N > 1$, $r > 0$, $0 < q = 1 - p < 1$, $0 < \bar{q} = 1 - \bar{p} < 1$, $\delta = \sum_{k=0}^{\lfloor N \rfloor} \binom{N}{k} p^k q^{N-k}$, $\binom{N}{k} = N(N-1)\cdots(N-k+1)/k!$, and $\lfloor N \rfloor$ is the greatest integer function of N. The study of asymptotic behavior for weighted sums of rvs is discussed widely in the literature under certain conditions on weights; for example, the sum of squares of weights is finite (see Chow and Lai [7]), weights are normalized (see Etemadi [9]) and geometrically weighted (see Bhati and Rattihalli [6]), among many others. However, we consider weights are natural numbers which do not satisfy these types of conditions and obtain error bounds for pseudo-binomial and negative binomial approximations. This study of proximity is useful to identify the behavior of such distributions over a finite set. We use the total variation distance metric and Stein's method to derive our approximation results.

Next, let $\mathcal{G} = \{g : \mathbb{Z}_+ \to \mathbb{R} |\, g \text{ is bounded}\}$ and $\mathcal{G}_Y = \{g \in \mathcal{G} |\, g(0) = 0 \text{ and } g(y) = 0, \text{ for } y \notin \mathrm{S}(Y)\}$, for a $\mathbb{Z}_+$-valued rv Y, where $\mathbf{S}(Y)$ is the support of the rv Y. We discuss briefly Stein's method (Stein [20]) which can be carried out mainly in the three steps. First, compute a Stein operator $\mathcal{A}_Y$ which satisfies $\mathbb{E}[\mathcal{A}_Y g(Y)] = 0$, for $g \in \mathcal{G}_Y$. Second, find the solution of Stein equation

$$\mathcal{A}_Y g(k) = f(k) - \mathbb{E} f(Y), \ \ f \in \mathcal{G} \text{ and } g \in \mathcal{G}_Y. \tag{1.1}$$

Finally, use a rv Z in place of k in (1.1) and take expectation and supremum which leads to the total variation distance between Y and Z as follows:

$$d_{TV}(Y, Z) := \sup_{f \in \mathcal{H}} |\mathbb{E} f(Y) - \mathbb{E} f(Z)| = \sup_{f \in \mathcal{H}} |\mathbb{E}\mathcal{A}_Y g(Z)|, \tag{1.2}$$

where $\mathcal{H} = \{I(A) | A \subseteq \mathbb{Z}_+\}$ and $I(A)$ is the indicator function of the set A.
Next, consider a rv X and its Stein operator of the form

$$\mathcal{A}_X g(k) = (\alpha + \beta k) g(k+1) - k g(k), \quad k \in \mathbb{Z}_+, \ g \in \mathcal{G}_X, \tag{1.3}$$

which represent pseudo-binomial Stein operator if $\alpha = Np/q$ and $\beta = -p/q$ and negative binomial Stein operator if $\alpha = r\bar{q}$ and $\beta = \bar{q}$, respectively. For details, see (5) and (6) of Upadhye et al. [23]. Also, the upper bound for the solution of (1.1) (say g_f) is given by

$$\|\Delta g_f\| \leq \begin{cases} 1/\lfloor N \rfloor p, & \text{if } X \sim \text{PB}(N, p); \\ 1/r\bar{q}, & \text{if } X \sim \text{NB}(r, \bar{p}), \end{cases} \tag{1.4}$$

where $\Delta g_f(k) = g_f(k+1) - g_f(k)$ and $\|\Delta g_f\| = \sup_k |\Delta g_f(k)|$. See (2.8), (2.10), and (2.11) of Kumar *et al.* [14] and (57) of Čekanavičius and Roos [24] for more details. Observe that

$$\frac{\alpha}{1-\beta} = \begin{cases} Np, & \text{if } X \sim \text{PB}(N, p); \\ r\bar{q}/\bar{p}, & \text{if } X \sim \text{NB}(r, \bar{p}) \end{cases} \quad \text{and} \quad \frac{\alpha}{(1-\beta)^2} = \begin{cases} Npq, & \text{if } X \sim \text{PB}(N, p); \\ r\bar{q}/\bar{p}^2, & \text{if } X \sim \text{NB}(r, \bar{p}) \end{cases}$$

are mean and variance of pseudo-binomial and negative binomial distributions, respectively. For more details, we refer the reader to Brown and Xia [5], Eichelsbacher and Reinert [8], Kumar et al. [14], Ley et al. [16], Upadhye and Barman [22], Upadhye et al. [23], and references therein.

This paper is organized as follows: In Sect. 2, we present our main results and discuss some relevant remarks and applications. In Sect. 3, we give the proofs of our main results.

2 Main Results

Let $J \subset \mathbb{N} = \{1, 2, \ldots\}$ be finite and $\{\eta_i, i \in J\}$ be a collection of $\mathbb{Z}_+$-valued random variables. Also, for each i, let $w_i \in \mathbb{N}$, $\mathbb{E}(\omega_i \eta_i)^3 < \infty$, and $i \in A_i \subseteq B_i \subset J$ be such that η_i is independent of $\eta_{A_i^c}$ and η_{A_i} is independent of $\eta_{B_i^c}$, where η_A is the collection of random variables $\{\eta_i, i \in A\}$ and A^c denotes the complement of the set A. See Section 3 of Röllin [19] for a similar type of locally dependent structure. In addition, if $A_i = B_i = \{i\}$, then our locally dependent structure reduced to the independent collection of random variables. Now, let $w_i = 1$ for at least one $i \in J$ and define

$$W := \sum_{i \in J} \omega_i \eta_i, \tag{2.1}$$

the weighted sum of locally dependent random variables. For a set $A \subset J$, we let $\eta_A^* = \sum_{i \in A} \omega_i \eta_i$. For any random variables Z, we define $\mathcal{D}(Z) := 2d_{TV}(\mathcal{L}(Z), \mathcal{L}(Z+1))$. Throughout this section, let X be a random variable having Stein operator (1.3) and

$$\alpha = \frac{(\mathbb{E}W)^2}{\text{Var}(W)} \quad \text{and} \quad \beta = \frac{\text{Var}(W) - \mathbb{E}W}{\text{Var}(W)} \tag{2.2}$$

so that $\mathbb{E}X = \mathbb{E}W$ and $\text{Var}(X) = \text{Var}(W)$.

2.1 Locally Dependent Random Variables

In this subsection, we consider $\{i\} \subset A_i \subset B_i$ and discuss the approximation result for the weighted sum of locally dependent random variables.

