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A Remark on the Meromorphic Solutions in the FitzHugh–Nagumo Model

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Abstract

Due to the Nevanlinna theory, the paper gives the general structure of transcendental meromorphic solutions of a certain ordinary differential equation with rational coefficients. As an application, the meromorphic solutions of the FitzHugh–Nagumo system are obtained in explicit form.

Keywords FitzHugh–Nagumo model · Nevalinna theory · Complex differential equation · Meromorphic solutions

Mathematics Subject Classification 34M05 · 30D35 · 39A32

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1 Introduction

The FitzHugh–Nagumo model (FHN) is given by the following famous FitzHugh–Nagumo system of polynomial ordinary differential equations

$$\begin{cases} v_t = v - v^3 - u + \sigma \\ \tau u_t = v - \beta u - \alpha \end{cases}$$

The FitzHugh–Nagumo model (FHN), named after Richard FitzHugh [1,2] who suggested the system in 1961 and Nagumo et al. [3] who created the equivalent circuit the following year, describes a prototype of an excitable system (e.g., a neuron). The FitzHugh–Nagumo model is a simplified version of the Hodgkin–Huxley model which models in a detailed manner activation and deactivation dynamics of a spiking neuron. Many works handle numerical simulation and dynamical properties of this model.

In [4], N. Kudryashov considered the analytical properties of the well-known FitzHugh–Nagumo model which is used for description of potential on the neuron membrane. Solving the first equation of the system yields

$$u = v - v^3 - v_t + \sigma.$$

Substituting this relation into the second equation of this system, one has

$$3\tau v^2 v_t + \beta v^3 + \tau v_{tt} + (\beta - \tau)v_t + (1-\beta)v - (\alpha + \beta\sigma) = 0. \tag{1.1}$$

Obviously, the system of FitzHugh–Nagumo ordinary differential equations can be transformed to the second nonlinear differential equation in the form (1.1). N. Kudryashov applied the Painlevé test to Eq. (1.1) to understand the integrability of this equation. However, Eq. (1.1) does not pass the Painlevé test, and the attempt to look for the general solution of Eq. (1.1) has not met with success. The question about the meromorphic solution for the transformed question requires further investigations (see p. 154 in [4]).

In 2018, Demina and Kudryashov [5] studied local properties of solutions of a second-order ordinary differential equation related to the FitzHugh–Nagumo model in the special case. By introducing the new function $w(t) = v^2(t)$, they obtained the algebraic second-order ordinary differential equation

$$2\tau w w_{tt} - \tau w_t^2 + 2\{3\tau w^2 + (\beta - \tau)w\}w_t + 4\beta w^3 + 4(1-\beta)w^2 = 0, \tag{1.2}$$

whenever $\alpha = -\beta\sigma$. The time t and parameters $(\alpha, \beta, \sigma, \tau)$ are supposed to be complex variables. The functions $u(t)$ and $v(t)$ are supposed to be complex-valued. Due to the Nevanlinna theory and the Painlevé test, the general structure of transcendental meromorphic solutions of Eq. (1.2) is obtained. Furthermore, with the method in [4,6–11], they constructed all the transcendental meromorphic solutions in explicit form and obtained the following result.

Theorem A Any transcendental meromorphic solutions of Eq. (1.2) takes the form

$$w(t) = \frac{\varepsilon}{2}\sqrt{L}\cot\left(\sqrt{L}\{t-t_0\}\right) + h_0 + h_1\exp\left(-\frac{2\beta\{t-t_0\}}{3\tau}\right),$$

where h_0, h_1, L are constants, t_0 is an arbitrary constants, $\varepsilon = 0$ or $\varepsilon = 1$, and there exists $q \in \mathbb{Q}/\{0\}$ such that $\sqrt{L} = i\beta q/(3\pi)$ whenever $h_1\beta \neq 0$.

Motivated by the work of Demina and Kudryashov in [5], we will use the Nevanlinna theory to find all the transcendental meromorphic solutions of Eq. (1.1) which posed by M. Demina and N. Kudryashov.

