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# Codimension Two Spacelike Submanifolds Through a Null Hypersurface in a Lorentzian Manifold

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**Abstract**

Most important examples of null hypersurfaces in a Lorentzian manifold admit an integrable screen distribution, which determines a spacelike foliation of the null hypersurface. In this paper, we obtain conditions for a codimension two spacelike submanifold contained in a null hypersurface to be a leaf of the (integrable) screen distribution. For this, we use the rigging technique to endow the null hypersurface with a Riemannian metric, which allows us to apply the classical Eschenburg maximum principle. We apply the obtained results to classical examples as generalized Robertson–Walker spaces and Kruskal space.

**Keywords** Rigging technique · Null hypersurface · Maximum principle

**Mathematics Subject Classification** 53C50 · 53B30 · 53C40

## 1 Introduction

If $L$ is a hypersurface of a $n$-dimensional Lorentzian manifold $(M, g)$, then the dimension of the radical $Rad\left(T_pL\right) = T_pL \cap \left(T_pL\right)^{\perp}$ is one or zero for each point $p \in L$. If $Rad\left(T_pL\right)$ is one-dimensional for all $p \in L$, then we say that $L$ is a null hypersurface

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and in this case $Rad\left(T_pL\right) = \left(T_pL\right)^{\perp}$. It is well-known that the main drawback to study them is the impossibility of defining a natural projection onto the tangent space. This is overcomed by making some arbitrary choices, which allows to construct all geometrical objects in a null hypersurface. Of course, they depend on the choices we have made, but some fundamental concepts, like being totally geodesic or umbilical, are independent of any choice, [5, Proposition 2.3.1, pg. 58].

The rigged technique introduced in [9] has shown to be an effective tool to study a null hypersurface. Briefly, the main idea consists of choosing a vector field $\zeta$ such that $\zeta_p \notin T_pL$ for all $p \in L$, which is called a rigging for $L$. From this unique arbitrary choice, we derive all the geometric objects needed to handle a null hypersurface: a null section of $Rad(TL)$, a screen distribution in $TL$, a transverse null section, and all the associated tensors. Moreover, a Riemannian metric, called rigged metric, can be also constructed on $L$, which suggests that we can use Riemannian techniques to obtain some results about null hypersurfaces, see [2,8,9].

In this paper, we follow the same spirit as the above papers: we apply Riemannian tools to get some results in a null hypersurface. Concretely, using the rigged metric, we can consider a codimension two spacelike submanifold contained in a null hypersurface as a hypersurface of a Riemannian manifold. The relationship between its mean curvature as a hypersurface and the mean curvature as a codimension two submanifold is established in order to apply the Eschenburg maximum principle, which allows us to obtain conditions to ensure that a codimension two spacelike submanifold through a null hypersurface is a leaf of the (integrable) screen distribution.

There is quite enough literature about codimension two spacelike submanifolds through a null hypersurface, but they are mainly focused in null cones and constant curvature ambient. For example, in [14,15] the authors obtain conditions for a spacelike surfaces through a null cone of the four-dimensional Minkowski space to be a totally umbilical round sphere. Recently, in [1] the authors have proved that a compact codimension two spacelike trapped submanifold through the null cone of the De Sitter space is conformally diffeomorphic to the round sphere. See also [3,11,12] for another results about this topic.

The interest of these ideas is twofold. Null hypersurface is a concept of Lorentzian geometry without Riemannian counterpart, so it is important to study them. As far as we know, the rigging technique is the most promising technique to do that and the notion of screen distribution is inherent to it. On the other hand, trapped surfaces are close related with the idea of marginally outer trapped surfaces (MOTS in the literature) which is a leaf of a screen distribution in a black hole horizon. As it is well-known, trapped surfaces are a key ingredient in the celebrated Penrose's singularity theorem [13].

## 2 Rigged Metric on a Null Submanifold

We review some important facts about null hypersurfaces and the rigging technique introduced in [7,9]. A rigging vector field $\zeta$ for a null hypersurface $L$ in a Lorentzian manifold $(M, g)$ is a vector field defined in some open subset containing $L$ such that $\zeta_p \notin T_pL$ for all $p \in L$. Fixed a rigging vector field for a null hypersurface, it induces the rigged vector field $\xi$, which is the unique (globally defined) null section in $L$ such that $g(\xi, \zeta) = 1$ and the null transverse vector field

$$N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi, \tag{1}$$

which is a null vector field over $L$ such that $g(N, \xi) = 1$.

Define the screen distribution $\mathcal{S} = \zeta^{\perp} \cap TL$. The following decompositions hold

$$T_p M = T_p L \oplus span(N_p), \tag{2}$$
$$T_p L = \mathcal{S}_p \oplus_{orth} span(\xi_p) \tag{3}$$

for all $p \in L$. Moreover, $\mathcal{S}$ is orthogonal to the transverse vector field $N$.

Given $U, V \in \mathfrak{X}(L)$ and $X \in \mathcal{S}$, we can decompose

$$\nabla_U V = \nabla^L_U V + B(U, V)N, \tag{4}$$
$$\nabla_U \xi = -A^*(U) - \tau(U)\xi, \tag{5}$$
$$\nabla^L_U X = \nabla^*_U X + C(U, X)\xi, \tag{6}$$

where $\nabla^L_U V \in \mathfrak{X}(L)$ and $A^*(U), \nabla^*_U X \in \mathcal{S}$. The tensor $B$ is called null second fundamental form of $L$. $B$ is symmetric, but $C$ is symmetric if and only if $\mathcal{S}$ is integrable. Indeed, we have

$$C(X, Y) - C(Y, X) = g([X, Y], N)$$

for all $X, Y \in \mathcal{S}$. Moreover, the following equations hold

$$B(U, V) = -g(\nabla_U \xi, V),$$
$$B(U, \xi) = 0,$$
$$C(U, X) = -g(\nabla_U N, X),$$
$$\tau(U) = g(\nabla_U N, \xi)$$

for all $X \in \mathcal{S}$ and $U, V \in \mathfrak{X}(L)$. On the other hand, it always holds $\nabla_\xi \xi = -\tau(\xi)\xi$, i.e., $\xi$ is a pregeodesic vector field, but if we choose a conformal rigging, then the associated rigged vector field is geodesic [9, Lemma 3.10].

We say that the rigging vector field is *screen conformal* if there exists $\varphi \in C^\infty(L)$ such that $C = \varphi B$. In this case, since $B$ is symmetric, the screen distribution is integrable. We say that the rigging vector field is *distinguished* if the one-form $\tau$ vanishes. In [4], some conditions for the existence of a distinguished rigging are given. There are important examples of null hypersurfaces which admit a screen conformal and distinguished rigging vector field, for instance in generalized Robertson–Walker spaces (in particular spaces of constant curvature), plane fronted waves, the Kruskal space.