Theorem 2.1 *Let W be the weighted sum of locally random variables as defined in (2.1) and X be a random variable having Stein operator (1.3) satisfying (2.2). Then,*

$$\begin{aligned} d_{TV}(\mathcal{L}(W),\mathcal{L}(X)) \leq & \|\Delta g\| \left\{ \frac{(1-\beta)}{2} \left[\sum_{i\in J} \omega_i \mathbb{E}\eta_i \mathbb{E}[\eta^*_{A_i}(2\eta^*_{B_i} - \eta^*_{A_i} - 1)\mathcal{D}(W|\eta_{A_i}, \eta_{B_i})] \right.\right. \\ & \left.\left. + \sum_{i\in J} \omega_i \mathbb{E}[\eta_i(\eta^*_{A_i} - 1)(2\eta^*_{B_i} - \eta^*_{A_i} - 2)\mathcal{D}(W|\eta_i, \eta_{A_i}, \eta_{B_i})] \right] \right. \\ & + \sum_{i\in J} \omega_i \big| (1-\beta)[\mathbb{E}(\eta_i)\mathbb{E}(\eta^*_{A_i}) \\ & - \mathbb{E}(\eta_i(\eta^*_{A_i} - 1))] + \beta \mathbb{E}(\eta_i) \big| \mathbb{E}[\eta^*_{B_i} \mathcal{D}(W|\eta_{B_i})] \\ & \left. + |\beta| \sum_{i\in J} \omega_i \mathbb{E}[\eta_i(\eta^*_{B_i} - 1)\mathcal{D}(W|\eta_{B_i})] \right\}, \end{aligned} \tag{2.3}$$

where the upper bound of $\|\Delta g\|$ is given in (1.4).

Remark 2.1 (i) Observe that W can be represented as a conditional sum of independent random variables. Therefore, Subsections 5.3 and 5.4 of Röllin [19] are useful to find the upper bound of $\mathcal{D}(W|\cdot)$.

(ii) The choice of parameters in (2.2) is valid if $\mathbb{E}W > \mathrm{Var}(W)$ and $\mathbb{E}W < \mathrm{Var}(W)$ for pseudo-binomial and negative binomial approximations, respectively.

Next, we discuss some applications of Theorem 2.1.

Example 2.1 *((1,1)-runs)* Let $J = \{1, 2, \ldots, n\}$ and $\{\zeta_i, i \in J\}$ be a sequence of independent Bernoulli trials with success probability $p_i = \mathbb{P}(\zeta_i = 1) = 1 - \mathbb{P}(\zeta_i = 0)$. Also, let $A_i = \{j : |j - i| \leq 1\} \cap J$, $B_i = \{j : |j - i| \leq 2\} \cap J$, $\bar{\zeta}_i = (1 - \zeta_{i-1})\zeta_i$, and $W_n = \sum_{i=2}^{n} \bar{\zeta}_i$. Then, the distribution of W_n is known as the distribution of (1, 1)-runs and it adopted our locally dependent structure with $\omega_i = 1$. For more details, see Huang and Tsai [12], Upadhye et al. [23], Vellaisamy [25], and reference therein.

Next, it can be easily verified that

$$\begin{aligned} & \sum_{i=2}^{n} (1 - p_{i-1})p_i = \mathbb{E}(W_n) > \mathrm{Var}(W_n) \\ & = \sum_{i=2}^{n} (1 - p_{i-1})p_i - \sum_{i=2}^{n} \sum_{j\in A_i} (1 - p_{i-1})(1 - p_{j-1})p_i p_j. \end{aligned}$$

Therefore, the pseudo-binomial approximation to W_n is suitable in the view of the valid choice of parameters. Now, let $\mathcal{D}(W_i^\star) = \mathcal{D}(W_n|\bar{\zeta}_{B_i})$ and $\bar{\zeta}_e = \{\bar{\zeta}_{2k}, 1 \leq k \leq \lfloor n/2 \rfloor\}$ then $\mathcal{L}(W_i^\star|\bar{\zeta}_e = k)$ can be represented as the sum of independent random variables (say ζ_i^k, $i \in \{1, 2, \ldots, n_k\} =: \mathcal{F}_k$ and $k \in \{0, 1\}^{\lfloor n/2 \rfloor}$), and therefore, from (5.11) if Röllin [19], we have

$$\mathcal{D}(W_i^\star) \leq \mathbb{E}\{\mathbb{E}[W_i^\star|\bar{\zeta}_e]\} \leq \mathbb{E}\left[\frac{2}{V_{\bar{\zeta}_e}^{1/2}} \right],$$

where $V_{\bar{\zeta}_e=k} = \sum_{j\in\mathcal{F}_k} \min\left\{1/2, 1-\mathcal{D}(\zeta_j^k)\right\}$. Let $1/2 \geq \mathbb{P}(\zeta_i^k = 1) = 1 - \mathbb{P}(\zeta_i^k = 0) = \sum_{k_1,k_2\in\{0,1\}} \mathbb{P}(\bar{\zeta}_i = 1|\bar{\zeta}_{i-1} = k_1, \bar{\zeta}_{i+1} = k_2) = \mathbb{P}(\bar{\zeta}_i = 1|\bar{\zeta}_{i-1} = 0, \bar{\zeta}_{i+1} = 0) = (1-p_{i-1})p_i$. Therefore,

$$\begin{aligned} 1-\mathcal{D}(\zeta_i^k) &= 1 - \frac{1}{2}\sum_{m=0}^{1} |\mathbb{P}(\zeta_i^k = m-1) - \mathbb{P}(\zeta_i^k = m)| \\ &= 1 - \frac{1}{2}\{2\mathbb{P}(\zeta_i^k = 0) - \mathbb{P}(\zeta_i^k = 1)\} = \frac{3}{2}(1-p_{i-1})p_i. \end{aligned}$$

Hence, $V_{\bar{\zeta}_e=k} = \frac{1}{2}\sum_{j\in\mathcal{F}_k} \min\{1, 3(1-p_{i-1})p_i\}$ and, from Theorem 2.1 with the pseudo-binomial setting, we have

$$\begin{aligned} d_{TV}(\mathcal{L}(W_n), \mathrm{PB}(\hat{n},\hat{p})) \leq & \frac{1}{\lfloor\hat{n}\rfloor\hat{p}\hat{q}} \left\{ \sum_{i=2}^{n} \left[\mathbb{E}\bar{\zeta}_i \mathbb{E}[\bar{\zeta}_{A_i}^* (2\bar{\zeta}_{B_i}^* - \bar{\zeta}_{A_i}^* - 1)] + \mathbb{E}[\bar{\zeta}_i(\bar{\zeta}_{A_i}^* - 1)(2\bar{\zeta}_{B_i}^* - \bar{\zeta}_{A_i}^* - 2)]\right] \right. \\ & + 2\sum_{i=2}^{n} \left|\mathbb{E}(\bar{\zeta}_i)\mathbb{E}(\bar{\zeta}_{A_i^*}) - \mathbb{E}(\bar{\zeta}_i(\bar{\zeta}_{A_i}^* - 1)) - \hat{p}\mathbb{E}(\bar{\zeta}_i)\right| \mathbb{E}[\bar{\zeta}_{B_i}^*] \\ & \left. + 2\hat{p}\sum_{i=2}^{n} \mathbb{E}[\bar{\zeta}_i(\bar{\zeta}_{B_i}^* - 1)] \right\} \left(\frac{1}{2}\sum_{j\in\mathcal{F}} \min\{1, 3(1-p_{i-1})p_i\}\right)^{-1/2}, \end{aligned}$$

where $\hat{n} = \left(\sum_{i=2}^{n}(1-p_{i-1})p_i\right)^2 / \sum_{i=2}^{n}\sum_{j\in A_i}(1-p_{i-1})(1-p_{j-1})p_i p_j$, $\hat{p} = \sum_{i=2}^{n}\sum_{j\in A_i}(1-p_{i-1})(1-p_{j-1})p_i p_j / \sum_{i=2}^{n}(1-p_{i-1})p_i$, and $\mathcal{F} = \min_k \mathcal{F}_k$. Note that the above bound is of $O(n^{-1/2})$ which is an improvement over (77) of Upadhye et al. [23], Theorem 2.1 of Vellaisamy [25], which are of $O(1)$, and Theorem 2.1 of Godbole [11], which is of order $O(n)$.

