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Radial Solutions for p-Laplacian Neumann Problems Involving Gradient Term Without Growth Restrictions

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Abstract

We study the existence of radial solutions for the p-Laplacian Neumann problem with gradient term of the type

$$\begin{cases} -\Delta_p u = f(|x|, u, x \cdot \nabla u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator with $p > 1$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a ball. We do not impose any growth restrictions on the nonlinearity. By using the topological transversality method together with the barrier strip technique, the existence of radial solutions to the above problem is obtained.

Keywords p-Laplacian Neumann problem · Existence of radial solutions · Barrier strip technique · Topological transversality method

Mathematics Subject Classification 35J92 · 35J62 · 34B15

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1 Introduction

In this paper, we study the existence of radial solutions to the p-Laplacian Neumann problem with gradient term of the form

$$\begin{cases} -\Delta_p u = f(|x|, u, x \cdot \nabla u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator with $p > 1$, $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$ with $N \geq 2$, the function $f : [0, R] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $|\cdot|$ indicates the Euclidean norm, and $\mathbf{n}$ is the outward unit normal vector of the boundary $\partial\Omega$.

The typical model equation is, for suitable a, b, g,

$$-\Delta_p u + b(|x|)x \cdot \nabla u + |u|^{p-2}u = a(|x|)g(u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$ and $p > 1$.

This kind of equations with Neumann boundary conditions and $p = 2$ has been studied extensively via various methods in the literature. Particularly, for the case of $p = 2$ and $b(\cdot) \equiv 0$, see Serra and Tilli [13], Bonheure et al. [4] and the references therein, for the case of $p = 2$ and $b(\cdot) \not\equiv 0$, see Bonheure et al. [3], Ma et al. [8] and the references therein. However, Eq. (1.2) with Neumann boundary conditions and $p \neq 2$ does not seem to have been deeply investigated. The only result that we are aware of is that of Secchi [12] in case of $p \neq 2$ and $b(\cdot) \equiv 0$. Up to now, we have not seen the solvability results of the radial solution of Eq. (1.2) with Neumann boundary conditions when $p \neq 2$ and $b(\cdot) \not\equiv 0$. For other works concerned with Eq. (1.2) or more general equations on infinite domains, we refer the readers to Yin [16] or Zhang [17], and for the works concerned with Neumann problems involving gradient term, we refer the readers to Cianciaruso [5] and references therein. In addition, see [2,9–11,14,15] and references therein for works concerned with more general equations driven by the (p, q)-Laplace operator or fractional integral operator.

Inspired by [1] and the above literature, in this paper, we establish the existence results of radial solutions of the general p-Laplacian Neumann problem (1.1) with gradient dependence in a ball by using topological transversality method together with barrier strip technique.

It is worth mentioning that since our results require no growth restrictions on the nonlinearity, it can also be applied to strongly nonlinear systems with the term $x \cdot \nabla u$ being super-quadratic. Also we remark here that the p-Laplacian Neumann problem ($p \neq 2$) on the ball with $x \cdot \nabla u$ has not been considered in the literature.

Throughout this paper, we use the following assumptions:

(H_1) There exists $M > 0$ such that

$$sf(r, s, 0) < 0, \quad \forall r \in [0, R], \ |s| > M.$$

(H$_2$) There exist constants $L_i, i = 1, 2, 3, 4$ with $L_3 < L_4 < 0 < L_1 < L_2$, such that

$$f(r, s, rt) + \frac{N-1}{R}\phi_p(t) \geq 0, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [L_1, L_2]$$

and

$$f(r, s, rt) + \frac{N-1}{R}\phi_p(t) \leq 0, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [L_3, L_4],$$

where $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$.

2 Main Results

In order to obtain the existence of radial solutions of problem (1.1), we set $r = |x|$ and $u(x) = v(r)$; then problem (1.1) becomes the following singular scalar p-Laplacian Neumann problem

$$\begin{cases} -(\phi_p(v'(r)))' = f(r, v(r), rv'(r)) + \dfrac{N-1}{r}\phi_p(v'(r)), \quad 0 < r \leq R, \\ v'(0) = 0, \quad v'(R) = 0. \end{cases} \tag{2.1}$$

We will obtain the existence of p-Laplacian Neumann problem (2.1) by using the topological transversality method, which we state here for the convenience of the reader. Let U be a convex subset of a Banach space X and $\mathcal{D} \subset U$ be an open set. Denote by $H_{\partial\mathcal{D}}(\overline{\mathcal{D}}, U)$ the set of compact operators $F : \overline{\mathcal{D}} \to U$ which are fixed point free on $\partial\mathcal{D}$.