The connections $\nabla^L$ and $\nabla^*$ are called the induced connection on $L$ and $\mathcal{S}$, respectively. If $\mathcal{S}$ is integrable, then $\nabla^*$ is the Levi–Civita connection of the leaves induced from the ambient. Moreover, in this case, the second fundamental form of the leaves is

$$\mathbb{I}_{\mathcal{S}}(X,Y) = C(X,Y)\xi + B(X,Y)N \tag{7}$$

for all $X, Y \in \mathcal{S}$.

We define the null mean curvature $H$ and the screen mean curvature $\Omega$ as

$$H_p = \sum_{i=1}^{n-2} B(e_i, e_i),$$
$$\Omega_p = \sum_{i=1}^{n-2} C(e_i, e_i)$$

where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $\mathcal{S}_p$. $\Omega$ depends on the fixed rigging vector field, but the null mean curvature only depends on the rigged vector field. From Eq. (7), if the screen is integrable, then the mean curvature vector of the leaves (as a submanifolds of $M$) is

$$\mathbf{H}_{\mathcal{S}} = \Omega \cdot \xi + H \cdot N. \tag{8}$$

The null hypersurface $L$ is called totally geodesic if $B = 0$ and totally umbilical if $B = \frac{H}{n-1} g$. These definitions do not depend on the chosen rigging vector field. On the other hand, we say that the screen is totally umbilical if $C = \frac{\Omega}{n-2} g$. In this case, $C(\xi, X) = 0$ for all $X \in \mathcal{S}$ and the screen distribution is integrable.

The rigged metric is a Riemannian metric on $L$ defined by

$$\widetilde{g} = g + \omega \otimes \omega,$$

where $\omega \in \Lambda^1(L)$ is the one-form given by $\omega(U) = g(U, \zeta) = g(U, N)$ for all $U \in \mathfrak{X}(L)$.

In [7], it is shown how all theses tensors change when we change the rigging. For our purpose, we recall the following special case.

**Lemma 1** *Let $L$ be a null hypersurface of a Lorentzian manifold and $\zeta$ a rigging vector field for it. Take $\Phi \in C^\infty(L)$ a never vanishing function and call $\zeta' = \Phi\zeta$. If we denote with a $'$ the derived objects from the rigging vector field $\zeta'$, then*

1. $\xi' = \frac{1}{\Phi}\xi$ *and* $N' = \Phi N$.
2. $B' = \frac{1}{\Phi} B$ *and* $H' = \frac{1}{\Phi} H$.
3. $C' = \Phi C$ *and* $\Omega' = \Phi\Omega$.
4. $\tau' = \tau + \frac{1}{\Phi} d\Phi$.
5. $\omega' = \Phi\omega$.

On the other hand, if we denote with a $\sim$ the derived objects from the rigged metric $\widetilde{g}$, then it can be shown the following proposition.

**Proposition 1** [7,9] *Let $L$ be a null hypersurface of a Lorentzian manifold and $\zeta$ a rigging vector field for it. Given $X, Y, Z \in \mathcal{S}$ and $U, V, W \in \mathfrak{X}(L)$, it holds.*

1. $g(D(U,V),W) = \frac{1}{2}\big(\omega(W)(L_\xi \widetilde{g})(U,V) + \omega(U)d\omega(V,W) + \omega(V)d\omega(U,W)\big) + \omega(W)B(U,V)$, *where* $D(U,V) = \widetilde{\nabla}_U V - \nabla^L_U V$.
2. $\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z)$.
3. $\left(L_\xi \widetilde{g}\right)(X,Y) = -2B(X,Y)$. *In particular,* $H = -\widetilde{div}\xi$.
4. $\tau(U) = g(\nabla_U \zeta, \xi)$.
5. $2C(U,X) + d\omega(U,X) + \left(L_\zeta g\right)(U,X) + g(\zeta,\zeta)B(U,X) = 0$.

It easily follows that $d\omega = 0$ if and only if the screen distribution is integrable and $\xi$ is $\widetilde{g}$-geodesic.

**Proposition 2** [7,9] *Let $L$ be a null hypersurface of a Lorentzian manifold and $\zeta$ a rigging vector field for it such that $d\omega = 0$. If $X, Y \in \mathcal{S}$ and $U, V, W \in \mathfrak{X}(L)$, then*

1. *The second fundamental form of the leaves of the screen distribution as a hypersurface of $(L,\widetilde{g})$ is $\widetilde{\mathbb{I}}(X,Y) = B(X,Y)\xi$.*
2. *The mean curvature of the leaves respect to $\xi$ as a hypersurface of $(L,\widetilde{g})$ is $\widetilde{H}_{\mathcal{S}} = H$.*
3. $C(\xi, X) = -\tau(X)$.
4. $\widetilde{g}(\widetilde{\nabla}_U V, W) = g(\nabla_U V, W) + \omega(W)U(\omega(V))$.

***Proof*** We sketch the proof for completeness. The first point follows from point 3 of the above proposition. The second point is an immediate consequence of the first one. Using the last point of Proposition 1 and formulas (4), (5) and (6), we have

$$
\begin{aligned}
-2C(\xi,X) &= \left(L_\zeta g\right)(\xi,X) = g(\nabla_\xi \zeta, X) + g(\xi, \nabla_X \zeta) \\
&= g(\nabla_\xi N, X) + g(\xi, \nabla_X N) = -C(\xi,X) + \tau(X)
\end{aligned}
$$

and we obtain point 3.

From the first point of Proposition 1, we have $\widetilde{g}(\widetilde{\nabla}_U V, Z) = g(\nabla_U V, Z)$ for all $U, V \in \mathfrak{X}(L)$ and $Z \in \mathcal{S}$. Using this equation and point 1, it can be checked that $\widetilde{g}(\widetilde{\nabla}_U Y, W) = g(\nabla_U Y, W)$ and $\widetilde{g}(\widetilde{\nabla}_U \xi, W) = g(\nabla_U \xi, W)$ for all $U, W \in \mathfrak{X}(L)$ and $Y \in \mathcal{S}$. Using that $\omega$ is $\widetilde{g}$-equivalent to $\xi$, any $V \in \mathfrak{X}(L)$ can be written $V = \omega(V)\xi + Y$. Then,

$$
\begin{aligned}
\widetilde{g}(\widetilde{\nabla}_U V, W) &= U(\omega(V))\omega(W) + \omega(V)\widetilde{g}(\widetilde{\nabla}_U \xi, W) + \widetilde{g}(\widetilde{\nabla}_U Y, W) \\
&= U(\omega(V))\omega(W) + g(\nabla_U V, W).
\end{aligned}
$$

□

Finally, we prove the following relation between screen conformal and distinguished riggings.