Example 2.2 (*Collateralized Debt Obligation (CDO)*) A CDO is a type of asset-backed security that transferred pool of assets into a product and sold to investors. These assets divided into a set of repayment which is called tranches. The tranches have different payment priorities and interest rates. The basic tranches used in CDO are senior, mezzanine, and equity. Investors can invest in their interested tranches. For more details, see Neammanee and Yonghint [17], Yonghint et al. [27], and reference therein.

In Yonghint et al. [27], it is demonstrated that the locally dependent CDO occurs in the borrower's related assets that arise from several groups. If the element of groups have weights in terms of economy. Then, the weighted locally dependent CDO is also useful in applications.

We consider the CDO similar to discussed by Yonghint et at. [27]. Let the CDO tranche pricing is based on $\bar{n}$ assets and the recovery rate of ith assets is $R_i > 0$. The percentage cumulative loss in CDO up to time T is

$$L(T) = \frac{1}{\bar{n}}\sum_{i=1}^{\bar{n}}(1-R_i)\omega_i I_i,$$

where $I_i = I(\tau_i \le T)$, and τ_i is the default time of the ith asset. Assume the recovery rate is constant, say R; then, the CDO pricing problem is reduced to calculate

$$\mathbb{E}[(L(T) - z^*)^+] = \frac{1-R}{\bar{n}}\mathbb{E}[(\overline{W}_{\bar{n}} - z)^+], \tag{2.4}$$

where $z^* = (1-R)z/\bar{n} > 0$ is the attachment or the detachment point of the tranche, $\overline{W}_{\bar{n}} = \sum_{i=1}^{\bar{n}} \omega_i I_i$, and $(a)^+ = \max(a, 0)$. Note that, from (2.4), it is sufficient to deal with $\mathbb{E}[(\overline{W}_{\bar{n}} - z)^+]$. For additional details, see Yonghint et al. [27] and reference therein.

We are interested to approximate $\mathbb{E}[(\overline{W}_{\bar{n}} - z)^+]$ by $\mathbb{E}[(\mathrm{PB}(N, p) - z)^+]$. First, let us modify the Stein equation (1.1) as

$$(k - z)^+ - \mathbb{E}[(\mathrm{PB}(N, p) - z)^+] = \mathcal{A}g(k). \tag{2.5}$$

Here, $f : \mathbb{Z}_+ \to \mathbb{R}$ such that $f(k) = (k - z)^+$. Using the rv $\overline{W}_{\bar{n}}$ in place of k and taking expectation, we get

$$\mathbb{E}[(\overline{W}_{\bar{n}} - z)^+] - \mathbb{E}[(\mathrm{PB}(N, p) - z)^+] = \mathbb{E}[\mathcal{A}g(\overline{W}_{\bar{n}})].$$

Therefore, it is enough to deal with the right-hand side, that is, $\mathbb{E}[\mathcal{A}g(\overline{W}_{\bar{n}})]$.

Next, we move to find the upper bound for $\|\Delta g\|$. Following the steps similar to Lemma 1 of Neammanee and Yonghint [17], for $z \ge 0$, we have

$$\begin{aligned}\mathbb{E}[(\mathrm{PB}(N, p) - z)^+] &= \frac{1}{\delta}\sum_{m=1}^{\lfloor N \rfloor}(m - z)^+ \binom{N}{m} p^m q^{N-m} \\ &\le \frac{1}{\delta}\sum_{m=1}^{\lfloor N \rfloor} m \binom{N}{m} p^m q^{N-m} = Np. \end{aligned} \tag{2.6}$$

It can be easily verified that (2.5) has a solution

$$\begin{aligned} g(k) = &-\sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k} \\ &\times \big((j-z)^+ - \mathbb{E}[(\mathrm{PB}(N, p) - z)^+]\big), \end{aligned}$$

for $k \ge 1$. For details, see (2.6) of Eichelsbacher and Reinert [8]. Now, following the steps similar to Lemma 2 Neammanee and Yonghint [17], for $k \ge 1$, we get

$$\begin{aligned} 0 < \sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k}(j-z)^+ &\le 1 + \sum_{j=1}^{\lfloor N-k \rfloor}\binom{N-k}{j}\left(\frac{p}{q}\right)^j \\ &\le q^{\lceil k-N \rceil} \le q^{\lceil 1-N \rceil}, \end{aligned} \tag{2.7}$$

where $\lceil x \rceil$ denote the least integer more than or equal to x. Also, for $k \le Np$, we have

$$\begin{aligned}\sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k} &\le \sum_{j=k}^{\lfloor N \rfloor} \frac{(N-k)\cdots(N-j+1)}{(j-k+1)!}\left(\frac{p}{q}\right)^{j-k} \\ &\le \frac{1}{N-k+1}\sum_{j=0}^{\lfloor N-k \rfloor}\binom{N-k+1}{j+1}\left(\frac{p}{q}\right)^{j} \\ &\le \frac{q^{\lceil k-N \rceil}-q}{(N-k)p} \le \frac{q^{-\lceil N \rceil}}{Np} \qquad (2.8)\end{aligned}$$

and, for $k \ge np$,

$$\begin{aligned}\sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k} &= \frac{1}{k}\left(1+\sum_{j=k+1}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)]k!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k}\right) \\ &\le \frac{1}{k}\left(1+\sum_{j=1}^{\lfloor N-k \rfloor}\binom{N-k}{j}\left(\frac{p}{q}\right)^{j}\right) \le \frac{q^{-\lceil N \rceil}}{Np}. \qquad (2.9)\end{aligned}$$

Combining (2.8) and (2.9), for $k \ge 1$, we have

$$0 < \sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k} \le \frac{q^{-\lceil N \rceil}}{Np}. \qquad (2.10)$$

Therefore, from (2.6) and (2.10), we have

$$\sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k} \mathbb{E}[(\mathrm{PB}(N,p)-z)^{+}] \le q^{-\lceil N \rceil}. \qquad (2.11)$$

Next, observe that

$$\Delta g(k) = g(k+1) - g(k) = C_k + D_k,$$

where

$$\begin{aligned}C_k = &\sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!}\left(\frac{p}{q}\right)^{j-k}(j-z)^{+} \\ &- \sum_{j=k+1}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)]k!}{[N(N-1)\cdots(N-k)]j!}\left(\frac{p}{q}\right)^{j-k-1}(j-z)^{+}\end{aligned}$$

and

$$D_k = \sum_{j=k+1}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)]k!}{[N(N-1)\cdots(N-k)]j!} \left(\frac{p}{q}\right)^{j-k-1} \mathbb{E}[(\mathrm{PB}(N,p)-z)^+] - \sum_{j=k}^{\lfloor N \rfloor} \frac{[N(N-1)\cdots(N-j+1)](k-1)!}{[N(N-1)\cdots(N-k+1)]j!} \left(\frac{p}{q}\right)^{j-k} \mathbb{E}[(\mathrm{PB}(N,p)-z)^+].$$

