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Approximation by Modified Meyer–König and Zeller Operators via Power Series Summability Method

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Abstract
In this paper, we study the Korovkin-type theorem for modified Meyer–König and Zeller operators via A-statistical convergence and power series summability method. The rate of convergence for this new summability methods is also obtained with the help of the modulus of continuity. Further, we establish Voronovskaya-type and Grüss–Voronovskaya-type theorems for A-statistical convergence.

Keywords A-statistical convergence · Modified Meyer–König and Zeller operators · Power series summability method · Korovkin-type theorem · Voronovskaya-type theorem · Rate of convergence

Mathematics Subject Classification 40G10 · 40C15 · 41A36 · 40A35

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1 Introduction

The *asymptotic density of* $A \subseteq \mathbb{N}$ (the set of all natural numbers) is defined by

$$\delta(A) = \lim_{p} \frac{1}{p} |\{r \leq p : r \in A\}|,$$

where $|.|$ denotes the cardinality of the enclosed set. A sequence $\xi = (\xi_r)$ is said to be *statistically convergent* (see [15]) to the number $\mathfrak{L}$ if $\delta(A(\varepsilon)) = 0$ for each $\varepsilon > 0$, where

$$A(\varepsilon) = \{r \leq p : |\xi_r - \mathfrak{L}| > \varepsilon\}$$

and we write $st\text{-}\lim \xi = \mathfrak{L}$. Note that every convergent sequence is statistically convergent but not conversely.

Let $A = (a_{nj})$ be an infinite matrix and $x = (x_j)$ be a sequence. If the series

$$(Ax)_n = \sum_j a_{nj} x_j$$

converge for every $n \in \mathbb{N}$, then we say that $(Ax)_n$ is the *A-transform* of the sequence $x = (x_n)$. If the $(Ax)_n$ converges to a number L, we say that x is *A-summable to L*. The summability matrix A is *regular* whenever $\lim_j x_j = L$; then, $\lim_n (Ax)_n = L$.

Let $A = (a_{nj})$ be a nonnegative regular summability matrix. The sequence $x = (x_j)$ is said to be *A-statistically convergent* (see [16]) to real number $\mathfrak{L}$ if for any $\epsilon > 0$,

$$\lim_n \sum_{j : |x_j - \mathfrak{L}| \geq \epsilon} a_{nj} = 0.$$

In this case, we write $st_A - \lim x = \mathfrak{L}$. The A-statistical convergence is generalization of the statistical convergence (see [10,30]).

The second summability method which is used in this paper is power summability method. Let (p_j) be a real sequence with $p_0 > 0$ and $p_1, p_2, \ldots \geq 0$, and such that the corresponding power series $p(t) = \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0, R)$,

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L,$$

then we say that $x = (x_j)$ is *convergent in the sense of power series method* (see [19,33]). Power series method includes many known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and they are not matrix methods (see [4,7,36]). Note that the power series method is more affective than ordinary convergence (see [35]).

Note that the power series method is regular if and only if $\lim_{t\to R^-} \frac{p_j t^j}{p(t)} = 0$ holds for each $j \in \{0, 1, 2, 3, \ldots\}$ (see [6]). Throughout the paper, we assume that power series method is regular.

It is known that A-statistical convergence is generalization of the statistical convergence as it is shown in Example 1.1 of [11]. In this paper, we will prove the Korovkin-type theorem for the modified Meyer–König and Zeller operators via A-statistical convergence and power summability method. Then, we present the rate of the convergence related to the above summability methods. In the last sections, we give a kind of Voronovskaya-type theorem for A-statistical convergence and Grüss–Voronovskaya-type theorem.

The modified Meyer–König and Zeller operators introduced by Cardenas et al. [13] are defined by

$$R_n(f,x) = \sum_{k=0}^{\infty} f\left(\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}}\right)\binom{n+k}{k} x^k(1-x)^{n+1}, \tag{1.1}$$

for $x \in [0, 1]$.

The first few moments of the modified Meyer–König and Zeller operators are given in the following lemma.

