Bull. Malays. Math. Sci. Soc. (2021) 44:1979–2004
https://doi.org/10.1007/s40840-020-01043-1

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

On $\mathcal{I}$-neighborhood Spaces and $\mathcal{I}$-quotient Spaces

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Received: 12 June 2020 / Revised: 19 October 2020 / Accepted: 22 October 2020 /
Published online: 9 November 2020

Abstract

An ideal on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under the operations of taking finite unions and subsets of its elements. The $\mathcal{I}$-open sets of topological spaces, which are determined by an ideal $\mathcal{I}$ on $\mathbb{N}$ and the topology of the spaces, are a basic concept of ideal topological spaces. However, it encounters some difficulties in the study of certain structures and mappings of topological spaces. In this paper, we discuss some properties of ideal topological spaces based on $\mathcal{I}_{sn}$-open sets, study the problem generating new topological spaces from ideals, characterize the mappings preserving $\mathcal{I}$-convergence and structure special $\mathcal{I}$-quotient spaces. The following main results are obtained.

(i) A mapping $f : X \to Y$ preserves $\mathcal{I}$-convergence if and only if provided U is an $\mathcal{I}_{sn}$-open subset of Y, then $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X.
(ii) A topological space X is an $\mathcal{I}$-neighborhood space if and only if every $\mathcal{I}$-continuous mapping on the space X preserves $\mathcal{I}$-convergence.
(iii) Suppose that both X, Y are topological spaces and $f : X \to Y$ is a surjective mapping. Then the topology μ of the space Y is the finest topology that makes f preserve $\mathcal{I}$-convergence if and only if $\mu = \tau_{f,\mathcal{I}_{sn}}$, if and only if f is an $\mathcal{I}_{sn}$-quotient mapping and $\mu = \mu_{\mathcal{I}_{sn}}$.
(iv) Let X be an $\mathcal{I}$-neighborhood space and $f : X \to Y$ be a surjective mapping. Then the topology μ of the space Y is the finest topology that makes f be $\mathcal{I}$-continuous if and only if $\mu = \tau_{f,\mathcal{I}}$, if and only if f is an $\mathcal{I}$-quotient mapping and Y is an $\mathcal{I}$-sequential space.

These show the unique role of $\mathcal{I}$-neighborhood spaces in the study of ideal topological spaces and present a version using the notion of ideals.

Communicated by Rosihan M. Ali.

This research is supported by NSFC (Nos. 11801254, 11471153).

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Keywords Ideal convergence · G-method · $\mathcal{I}$-open set · $\mathcal{I}_{sn}$-open set · $\mathcal{I}$-continuous mapping · Preserving $\mathcal{I}$-convergence · $\mathcal{I}_{sn}$-continuous mapping · $\mathcal{I}$-sequential space · $\mathcal{I}$-neighborhood space · $\mathcal{I}$-quotient space · $\mathcal{I}_{sn}$-quotient space · $\mathcal{I}$-topological space

Mathematics Subject Classification 54A05 · 54A20 · 54B15 · 54C08 · 54D55

1 Introduction

Convergence of sequences in a topological space is a basic and important concept in mathematics. In addition to the usual convergence of sequences, statistical convergence, ideal convergence and even the general G-convergence have attracted extensive attention [12,16,23].

Let X be a set, $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x} \in s(X)$ if and only if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. A *method* on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$ [26, Definition 1.1]. A sequence $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be *G-convergent* to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$. Let G be a method on a set X and $A \subset X$. The set A is called a *G-closed subset* of X if, whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$ [26, Definition 2.1]; A is called a *G-open subset* of X if $X \setminus A$ is G-closed in X; and A is called a *G-neighborhood* of a point $x \in X$ if there exists a G-open subset U with $x \in U \subset A$ [26, Definition 3.1]. In the paper [26], the authors studied G-closures and G-interiors in topological spaces, discuss G-sequential spaces and G-Fréchet spaces, compare the topology of a topological space and the family of all G-open subsets on the space, and give the mutual relationship between continuity and G-continuity. These show that G-methods really become a method to study convergence and continuity in general topology.

As the basic relationship between topological spaces and mappings, the following four questions are before us. First of all, the G-open sets are generated by G-convergence, and the family of all G-open subsets of a topological space generally forms a generalized topology, not a topology [26, part (1) of Example 2.14]. This is not conducive to further discussion of topological properties defined by G-open sets.