Using (2.7) and (2.11), we have

$$|\Delta g| \le |C_k| + |D_k| \le q^{-\lceil N \rceil}(1+q).$$

Hence, from (3.5), we have

$$\begin{aligned} \left|\mathbb{E}[\mathscr{A}g(\overline{W}_{\bar{n}})]\right| \le & \frac{(1+q)}{q^{\lceil N \rceil+1}} \left\{ \frac{1}{2} \left[\sum_{i=1}^{\bar{n}} \omega_i \mathbb{E} I_i \mathbb{E}[I^*_{A_i}(2I^*_{B_i} - I^*_{A_i} - 1)\mathcal{D}(\overline{W}_{\bar{n}} | I_{A_i}, I_{B_i})] \right.\right. \\ & \left.\left. + \sum_{i=1}^{\bar{n}} \omega_i \mathbb{E}[I_i(I^*_{A_i} - 1)(2I^*_{B_i} - I^*_{A_i} - 2)\mathcal{D}(\overline{W}_{\bar{n}} | I_i, I_{A_i}, I_{B_i})] \right] \right. \\ & + \sum_{i=1}^{\bar{n}} \omega_i \left| \mathbb{E}(I_i)\mathbb{E}(I_{A_i^*}) - \mathbb{E}(I_i(I^*_{A_i} - 1)) - p\mathbb{E}(I_i) \right| \mathbb{E}[I^*_{B_i} \mathcal{D}(W | I_{B_i})] \\ & \left. + p \sum_{i=1}^{\bar{n}} \omega_i \mathbb{E}[I_i(I^*_{B_i} - 1)\mathcal{D}(\overline{W}_{\bar{n}} | I_{B_i})] \right\}, \end{aligned} \tag{2.12}$$

where $q = 1 - p = \mathrm{Var}(\overline{W}_{\bar{n}})/\mathbb{E}\overline{W}_{\bar{n}}$, $N = (\mathbb{E}\overline{W}_{\bar{n}})^2/(\mathbb{E}\overline{W}_{\bar{n}} - \mathrm{Var}(\overline{W}_{\bar{n}}))$, $I^*_A = \sum_{i \in A} \omega_i I_i$, I_A is the collection of random variables $\{I_i : i \in A\}$, for $A \subset \{1, 2, \ldots, \bar{n}\}$, and $\mathcal{D}(\overline{W}_{\bar{n}} | \cdot)$ can be computed subject to the exact structure of dependency. For example, if the dependency structure is the same as discussed in Example 2.1 and $\omega_i = 1$, $1 \le i \le n$, then

$$\begin{aligned} \left|\mathbb{E}[\mathscr{A}g(\overline{W}_{\bar{n}})]\right| \le & \frac{(1+q)}{q^{\lceil N \rceil+1}} \left\{ \sum_{i=1}^{\bar{n}} \left[\mathbb{E} I_i \mathbb{E}[I^*_{A_i}(2I^*_{B_i} - I^*_{A_i} - 1)] + \mathbb{E}[I_i(I^*_{A_i} - 1)(2I^*_{B_i} - I^*_{A_i} - 2)] \right] \right. \\ & + 2 \sum_{i=1}^{\bar{n}} \left| \mathbb{E}(I_i)\mathbb{E}(I_{A_i^*}) - \mathbb{E}(I_i(I^*_{A_i} - 1)) - p\mathbb{E}(I_i) \right| \mathbb{E}[I^*_{B_i}] \\ & \left. + 2p \sum_{i=1}^{\bar{n}} \mathbb{E}[I_i(I^*_{B_i} - 1)] \right\} \left(\frac{1}{2} \sum_{j \in \mathcal{F}} \min\{1, 3(1 - \mathbb{E}(I_{i-1}))\mathbb{E} I_i\} \right)^{-1/2}, \end{aligned}$$

which is an improvement over the bound given in Theorem 2(1) of Yonghint et al. [27].

2.2 Independent Random Variables

In this subsection, we consider $B_i = A_i = \{i\}$ in the earlier discussed setup and obtain approximation results for $W^* = \sum_{i\in J} \omega_i \eta_i$, the weighted sum of independent random variables. To simplify the presentation, let us define $p_i(k) := \mathbb{P}(\eta_i = k)$ and $\gamma := 2\max_{i\in J} d_{TV}(W_i, W_i + 1)$ where $W_i = W^* - \omega_i\eta_i$.

Theorem 2.2 *Let W^* be the weighted sum of independent random variables and X be a random variable having Stein operator (1.3) satisfying (2.2). Then,*

$$d_{TV}(\mathcal{L}(W^*), \mathcal{L}(X)) \le \gamma \|\Delta g\| \sum_{i\in J} \omega_i \left(\sum_{k=1}^{\infty} h_i(k) + d_i \right), \tag{2.13}$$

where $d_i = \mathbb{E}(\omega_i\eta_i)|(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] + \beta\mathbb{E}(\eta_i)|$ and

$$h_i(k) = \begin{cases} \frac{k(k-1)}{2}|(1-\beta)\mathbb{E}\eta_i\, p_i(k) + \beta k p_i(k) - (k+1)p_i(k+1)|, & \text{if } \omega_i = 1; \\ \sum_{\ell=1}^{\omega_i k-1} |(1-\beta)\ell\mathbb{E}\eta_i + \beta\ell k - (\ell-1)k|p_i(k), & \text{if } \omega_i \ge 2. \end{cases}$$

Remark 2.1 Note that Remark 4.1 of Vellaisamy et al. [26] can be applied to our results, and hence for $\gamma_j = \min\{1/2, 1 - d_{TV}(\omega_j\eta_j, \omega_j\eta_j + 1)\}$, $\gamma^* = \max_{j\in J}\gamma_j$, we have

$$\gamma \le \sqrt{\frac{2}{\pi}} \left(\frac{1}{4} + \sum_{j\in J} \gamma_j - \gamma^* \right)^{-1/2}.$$

Therefore, if η_J is of $O(n)$, then the bound (2.13) is of $O(n^{-1/2})$. Observe that the above bound for γ is useful when $w_j = 1$ for many values of j.

Corollary 2.1 *Assume the conditions of Theorem 2.2 hold with $X \sim PB(N, p)$ and $\mathbb{E}W^* > \mathrm{Var}(W^*)$. Then,*

$$d_{TV}(\mathcal{L}(W^*), PB(N, p)) \le \frac{\gamma}{\lfloor N \rfloor pq} \sum_{i\in J} \omega_i \left(\sum_{k=1}^{\infty} h_i(k) + d_i \right),$$

where $d_i = \mathbb{E}(\omega_i\eta_i)|\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1)) - p\mathbb{E}(\eta_i)|$ and

$$h_i(k) = \begin{cases} \frac{k(k-1)}{2}|\mathbb{E}\eta_i\, p_i(k) - pkp_i(k) - q(k+1)p_i(k+1)|, & \text{if } \omega_i = 1; \\ \sum_{\ell=1}^{\omega_i k-1} |\ell\mathbb{E}\eta_i - p\ell k - q(\ell-1)k|p_i(k), & \text{if } \omega_i \ge 2. \end{cases}$$

Remark 2.2 (i) If $J = \{1, 2, \ldots, n\}$, $\omega_i = 1$, and $\eta_i \sim \mathrm{Ber}(p)$, for all $1 \le i \le n$. Then, $h_i(k) = d_i = 0$, and hence, $d_{TV}(\mathcal{L}(W^*), \mathrm{PB}(N, p)) = 0$, as expected.