2 Main Result

In this section, we first concern the transcendental meromorphic solutions of the following ordinary differential equation (ODE)

$$P_1 v^2 v_t + P_2 v^3 + P_3 v_{tt} + P_4 v_t + P_5 v + P_6 = 0, \tag{2.1}$$

with rational coefficients $P_1(\not\equiv 0), P_2, \ldots, P_6$. Obviously, Eq. (2.1) is a generalization of Eq. (1.1) from the constant coefficients to the rational coefficients. In this following, we pay an attention to the meromorphic solutions of Eq. (2.1). Our main result is the following theorem.

Theorem 1 *Any transcendental meromorphic solutions of Eq.* (2.1) *takes the form*

$$v(t) = \prod_{j=1}^{q}(t - z_j)^{m_j} e^{P(t)},$$

where P is a polynomial, $m_1, m_2, \ldots, m_q$ are integers and $z_1, z_2, \ldots, z_q$ are distinct complex numbers. In particular,

$$P_1 v_t + P_2 v \equiv 0, \quad P_3 v_{tt} + P_4 v_t + P_5 v + P_6 = 0.$$

Remark It is pointed out that $P_1 \not\equiv 0$ is necessary in the main theorem. Consider the function $v(t) = \tan t$. Then, $v(t)$ is a meromorphic solution of ODE

$$2v^3 - v_{tt} + 2v = 0.$$

But, the form of v does not satisfy the conclusion of Main theorem. Some further information about the coefficient $P_1 \equiv 0$ in Eq. (2.1) will be found in Sect. 3.

Example 1 Let $f = \frac{e^t}{t-1}$. Then f is a transcendental meromorphic solution of

$$(t-1)v^2 v_t - (t-2)v^3 + (t-1)^2 v_{tt} - (t-1)(t-2)v_t - v = 0.$$

As an application, we get the form of the meromorphic solutions of Eq. (1.1).

Corollary 1 *Any transcendental meromorphic solutions of Eq.* (1.1) *takes the form*

$$v(t)=e^{-\frac{\beta}{3\tau}t+A},$$

where A is a constant. In particular,

$$\frac{2\beta^2}{9\tau}+\frac{2\beta}{3}=1,\ \alpha+\beta\sigma=0.$$

From the proof of Theorem 1, we can obtain the transcendental meromorphic solution of the following nonlinear differential equation.

Theorem 2 *If f is a transcendental meromorphic solutions of*

$$Av^{n-1}v_t+Bv^n+L(v)=0, \tag{2.2}$$

where $n\geq 3$, $L(v)=P_kv^{(k)}+\cdots+P_1v_t'+P_0$ and $P_k,\ldots,P_0$ are polynomials, then f has the form

$$v(t)=\prod_{j=1}^{q}(t-z_j)^{m_j}e^{P(t)},$$

where P is a polynomial, $m_1,m_2,\ldots,m_q$ are integers and $z_1,z_2,\ldots,z_q$ are distinct complex numbers. In particular,

$$Av_t+Bv\equiv 0,\quad L(v)=0. \tag{2.3}$$

3 Proof of the Main Results

In this following, we will employ the Nevanlinna theory to prove the main theorem. Firstly, some basic results in Nevanlinna theory will be introduced. For a meromorphic function f, we will use the notations the proximity function $m(r,f)$ and $N(r,f)$ the integrated counting function, which are defined as

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\mathrm{d}\theta,$$
$$N(r,f)=\int_0^{\infty}\frac{n(t,f)-n(0,f)}{t}\mathrm{d}t+n(0,f)\log r,$$

where $\log^+ s=\max\{\log s,0\}$ and $n(t,f)$ denotes the number of poles of f in the disk $|z|\leq t$, each pole is counted according to its multiplicity. We also need the characteristic function $T(r,f):=m(r,f)+N(r,f)$, and utilize five results in Nevanlinna theory. (see e.g., [12–15]):

Lemma 1 (The Nevanlinna First Fundamental Theorem) *Let f be a transcendental function. Then*

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1).$$

Lemma 2 (The Logarithmic Derivative Lemma) *Let f be a transcendental function. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log\{rT(r, f)\}),$$

possibly outside a set of finite linear measure, where $f^{(k)}$ is the k-th derivative of f.