**Lemma 2** *Let $L$ be a null hypersurface and $\zeta$ a rigging vector field for it.*

1. *If $\zeta$ is screen conformal and distinguished, then $d\omega = 0$.*

2. *It the screen is totally umbilical and $\zeta$ is distinguished, then $d\omega = 0$.*
3. *If $\zeta$ is a conformal vector field and screen conformal, then it is distinguished. Moreover, if the conformal factor of $\zeta$ never vanishes, then $L$ is totally umbilical*

***Proof*** 1. Since $\zeta$ is screen conformal, then the screen distribution is integrable, so $d\omega(X, Y) = 0$ for all $X, Y \in \mathcal{S}$ and it only remains to check that $d\omega(\xi, X) = 0$ for all $X \in \mathcal{S}$. From the definition of $\omega$ and Proposition 1, we have

$$
\begin{aligned}
d\omega(\xi, X) &= g(\nabla_\xi \zeta, X) - g(\xi, \nabla_X \zeta) = -g(\zeta, \nabla_\xi X) - \tau(X) \\
&= -C(\xi, X) = -\varphi B(\xi, X) = 0.
\end{aligned}
$$

2. The same proof as before, using at the final step $C = \frac{\Omega}{n-2} g$, is valid.
3. Suppose that the rigging vector field is conformal, that is, $L_\zeta g = \lambda g$. From point (4) of Proposition 1, we have

$$
\begin{aligned}
\tau(\xi) &= g(\nabla_\xi \zeta, \xi) = \frac{1}{2}\left(L_\zeta g\right)(\xi, \xi) = 0, \\
\tau(X) &= g(\nabla_X \zeta, \xi) = \left(L_\zeta g\right)(X, \xi) - g(X, \nabla_\xi \zeta) = -g(X, \nabla_\xi \zeta)
\end{aligned}
$$

for all $X \in \mathcal{S}$. Formula (1) implies

$$
g(\nabla_\xi \zeta, X) = g(\nabla_\xi N, X) = C(\xi, X) = 0.
$$

Finally, using point (5) of Proposition 1, we have

$$
\lambda g(X, Y) + 2\left(\varphi + g(\zeta, \zeta)\right) B(X, Y) = 0
$$

for all $X, Y \in \mathcal{S}$. If $\varphi + g(\zeta, \zeta)$ vanishes, then $\lambda$ also vanishes, so

$$
B(X, Y) = -\frac{\lambda}{2(\varphi + g(\zeta, \zeta))} g(X, Y)
$$

and thus $L$ is totally umbilical.

□

## 3 Codimension Two Spacelike Submanifold Through a Null Hypersurface

Throughout this section, we suppose that $L$ is a null hypersurface of a Lorentzian manifold, $\zeta$ is a rigging vector field for it and $\Sigma$ is a codimension two spacelike submanifold of $M$ contained in $L$. Since $(T_x\Sigma)^\perp$ is a Lorentzian plane, we can suppose that

$$
(T\Sigma)^\perp = span\{\xi, \eta\},
$$

where $\eta$ is the unique null vector field over the inclusion $i : \Sigma \to M$, orthogonal to $\Sigma$ with $g(\xi, \eta) = 1$. If we use the decompositions (2) and (3), then

$$\eta = X_0 + \alpha\xi + N, \tag{9}$$

where $X_0 \in \mathcal{S}$ and $\alpha = -\frac{1}{2}g(X_0, X_0) = g(\eta, N)$. It is easy to check that $X_0 \in (T\Sigma \cap \mathcal{S})^{\perp}$ and

$$T\Sigma = (T\Sigma \cap \mathcal{S}) \oplus_{orth(g)} span(V_0),$$

where $V_0 = X_0 + 2\alpha\xi$. Moreover, $V_0 = -\mathcal{P}_\Sigma(N) = -\mathcal{P}_\Sigma(\zeta)$ and $X_0 = \mathcal{P}_\mathcal{S}(V_0)$, being

$$\mathcal{P}_\Sigma : TM|_\Sigma \to T\Sigma \;\; and \;\; \mathcal{P}_\mathcal{S} : TL \to \mathcal{S} \tag{10}$$

the canonical projections according to decompositions (2) and (3).

Observe that $\alpha \leq 0$ because $N$ and $\eta$ are in the same cone and given $p \in \Sigma$, it holds $X_0(p) = V_0(p) = 0$ if and only if $T_p\Sigma = \mathcal{S}_p$.

The following lemma is straightforward.

**Lemma 3** *The second fundamental form of $\Sigma$ in $(M, g)$ is given by*

$$\mathbb{I}_\Sigma(U, V) = g(A_\eta(U), V)\xi + B(U, V)\eta \tag{11}$$

*for all $U, V \in \mathfrak{X}(\Sigma)$, where $A_\eta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by $A_\eta(U) = -(\nabla_U \eta)^{T\Sigma}$.*

Therefore, the mean curvature vector field $\mathbf{H}_\Sigma$ of $\Sigma$ as a submanifold of $M$ is

$$\mathbf{H}_\Sigma = tr_\Sigma A_\eta \cdot \xi + H \cdot \eta.$$

Observe that if $\Sigma$ is totally umbilical (geodesic), then $L$ has not to be totally umbilical (geodesic), since Eq. (11) only holds along $\Sigma$. However, the geometric properties of $\Sigma$ and $L$ are related. For example, we have the following.

**Proposition 3** *Let $L$ be a totally umbilical null hypersurface of a Lorentzian manifold admitting a rigging vector field $\zeta$, and $\Sigma$ a codimension two spacelike submanifold through $L$.*

1. *The scalar curvature $\tau_\Sigma$ of $\Sigma$ is*

$$\tau_\Sigma = \sum_{i,j=1}^{n-2} g(R_{e_i e_j} e_j, e_i) + \frac{n-3}{n-2} g(\mathbf{H}_\Sigma, \mathbf{H}_\Sigma)$$

   *where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $T\Sigma$.*
2. *If $M$ is four-dimensional, $K(\Pi) \leq 0$ for all spacelike plane $\Pi$ and $\Sigma$ is compact and simply connected, then the mean curvature vector field $\mathbf{H}_\Sigma$ is spacelike at some point.*

***Proof*** 1. Given $v \in T\Sigma$, Gauss's lemma implies

$$g(R_{ve_i}e_i, v) = g(R^{\Sigma}_{ve_i}e_i, v) - g(\mathbb{I}_{\Sigma}(v, v), \mathbb{I}_{\Sigma}(e_i, e_i)) + g(\mathbb{I}_{\Sigma}(e_i, v), \mathbb{I}_{\Sigma}(e_i, v)).$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n-2} g(R_{ve_i}e_i, v) &= Ric^{\Sigma}(v, v) - g(\mathbb{I}_{\Sigma}(v, v), \mathbf{H}_{\Sigma}) + 2\sum_{i=1}^{n-2} g(A_{\eta}(v), e_i)B(v, e_i) \\ &= Ric^{\Sigma}(v, v) - tr_{\Sigma}A_{\eta}B(v, v) - Hg(A_{\eta}(v), v) \\ &\quad + 2B(v, A_{\eta}(v)).\end{aligned}$$

Since $L$ is totally umbilical, if we take $v = e_i$ and sum the above expression from $i = 1$ to $n - 2$, we get

$$\begin{aligned}\sum_{i,j=1}^{n-2} g(R_{e_ie_j}e_j, e_i) &= \tau^{\Sigma} - 2Htr_{\Sigma}A_{\eta} + \frac{2H}{n-2}tr_{\Sigma}A_{\eta} \\ &= \tau^{\Sigma} - \frac{n-3}{n-2}g(\mathbf{H}_{\Sigma}, \mathbf{H}_{\Sigma}).\end{aligned}$$

2. If $g(\mathbf{H}_{\Sigma}, \mathbf{H}_{\Sigma}) \leq 0$ for all points, then $\tau^{\Sigma} \leq 0$, which is a contradiction because $\Sigma$ is compact and simply connected.