(ii) Let $\omega_i = 1$ and $\eta_i \sim \text{Ber}(p_i)$, for $i \in J = \{1, 2, \ldots, n\}$. Then, from Corollary 2.1, we have

$$d_{TV}(\mathcal{L}(W^*), PB(N, p)) \leq \sqrt{\frac{2}{\pi}} \left(\frac{1}{4} + \sum_{i=1}^{n} \bar{\gamma}_i - \bar{\gamma}^* \right)^{-1/2} \frac{1}{\lfloor N \rfloor pq} \sum_{i=1}^{n} p_i^2 |p - p_i|, \tag{2.14}$$

where $\bar{\gamma}_j = \frac{1}{2} \min\{1, 1 + p_i - |1 - 2p_i|\}$ and $\bar{\gamma}^* = \max_{1 \leq i \leq n} \bar{\gamma}_i$. The bound given in (2.14) is of $O(n^{-1/2})$ and is an order improvement over Theorem 1 of Barbour and Hall [2], Theorem 9.E of Barbour et al. [3], and the bounds discussed by Kerstan [13] and Le Cam [15].

(iii) Consider the setup of CDO discussed in Example 2.2 under independent Bernoulli trials and unit weights, that is, $\overline{W}_n^* = \sum_{i=1}^{n} I_i$ where I_i, for $1 \leq i \leq n$, are independent Bernoulli trials. Using $A_i = B_i = \{i\}$ in (3.1), routine calculations lead to

$$\left| \mathbb{E}[\mathscr{A}g(\overline{W}_n^*)] \right| \leq \frac{(1+q)}{q^{\lceil N \rceil + 1}} \sum_{i=1}^{n} p_i |p - p_i|, \tag{2.15}$$

where $p = \frac{1}{N} \sum_{i=1}^{n} p_i$. Note that if $p_i = p$ in (2.15), for $1 \leq i \leq n$, then $\left| \mathbb{E}[\mathscr{A}g(\overline{W}_n^*)] \right| = 0$, as expected. Also, from (3.8) with $\omega_i = 1$, for $1 \leq i \leq n$, we have

$$\left| \mathbb{E}[\mathscr{A}g(\overline{W}_n^*)] \right| \leq \sqrt{\frac{2}{\pi}} \left(\frac{1}{4} + \sum_{i=1}^{n} \tilde{\gamma}_i - \tilde{\gamma}^* \right)^{-1/2} \frac{1+q}{q^{\lceil N \rceil + 1}} \sum_{i=1}^{n} p_i^2 |p - p_i|, \tag{2.16}$$

where $\tilde{\gamma}_i = \frac{1}{2} \min\{1, 1 + p_i - |1 - 2p_i|\}$ and $\tilde{\gamma}^* = \max_{1 \leq i \leq n} \tilde{\gamma}_i$, $q = 1 - p = \sum_{i=1}^{n} p_i q_i / \sum_{i=1}^{n} p_i$, and $N = (\sum_{i=1}^{n} p_i)^2 / \sum_{i=1}^{n} p_i^2$. For Poisson approximation, the existing bound given in (4) of Neammanee and Yonghint [17] is

$$\left| \mathbb{E}[\mathscr{A}g(\overline{W}_n^*)] \right| \leq \left(2 \exp\left(\sum_{i=1}^{n} p_i \right) - 1 \right) \sum_{i=1}^{n} p_i^2. \tag{2.17}$$

Note that, for small values of p_i, the bound given in (2.15) is better than the bound given in (2.17). For instance, let $n = 50$ and p_i, $1 \leq i \leq 50$, be defined as follows:

i	p_i	i	p_i	i	p_i
0-10	0.05	21-30	0.15	41-50	0.25
11-20	0.10	31-40	0.20		

Next, the following table gives a comparison between (2.15), (2.16), and (2.17).

n	From (2.15)	From (2.16)	From (2.17)
10	7.14×10^{-17}	6.00×10^{-18}	0.0574
20	0.3711	0.01630	0.9954
30	4.9800	0.36415	13.7099
40	111.8440	11.7054	221.8700
50	3311.4600	897.600	4970.7400

Observe that our bounds are better than existing bounds for various values of N and p_i.

Corollary 2.2 *Assume the conditions of Theorem 2.2 hold with* $X \sim NB(r, \bar{p})$ *and* $\mathbb{E}W^* < \mathrm{Var}(W^*)$. *Then,*

$$d_{TV}(\mathcal{L}(W^*), NB(r, \bar{p})) \leq \frac{\gamma}{r\bar{q}} \sum_{i \in J} \omega_i \left(\sum_{k=1}^{\infty} h_i(k) + d_i \right), \tag{2.18}$$

where $d_i = \mathbb{E}(\omega_i \eta_i) | \bar{p}[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i - 1))] + \bar{q}\mathbb{E}(\eta_i)|$ *and*

$$h_i(k) = \begin{cases} \frac{k(k-1)}{2} | \bar{p}\mathbb{E}\eta_i \, p_i(k) + \bar{q}kp_i(k) - (k+1)p_i(k+1)|, & \text{if } \omega_i = 1; \\ \sum_{\ell=1}^{\omega_i k - 1} |\bar{p}\ell\mathbb{E}\eta_i + \bar{q}\ell k - (\ell - 1)k| p_i(k), & \text{if } \omega_i \geq 2. \end{cases}$$

Remark 2.3 (i) If $J = \{1, 2, \ldots, n\}$ and $\omega_i = 1$, for all i, then, from Corollary 2.2, we have

$$\begin{aligned} &d_{TV}(\mathcal{L}(W^*), \mathrm{NB}(r, \bar{p})) \\ &\leq \frac{\gamma}{r\bar{q}} \sum_{i=1}^{n} \left(\sum_{k=2}^{\infty} \frac{k(k-1)}{2} | \bar{p}\mathbb{E}\eta_i \, p_i(k) + \bar{q}kp_i(k) - (k+1)p_i(k+1)| + d_i \right), \end{aligned} \tag{2.19}$$

which is an improvement over the bound given in (17) of Vellaisamy et al. [26]. Also, if $\eta_i \sim \mathrm{Geo}(\bar{p})$, the geometric distribution, for $1 \leq i \leq n$, then $d_{TV}(\mathcal{L}(W^*), \mathrm{NB}(n, \bar{p})) = 0$, as expected.