Definition 2.1 An operator $F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}}, U)$ is said to be essential if every operator in $H_{\partial\mathcal{D}}(\overline{\mathcal{D}}, U)$ which agrees with F on $\partial\mathcal{D}$ has a fixed point in $\mathcal{D}$.

The next two lemmas can be found in [6].

Lemma 2.1 *If $q \in \mathcal{D}$ and $F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}}, U)$ is a constant operator, $F(x) = q$ for $x \in \overline{\mathcal{D}}$, then F is essential.*

Lemma 2.2 *Let*

(i) *$F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}}, U)$ be essential;*
(ii) *$H : \overline{\mathcal{D}} \times [0, 1] \to U$ be a compact homotopy, $H(\cdot, 0) = F$ and $H(x, \lambda) \neq x$ for $x \in \partial\mathcal{D}$ and $\lambda \in [0, 1]$.*

Then, $H(\cdot, 1)$ is essential and therefore it has a fixed point in $\mathcal{D}$.

Consider the family of the following modified Neumann problem

$$-(\phi_p(v'(r)))' = \lambda\left(f(r, v(r), rv'(r)) + \frac{N-1}{h_n(r)}\phi_p(v'(r))\right), \quad 0 \leq r \leq R, \tag{2.2}$$

$$v'(0) = 0, \quad v'(R) = 0, \tag{2.3}$$

where $\lambda \in (0, 1]$, $n \geq [1/R] + 1 =: n_0$, and

$$h_n(r) = \begin{cases} \frac{1}{n}, & r \in \left[0, \frac{1}{n}\right]; \\ r, & r \in \left(\frac{1}{n}, R\right]. \end{cases}$$

A priori bounds for solutions of Neumann problem (2.2), (2.3) are presented in the following lemmas.

Lemma 2.3 *Assume that* (H_1) *holds. Let* v *be a solution of problem* (2.2), (2.3) *for some* $\lambda \in (0, 1]$, $n \geq n_0$. *Then,*

$$|v(r)| \leq M, \quad \forall r \in [0, R]. \tag{2.4}$$

Proof Suppose on the contrary that there exist $r_0 \in [0, R]$ such that $|v(r_0)| > M$. We may assume that $v(r_0) > M$. Let $r_1 \in [0, R]$ be such that

$$v(r_1) = \max_{r \in [0, R]} v(r) > M. \tag{2.5}$$

Without loss of generality, we assume that $r_1 \in (0, R)$, then $v'(r_1) = 0$. It follows from the condition (H_1) that

$$(\phi_p(v'(r)))'|_{r=r_1} = -\lambda f(r_1, v(r_1), 0) > 0,$$

and thus there exist $\delta > 0$ such that $\phi_p(v'(r))$ is increasing on $(r_1 - \delta, r_1 + \delta) \subset (0, R)$. This together with the monotonicity of $\phi_p(\cdot)$ implies that $v'(r) > 0$, $\forall r \in (r_1, r_1 + \delta)$, which contradicts (2.5). This completes the proof of the lemma. □

We now obtain a priori bounds for $v'(r)$ by applying barrier strip technique due to [7].

Lemma 2.4 *Assume that* (H_1) *and* (H_2) *hold. Let* v *be a solution of problem* (2.2), (2.3) *for some* $\lambda \in (0, 1]$, $n \geq n_0$. *Then,*

$$|v'(r)| \leq M_1 := \max\{L_1, -L_4\}, \quad \forall r \in [0, R]. \tag{2.6}$$

Proof From Lemma 2.3, it follows that

$$|v(r)| \leq M, \quad \forall r \in [0, R].$$

Let

$$S_0 = \{r \in [0, R] : L_1 < v'(r) \leq L_2\}, \quad S_1 = \{r \in [0, R] : L_3 \leq v'(r) < L_4\}.$$