□

If $M$ is time-orientable, it is said that $\Sigma$ is future (past) weakly trapped if $\mathbf{H}_{\Sigma}$ is causal and future-pointing (past-pointing). The second point of Proposition 3 says that $\Sigma$ is not weakly trapped.

**Lemma 4** *The vector field* $E = \frac{1}{\sqrt{1-2\alpha}}(X_0 + \xi) \in \mathfrak{X}(L)$ *is a* $\widetilde{g}$*-unitary and normal vector field to* $\Sigma$ *as a hypersurface of* $(L, \widetilde{g})$.

***Proof*** Given $v \in T\Sigma$, we have $g(v, X_0) = g(v, \eta - \alpha\xi - N) = -g(v, N) = -\omega(v) = -\widetilde{g}(v, \xi)$. Therefore, $\widetilde{g}(v, X_0 + \xi) = 0$. □

It immediately follows that the function $\alpha$ is given by

$$\alpha = -\frac{1}{2}\tan^2\theta, \tag{12}$$

where $\theta \in [0, \frac{\pi}{2})$ is the $\widetilde{g}$-angle between $T_x\Sigma$ and $\mathcal{S}_x$. Therefore, we have $E = \cos\theta\ (X_0 + \xi)$.

Now, we compute the mean curvature $\widetilde{H}_{\Sigma}$ of $\Sigma$ as a hypersuperface of $(L, \widetilde{g})$.

**Proposition 4** *Let* $L$ *be a null hypersurface of a Lorentzian manifold and* $\zeta$ *a rigging vector field for it such that* $d\omega = 0$. *If* $\Sigma$ *is a spacelike codimension two submanifold* *of $M$ through $L$, then the mean curvature $\widetilde{H}_\Sigma$ of $\Sigma$ respect to $E$ holds*

$$\frac{\widetilde{H}_\Sigma}{\cos\theta} = g(\mathbf{H}_\Sigma, N) - \Omega - B(X_0, X_0) + \frac{1}{\cos^2\theta} H \\ + \cos^2\theta\,(C(X_0, X_0) - g(\mathbb{I}_\Sigma(V_0, V_0), N) - \tau(X_0 + V_0))\,,$$

*where $\theta$ is the $\widetilde{g}$-angle between $T_p\Sigma$ and $\mathcal{S}_p$, and $V_0 = -\mathcal{P}_\Sigma(\zeta)$ and $X_0 = \mathcal{P}_\mathcal{S}(V_0)$, where $\mathcal{P}_\Sigma$ and $\mathcal{P}_\mathcal{S}$ are the canonical projection given in (10).*

***Proof*** We know that $T\Sigma = (T\Sigma \cap \mathcal{S}) \oplus_{orth(\widetilde{g})} span(V_0)$, where

$$V_0 = -\mathcal{P}_\Sigma(\zeta) = X_0 + 2\alpha\xi \in T\Sigma.$$

Fix a point $p \in \Sigma$ and suppose that $V_0(p) \neq 0$, that is, $\Sigma$ is not tangent to the leaf of $\mathcal{S}$ at $p$. If $\{v_1, \ldots, v_{n-3}\}$ is an orthonormal basis of $T_p\Sigma \cap \mathcal{S}_p$, then $\{v_1, \ldots, v_{n-3}, v_{n-2} = \frac{v_0}{\sqrt{\widetilde{g}(v_0,v_0)}}\}$ is a $\widetilde{g}$-orthonormal basis of $T_p\Sigma$, where $v_0 = V_0(p)$. From Proposition 2

$$\widetilde{g}(\widetilde{\nabla}_U\xi, V) = g(\nabla_U\xi, V) = -B(U, V), \\ \widetilde{g}(\widetilde{\nabla}_U X, V) = g(\nabla_U X, V)$$

for all $U, V \in \mathfrak{X}(L)$ and $X \in \mathcal{S}$. Therefore,

$$\begin{aligned}\sqrt{1-2\alpha}\sum_{i=1}^{n-3}\widetilde{g}\left(\widetilde{\nabla}_{v_i}E, v_i\right) &= \sum_{i=1}^{n-3}\widetilde{g}\left(\widetilde{\nabla}_{v_i}(X_0+\xi), v_i\right)\\ &= \sum_{i=1}^{n-3} g(\nabla_{v_i}X_0, v_i) - B(v_i, v_i)\\ &= \sum_{i=1}^{n-3} g(\nabla_{v_i}\eta, v_i) - \alpha g(\nabla_{v_i}\xi, v_i) - g(\nabla_{v_i}N, v_i)\\ &\quad - B(v_i, v_i)\\ &= \sum_{i=1}^{n-3} g(\nabla_{v_i}\eta, v_i) + (\alpha-1)B(v_i, v_i) + C(v_i, v_i). \quad (13)\end{aligned}$$

Observe that $\{v_1, \ldots, v_{n-3}, \frac{v_0}{\sqrt{g(v_0,v_0)}}\}$ is a $g$-orthonormal basis of $T_p\Sigma$ and $\{v_1, \ldots, v_{n-3}, \frac{X_0}{\sqrt{g(X_0,X_0)}}\}$ is a $g$-orthonormal basis of $\mathcal{S}_p$, so we have

$$-trA_\eta = \sum_{i=1}^{n-3} g(\nabla_{v_i}\eta, v_i) + \frac{1}{g(v_0, v_0)} g(\nabla_{v_0}\eta, v_0), \\ \Omega = \sum_{i=1}^{n-3} C(v_i, v_i) + \frac{1}{g(X_0, X_0)} C(X_0, X_0),$$

$$H = \sum_{i=1}^{n-3} B(v_i, v_i) + \frac{1}{g(X_0, X_0)} B(X_0, X_0).$$

Since $g(v_0, v_0) = g(X_0, X_0)$, we get

$$\sqrt{1-2\alpha} \sum_{i=1}^{n-3} \widetilde{g}\left(\widetilde{\nabla}_{v_i} E, v_i\right) = -tr A_\eta + (\alpha - 1)H + \Omega$$
$$- \frac{1}{g(X_0, X_0)} \Big( g(\nabla_{v_0}\eta, v_0) + (\alpha - 1)B(X_0, X_0) + C(X_0, X_0) \Big).$$