Lemma 3 (Clunie's Lemma) *Let f be a transcendental meromorphic function satisfying the following equation*

$$f^n P(z, f, f_z, \ldots) = Q(z, f, f_z, \ldots),$$

where P, Q are polynomials in f and its derivatives with rational coefficients. If the degree of Q is at most n, then

$$m(r, P(z, f, f_z, \ldots)) = O(\log rT(r, f)), r \to \infty$$

possibly outside a set of finite linear measure.

Lemma 4 *A meromorphic function f is transcendental if and only if*

$$\liminf_{r\to\infty} \frac{T(r, f)}{\log r} = \infty$$

and if f is a non-constant rational function, then

$$T(r, f) = d \log r + O(1)$$

for some $d \neq 0$.

Lemma 5 *Let $f(z)$ be a non-constant meromorphic function and let $a_0, a_1, \ldots, a_n (\neq 0)$ be constants. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0) = nT(r, f) + O(1).$$

Proof of Theorem 1 Rewrite (2.1) as follows

$$v^2(P_1 v_t + P_2 v) = -[P_3 v_{tt} + P_4 v_t + P_5 v + P_6] = L(v). \tag{3.1}$$

Obviously, $L(v)$ is a differential polynomial in v of degree 1 with rational coefficients. Suppose that $P_1v_t + P_2v \not\equiv 0$. Then, by Clunie's Lemma, we see that

$$m(r, P_1v_t + P_2v) = O(\log\{rT(r, v)\}), \tag{3.2}$$

possibly outside a set of finite linear measure. From Eq. (3.1), it is easy to derive that all the poles of v must be the zeros of $P_1(z)$. So, v has finitely many poles. Further,

$$N(r, P_1v_t + P_2v) \leq 2N(r, v) = O(\log r). \tag{3.3}$$

Combining (3.2) and (3.3) yields that

$$T(r, P_1v_t + P_2v) = m(r, P_1v_t + P_2v) + N(r, P_1v_t + P_2v) = O(\log\{rT(r, v)\}),$$

possibly outside a set of finite linear measure. By the Nevanlinna first fundamental theorem, one has

$$T\left(r, \frac{1}{P_1v_t + P_2v}\right) = T(r, P_1v_t + P_2v) = O(\log\{rT(r, v)\}), \tag{3.4}$$

possibly outside a set of finite linear measure. We will estimate the proximity function of $L(v)$. Again by the logarithmic derivative lemma, one has

$$\begin{aligned} m(r, L(v)) &= m(r, P_3v_{tt} + P_4v_t + P_5v + P_6) \\ &\leq m(r, P_3v_{tt} + P_4v_t + P_5v) + m(r, P_6) \\ &\leq m(r, \frac{P_3v_{tt} + P_4v_t + P_5v}{v}) + m(r, v) + O(\log r) \\ &\leq m(r, v) + O(\log\{rT(r, v)\}), \end{aligned} \tag{3.5}$$

possibly outside a set of finite linear measure. Rewrite (3.1) as

$$v^2 = \frac{L(v)}{P_1v_t + P_2v}.$$

By the above function and (3.5), we derive that

$$\begin{aligned} &2T(r, v) + O(1) \\ &\quad = T(r, v^2) = m(r, v^2) + N(r, v^2) = m(r, v^2) + O(\log r) \\ &\quad = m\left(r, \frac{L(v)}{P_1v_t + P_2v}\right) + O(\log r) \leq m\left(r, L(v)) + m(r, \frac{1}{P_1v_t + P_2v}\right) + O(\log r) \\ &\quad \leq m(r, L(v)) + T\left(r, \frac{1}{P_1v_t + P_2v}\right) + O(\log r) \leq m(r, v) + O(\log\{rT(r, v)\}) \end{aligned}$$

which implies that

$$T(r,v)=O(\log\{rT(r,v)\}),\tag{3.6}$$

possibly outside a set of finite linear measure. By the Lemma 4, we have

$$\liminf_{r\to\infty}\frac{T(r,v)}{\log r}=\infty,$$

which contradicts with (3.6). Then, all the above discussion yields that

$$P_1v_t+P_2v\equiv 0,\quad P_3v_{tt}+P_4v_t+P_5v+P_6=0.$$

From the form

$$\frac{v_t}{v}=-\frac{P_2}{P_1}$$

yields that $-\frac{P_2}{P_1}$ just has simple poles. Then, we can set

$$\frac{v_t}{v}=-\frac{P_2}{P_1}=P_0+\sum_{j=1}^{q}\frac{m_j}{t-z_j},\tag{3.7}$$

where P_0 is a polynomial, $q\geq 0$ and $m_1,m_2,\ldots,m_q$ are integers such that if $q=0$, then