We now assert that the sets S_0 and S_1 are empty. Indeed, suppose on the contrary that $S_0 \neq \emptyset$. Taking $r_0 \in S_0$, then $L_1 < v'(r_0) \leq L_2$ and $0 < r_0 < T$. From the continuity of $v'(r)$ on $[0, R]$, there exist $0 < r_1 < r_2 \leq r_0$ such that

$$L_1 < v'(r_1) < v'(r_2) = v'(r_0) \leq L_2, \tag{2.7}$$

and

$$v'(r_1) \leq v'(r) \leq v'(r_2), \quad \forall r \in [r_1, r_2].$$

Thus, $[r_1, r_2] \subset S_0$, whereas, from assumption (H$_2$), we have

$$\begin{aligned}(\phi_p(v'(r)))' &= -\lambda \left(f(r, v(r), rv'(r)) + \frac{N-1}{h_n(r)} \phi_p(v'(r)) \right) \\ &\leq -\lambda \left(f(r, v(r), rv'(r)) + \frac{N-1}{R} \phi_p(v'(r)) \right) \\ &\leq 0, \quad \forall r \in S_0.\end{aligned}$$

Consequently, $v'(r_2) \leq v'(r_1)$, which contradicts (2.7). This implies that $S_0 = \emptyset$. Similarly, we can show that $S_1 = \emptyset$. Therefore, by the facts that (2.3) and the continuity of $v'(r)$ on $[0, R]$, we obtain

$$L_4 \leq v'(r) \leq L_1, \quad \forall r \in [0, R].$$

This means that (2.6) holds. This completes the proof of the lemma. □

Now, we denote $X = C^1[0, R] \times \mathbb{R}$ the Banach space equipped with the norm $\|(v, \rho)\| = \|v\|_\infty + \|v'\|_\infty + |\rho|$. Set

$$U = \{(v, \rho) \in X : v(0) = 0, \rho \in \mathbb{R}\}$$

and

$$\mathcal{D} = \{(v, \rho) \in U : \|v\|_\infty < 2M + 1, \|v'\|_\infty < M_1 + 1, |\rho| < M + 1\}.$$

Then, U is a closed and convex subset of X and $\mathcal{D}$ is an open subset of U.

Lemma 2.5 *Assume that (H$_1$) holds. For each fixed $n \geq n_0$, let the operator $F : \overline{\mathcal{D}} \to U$ be defined by*

$$F(v, \rho) = \left(0, \rho + \int_0^R \left(f(\tau, v(\tau) + \rho, \tau v'(\tau)) + \frac{N-1}{h_n(\tau)} \phi_p(v'(\tau)) \right) \mathrm{d}\tau \right).$$

Then, F is essential.

Proof Define $H : \overline{\mathcal{D}} \times [0, 1] \to U$ by

$$H(v, \rho, \lambda) = \left(0, \lambda\rho + \lambda \int_0^R \left(f(\tau, v(\tau) + \rho, \tau v'(\tau)) + \frac{N-1}{h_n(\tau)} \phi_p(v'(\tau)) \right) \mathrm{d}\tau \right).$$

Then, $H(\cdot, \cdot, 1) = F(\cdot, \cdot)$, $H(v, \rho, 0) = (0, 0) \in \mathcal{D}$ for $(v, \rho) \in \overline{\mathcal{D}}$, and thus from Lemma 2.1 it follows that $H(v, \rho, 0)$ is essential. Meanwhile, it is easy to show that $H(v, \rho, \lambda)$ is compact by using the Arzelà–Ascoli theorem.

We now show that

$$H(v, \rho, \lambda) \neq (v, \rho), \quad \forall (v, \rho) \in \partial\mathcal{D},\ \lambda \in [0, 1]. \tag{2.8}$$

Obviously, $H(v, \rho, 0) \neq (v, \rho)$ for all $(v, \rho) \in \partial\mathcal{D}$. Suppose that $H(v_0, \rho_0, \lambda_0) = (v_0, \rho_0)$ for some $(v_0, \rho_0) \in \partial\mathcal{D}$ and $\lambda_0 \in (0, 1]$. Then, $v_0 = 0$ and

$$\int_0^R f(\tau, \rho_0, 0)\mathrm{d}\tau = \left(\frac{1}{\lambda_0} - 1\right)\rho_0.$$