Question 1.1 *Which families of subsets related to G-open sets make up a topology?*

Secondly, it is well known that mappings are an important tool to study topological spaces. Generally speaking, every mapping preserving G-convergence is a G-continuous mapping, but the inverse is not true [26, Example 7.4]. In addition, for the usual convergence, every sequentially continuous mapping is equivalent to the mapping which preserves convergent sequences [7, Theorem 3.1].

Question 1.2 *How to characterize the spaces with the following property: every G-continuous mapping on the space preserves G-convergence?*

Thirdly, G-continuous mappings are defined by the pre-image of each G-open subset being a G-open subset, which we have accepted and are familiar with this definition.

Question 1.3 *How to characterize the mappings preserving G-convergence by the special properties of the pre-image of certain subsets on the range space?*

Finally, the quotient topology is the finest topology of the range that makes the mapping be surjective and continuous.

Question 1.4 *How to characterize the finest topology of the range that makes the mapping preserve G-convergence or be G-continuous? In particular, how to characterize the finest topology of the range so that the mapping is sequentially continuous?*

Ideals are a very useful notion in topology, analysis and set theory and have been studied for along time. A broad perspective of study concerning the analysis of algebraic, topological and combinatorial structures by means of an accurate investigation of the properties of specific set systems arises from them. More in general, such an analysis may be extended to the interrelations of set operators, binary set relations and set systems on the same ground set [1–3,6,9–11,15,17,18]. In recent years, ideal convergence has become one of the hotspots of general topology and set theory [8,14,20,21].

In this paper, we discuss Questions 1.1–1.4 in ideal topological spaces. Let $\mathcal{I}$ be an ideal on the set $\mathbb{N}$ of all natural numbers. We introduce the notion of $\mathcal{I}_{sn}$-open sets between open sets and $\mathcal{I}$-open sets, show that the family of all $\mathcal{I}_{sn}$-open subsets of a topological space forms a topology, and then, $\mathcal{I}$-neighborhood spaces and $\mathcal{I}_{sn}$-open topological spaces are defined. Secondly, we prove that the property that the pre-image of each $\mathcal{I}_{sn}$-open subset on the range space is an $\mathcal{I}_{sn}$-open subset describes exactly the mapping preserving $\mathcal{I}$-convergence, and the $\mathcal{I}$-neighborhood space characterizes the space such that every $\mathcal{I}$-continuous mapping on the space is the mapping preserving $\mathcal{I}$-convergence. $\mathcal{I}$-neighborhood spaces are general. For example, the spaces with the usual convergence of sequences and $\mathcal{I}$-sequential spaces are $\mathcal{I}$-neighborhood spaces. Thirdly, we obtain the necessary and sufficient condition such that the mapping from a topological space X onto a topological space Y satisfies that the topology of Y is the finest topology of the range so that the mapping preserves $\mathcal{I}$-convergence. Fourthly, we introduce $\mathcal{I}$-topological spaces and discuss the relationship between $\mathcal{I}$-quotient topology and the topology of the range which is the finest topology that makes the mapping be $\mathcal{I}$-continuous. These studies deepen our understanding for ideal topological spaces and the quotient spaces, present a version using the notion of ideals and provide a new research path for revealing the mutual relationship of spaces and mappings.

2 Preliminaries

The main purpose of this section is to introduce the $\mathcal{I}_{sn}$-interior and $\mathcal{I}_{sn}$-closure operators for an ideal $\mathcal{I}$ on $\mathbb{N}$ and a topological space, and recall some related concepts and results to be discussed in this paper. Our topological terminology and notation are as in the book [19].

Throughout this paper, the set of all natural numbers is denoted by $\mathbb{N}$ and $\omega = \{0\} \cup \mathbb{N}$. Let X be a topological space and $P \subset X$. P is called a *sequential neighborhood* of a point $x \in X$ if every sequence $\{x_n\}$ in X converging to x is eventually in P, i.e., $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$. P is called a *sequentially open set* of X if P is a sequential neighborhood of x for each $x \in P$. The complement set of a sequentially open set is called a *sequentially closed set*. A set P is sequentially closed in X if and only if the limit point $x \in P$ provided a sequence $\{x_n\}$ in P converges to a point x in X.

Let $\mathcal{I}$ be a family of subsets on $\mathbb{N}$. Consider the following conditions.