(ii) If $\eta_i \sim \mathrm{NB}(n_i, p_i)$, $1 \leq i \leq n$, then the bound given in (2.19) leads to

$$d_{TV}(\mathcal{L}(W^*), \mathrm{NB}(r, \bar{p})) \leq \frac{\gamma^*}{r\bar{q}} \sum_{i=1}^{n} \left(\bar{p}(n_i q_i + 1) \left| \frac{q_i}{p_i} - \frac{\bar{q}}{\bar{p}} \right| \frac{n_i(n_i+1)q_i^2}{p_i^2} + d_i \right),$$

where $\gamma^* \leq \sqrt{\frac{2}{\pi}} (\frac{1}{4} + \sum_{i=1}^{n} \mathbb{P}(\eta_i = \lfloor (n_i - 1)q_i / p_i \rfloor) - \max_{1 \leq i \leq n} \mathbb{P}(\eta_i = \lfloor (n_i - 1)q_i / p_i \rfloor))^{-1/2}$ and $q_i = 1 - p_i$, which is an order improvement over the bound given in Theorem 3.1 of Teerapabolarn [21].

Example 2.3 (*Compound Poisson Distribution*) Let $w_i = i$, $\eta_i \sim \text{Po}(\lambda_i)$, the Poisson distribution, for $i \in J = \{1, 2, \ldots, n\}$, and $S_n = \sum_{i=1}^{n} i\eta_i$. The distribution of S_∞ is known as compound Poisson distribution. The mean and variance of S_n satisfy $\sum_{i=1}^{n} i\lambda_i = \mathbb{E}S_n < \text{Var}(S_n) = \sum_{i=1}^{n} i^2\lambda_i$. Therefore, the negative binomial approximation is suitable in the view of the applicability of parameters. Hence, from (2.18), we have

$$d_{TV}(\mathcal{L}(S_n), \text{NB}(r, \bar{p})) \leq \sqrt{\frac{2}{\pi}}\frac{2}{r\bar{q}}\sum_{i=1}^{n} E_i,$$

where $\bar{q} = \sum_{i=1}^{n} i(i-1)\lambda_i \big/ \sum_{i=1}^{n} i^2\lambda_i$ and $r = \left(\sum_{i=1}^{n} i\lambda_i\right)^2 \big/ \sum_{i=1}^{n} i(i-1)\lambda_i$, and

$$E_i \leq \begin{cases} \bar{q}\lambda_i^2(\lambda_i + 1) + \bar{q}(i\lambda_i)^2, & \text{if } i = 1; \\ \frac{1}{2}\mathbb{E}[i\eta_i(i\eta_i - 1)(\bar{p}(i\eta_i + i\lambda_i) + 2)] + \bar{q}(i\lambda_i)^2, & \text{if } i \geq 2. \end{cases}$$

Note that if $i\lambda_i$ is decreasing in i, then the bound is useful in practice. For similar conditions, see Barbour et al. [1] and Gan and Xia [10].

3 Proofs

In this section, we prove the main results presented in Sect. 2.

Proof of Theorem 2.1 Consider the Stein operator (1.3), and taking expectation with respect to W, we have

$$\begin{aligned}\mathbb{E}[\mathscr{A}_X g(W)] &= \alpha\mathbb{E}[g(W+1)] + \beta\mathbb{E}[Wg(W+1)] - \mathbb{E}[Wg(W)] \\ &= (1-\beta)\left[\frac{\alpha}{1-\beta}\mathbb{E}[g(W+1)] - \mathbb{E}[Wg(W)]\right] + \beta\mathbb{E}[W\Delta g(W)].\end{aligned}$$

Using (2.2), the above expression leads to

$$\mathbb{E}[\mathscr{A}_X g(W)] = (1-\beta)\left[\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}[g(W+1)] - \sum_{i\in J}\omega_i\mathbb{E}[\eta_i g(W)]\right] + \beta\mathbb{E}[W\Delta g(W)]. \tag{3.1}$$

Let $W_i = W - \sum_{j\in A_i}\omega_i\eta_i = W - \eta^*_{A_i}$ so that η_i and W_i are independent random variables. Therefore,

$$\begin{aligned}\mathbb{E}[\mathscr{A}_X g(W)] &= (1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}[g(W+1) - g(W_i+1)] \\ &\quad - (1-\beta)\sum_{i\in J}\omega_i\mathbb{E}[\eta_i(g(W) - g(W_i+1))] + \beta\mathbb{E}[W\Delta g(W)]\end{aligned}$$

$$= (1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}\Bigg[\sum_{j=1}^{\eta^*_{A_i}}\Delta g(W_i+j)\Bigg]$$
$$-(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{j=1}^{\eta^*_{A_i}-1}\Delta g(W_i+j)\Bigg]+\beta\sum_{i\in J}\omega_i\mathbb{E}[\eta_i\Delta g(W)]. \tag{3.2}$$

Next, let $W_i^* = W - \sum_{j\in B_i}\omega_i\eta_i = W - \eta^*_{B_i}$ so that η_i and η_{A_i} are independent of W_i^*. Also, observe that

$$\begin{aligned}
&(1-\beta)\left\{\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}\Bigg[\sum_{j=1}^{\eta^*_{A_i}}1\Bigg]-\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{j=1}^{\eta^*_{A_i}-1}1\Bigg]\right\}+\beta\sum_{i\in J}\omega_i\mathbb{E}\eta_i\\
&=(1-\beta)\left\{\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}[\eta^*_{A_i}]-\sum_{i\in J}\omega_i\mathbb{E}[\eta_i(\eta^*_{A_i}-1)]\right\}+\beta\sum_{i\in J}\omega_i\mathbb{E}\eta_i\\
&=(1-\beta)\left[\frac{1}{1-\beta}\sum_{i\in J}\omega_i\mathbb{E}\eta_i-\sum_{i\in J}\sum_{j\in A_i}\omega_i\omega_j(\mathbb{E}[\eta_i\eta_j]-\mathbb{E}\eta_i\mathbb{E}\eta_j)\right]\\
&=(1-\beta)\left[\frac{1}{1-\beta}\mathbb{E}(W_n)-\mathrm{Var}(W_n)\right]=0.
\end{aligned} \tag{3.3}$$

Multiply $\mathbb{E}[\Delta g(W+1)]$ in (3.3) and using the corresponding expression in (3.2), we get