Analogously,

$$\frac{\sqrt{1-2\alpha}}{\widetilde{g}(v_0, v_0)} \widetilde{g}\left(\widetilde{\nabla}_{v_0} E, v_0\right) = \frac{1}{\widetilde{g}(v_0, v_0)} \Big( g(\nabla_{v_0}\eta, v_0) + (\alpha - 1)B(v_0, v_0) - g(\nabla_{v_0} N, v_0) \Big)$$
$$= \frac{1}{\widetilde{g}(v_0, v_0)} \Big( g(\nabla_{v_0}\eta, v_0) + (\alpha - 1)B(X_0, X_0) - 2\alpha\tau(v_0) + C(v_0, X_0) \Big).$$

By Proposition 2, we have that $C(\xi, X_0) = -\tau(X_0)$, so the above expression is

$$\frac{1}{\widetilde{g}(v_0, v_0)} \big( g(\nabla_{v_0}\eta, v_0) + (\alpha - 1)B(X_0, X_0) + C(X_0, X_0) - 2\alpha\tau(X_0 + v_0) \big).$$

Now,

$$\sqrt{1-2\alpha}\, \widetilde{H}_\Sigma = tr A_\eta - (\alpha - 1)H - \Omega + \frac{2\alpha}{\widetilde{g}(v_0, v_0)} \tau(X_0 + v_0)$$
$$+ \kappa \left( g(\nabla_{v_0}\eta, v_0) + (\alpha - 1)B(X_0, X_0) + C(X_0, X_0) \right),$$

where $\kappa = \frac{1}{g(v_0,v_0)} - \frac{1}{\widetilde{g}(v_0,v_0)} = \frac{1}{1-2\alpha} = \cos^2\theta$. Since $\frac{2\alpha}{\widetilde{g}(v_0,v_0)} = \frac{1}{2\alpha-1} = -\cos^2\theta$, $g(\mathbb{I}_\Sigma(v_0, v_0), N) = -g(\nabla_{v_0}\eta, v_0) + \alpha B(X_0, X_0)$ and $g(\mathbf{H}_\Sigma, N) = tr A_\eta + \alpha H$, we get the desired formula.

Finally, if $p \in \Sigma$ is a point such that $V_0(p) = 0$, then we can take an orthonormal basis of $T_p\Sigma = \mathcal{S}_p$ to compute the mean curvature as in formula (13). Since in this case $V_0(p) = X_0(p) = \alpha(p) = \theta(p) = 0$, we obtain the same formula as in the proposition. □

For the following definition, recall that the rigged vector field $\xi$ is always pre-geodesic, so the null geodesics with initial velocity given by $\xi$ are contained, at least locally, in the null hypersurface.

**Definition 1** Suppose that $\zeta$ is a rigging vector field for a null hypersurface $L$ with integrable screen distribution and take $S$ a leaf of the screen distribution. The signed distance function of $S$ respect to $\zeta$ is

$$d_S^\zeta = \Pi \circ \Phi^{-1},$$

where $\Phi$ is the diffeomorphism $\Phi : (-\varepsilon, \varepsilon) \times U \to V$ given by $\Phi(t, p) = \exp_p(t\xi_p)$, being $U \subset S$ and $V \subset L$ open neighborhoods, and $\Pi$ is the projection onto the first factor.

Take $p_0 \in \Sigma$ and $S$ the leaf of the screen distribution through $p_0$. If $d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ is "on one side of $S$" at least locally. Moreover, $\Sigma$ and $S$ are tangent at the point $p_0$, since $p_0$ is a local minimum of $d_S^\zeta$. In particular, it holds $E_{p_0} = \xi_{p_0}$.

Now, we can combine Propositions 2, 4 and [6, Theorem 1] to get the following.

**Proposition 5** *Let $L$ be a null hypersurface of a Lorentzian manifold and $\zeta$ a rigging vector field for it such that $d\omega = 0$ and $\Sigma$ is a spacelike codimension two submanifold of $M$ through $L$. Take a point $p_0 \in \Sigma$, $S$ the leaf of the screen distribution through $p_0$ and suppose that:*

1. *$d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$.*
2. *$a \leq H$ for some $a \in \mathbb{R}$ in a neighborhood of $p_0$ in $S$.*
3. *In a neighborhood of $p_0$ in $\Sigma$, it holds*

$$\begin{aligned} & g(\mathbf{H}_\Sigma, N) - \Omega - B(X_0, X_0) + \frac{1}{\cos^2\theta} H \\ & \quad + \cos^2\theta \left( C(X_0, X_0) - g(\mathbb{I}_\Sigma(V_0, V_0), N) - \tau(X_0 + V_0) \right) \leq \frac{a}{\cos\theta}, \end{aligned} \tag{14}$$

*where $\theta$ is the $\widetilde{g}$-angle between $T_p\Sigma$ and $\mathcal{S}_p$ and $V_0 = -\mathcal{P}_\Sigma(\zeta)$ and $X_0 = \mathcal{P}_\mathcal{S}(V_0)$ where $\mathcal{P}_\Sigma$ and $\mathcal{P}_\mathcal{S}$ are defined in (10).*

*Then, $\Sigma$ coincides with the leaf $S$ and $H = a$ in a neighborhood of $p_0$.*

This proposition is not directly applicable because to check inequality (14), we have to know $X_0$ and $V_0$, but recall that $X_0 = 0$ if and only if $\Sigma$ is tangent to the screen distribution and this is just the conclusion that we want to obtain. However, under some additional hypotheses we can avoid this issue, as in the following theorems which are the main results of this paper.

**Theorem 1** *Let $L$ be a null hypersurface of a Lorentzian manifold, $\zeta$ a rigging vector field for it and $\Sigma$ a spacelike totally geodesic codimension two submanifold of $M$ through $L$. Take a point $p_0 \in \Sigma$ and let $S$ be the leaf of the screen distribution through $p_0$. Suppose that*

1. *$\zeta$ is distinguished.*
2. *$\zeta$ is screen conformal.*
3. *$d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$.*
4. *$H(p) \geq 0$ for all $p$ in a neighborhood of $p_0$ in $S$.*

*Then, $\Sigma$ coincides with the leaf $S$ in a neighborhood of $p_0$.*

***Proof*** Being $\zeta$ screen conformal and distinguished, we have from Lemma 2 that $d\omega = 0$. Since $\Sigma$ is totally geodesic, we have from Eq. (11) that $B(U, V) = 0$ for all $U, V \in \mathfrak{X}(\Sigma)$ and $H(p) = 0$ for all $p \in \Sigma$. Thus, $B(X_0, X_0) = B(V_0, V_0) = 0$ and so the left side of inequality (14) is zero. Now, we apply Proposition 5 with $a = 0$. □

**Theorem 2** *Let $L$ be a null hypersurface of a Lorentzian manifold, $\zeta$ a rigging vector field for it and $\Sigma$ a spacelike totally umbilical codimension two submanifold of $M$ through $L$. Take a point $p_0 \in \Sigma$ and let $S$ be the leaf of the screen distribution through $p_0$. Suppose that*