Hence, from (H_1), it follows that $|\rho_0| \leq M$, which contradicts $(v_0, \rho_0) \in \partial\mathcal{D}$. This implies that (2.8) holds. Hence, from Lemma 2.2, $F(\cdot, \cdot) = H(\cdot, \cdot, 1)$ is essential. This completes the proof of the lemma. □

Lemma 2.6 *Assume that (H_1) and (H_2) hold. Then, for each fixed $n \geq n_0$, problem* (2.2), (2.3) *with $\lambda = 1$ has a solution $v = v(r)$ satisfying* (2.4), (2.6).

Proof Define the operator $G : \overline{\mathcal{D}} \times [0, 1] \to U$ by

$$G(v, \rho, \lambda) = \begin{pmatrix} -\displaystyle\int_0^r \phi_p^{-1}\left(\lambda \int_0^s \left(f(\tau, v(\tau) + \rho, \tau v'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v'(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ \rho + \displaystyle\int_0^R \left(f(\tau, v(\tau) + \rho, \tau v'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v'(\tau))\right)\mathrm{d}\tau \end{pmatrix}^*,$$

where the symbol "$*$" denotes the transpose of vector.

Suppose that (v_1, ρ_1) is a fixed point of $G(\cdot, \cdot, 1)$. Then, for $r \in [0, R]$,

$$v_1(r) = -\int_0^r \phi_p^{-1}\left(\int_0^s \left(f(\tau, v_1(\tau) + \rho_1, \tau v_1'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_1'(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s$$

and

$$\int_0^R \left(f(\tau, v_1(\tau) + \rho_1, \tau v_1'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_1'(\tau))\right)\mathrm{d}\tau = 0. \tag{2.9}$$

It follows that

$$-\phi_p(v_1'(r)) = \int_0^r \left(f(\tau, v_1(\tau) + \rho_1, \tau v_1'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_1'(\tau))\right)\mathrm{d}\tau, \quad r \in [0, R],$$

and so by (2.9),

$$v_1'(0) = 0, \quad v_1'(R) = 0.$$

Setting $v_2(r) = v_1(r) + \rho_1$ for $r \in [0, R]$, it is easy to see that $v_2(r)$ is a solution of problem (2.2), (2.3) with $\lambda = 1$, and validity of (2.4) and (2.6) now follows from

Lemmas 2.3 and 2.4. Therefore, to prove the existence of a solution of problem (2.2), (2.3) with $\lambda = 1$ satisfying (2.4) and (2.6), it is enough to show that the operator $G(\cdot, \cdot, 1)$ has a fixed point. Since $G(\cdot, \cdot, 0) = F(\cdot, \cdot)$ and F is essential by Lemma 2.5, for the existence of a fixed point of $G(\cdot, \cdot, 1)$ it is sufficient to verify the condition (ii) of Lemma 2.2. Indeed, by the dominated convergence theorem and the Arzelà–Ascoli theorem, it is easy to show that G is continuous and $G(\overline{\mathcal{D}} \times [0, 1])$ is relatively compact in U. Let $G(v_0, \rho_0, \lambda_0) = (v_0, \rho_0)$ for some $(v_0, \rho_0) \in \partial\mathcal{D}$ and $\lambda_0 \in [0, 1]$. If $\lambda_0 = 0$, then $(v_0, \rho_0) \notin \partial\mathcal{D}$, which has been proved in the proof of Lemma 2.5. Let $\lambda_0 \in (0, 1]$, then for $r \in [0, R]$,

$$v_0(r) = -\int_0^r \phi_p^{-1}\left(\lambda_0 \int_0^s \left(f(\tau, v_0(\tau) + \rho_0, \tau v_0'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_0'(\tau))\right) \mathrm{d}\tau\right) \mathrm{d}s$$

and

$$\int_0^R \left(f(\tau, v_0(\tau) + \rho_0, \tau v_0'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_0'(\tau))\right) \mathrm{d}\tau = 0.$$