$$\begin{aligned}
\mathbb{E}[\mathscr{A}_X g(W)] =&(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}\Bigg[\sum_{j=1}^{\eta^*_{A_i}}(\Delta g(W_i+j)-\Delta g(W_i^*+1))\Bigg]\\
&-(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{j=1}^{\eta^*_{A_i}-1}(\Delta g(W_i+j)-\Delta g(W_i^*+1))\Bigg]\\
&+\beta\sum_{i\in J}\omega_i\mathbb{E}[\eta_i(\Delta g(W)-\Delta g(W_i^*+1))]\\
&-\sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)\mathbb{E}(\eta^*_{A_i})-\mathbb{E}(\eta_i(\eta^*_{A_i}\\
&-1))]+\beta\mathbb{E}(\eta_i)\}\mathbb{E}[\Delta g(W+1)-\Delta g(W_i^*+1)]\\
&=(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}\Bigg[\sum_{j=1}^{\eta^*_{A_i}}\sum_{\ell=1}^{\eta^*_{B_i\setminus A_i}+j-1}\Delta^2 g(W_i^*+\ell)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- (1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{j=1}^{\eta^*_{A_i}-1}\sum_{\ell=1}^{\eta^*_{B_i\setminus A_i}+j-1}\Delta^2 g(W_i^*+\ell)\Bigg]\\
&+\beta\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{\ell=1}^{\eta^*_{B_i}-1}\Delta^2 g(W_i^*+\ell)\Bigg]\\
&-\sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)\mathbb{E}(\eta^*_{A_i})-\mathbb{E}(\eta_i(\eta^*_{A_i}-1))]\\
&+\beta\mathbb{E}(\eta_i)\}\mathbb{E}\Bigg[\sum_{\ell=1}^{\eta^*_{B_i}}\Delta^2 g(W_i^*+\ell)\Bigg]\\
=&(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}\Bigg[\sum_{j=1}^{\eta^*_{A_i}}\sum_{\ell=1}^{\eta^*_{B_i\setminus A_i}+j-1}\mathbb{E}[\Delta^2 g(W_i^*+\ell)|\eta_{A_i},\eta_{B_i}]\Bigg]\\
&-(1-\beta)\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{j=1}^{\eta^*_{A_i}-1}\sum_{\ell=1}^{\eta^*_{B_i\setminus A_i}+j-1}\mathbb{E}[\Delta^2 g(W_i^*+\ell)|\eta_i,\eta_{A_i},\eta_{B_i}]\Bigg]\\
&+\beta\sum_{i\in J}\omega_i\mathbb{E}\Bigg[\eta_i\sum_{\ell=1}^{\eta^*_{B_i}-1}\mathbb{E}[\Delta^2 g(W_i^*+\ell)|\eta_{B_i}]\Bigg]\\
&-\sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)\mathbb{E}(\eta^*_{A_i})-\mathbb{E}(\eta_i(\eta^*_{A_i}-1))]\\
&+\beta\mathbb{E}(\eta_i)\}\mathbb{E}\Bigg[\sum_{\ell=1}^{\eta^*_{B_i}}\mathbb{E}[\Delta^2 g(W_i^*+\ell)|\eta_{B_i}]\Bigg]. \qquad (3.4)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\mathbb{E}[\mathscr{A}_X g(W)]| \le&\|\Delta g\|\Bigg\{\frac{(1-\beta)}{2}\Bigg[\sum_{i\in J}\omega_i\mathbb{E}\eta_i\mathbb{E}[\eta^*_{A_i}(2\eta^*_{B_i}-\eta^*_{A_i}-1)\mathcal{D}(W|\eta_{A_i},\eta_{B_i})]\\
&+\sum_{i\in J}\omega_i\mathbb{E}[\eta_i(\eta^*_{A_i}-1)(2\eta^*_{B_i}-\eta^*_{A_i}-2)\mathcal{D}(W|\eta_i,\eta_{A_i},\eta_{B_i})]\Bigg]\\
&+\sum_{i\in J}\omega_i\big|(1-\beta)[\mathbb{E}(\eta_i)\mathbb{E}(\eta^*_{A_i})-\mathbb{E}(\eta_i(\eta^*_{A_i}-1))]\\
&+\beta\mathbb{E}(\eta_i)\big|\mathbb{E}[\eta^*_{B_i}\mathcal{D}(W|\eta_{B_i})]\\
&+|\beta|\sum_{i\in J}\omega_i\mathbb{E}[\eta_i(\eta^*_{B_i}-1)\mathcal{D}(W|\eta_{B_i})]\Bigg\}. \qquad (3.5)
\end{aligned}
$$

Using (1.2), the proof follows.

Proof of Theorem 2.2 Let $A_i = B_i = \{i\}$, then $\{\eta_i, i \in J\}$ becomes independent random variables, and $W_i = W_i^*$ is independent of $\eta_{A_i} = \eta_{B_i} = \eta_i$. Therefore, from (3.4), we have

$$
\begin{aligned}
\mathbb{E}[\mathscr{A}_X g(W^*)] =& (1-\beta)\sum_{i\in J}\omega_i \mathbb{E}\eta_i \mathbb{E}\Bigg[\sum_{j=1}^{\omega_i\eta_i}\sum_{\ell=1}^{j-1}\mathbb{E}[\Delta^2 g(W_i+\ell)]\Bigg] \\
&- (1-\beta)\sum_{i\in J}\omega_i \mathbb{E}\Bigg[\eta_i \sum_{j=1}^{\omega_i\eta_i-1}\sum_{\ell=1}^{j-1}\mathbb{E}[\Delta^2 g(W_i+\ell)]\Bigg] \\
&+ \beta\sum_{i\in J}\omega_i \mathbb{E}\Bigg[\eta_i \sum_{\ell=1}^{\omega_i\eta_i-1}\mathbb{E}[\Delta^2 g(W_i+\ell)]\Bigg] \\
&- \sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] \\
&+ \beta\mathbb{E}(\eta_i)\}\mathbb{E}\Bigg[\sum_{\ell=1}^{\omega_i\eta_i}\mathbb{E}[\Delta^2 g(W_i+\ell)]\Bigg] \\
=& (1-\beta)\sum_{i\in J}\omega_i \mathbb{E}\eta_i \sum_{k=1}^{\infty}\sum_{j=1}^{\omega_i k}\sum_{\ell=1}^{j-1}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
&- (1-\beta)\sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{j=1}^{\omega_i k-1}\sum_{\ell=1}^{j-1} k\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
&+ \beta\sum_{i\in J}^{n}\omega_i \sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-1} k\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
&- \sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] \\
&+ \beta\mathbb{E}(\eta_i)\}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
=& \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{j=1}^{\omega_i k}\sum_{\ell=1}^{j-1}[(1-\beta)\mathbb{E}\eta_i + \beta k]\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
&- \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{j=1}^{\omega_i k-1}\sum_{\ell=1}^{j-1} k\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k) \\
&- \sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] \\
&+ \beta\mathbb{E}(\eta_i)\}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-1} \ell[(1-\beta)\mathbb{E}\eta_i + \beta k]\mathbb{E}[\Delta^2 g(W_i+\omega_i k-\ell)]p_i(k)\\
&\quad - \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-2} \ell k\mathbb{E}[\Delta^2 g(W_i+\omega_i k-\ell-1)]p_i(k)\\
&\quad - \sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))]\\
&\quad + \beta\mathbb{E}(\eta_i)\}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k).
\end{aligned}
$$

Case I: If $\omega_i = 1$, then

$$
\begin{aligned}
\mathbb{E}[\mathscr{A}_X g(W^*)] &= \sum_{i\in J}\sum_{k=2}^{\infty}\sum_{\ell=1}^{k-1} \ell[(1-\beta)\mathbb{E}\eta_i + \beta k]\mathbb{E}[\Delta^2 g(W_i+k-\ell)]p_i(k)\\
&\quad - \sum_{i\in J}\sum_{k=3}^{\infty}\sum_{\ell=1}^{k-2} \ell k\mathbb{E}[\Delta^2 g(W_i+k-\ell-1)]p_i(k)\\
&\quad - \sum_{i\in J}\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] + \beta\mathbb{E}(\eta_i)\}\\
&\quad \times \sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k)\\
&= \sum_{i\in J}\sum_{k=2}^{\infty}\sum_{\ell=1}^{k-1} \ell[(1-\beta)\mathbb{E}\eta_i\, p_i(k) + \beta k p_i(k)\\
&\quad - (k+1)p_i(k+1)]\mathbb{E}[\Delta^2 g(W_i+k-\ell)]\\
&\quad - \sum_{i\in J}\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] + \beta\mathbb{E}(\eta_i)\}\\
&\quad \times \sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k). \qquad (3.6)
\end{aligned}
$$