1. *$\zeta$ is distinguished.*
2. *$\zeta$ is screen conformal with conformal factor $\varphi$.*
3. *$dH = c\omega$ for some non-positive function $c \in C^\infty(L)$.*
4. *$d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$.*
5. *$H(p_0) \leq 0$.*
6. *$g(\mathbf{H}_\Sigma, N) \leq \varphi H$ in a neighborhood of $p_0$ in $\Sigma$.*

*Then, $\Sigma$ coincides with the leaf $S$ in a neighborhood of $p_0$.*

***Proof*** We know that $d\omega = 0$ from Lemma 2. Since $\mathbb{I}_\Sigma = \frac{\mathbf{H}_\Sigma}{n-2} g$ and $g(V_0, V_0) = g(X_0, X_0) = \tan^2\theta$, then from Eqs. (11) and (13), we get $B(X_0, X_0) = B(V_0, V_0) = \frac{\tan^2\theta}{n-2} H$. Therefore, the left side of inequality (14) is

$$g(\mathbf{H}_\Sigma, N) - \varphi H - \frac{\tan^2\theta}{n-2} H + \frac{1}{\cos^2\theta} H + \frac{\varphi \sin^2\theta}{n-2} H - \frac{\sin^2\theta}{n-2} g(\mathbf{H}_\Sigma, N)$$
$$= \left(1 - \frac{\sin^2\theta}{n-2}\right)(g(\mathbf{H}_\Sigma, N) - \varphi H) + \left(\frac{1}{\cos^2\theta} - \frac{\tan^2\theta}{n-2}\right) H$$
$$\leq \left(1 + \frac{(n-3)\tan^2\theta}{n-2}\right) H.$$

Since $dH = c\omega$ with $c \leq 0$, the null mean curvature is constant through the leaves of the screen distribution and it is decreasing along the integral curves of $\xi$. Thus, since $d_S^\zeta(p) \geq 0$ for all $p$ in a neighborhood of $p_0$ in $\Sigma$, then it holds $H(p) \leq H(p_0)$ and the above expression is

$$\leq \left(1 + \frac{(n-3)\tan^2\theta}{n-2}\right) H(p_0).$$

Finally, since $H(p_0) \leq 0$ and $\frac{1}{\cos\theta} \leq 1 + \frac{(n-3)\tan^2\theta}{n-2}$ (to check this last inequality just observe that $1 - \frac{1}{\cos\theta} + \frac{(n-3)\tan^2\theta}{n-2}$ is non-decreasing for $\theta \in [0, \frac{\pi}{2})$ and it takes the value zero when $\theta = 0$), we get

$$\left(1 + \frac{(n-3)\tan^2\theta}{n-2}\right) H(p_0) \leq \frac{1}{\cos\theta} H(p_0).$$

Now, apply Proposition 5 with $a = H(p_0)$ to obtain the result. □

Using Lemma 1, it is easy to check that if we change the sign of the rigging vector field, then conditions 1, 2 and 3 of the above theorem still hold, but the inequalities in conditions 4, 5 and 6 change. On the other hand, recall that if $\xi$ is geodesic, i.e., $\tau(\xi) = 0$, using the null Raychaudhuri equation

$$dH(\xi) = \xi(H) = Ric(\xi, \xi) + tr((A^*)^2) \geq Ric(\xi, \xi).$$

Thus, if $Ric(\xi, \xi) \geq 0$ and $H$ is not constant, condition 3 can not hold. This is why Theorem 2 cannot be used, for example, in the case of a null cone in a constant curvature Lorentzian manifold.

**Corollary 1** *Let $L$ be a null hypersurface with zero null mean curvature of a Lorentzian time-orientable manifold and $\zeta$ a rigging vector field for it such that:*

1. *$\zeta$ is distinguished.*
2. *$\zeta$ is screen conformal.*
3. *$\xi$ is future-pointing.*

*Suppose that $\Sigma$ is a spacelike totally umbilical codimension two submanifold of $M$ through $L$. If $d_S^\zeta \geq 0$ in a neighborhood of a point $p_0 \in \Sigma$, where $S$ is the leaf of the screen distribution through $p_0$, then $\Sigma$ is not past weakly trapped*

***Proof*** If $\mathbf{H}_\Sigma$ is causal and past-pointing, then $g(\mathbf{H}_\Sigma, N) \leq 0$ and from Theorem 2, $\Sigma$ coincides with the leaf $S$ in a neighborhood of $p_0$. Using (8) and $C = \varphi B$, we get $\mathbf{H}_\Sigma = 0$, which is a contradiction. □

We can avoid the hypothesis about $d_S^\zeta$ in Theorem 2 as follows.

**Corollary 2** *Let $L$ be a null hypersurface of a Lorentzian manifold, $\zeta$ a rigging vector field for it and $\Sigma$ a spacelike totally umbilical codimension two submanifold of $M$ through $L$. Take a point $p_0 \in \Sigma$ and suppose that*

1. *$\zeta$ is distinguished.*
2. *$\zeta$ is screen conformal with conformal factor $\varphi$.*
3. *$dH = c\omega$ for some negative function $c \in C^\infty(L)$.*
4. *$H$ restricted to $\Sigma$ attains a local maximum at $p_0$.*
5. *$H(p_0) \leq 0$.*
6. *$g(\mathbf{H}_\Sigma, N) \leq \varphi H$ in a neighborhood of $p_0$ in $\Sigma$.*

*Then, $\Sigma$ coincides with a leaf of the screen distribution in a neighborhood of $p_0$.*

***Proof*** The null mean curvature $H$ is strictly decreasing along the integral curves of $\xi$ because the function $c$ is negative. Since $H$ is constant through the leaves of the screen distribution and $H|_\Sigma$ attains a local maximum at $p_0$, then $d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$, being $S$ the leaf of the screen distribution through $p_0$. Now, apply Theorem 2. □

**Corollary 3** *Let $L$ be a null hypersurface of a Lorentzian manifold and $\zeta$ a rigging vector field for it such that*

1. *$\zeta$ is distinguished.*
2. *$\zeta$ is screen conformal with conformal factor $\varphi$.*

3. *$dH = c\omega$ for some negative function $c \in C^\infty(L)$.*
4. *$H \leq 0$.*

*If $\Sigma$ is a compact spacelike totally umbilical codimension two submanifold of $M$ through $L$ such that $g(\mathbf{H}_\Sigma, N) \leq \varphi H$ for all $p \in \Sigma$, then $\Sigma$ coincides with a leaf of the screen distribution.*

**Theorem 3** *Let $L$ be a totally geodesic null hypersurface of a Lorentzian manifold and $\zeta$ a rigging for it such that*

1. *$\zeta$ is distinguished.*
2. *The screen is totally umbilical.*

*Suppose that $\Sigma$ is a spacelike totally umbilical codimension two submanifold of $M$ through $L$ and there is a point $p_0 \in \Sigma$ such that $d_S^\zeta \geq 0$ in a neighborhood of $p_0$ in $\Sigma$, where $S$ is the leaf of the screen distribution through $p_0$.*