Hence,

$$\begin{aligned} -\phi_p(v_0'(r)) = \lambda_0 \int_0^r \Big(& f(\tau, v_0(\tau) + \rho_0, \tau v_0'(\tau)) \\ &+ \frac{N-1}{h_n(\tau)}\phi_p(v_0'(\tau))\Big)\mathrm{d}\tau, \quad r \in [0, R], \end{aligned}$$

and thus

$$v_0'(0) = 0, \quad v_0'(R) = 0.$$

Setting $v(r) = v_0(r) + \rho_0$ for $r \in [0, R]$, then we can see that $v(r)$ is a solution of problem (2.2), (2.3) with $\lambda = \lambda_0$. Therefore, it follows from Lemmas 2.3 and 2.4 that

$$\|v_0 + \rho_0\|_\infty = \|v\|_\infty \le M, \quad \|v_0'\|_\infty = \|v'\|_\infty \le M_1 < M_1 + 1. \tag{2.10}$$

Since $v_0(0) = 0$, (2.10) yields $|\rho_0| \le M$, and thus $\|v_0\|_\infty < 2M + 1$. Hence, $(v_0, \rho_0) \notin \partial\mathcal{D}$, and so the condition (ii) is satisfied. This completes the proof of the lemma. □

With the above preparations, now we can prove our main result.

Theorem 2.1 *Assume that* (H_1) *and* (H_2) *hold. Then, problem* (1.1) *has at least one radial solution* $u(x) = v(|x|)$ *satisfying*

$$|u(x)| \le M, \quad |\nabla u(x)| \le \max\{L_1, -L_4\}, \quad \forall x \in \Omega. \tag{2.11}$$

Proof It follows from Lemma 2.6 that for each $n \geq n_0$, problem (2.2), (2.3) with $\lambda = 1$ has a solution denoted by $v_n(r)$ satisfying

$$|v_n(r)| \leq M, \quad \forall r \in [0, R], \tag{2.12}$$

$$|v_n'(r)| \leq M_1 := \max\{L_1, -L_4\}, \quad \forall r \in [0, R]. \tag{2.13}$$

So from the Arzelà–Ascoli theorem, $\{v_n(r)\}$ has a uniformly convergent subsequence. Without loss of generality, we assume that $\{v_n(r)\}$ converge to $v(r)$ uniformly on $[0, R]$. For each fixed $\varepsilon \in (0, R]$, we let $n_1 = [1/\varepsilon] + 1$. Then, $h_n(r) \geq \varepsilon$ on $[\varepsilon, R]$ for $n \geq n_1$, and thus from (2.2) with $\lambda = 1$, (2.12) and (2.13), it follows that $\{(\phi_p(v_n'(r)))'\}_{n=n_1}^{\infty}$ is uniformly bounded on $[\varepsilon, R]$. Since $\{\phi_p(v_n'(r))\}_{n=n_1}^{\infty}$ is uniformly bounded on $[\varepsilon, R]$, from the Arzelà–Ascoli theorem, $\{\phi_p(v_n'(r))\}_{n=n_1}^{\infty}$ has a uniformly convergent subsequence. We can assume that $\{\phi_p(v_n'(r))\}_{n=n_1}^{\infty}$ converge uniformly on $[\varepsilon, R]$, and thus $\{v_n'(r)\}_{n=n_1}^{\infty}$ converge to $v'(r)$ uniformly on $[\varepsilon, R]$. From this together with the arbitrariness of ε, we know that $v \in C[0, R] \cap C^1(0, R]$ and $|v'(r)| \leq M_1$ on $(0, R]$. Notice that

$$\begin{cases} -(\phi_p(v_n'(r)))' = f(r, v_n(r), rv_n'(r)) + \dfrac{N-1}{h_n(r)}\phi_p(v_n'(r)), \quad r \in [0, R], \\ v_n'(0) = 0, \quad v_n'(R) = 0. \end{cases} \tag{2.14}$$

Integrating both sides of the equation in (2.14) over $[r, R] \subset (0, R]$, we get

$$\phi_p(v_n'(r)) = \int_r^R \left(f(\tau, v_n(\tau), \tau v_n'(\tau)) + \frac{N-1}{h_n(\tau)}\phi_p(v_n'(\tau)) \right) \mathrm{d}\tau, \quad r \in (0, R].$$