Lemma 1.1 (see [13]) *We have*

(1) $R_n(e_0, x) = 1$,
(2) $R_n(e_2, x) = x^2$,
(3) $R_n(e_4, x) = \frac{4nx^3}{(n+2)(n+3)} + \frac{n^2(n+1)x^4}{(n+2)(n+3)(n+4)} + \frac{4n^3x^5}{(n+2)(n+3)(n+4)(n+5)} + O(n^{-2})$.

An estimation related to the $R_n(e_1, x)$ is given in [31].

Lemma 1.2 (see [31]) *For all $n \in \mathbb{N}$ and $x \in [0, 1)$, we have*

$$x - \frac{1-x}{n} \le R_n(e_1, x) \le x + \frac{x(1+x)}{n}.$$

Now, we can prove the estimation for the central moments for the modified Meyer–König and Zeller operators.

Proposition 1.3 *The estimations for the central moments for modified Meyer–König and Zeller operators are given by*

$$R_n((t-x), x) \le \frac{x(1+x)}{n},$$
$$R_n((t-x)^2, x) \le \frac{2x(1-x)}{n},$$
$$R_n((t-x)^4, x) \le \frac{4n^3}{(n+2)(n+3)(n+4)(n+5)}x^5 + \left(\frac{n^2(n+1)}{(n+2)(n+3)(n+4)} - \frac{4}{n} + 3\right)x^4$$
$$+ \left(\frac{4}{n} + \frac{4n}{(n+2)(n+3)}\right)x^3 - \frac{1}{4n^2}\left(\frac{1}{n}+1\right)x^2 + \frac{1}{4n^3}x + O\left(\frac{1}{n^2}\right),$$

$$R_n((t-x)^6,x) \le \left(\frac{60n^3}{\prod_{j=2}^{5}(n+j)} + \frac{12n^{12}}{(n+1)\prod_{j=2}^{7}(n+j)^2}\right)x^7$$
$$+\left(\frac{15n^2(n+1)}{\prod_{j=2}^{4}(n+j)} - \frac{6}{n} + \frac{n^{10}(n+17)}{(n+1)\prod_{j=2}^{6}(n+j)^2} + 10\right)x^6$$
$$+\left(\frac{6}{n} + \frac{60n}{(n+2)(n+3)} + \frac{12n^8}{(n+1)\prod_{j=2}^{5}(n+j)^2}\right)x^5 - \frac{5\left(\frac{1}{n}+1\right)}{4n^2}x^4$$
$$+\frac{5}{4n^3}x^3 - \frac{3\left(\frac{1}{n}+1\right)}{8n^4}x^2 + \frac{3}{8n^5}x + O\left(\frac{1}{n^2}\right),$$

for every $x \in [0, 1)$.

Proof Let us start from the first central moment. From Lemma 1.2, we get that

$$R_n((t-x),x) = R_n(t,x) - xR_n(1,x) \le x + \frac{x(1+x)}{n} - x.$$

For the second central moment, based on Lemmas 1.1 and 1.2, we have this estimation:

$$R_n((t-x)^2,x) = R_n(t^2,x) - 2xR_n(t,x) + x^2R_n(1,x) \le x^2 - 2x\left[x - \frac{1-x}{n}\right] + x^2$$
$$= \frac{2x(1-x)}{n}.$$

And now we will estimate the third central. Firstly, we will give estimation for $R_n(e_3, x)$:

$$R_n(e_3,x) = \sum_{k=0}^{\infty}\left(\frac{k(k-1)}{(n+k)(n+k-1)}\right)^{\frac{3}{2}}\binom{n+k}{k}x^k(1-x)^{n+1}.$$

By the fact that

$$\left(\frac{k(k-1)}{(n+k)(n+k-1)}\right)^{\frac{3}{2}} \le \frac{k^3}{(n+k-1)^3},$$

for every $k \in \{0, 1, 2, 3\ldots\}$ and $n \in \mathbb{N}$, we have

$$R_n(e_3,x) = \sum_{k=0}^{\infty}\left(\frac{k(k-1)}{(n+k)(n+k-1)}\right)^{\frac{3}{2}}\binom{n+k}{k}x^k(1-x)^{n+1}$$
$$\le \sum_{k=1}^{\infty}\frac{k^3}{(n+k-1)^3}\binom{n+k}{k}x^k(1-x)^{n+1}$$
$$\le \sum_{k=1}^{\infty}\frac{k}{(n+k-1)}\binom{n+k}{k}x^k(1-x)^{n+1} = x + \frac{x(1-x)}{n}.$$