*If $g(\mathbf{H}_\Sigma, \zeta) \leq \Omega$ in a neighborhood of $p_0$, then $\Sigma$ coincides with the leaf $S$ of the screen distribution in a neighborhood of $p_0$.*

***Proof*** From Lemma 2, we have that $d\omega = 0$. The left side of inequality (14) is

$$\left(1 - \frac{\sin^2\theta}{n-2}\right)(g(\mathbf{H}_\Sigma, N) - \Omega).$$

Since $L$ is totally geodesic, $g(\mathbf{H}_\Sigma, N) = g(\mathbf{H}_\Sigma, \zeta)$ and we can apply Proposition 5 with $a = 0$. □

The conditions of being screen conformal and distinguished assumed above are intended to simplify the geometric objects involved in a null hypersurface. They mean that the extrinsic geometry of the null hypersurface $L$ and that of the leaves of the screen distributions as submanifolds of the ambient space $M$, are codified by the same tensor $B$, see Eqs. (4) (5) and (6). On the other hand, conditions 3, 5 and 6 in Theorem 2 are imposed to get the inequality between the mean curvature $\widetilde{H}_\Sigma$ and $H$ in order to apply the Eschenburg maximum principle. If one of them is not fulfilled, then the inequalities sequence in the proof is broken and we cannot get the conclusion.

An example is the following. Consider $\mathbb{L}^3 = \left(\mathbb{R}^3, -dt^2 + dx^2 + dy^2\right)$, $L = \{(t, x, y) \in \mathbb{R}^3 : t = x\}$ a totally geodesic null plane and the rigging $\zeta = -\partial_t$. The leaf $S$ of the screen distribution through the origin is the $z$-axis. If we take $\Sigma$ some curve with $d_S^\zeta \geq 0$ and tangent to $S$ at the origin, then it is totally umbilic because it is one-dimensional and all conditions in Theorem 2, except condition 6, are fulfilled, obtaining a counterexample in this case.

Certainly, we need to assume many conditions in the above results, but there are examples where we can apply them.

***Example 1*** Let $(F, g_0)$ be a Riemannian manifold with dimension $n-1$ and define a generalized Robertson–Walker space

$$(M, g) = \left(I \times F, -dt^2 + \phi(t)^2 g_0\right).$$

Suppose that $(F, g_0)$ can be decomposed as a warped product with one-dimensional base, $(F, g_0) = (J \times K, ds^2 + \mu(s)^2 h_0)$, where $J \subset \mathbb{R}$ and $(K, h_0)$ are a Riemannian manifold. The hypersurface $L$ given by

$$L = \left\{ (t, s, x) \in I \times J \times K : s = \int_{t_*}^{t} \frac{1}{\phi(r)} \mathrm{d}r \right\}$$

for some fixed $t_* \in I$ is a totally umbilical null hypersurface, i.e., $B = \frac{H}{n-2} g$, [10]. Moreover, if we consider the rigging vector field $\zeta = \phi \partial_t$, then the null mean curvature is

$$H_{(t,s,x)} = \frac{n-2}{\phi(t)^2} \left( \phi'(t) + \frac{\mu'(s)}{\mu(s)} \right). \tag{15}$$

Since $\zeta$ is a closed and conformal vector field, then $\nabla_U \zeta = \Phi' U$ for all $U \in \mathfrak{X}(M)$ and from Proposition 1 we have that

$$\begin{aligned} C &= \left( \frac{H\phi^2}{2(n-2)} - \phi' \right) g, \\ \Omega &= \frac{H\phi^2}{2} - (n-2)\phi', \\ \tau &= 0. \end{aligned}$$

Therefore, the rigging vector field $\zeta$ is distinguished and if $H \neq 0$, then it is also screen conformal with factor

$$\varphi = \frac{\phi^2}{2} - \frac{(n-2)\phi'}{H}.$$

On the other hand, the leaf of the screen distribution through a point $p_0 = (t_0, s_0, x_0) \in L$ is given by $S = \{(t, s, x) \in I \times J \times K : t = t_0, s = s_0\}$, thus from Eq. (15) we see that $H$ is constant on the leaves and then $dH = c\omega$ for some $c \in C^\infty(L)$.

Since $\xi = -\frac{1}{\phi}\partial_t - \frac{1}{\phi^2}\partial_s$, if we fix $p_0 \in L$ and $S$ the leaf through $p_0$, the condition $d_S^\zeta(p) \geq 0$ is equivalent to $t(p) \leq t(p_0)$, where $t : M \to \mathbb{R}$ is the canonical projection onto the first factor. Moreover, the transverse vector field is $N = \frac{1}{2}(\phi\partial_t - \partial_s)$.

We particularize the above situation to the case of the Lorentzian manifold $(M, g) = \left(\mathbb{R} \times \mathbb{H}^{n-1}, -dt^2 + g_0\right)$. The hyperbolic space $\mathbb{H}^{n-1}$ can be decomposed as

$$\left( \mathbb{R} \times \mathbb{R}^{n-2}, ds^2 + e^{-2s} h_0 \right),$$

being $h_0$ the Euclidean metric. The null hypersurface $L$ is given in this case by

$$L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\}$$

and its null mean curvature respect to the rigging vector field $\zeta = \partial_t$ is $H = 2 - n$. Therefore, conditions 1, 2 and 3 in Theorem 2 are fulfilled and we can apply it to get the following.

Suppose that $\Sigma$ is a codimension two totally umbilic spacelike hypersurface in $\mathbb{R} \times \mathbb{H}^{n-1}$ contained in $L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\}$. If there is a point $p_0 = (t_0, t_0, x_0) \in \Sigma$ such that $t(p) \leq t_0$ and $g(\mathbf{H}_\Sigma, \partial_t - \partial_s) \leq 2 - n$ for all $p$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ is locally contained in $\{(t_0, t_0, x) : x \in \mathbb{R}^{n-2}\}$.

***Example 2*** Let $\varpi > 0$ be a constant and $Q = \{(u, v) \in \mathbb{R}^2 : -\frac{2\varpi}{e} < uv\}$. Take the functions $F(r) = \frac{8\varpi^2}{r} e^{1-\frac{r}{2\varpi}}$, $f(r) = (r - 2\varpi)e^{\frac{r}{2\varpi}-1}$ for $0 \leq r$ and $r(u, v) = f^{-1}(uv)$ for $(u, v) \in Q$. The Kruskal space is the product $Q \times \mathbb{S}^{n-2}$ endowed with the metric

$$2F(r)dudv + r^2 g_0,$$

where $g_0$ is the standard metric in $\mathbb{S}^{n-2}$. We call $u, v : Q \times \mathbb{S}^{n-2} \to \mathbb{R}$ the canonical projections. The hypersurface

$$L = \{p \in Q \times \mathbb{S}^{n-2} : u(p) = 0\}$$

is a null hypersurface, and $\zeta = \partial_u$ is a rigging vector field for it. The rigged vector field is $\xi = \frac{1}{F}\partial_v$ and the null transverse vector field is $N = \zeta$. Observe that $d\omega = 0$, although $\zeta$ is not closed, and the leaf of the screen distribution through a point $p_0 = (0, v_0, x_0) \in L$ is $S = \{(0, v_0, x) : x \in \mathbb{S}^{n-2}\}$. Therefore, in this case, the condition $d_S^\zeta(p) \geq 0$ is equivalent to $v(p_0) \leq v(p)$.