Case II: If $\omega_i \geq 2$, then

$$
\begin{aligned}
\mathbb{E}[\mathscr{A} g(W)] &= \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-1} \ell[(1-\beta)\mathbb{E}\eta_i + \beta k]\mathbb{E}[\Delta^2 g(W_i+\omega_i k-\ell)]p_i(k)\\
&\quad - \sum_{i\in J}\omega_i \sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-2} \ell k\mathbb{E}[\Delta^2 g(W_i+\omega_i k-\ell-1)]p_i(k)\\
&\quad - \sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2 - \mathbb{E}(\eta_i(\eta_i-1))] + \beta\mathbb{E}(\eta_i)\}
\end{aligned}
$$

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k)$$
$$=\sum_{i\in J}\omega_i\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k-1}[(1-\beta)\ell\mathbb{E}\eta_i+\beta\ell k$$
$$-(\ell-1)k]\mathbb{E}[\Delta^2 g(W_i+\omega_i k-\ell)]p_i(k)$$
$$-\sum_{i\in J}\omega_i\{(1-\beta)[\mathbb{E}(\eta_i)^2-\mathbb{E}(\eta_i(\eta_i-1))]+\beta\mathbb{E}(\eta_i)\}$$
$$\times\sum_{k=1}^{\infty}\sum_{\ell=1}^{\omega_i k}\mathbb{E}[\Delta^2 g(W_i+\ell)]p_i(k). \tag{3.7}$$

It is shown that in Barbour and Xia [5] (see also Barbour and Čekanavičius [4], p. 517) $|\mathbb{E}(\Delta^2 g(W_i+\cdot))|\le\gamma\|\Delta g\|$. Hence, from (3.6) and (3.7), we have

$$|\mathbb{E}[\mathscr{A}g(W)]|\le\gamma\|\Delta g\|\sum_{i\in J}\omega_i\left(\sum_{k=1}^{\infty}h_i(k)+d_i\right). \tag{3.8}$$

Using (1.2), the proof follows. □

Acknowledgements The author would like to thank Dr. Neelesh S. Upadhye for helpful comments and suggestions. The author is grateful to the associate editor and reviewers for many valuable suggestions and critical comments that improved the presentation of the paper.

References

1. Barbour, A.D., Chen, L.H.Y., Loh, W.L.: Compound poisson approximation for nonnegative random variables via Stein's method. Ann. Prob. **20**(4), 1843–1866 (1992)
2. Barbour, A.D., Hall, P.: On the rate of Poisson convergence. Math. Proc. Camb. Philos. Soc. **95**(3), 473–480 (1984)
3. Barbour, A.D., Holst, L., Janson, S.: Poisson Approximation. Oxford University Press, Oxford (1992)
4. Barbour, A.D., Čekanavičius, V.: Total variation asymptotes for sum of independent integer random variables. Ann. Prob. **30**(2), 509–545 (2002)
5. Barbour, A.D., Xia, A.: Poisson perturbation. ESAIM Probab. Stat. **3**, 131–150 (1999)
6. Bhati, D., Rattihalli, R.N.: Distributions of sums of geometrically weighted finite valued discrete random variables. ProbStat Forum **7**(January), 12–20 (2014)
7. Chow, Y.S., Lai, T.L.: Limiting behavior of weighted sums of independent random variables. Ann. Prob. **1**(5), 810–824 (1973)
8. Eichelsbacher, P., Reinert, G.: Stein's method for discrete Gibbs measures. Ann. Appl. Probab. **18**(4), 1588–1618 (2008)
9. Etemadi, N.: Convergence of weighted averages of random variables revisited. Proc. Am. Math. Soc. **134**(9), 2739–2744 (2006)
10. Gan, H.L., Xia, A.: Stein's method for conditional compound Poisson approximation. Stat. Probab. Lett. **100**, 19–26 (2015)
11. Godbole, A.P.: Approximate reliabilities of m-consecutive-k-out-of-n: failure systems. Stat. Sinica **1**, 321–327 (1993)
12. Huang, W.T., Tsai, C.S.: On a modified binomial distribution of order k. Stat. Probab. Lett. **11**(2), 125–131 (1991)

13. Kerstan, J.: Verallgemeinerung eines Satzes von Prochorow und Le Cam. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 173–179 (1964)
14. Kumar, A.N., Upadhye, N.S., Vellaisamy, P.: Approximations related to the sums of m-dependent random variables. Preprint arXiv:2005.00780 (2020)
15. Le Cam, L.: An approximation theorem for the Poisson binomial distribution. Pac. J. Math. **10**, 1181–1197 (1960)
16. Ley, C., Reinert, G., Swan, Y.: Stein's method for comparison of univariate distributions. Probab. Surv. **14**, 1–52 (2017)
17. Neammanee, K., Yonghint, N.: Poisson approximation for call function via Stein-Chen method. Bull. Malays. Math. Sci. Soc. **43**, 1135–1152 (2020)
18. Olvera-Cravioto, M.: Asymptotics for weighted random sums. Adv. Appl. Probab. **44**(4), 1142–1172 (2012). https://doi.org/10.1239/aap/1354716592
19. Röllin, A.: Symmetric and centered binomial approximation of sums of locally dependent random variables. Electron. J. Probab. **13**(24), 756–776 (2008)
20. Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: Proc. Sixth Berkeley Symp. Math. Statist. Probab. II. Probability Theory, pp. 583–602. Univ. California Press, Berkeley
21. Teerapabolarn, K.: Negative binomial approximation for independent negative binomial variables. Appl. Math. Sci. **9**(5), 239–241 (2015)
22. Upadhye, N.S., Barman, K.: A unified approach to Stein's method for stable distributions. Preprint arXiv:2004.07593 (2020)
23. Upadhye, N.S., Čekanavičius, V., Vellaisamy, P.: On Stein operators for discrete approximations. Bernoulli **23**(4A), 2828–2859 (2017)
24. Čekanavičius, V., Roos, B.: Two-parametric compound binomial approximations. Lith. Math. J. **14**(4), 354–374 (2004)
25. Vellaisamy, P.: Poisson approximation for (k_1, k_2)-events via the Stein-Chen method. J. Appl. Probab. **41**(4), 1081–1092 (2004)
26. Vellaisamy, P., Upadhye, N.S., Čekanavičius, V.: On negative binomial approximation. Theory Probab. Appl. **57**(1), 97–109 (2013)
27. Yonghint, N., Neammanee, K., Chaidee, N.: Poisson approximation for locally dependent CDO. Comm. Statist. Theory Methods (2020). https://doi.org/10.1080/03610926.2020.1759638
28. Zhengyan, L.: Laws of large numbers for weighted sums of random variables and random elements. Acta Math. Sin. **5**(2), 185–192 (1989)