On the other hand, for $k \in \{0, 1, 2, 3, \ldots\}$ and $n \in \mathbb{N}$, we have

$$\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}} \geq \frac{(k-1)}{(n+k)},$$

which implies

$$\begin{aligned} R_n(e_3, x) &= \sum_{k=0}^{\infty} \left(\frac{k(k-1)}{(n+k)(n+k-1)} \right)^{\frac{3}{2}} \binom{n+k}{k} x^k (1-x)^{n+1} \\ &\geq \sum_{k=1}^{\infty} \frac{k-1}{n+k} \left(\frac{k-1}{n+k} \right)^2 \binom{n+k}{k} x^k (1-x)^{n+1} \end{aligned}$$

For $k \geq 2$ and $n \geq 3$, the following inequality holds:

$$\frac{(k-1)^2}{(n+k)^2} \geq \frac{1}{16n^2},$$

from which we obtain

$$\sum_{k=2}^{\infty} \frac{k-1}{n+k} \left(\frac{k-1}{n+k} \right)^2 \binom{n+k}{k} x^k (1-x)^{n+1} \geq \frac{1}{16n^2} \left[x - \frac{1-x}{n} \right].$$

By Corollary 3.3 in [13], we have that $R_n(e_0, x) = 1$, $R_n(e_2, x) = x^2$ and

$$\begin{aligned} R_n(e_4, x) &= \frac{4nx^3}{(n+2)(n+3)} + \frac{n^2(n+1)x^4}{(n+2)(n+3)(n+4)} \\ &\quad + \frac{4n^3x^5}{(n+2)(n+3)(n+4)(n+5)} + O(1/n^2). \end{aligned}$$

Hence,

$$\begin{aligned} R_n((t-x)^4, x) &= R_n(e_4, x) - 4xR_n(e_3, x) + 6x^2R_n(e_2, x) - 4x^3R_n(e_1, x) + x^4 \\ &\leq \frac{4n^3}{(n+2)(n+3)(n+4)(n+5)} x^5 + \left(\frac{n^2(n+1)}{(n+2)(n+3)(n+4)} - \frac{4}{n} + 3 \right) x^4 \\ &\quad + \left(\frac{4}{n} + \frac{4n}{(n+2)(n+3)} \right) x^3 - \frac{1}{4n^2} \left(\frac{1}{n} + 1 \right) x^2 + \frac{1}{4n^3} x + O\left(\frac{1}{n^2} \right), \end{aligned}$$

as required. In similar way, we can get the estimation for the $R_n((t-x)^6, x)$. □

Corollary 1.4 *For the fourth and sixth central moment for modified Meyer–König and Zeller operators are valid the following relations*

$$R_n((t-x)^4, x) = O\left(\frac{1}{n^3}\right) \quad \text{and} \quad R_n((t-x)^6, x) = O\left(\frac{1}{n^5}\right),$$

for every $x \in [0, 1)$.

The theory of Korovkin-type theorems was studied in several function spaces, and further details reader can find in those papers (see [2,7–9,12,14,19,21,32,35–37]). In what follows, we define the following power series for sequence of operators R_n

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} R_n(f, x) p_n t^n = L,$$

and say that sequence of operators (R_n) converges to L in the sense of power series, for every $t \in (0, R)$, if the above series converges.

2 Korovkin-Type Results

In this section we obtain A-statistical convergence of the modified Meyer–König and Zeller operators to the identity operator. The Korovkin-type theorem for A-statistical convergence was given as follows. Let $B[0, 1)$ be the space of all bounded functions on the interval $[0, 1)$ and $C[0, 1)$ be the space of all continuous functions defined in the interval $[0, 1)$ (similarly, we define $B[0, 1]$ and $C[0, 1]$).