$$\sum_{j=1}^{q}\frac{m_j}{t-z_j}=0,$$

and if $q\geq 1$, then m_j $(1\leq j\leq q)$ are q nonzero integers, and $z_1,z_2,\ldots,z_q$ are the zeros and poles of v. By integrating two sides of (3.7), we have that

$$v=e^P\prod_{j=1}^{q}(t-z_j)^{m_j},$$

where P is a primitive function of P_0 such that $P'=P_0$. This finishes the proof of Main theorem. □

Proof of Corollary 1 If $\tau=0$. Then, Eq. (1.1) reduces to

$$\beta v^3+\beta v_t+(1-\beta)v-(\alpha+\beta\sigma)=0.$$

The above equation yields that v has no poles. Hence by Lemma 5, we have

$$\begin{aligned}3T(r,v)+O(1)&=T(r,\beta v^3)=m(r,v^3)\\&=m(r,\beta v_t+(1-\beta)v-(\alpha+\beta\sigma))\\&\le m\left(r,\frac{\beta v_t+(1-\beta)v}{v}\right)+m(r,v)+O(1)\\&\le m(r,v)+O(\log\{rT(r,v)\}),\end{aligned}$$

possibly outside a set of finite linear measure, which is impossible. Below, we assume that $\tau\neq 0$. Therefore, by the conclusion (2.3) and Eq. (1.1), one has

$$\beta v+3\tau v_t=0, \tag{3.8}$$

$$\tau v_{tt}+(\beta-\tau)v_t+(1-\beta)v-(\alpha+\beta\sigma)=0. \tag{3.9}$$

Then, integrating the equation of (3.8) yields that

$$v(t)=e^{-\frac{\beta}{3\tau}t+A}.$$

Substituting this form of v into the equation of (3.9) leads to

$$\left(\frac{-2\beta^2-6\beta\tau+9\tau}{9\tau}\right)e^{-\frac{\beta}{3\tau}t+A}-(\alpha+\beta\sigma)=0.$$

From the equality, we can obtain that

$$\frac{2\beta^2}{9\tau}+\frac{2\beta}{3}=1 \quad \text{and} \quad \alpha+\beta\sigma=0.$$

This finishes the proof of Corollary 1.

□

4 Discussion

In this section, we turn our attention to Eq. (2.1) when $P_1\equiv 0$. Then, Eq. (2.1) becomes

$$P_2v^3+P_3v_{tt}+P_4v_t+P_5v+P_6=0. \tag{4.1}$$

This is a nonlinear ordinary differential equation. As is well known, there is no foolproof way to solve this kind of differential equations. A slight variation of an equation might require a different method. So, it is hard to give the general structure of transcendental meromorphic solutions of the above equation.

Using the complex method, Yuan et al. [16] got the meromorphic solutions to a class special differential equations

$$P_2 v^3 + P_3 v_{tt} + P_5 v + P_6 = 0,$$

where P_2, P_3, P_5 and P_6 are constants. Also by the same method, they [17] also gave the meromorphic solution of

$$P_2 v^3 + P_3 v_{tt} + P_4 v_t + P_5 v = 0,$$

where P_2, P_3, P_4 and P_5 are constants, and the solution of the Fisher equation with degree three (see [18])

$$v^3 - v_{tt} - c v_t - v = 0,$$

where c is a constant.

We also observed that the Painlevé equation (II) is the special cases of Eq. 2.2. In mathematics, Painlevé transcendents are solutions to certain nonlinear second-order ordinary differential equations in the complex plane with the Painlevé property. The equation traditionally called Painlevé (II), are as follows:

$$\text{(II)}: \quad v_{tt} = 2v^3 + tv + \alpha.$$

It is well known that the Painlevé equations can all be represented as Hamiltonian systems. By the Painlevé test, one can give the representation of the meromorphic solutions of the Painlevé (II).

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Compliance with Ethical Standards

Conflict of interest The authors declare that none of the authors have any competing interests in the manuscript.

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