By Lebesgue's dominated convergence theorem, we have

$$\phi_p(v'(r)) = \int_r^R \left(f(\tau, v(\tau), \tau v'(\tau)) + \frac{N-1}{\tau}\phi_p(v'(\tau)) \right) \mathrm{d}\tau, \quad r \in (0, R],$$

and so

$$-(\phi_p(v'(r)))' = f(r, v(r), rv'(r)) + \frac{N-1}{r}\phi_p(v'(r)), \quad r \in (0, R]. \tag{2.15}$$

In addition, we have

$$v'(R) = \lim_{n\to\infty} v_n'(R) = 0. \tag{2.16}$$

Notice that Eq. (2.15) is equivalent to the following equation

$$(r^{N-1}\phi_p(v'(r)))' + r^{N-1} f(r, v(r), rv'(r)) = 0, \quad r \in (0, R]. \tag{2.17}$$

Integrating both sides of Eq. (2.17) over $[r, R](r > 0)$ and applying (2.16), we obtain

$$r^{N-1}\phi_p(v'(r)) = \int_r^R \tau^{N-1} f(\tau, v(\tau), \tau v'(\tau))\mathrm{d}\tau.$$

Hence, by the L'Hospital rule, one has

$$\begin{aligned}\lim_{r\to 0^+} \phi_p(v'(r)) &= \lim_{r\to 0^+} \frac{\int_r^R \tau^{N-1} f(\tau, v(\tau), \tau v'(\tau))\mathrm{d}\tau}{r^{N-1}} \\ &= -\lim_{r\to 0^+} \frac{rf(r, v(r), rv'(r))}{N-1} \\ &= 0,\end{aligned}$$

which implies that $v'(0) := \lim_{r\to 0^+} v'(r) = 0$. In summary, $v(\cdot) \in C^1[0, R]$ with $\phi_p(v'(\cdot)) \in C^1[0, R]$ is a solution of problem (2.1), and hence, $u(x) = v(|x|)$ is a radial solution of problem (1.1) satisfying (2.11). This completes the proof of the theorem. □

The following results are direct consequences of Theorem 2.1.

Corollary 2.1 *Assume that* (H_1) *holds. Suppose further that*

(H_2') *there exist constants* $L_i, i = 1, 2, 3, 4$ *with* $L_3 < L_4 < 0 < L_1 < L_2$*, such that*

$$f(r, s, t) + \frac{N-1}{R} L_1^{p-1} \geq 0, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [0, RL_2]$$

and

$$f(r, s, t) - \frac{N-1}{R} |L_4|^{p-1} \leq 0, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [RL_3, 0].$$

Then, problem (1.1) *has at least one radial solution* $u(x) = v(|x|)$ *satisfying* (2.11).

Proof It is sufficient to verify that condition (H_2) holds. Indeed, notice that

$$0 \leq rt \leq RL_2, \quad \forall (r, t) \in [0, R] \times [L_1, L_2]$$

and

$$\phi_p(t) = |t|^{p-2}t = t^{p-1} \geq L_1^{p-1}, \quad \forall t \in [L_1, L_2].$$

It follows from condition (H_2') that

$$f(r, s, rt) + \frac{N-1}{R}\phi_p(t) \geq f(r, s, rt) + \frac{N-1}{R} L_1^{p-1} \geq 0$$

for all $(r, s, t) \in [0, R] \times [-M, M] \times [L_1, L_2]$. Similarly, we can show that

$$f(r, s, rt) + \frac{N-1}{R}\phi_p(t) \le 0, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [L_3, L_4].$$

This completes the proof of the corollary. □

Corollary 2.2 *Assume that the function $f(r, s, t)$ has the decomposition*

$$f(r, s, t) = f_1(r, s) + f_2(r, t)$$

and satisfies the following conditions:

(i) *the function $f_1 : [0, R] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $M > 0$ such that*

$$sf_1(r, s) < 0, \quad \forall r \in [0, R], \ |s| > M;$$

(ii) *the function $f_2 : [0, R] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f_2(r, 0) \equiv 0$ on $[0, R]$ such that*

$$\liminf_{t \to \pm\infty} \frac{f_2(r, rt)}{\phi_p(t)} > -\frac{N-1}{R}$$

uniformly in $r \in [0, R]$.