Theorem 2.1 (see [14]) *Let* $A = (a_{nj})$ *be a nonnegative regular summability matrix and let* (B_j) *be a sequence of positive linear operators on* $C[0, 1]$ *such that for* $i = 0, 1, 2$,

$$st_A - \lim_n ||B_j e_i - e_i|| = 0.$$

Then for any $f \in C[0, 1]$,

$$st_A - \lim_n ||B_j f - f|| = 0,$$

where $||f|| = \max_{0\le t\le 1} |f(t)|$ *for any* $f \in C[0, 1]$.

Based on the above theorem, we give the following result for the modified Meyer–König and Zeller operators.

Theorem 2.2 *Let* $A = (a_{nj})$ *be a nonnegative regular summability matrix and let* (R_n) *be a sequence of positive linear operators* (1.1) *on* $C[0, 1)$ *such that for* $i = 1, 2$,

$$st_A - \lim_n ||R_n e_i - e_i|| = 0.$$

Then,

$$st_A - \lim_n ||R_n f - f|| = 0,$$

for any $f \in C[0,1)$*, where* $||f|| = \max_{0\leq t<1} |f(t)|$ *for any* $f \in C[0,1)$.

Proof Proof of the theorem follows directly from Lemmas 1.1, 1.2 and Theorem 2.1. □

Our theorem is an extension of Theorem 2.2 given in [31].

Example 2.3 Let us define the following type of operators

$$P_n(f;x) = (1+x_n)R_n(f;x),$$

where sequence (x_n) is defined as in Example 1.1 in [11]. Then, the following relations are fulfilled

$$P_n(e_0;x) = (1+x_n),$$
$$(1+x_n)\left(x - \frac{1-x}{n}\right) \leq P_n(e_1,x) \leq (1+x_n)\left(x + \frac{x(1+x)}{n}\right)$$

and

$$P_n(e_2;x) = (1+x_n)x^2.$$

By Theorem 2.2, we obtain that

$$st_A - \lim_n ||P_n f - f|| = 0.$$

But the above-defined operators, $P_n(f;x)$, do not satisfy Theorem 2.2, in [31].

Now, we give a Korovkin-type theorem for the modified Meyer–König and Zeller operators, by power series method. It is known that Korovkin-type theorems are proved by Abel summability method (for example, see [32,37]). For more on Korovkin-type approximation theorems in different settings, we refer to [1,5,20,22–29].

Theorem 2.4 *Let* (R_n) *be a sequence of positive linear operators* (1.1) *from* $C[0,1)$ *into* $B[0,1)$*. Then for any* $f \in C[0,1)$*, we have*

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} R_n(e_i,x)p_n t^n - e_i \right\| = 0, \; i = 0,1,2, \tag{2.1}$$

if and only if

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} R_n(f,x)p_n t^n - f \right\| = 0. \tag{2.2}$$

Proof Proof of the theorem is similar to the proof of Theorem 1 of [34], and we omit it. □

3 Rate of Convergence

In this section, we study the rate of $A-$statistical convergence for the modified Meyer–König and Zeller operators via power series summability method. We begin by presenting the following facts. The modulus of continuity for function $f(x) \in C[0,1)$ is defined as follows:

$$\omega(f,\delta) = \sup_{|h|<\delta} |f(x+h) - f(x)|.$$

It is well known that

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{3.1}$$

So, we have the following result.

Theorem 3.1 *Let $A = (a_{nj})$ be a nonnegative regular summability matrix and $f \in C[0,1)$. If (α_n) is a sequence of positive real numbers such that $\omega(f,\delta_n) = st_A - o(\alpha_n)$, then*

$$||R_n f - f|| = st_A - o(\alpha_n),$$

where

$$\delta_n = \sup_{\substack{0\leq x<1 \\ n\in\mathbb{N}}} \left\{\frac{2x(1-x)}{n}\right\}^2.$$

Proof Taking into consideration the linearity and positivity of $R_n f$ and relation (3.1), we have

$$|R_n(f;x) - f| \leq R_n(|f(t) - f(x)|;x) \leq \omega(f,\delta)R_n\left(1 + \frac{|t-x|}{\delta}\right)$$

Applying Cauchy–Schwarz inequality in the last expression, we get

$$|R_n(f;x) - f| \leq \omega(f,\delta)\left[1 + \frac{1}{\delta}\left(R_n\left(|t-x|^2;x\right)\right)^{\frac{1}{2}}\right].$$