A direct computation shows that $L$ is totally geodesic. Moreover, using that $r = 2\varpi$ through $L$ and Proposition 1, we have

$$\tau = 0,$$
$$C = -\frac{v}{2\varpi} g,$$
$$\Omega = -\frac{(n-2)v}{2\varpi}.$$

Using Theorem 3, if $\Sigma$ is a codimension two spacelike totally umbilical submanifold contained in $L$ and $p_0 \in \Sigma$ holds $v(p_0) \leq v(p)$ and $g(\mathbf{H}_\Sigma, \partial_u) \leq \frac{2-n}{2\varpi} v$ for all $p$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ is locally contained in the sphere $\{p \in Q \times \mathbb{S}^{n-2} : u(p) = 0, v(p) = v(p_0)\}$.

We can give another result ensuring the coincidence of a codimension two spacelike submanifold and a leaf of the screen without using the rigged metric, but we have to suppose that the null hypersurface has zero null mean curvature and that it exists a rigging vector field which is a gradient. In this case, the screen distribution is integrable and the leaves are given by the intersection of the level hypersurfaces of the function and the null hypersuperface.

**Proposition 6** *Let $L$ be a null hypersurface with zero null mean curvature and $f \in C^\infty(M)$ a function such that $\zeta = \nabla f$ is a distinguished rigging vector field for $L$. If $\Sigma$ is a codimension two spacelike submanifold through $L$ and $p_0 \in \Sigma$ is a point such that*

$$f(p_0) \le f(p),$$
$$g(\mathbf{H}_\Sigma, \nabla f) + \triangle f + \xi g(\nabla f, \nabla f) \le 0$$

*for all $p$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ coincides with a leaf of the screen distribution in a neighborhood of $p_0$.*

***Proof*** If we call $i : \Sigma \to M$ the canonical inclusion, then $\nabla f = \nabla^\Sigma(f \circ i) + g(\nabla f, \eta)\xi + \eta$. Take $\{e_1, \dots, e_{n-2}\}$ an orthonormal basis of $T_p\Sigma$. We have

$$\begin{aligned}\triangle^\Sigma(f \circ i) &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\nabla^\Sigma(f \circ i), e_i) = \sum_{i=1}^{n-2} g(\nabla_{e_i}\left(\nabla f - g(\nabla f, \eta)\xi - \eta\right), e_i) \\ &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\nabla f, e_i) - g(\nabla f, \eta) g(\nabla_{e_i}\xi, e_i) - g(\nabla_{e_i}\eta, e_i) \\ &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\nabla f, e_i) + g(\nabla f, \eta)H + g(\mathbf{H}_\Sigma, \eta).\end{aligned}$$

Since we are assuming that $H = 0$, then $\mathbf{H}_\Sigma = tr_\Sigma A_\eta \cdot \xi$ and so $g(\mathbf{H}_\Sigma, \eta) = g(\mathbf{H}_\Sigma, \nabla f)$. Therefore,

$$\triangle^\Sigma(f \circ i) = \triangle f + 2g(\nabla_\eta \nabla f, \xi) + g(\mathbf{H}_\Sigma, \nabla f).$$

Since $\tau = 0$, using Eqs. (1) and (9) and Proposition 1, we get that

$$2g(\nabla_\eta \nabla f, \xi) = 2g(\nabla_{\nabla f}\nabla f, \xi) = 2g(\nabla_\xi \nabla f, \nabla f) = \xi g(\nabla f, \nabla f).$$

Therefore, $f \circ i$ has a minimum at $p_0$ and $\triangle^\Sigma(f \circ i) \le 0$, so $f \circ i$ is constant in a neighborhood of $p_0$, i.e., $\Sigma$ coincides with the leaf of the screen distribution induced from $\nabla f$ in a neighborhood of $p_0$. □

***Example 3*** A plane fronted wave is the Lorentzian manifold $M = M_0 \times \mathbb{R}^2$ endowed with the metric

$$g = g_0 + 2dudv + \phi(x, u)du^2,$$

where $(M_0, g_0)$ is a Riemannian manifold and $\phi : M_0 \times \mathbb{R} \to \mathbb{R}$ is some function. It holds that $\partial_v$ is a parallel null vector field in $M$.

Call $u, v : M \to \mathbb{R}$ the canonical projections. We have that

$$L_{u_0} = \{p \in M : u(p) = u_0\}$$

is a totally geodesic null hypersurface for all $u_0 \in \mathbb{R}$. The vector field $\zeta = \nabla v = \partial_u - \Phi\partial_v$ is a rigging vector field for $L_{u_0}$, and its rigged vector field is $\xi = \partial_v$. Moreover, $\tau = 0$ and the leaf of the screen distribution through a point $p_0 \in L_{u_0}$ is $S_{p_0} = \{p \in M : u(p) = u_0, v(p) = v(p_0)\}$.

Since $\triangle v = 0$ and $\partial_v\,(g(\nabla v, \nabla v)) = 0$, from the above proposition, if $\Sigma$ is a codimension two spacelike submanifold contained in $L_{u_0}$ and there is a point $p_0 \in \Sigma$ such that $v(p_0) \leq v(p)$ and $g(\mathbf{H}_\Sigma, \partial_u) \leq 0$ for all $p$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ is locally contained in $S_{p_0}$.

***Example 4*** Suppose that $L$ is a null hypersurface with zero null mean curvature in a generalized Robertson–Walker space $(M, g) = \big(I \times F, -dt^2 + \phi(t)^2 g_0\big)$.

We call $f : M \to \mathbb{R}$ the function given by $f(p) = -\int_c^{t(p)} \phi(s)ds$, being $c \in I$ a fixed point and $t : M \to \mathbb{R}$ the canonical projection. We know that $\zeta = \nabla f = \phi\partial_t$ is a distinguished rigging vector field for $L$. Since $\triangle f + \xi(g(\nabla f, \nabla f)) = (n-2)\phi'$, Proposition 6 implies that if $\Sigma$ is a codimension two spacelike submanifold through $L$ and $p_0 \in \Sigma$ holds $t(p) \leq t(p_0)$ and $g(\mathbf{H}_\Sigma, \partial_t) \leq -\frac{(n-2)\phi'}{\phi}$ in a neighborhood of $p_0$ in $\Sigma$, then $\Sigma$ is contained in the slice $t = t(p_0)$ in a neighborhood of $p_0$.

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