Then, problem (1.1) has at least one radial solution.

Proof It is enough to verify conditions (H_1) and (H_2) hold. At first, from conditions (i) and(ii), we have

$$sf(r, s, 0) = sf_1(r, s) < 0, \quad \forall r \in [0, R], \ |s| > M.$$

Hence, condition (H_1) is satisfied.

Next, notice that function $f_1(r, s)$ is bounded on $[0, R] \times [-M, M]$; it follows from condition (ii) that

$$\liminf_{t \to \pm\infty} \frac{f(r, s, rt)}{\phi_p(t)} > -\frac{N-1}{R}$$

uniformly in $r \in [0, R]$ and $s \in [-M, M]$. Hence, there exit constants $L_i, i = 1, 2, 3, 4$ with $L_3 < L_4 < 0 < L_1 < L_2$, such that

$$\frac{f(r, s, rt)}{\phi_p(t)} > -\frac{N-1}{R}, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [L_1, L_2]$$

and

$$\frac{f(r, s, rt)}{\phi_p(t)} > -\frac{N-1}{R}, \quad \forall (r, s, t) \in [0, R] \times [-M, M] \times [L_3, L_4],$$

i.e.,

$$f(r,s,rt)+\frac{N-1}{R}\phi_p(t)>0,\quad \forall(r,s,t)\in[0,R]\times[-M,M]\times[L_1,L_2]$$

and

$$f(r,s,rt)+\frac{N-1}{R}\phi_p(t)<0,\quad \forall(r,s,t)\in[0,R]\times[-M,M]\times[L_3,L_4].$$

Thus, condition (H_2) is satisfied. This completes the proof of the corollary. □

3 An Example

In this section, we give an example to illustrate our main results.

Example 3.1 Consider p-Laplacian Neumann problem of the form

$$\begin{cases} -\Delta_p u + b(|x|)(x\cdot\nabla u)^m + |u|^{p-2}u = a(|x|)\displaystyle\sum_{i=0}^{n} c_i u^i \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p>1$, $\Omega=\{x\in\mathbb{R}^N : |x|<R\}$ with $N\geq 2$, m is an odd number, $a, b\in C[0,R]$, $c_i\in\mathbb{R}$ $(i=0,1,\ldots,n)$ with $c_0\neq 0$, $c_n=1$. If one of the following conditions holds:

(C_1) n is an odd number, $a(r)\leq 0$ on $[0,R]$, and one of the following conditions is satisfied

(i) $b(r)\leq 0$ on $[0,R]$;
(ii) $p>m+1$;
(iii) $|b(r)|<(N-1)/R^{m+1}$ on $[0,R]$ with $p=m+1$;

(C_2) n is an even number, $p>n+1$, and one of the conditions (i), (ii) and (iii) is satisfied;
(C_3) n is an even number, $p=n+1$, $|a(r)|<1$ on $[0,R]$, either $b(r)\leq 0$ on $[0,R]$ or $n>m$,

then p-Laplacian Neumann problem (3.1) has at least one radial solution.

For the sake of certainty, we assume that condition (C_1)-(i) holds. Let

$$f(r,s,t)=a(r)\sum_{i=0}^{n}c_i s^i-|s|^{p-2}s-b(r)t^m,\quad (r,s,t)\in[0,R]\times\mathbb{R}^2.$$

Then,

$$sf(r,s,0) = a(r)\sum_{i=0}^{n} c_i s^{i+1} - |s|^p \to -\infty \quad (s \to \pm\infty)$$

uniformly in $r \in [0, R]$, and thus, there exists $M > 0$ such that

$$sf(r,s,0) < 0, \quad \forall r \in [0, R],\ |s| > M,$$

that is, condition (H_1) is satisfied. On the other hand, we have

$$f(r,s,rt) + \frac{N-1}{R}\phi_p(t) \to +\infty \quad (t \to +\infty)$$

and

$$f(r,s,rt) + \frac{N-1}{R}\phi_p(t) \to -\infty \quad (t \to -\infty)$$

uniformly for $(r,s) \in [0, R]\times[-M, M]$, and so condition ($H_2$) is satisfied. Therefore, from Theorem 2.1, problem (3.1) has at least one radial solution.

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