On the other hand, by Lemma 1.1, we obtain

$$\begin{aligned} R_n&\left(|t-x|^2;x\right) \\ &= R_n(t^2;x) - 2xR_n(t;x) + x^2R_n(e_0;x) \\ &\leq \frac{2x(1-x)}{n}, \end{aligned}$$

for every $x \in [0, 1)$. Taking

$$\delta_n = \sup_{\substack{0 \leq x < 1 \\ n \in \mathbb{N}}} \left\{ \frac{2x(1-x)}{n} \right\}^2,$$

we get that $||R_n f - f|| \leq 2 \cdot \omega(f, \delta_n)$. Therefore for every $\epsilon > 0$, is valid the following relation

$$\frac{1}{\alpha_n} \sum_{||R_n f - f|| \geq \epsilon} a_{nj} \leq \frac{1}{\alpha_n} \sum_{2 \cdot \omega(f, \delta_n) \geq \epsilon} a_{nj}.$$

From conditions given in the theorem, we have that $||R_n f - f|| = st_A - o(\alpha_n)$, as desired. □

In what follows, we give the rate of convergence for power series summability method.

Theorem 3.2 *Let $f \in C[0, 1)$ and let ϕ be a positive real function defined on $[0, 1)$. If $\omega(f, \psi(t)) = o(\phi(t))$ as $t \to R^-$, then we have*

$$\frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} (R_n e_i - e_i) p_n t^n \right\| = o(\phi(t)),$$

i.e.,

$$\lim_{t \to R^-} \frac{1}{\phi(t) p(t)} \left\| \sum_{n=0}^{\infty} (R_n e_i - e_i) p_n t^n \right\| = 0,$$

where the function $\psi : (0, 1) \to \mathbb{R}$ is defined by

$$\psi(t) = \left\{ \sup_{\substack{0 \leq x < 1 \\ n \in \mathbb{N}}} \{R_n((t-x)^2; x)\} \right\}^{\frac{1}{2}}.$$

Proof We omit it, because it is similar to proof of Theorem 3.2 of [10]. □

4 Voronovskaya-Type Theorem

In [3,18], the following Voronovskaya theorem for the Meyer–König and Zeller operators $M_n(f, t)$ states as:

Theorem 4.1 (see [3,18]) *Let f be continuous and have second derivatives on $[0, \infty)$. Then,*

$$\lim_{n \to \infty} n[(M_n(f, t) - f(t))] = t(1-t)^2 \frac{f''(t)}{2}.$$

In [13] is proven the Voronovskaya-type theorem for the modified Meyer–König and Zeller operators, as follows:

Theorem 4.2 *For all $f \in C[0,1]$and $x \in (0,1)$, whenever $f''(x)$ exists,*

$$\lim_{n\to\infty} n[R_n(f,x) - f(x)] = -\frac{(1-x)^2}{2} f'(x) + \frac{x(1-x)^2}{s} f''(x).$$

In what follows, we will prove its A-statistically version of it, in the interval $[\alpha, 1)$, where $\alpha > 0$. First, we prove this.

Lemma 4.3 *For every $y \in [0,1)$, and for every $x \in [0,1)$, we have*

$$P_n(\Phi^4) \sim K(x)(st_A) \quad on \quad [0,1),$$

where $\Phi_x(y) = (y - x)$ and K depends only from x.

Proof Proof of the Lemma follows directly from Proposition 1.3. □

Theorem 4.4 *Let $f \in C[\alpha, 1)$, such that $f', f'' \in C[\alpha, 1)$, and $x \in [\alpha, 1)$ for any $\alpha = \frac{1}{B} > 0$, for enough big B. Then,*

$$|P_n f - f| \sim \left| [P_n(f,x) - f(x)] - Bxf'(x) - 2B\frac{x^2 \cdot f''(x)}{2} \right| (st_A),$$

on $x \in [\alpha, 1)$.

Proof Let us suppose that $f', f'' \in C[\alpha, 1)$ and $x \in [\alpha, 1)$. Without lose of generality, we can put $\alpha = \frac{1}{B} > 0$, for enough big constant B. By the Taylor's formula, we have:

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}(y-x)^2 f''(x) + (y-x)^2 \psi_x(y), \tag{4.1}$$

where

$$\psi_x(y) = \begin{cases} \frac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2} & \text{for} \quad x \neq y \\ o & \quad x = y. \end{cases}$$

We know that

$$P_n(1; x) = (1 + x_n).$$

And under conditions given in the theorem, we have that:

$$(1+x_n)\left(-Bx + \frac{x}{n}\right) \le P_n((y-x); x) \le (1+x_n)\frac{x(x+1)}{n} \le (1+x_n)\left(Bx + \frac{x^2}{n}\right),$$
$$(1+x_n)\left(-2Bx^2 - \frac{2x^3}{n}\right) \le P_n((y-x)^2; x) \le (1+x_n)\left(2Bx^2 - \frac{2x^2}{n}\right).$$

After operating in both sides of the relation (4.1), from the operator P_n, we obtain

$$P_n(f) \leq (1+x_n)f(x) + (1+x_n)f'(x)\left(Bx + \frac{x^2}{n}\right) + (1+x_n)\frac{f''(x)}{2}\left(2Bx^2 - \frac{2x^2}{n}\right) + (1+x_n)R_n(\Phi^2\psi_x; x),$$

which yields

$$\left|[P_n f - f] - Bxf'(x) - 2Bx^2\frac{f''(x)}{2}\right| \leq Mx_n + M_1 \cdot x_n\left(Bx + \frac{x^2}{n}\right) + \frac{x^2}{n}M_1 + \frac{x^2}{n}M_2 + x_n\frac{M_2}{2}\left|2Bx^2 - \frac{2x^2}{n}\right| + |R_n((y-x)^2\psi_x; x)| + x_n|R_n((y-x)^2\psi_x; x)|, \quad (4.2)$$

where $M = \sup|f(x)|$, $M_1 = \sup|f'(x)|$ and $M_2 = \sup|f''(x)|$.

After applying the Cauchy–Schwarz inequality in the last term of the relation (4.2), we find that

$$\left|R_n((y-x)^2\psi_x; x)\right| \leq \left[R_n(\Phi^4; x)\right]^{\frac{1}{2}} \cdot \left[R_n(\psi_x^2; x)\right]^{\frac{1}{2}}. \quad (4.3)$$

Also, by putting $\eta_x(y) = (\psi_x(y))^2$, we see that $\eta_x(x) = 0$ and $\eta_x(\cdot) \in C[\alpha, 1)]$. Clearly, it follows that

$$R_n(\eta_x) \to 0(st_A) \quad \text{on} \quad [\alpha, 1).$$

Now from the last relation, relations (4.2), (4.3) and Lemma 4.3, we obtain that

$$R_n(\Phi^2\psi_x; x) \to 0(st_A) \quad \text{on} \quad [\alpha, 1). \quad (4.4)$$

Hence, it is proved theorem. □

5 Grüss–Voronovskaya-Type Theorems

In this section, we show some kind of Grüss–Voronovskaya-type theorem for the modified Meyer–König and Zeller operators. This kind of result, for first time, is shown in [17]. Now, we are ready to prove the following result.

Theorem 5.1 *For $f, f', f'' \in C[0, 1)$ and any $x \in [0, 1)$. Then,*

$$\left|n\,[R_n(f, x) - f(x)] - nf'(x)R_n((t-x); x) - n\frac{f''(x)}{2}R_n((t-x)^2; x)\right| = O(1)\omega\left(f'', n^{-1}\right),$$

as $n \to \infty$.

Proof Taylor's theorem shows

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + R(t,x),$$

where $R(t,x) = \frac{f''(\theta)-f''(x)}{2}(t-x)^2$, for $\theta \in (t,x)$. Applying in both sides of the above relation, by operators $R_n(\cdot, x)$, we obtain

$$\left| n\left[R_n(f,x) - f(x)\right] - nf'(x)R_n((t-x);x) - n\frac{f''(x)}{2}R_n((t-x)^2;x)\right| \\ = nR_n(|R(t,x)|, x).$$

By properties of the modulus of continuity, we have

$$\left|\frac{f''(\theta)-f''(x)}{2!}\right| \le \frac{1}{2!}\left(1 + \frac{|\theta - x|}{\delta}\right)\omega(f'', \delta).$$

On the other hand,

$$\left|\frac{f''(\theta)-f''(x)}{2!}\right| \le \begin{cases} \omega(f'',\delta) & ; |t-x| \le \delta \\ \frac{(t-x)^4}{\delta^4}\omega(f'',\delta) & ; |t-x| \ge \delta \end{cases}$$

For $0 < \delta < 1$, we obtain that

$$\left|\frac{f''(\theta)-f''(x)}{2!}\right| \le \omega(f'',\delta)\left(1 + \frac{(t-x)^4}{\delta^4}\right),$$

which gives

$$|R(t,x)| \le \omega(f'',\delta)\left(1 + \frac{(t-x)^4}{\delta^4}\right)(t-x)^2 = \omega(f'',\delta)\left((t-x)^2 + \frac{(t-x)^6}{\delta^4}\right).$$

By the linearity of R_n and the above relation, we obtain

$$R_n(|R(t,x)|, x) \le \omega(f'',\delta)\left(R_n((t-x)^2, x) + \frac{1}{\delta^4}R_n((t-x)^6, x)\right).$$

Taking into consideration Corollary 1.4 and Proposition 1.3, we have

$$R_n(|R(t,x)|, x) \le \omega(f'',\delta)\left(O\left(\frac{1}{n}\right) + \frac{1}{\delta^4}O\left(\frac{1}{n^5}\right)\right) = O\left(\frac{1}{n}\right)\omega(f'',\delta).$$

For $\delta = n^{-1}$, we complete the proof. □

Theorem 5.2 *Let $f', g', f'', g'', (fg)', (fg)'' \in C[0, 1)$ and $f'g' \geq 0$. Then,*

$$\begin{aligned}-2x^2(1+x)f'(x)g'(x) &\leq \lim_{n\to\infty} n[R_n(fg,x) - R_n(f,x)\cdot R_n(g,x)] \\ &\leq 2x(1-x)f'(x)g'(x),\end{aligned}$$

for every $x \in (0, 1)$ and for $x = 0$, we get that

$$\lim_{n\to\infty} n[R_n(fg,x) - R_n(f,x)\cdot R_n(g,x)] = 0.$$

Proof We know that

$$\begin{aligned}&n\{R_n(fg,x) - R_n(f,x)R_n(g,x)\} \\ &\quad = n\Big\{R_n(fg,x) - (fg)(x) - (fg)'(x)R_n((t-x),x) - \frac{(fg)''(x)}{2}R_n((t-x)^2,x) \\ &\quad - g(x)\left[R_n(f,x) - f(x) - f'(x)R_n((t-x),x) - \frac{f''(x)}{2}R_n((t-x)^2,x)\right] \\ &\quad - R_n(f,x)\left[R_n(g,x) - g(x) - g'(x)R_n((t-x),x) - \frac{g''(x)}{2}R_n((t-x)^2,x)\right] \\ &\quad + \frac{g''(x)}{2}R_n((t-x)^2,x)[f(x) - R_n(f,x)] + f'(x)g'(x)R_n((t-x)^2,x) \\ &\quad - g'(x)R_n((t-x),x)[R_n(f,x) - f(x)]\Big\}.\end{aligned}$$

From relation

$$-\frac{2x^2(1+x)}{n}f'(x)g'(x) \leq f'(x)g'(x)R((t-x)^2,x) \leq \frac{2x(1-x)}{n}f'(x)g'(x),$$

and Theorem 5.1, Theorem 2.2 in [31], we get

$$\begin{aligned}-2x^2(1+x)f'(x)g'(x) &\leq \lim_{n\to\infty} n[R_n(fg,x) - R_n(f,x)\cdot R_n(g,x)] \\ &\leq 2x(1-x)f'(x)g'(x),\end{aligned}$$

for every $x \in (0, 1)$. If $x = 0$, then

$$\lim_{n\to\infty} n[R_n(fg,x) - R_n(f,x)\cdot R_n(g,x)] = 0$$

as required. □

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