(i) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
(ii) If $B \subset A \in \mathcal{I}$, then $B \in \mathcal{I}$.
(iii) $\mathcal{I} \nsubseteq \{\varnothing\}$ and $\mathbb{N} \notin \mathcal{I}$.
(iv) $\mathcal{I}$ is a cover of $\mathbb{N}$.

The family $\mathcal{I}$ is called an *ideal* on $\mathbb{N}$ if it satisfies the above conditions (i) and (ii); $\mathcal{I}$ is called a *non-trivial ideal* on $\mathbb{N}$ if it satisfies the above conditions (i)–(iii); $\mathcal{I}$ is called an *admissible ideal* on $\mathbb{N}$ if it satisfies the above conditions (i)–(iv). Only admissible ideals are discussed in this paper.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $\{x_n\}$ in X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ provided for any neighborhood U of x, we have the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$ [23]. Since ideal convergence is a special kind of G-convergence, the results of G-convergence are all applicable to ideal convergence [26]. The usual convergence of sequences in topological spaces can be extended to ideal convergence on $\mathbb{N}$. The family of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{I}_{fin}$. Then $\mathcal{I}_{fin}$ is the smallest non-trivial ideal contained in every admissible ideal, and the $\mathcal{I}$-convergence on a topological space X is just the usual convergence of sequences in X.

For the sake of brevity, if not specially mentioned, we always use $\mathcal{I}$ to express an admissible ideal on $\mathbb{N}$, and $\mathcal{I}_{fin}$ is called the minimal ideal on $\mathbb{N}$.

Let X be a topological space and $P \subset X$. A sequence $\{x_n\}$ in X is said to be *$\mathcal{I}$-eventually* in P if the set $\{n \in \mathbb{N} : x_n \notin P\} \in \mathcal{I}$ [32, Definition 3.15]; the set P is said to be an *$\mathcal{I}$-sequential neighborhood* of a point $x \in X$ if every sequence which is $\mathcal{I}$-convergent to x is $\mathcal{I}$-eventually in P [31]; the set P is said to be an *$\mathcal{I}_{sn}$-open set* of X if P is an $\mathcal{I}$-sequential neighborhood of x for each $x \in P$; the set P is said to be an *$\mathcal{I}_{sn}$-closed set* of X if the complement set $X \backslash P$ is an $\mathcal{I}_{sn}$-open set; the set P is said to an *$\mathcal{I}$-closed set* of X if whenever a sequence $\{x_n\}$ in P with $x_n \xrightarrow{\mathcal{I}} x$ in X, the $\mathcal{I}$-limit point $x \in P$; the set P is said to be an *$\mathcal{I}$-open set* of X if the complement set $X \backslash P$ is an $\mathcal{I}$-closed set.

We can further improve the concepts of certain neighborhoods of a point in a topological space. Given a topological space X, $P \subset X$ and $x \in X$, the set P is said to be an *$\mathcal{I}$-neighborhood* of x if there exists an $\mathcal{I}$-open set V of X such that $x \in V \subset P$; the set P is said to be an *$\mathcal{I}_{sn}$-neighborhood* of x if there exists an $\mathcal{I}_{sn}$-open set V of X such that $x \in V \subset P$.

The concepts of sequential neighborhoods and sequentially open sets have been widely used in general topology and have obtained rich results [25]. It is the first time to introduce $\mathcal{I}_{sn}$-open sets and study their properties in topological spaces. We expect that it will answer some problems related to ideal topological spaces and promote the further development for $\mathcal{I}$-convergence. The family of all $\mathcal{I}$-open subsets of a topological space forms a generalized topology [13,26]. Whether the family is a topology may be related to special ideals on $\mathbb{N}$. In this paper, one of the main purposes introducing

$\mathcal{I}_{sn}$-open sets is to show that the family of all $\mathcal{I}_{sn}$-open sets constitutes a topology, so we can study some properties of ideal topological spaces more deeply and further establish the relationship between spaces and mappings from the perspective of ideal convergence.

Lemma 2.1 *Consider the following conditions for a topological space X and a subset A of X.*

(i) *A is an open set of X.*
(ii) *A is an $\mathcal{I}_{sn}$-open set of X.*
(iii) *A is an $\mathcal{I}$-open set of X.*
(iv) *$\{n \in \mathbb{N} : x_n \in A\} \notin \mathcal{I}$ for each sequence $\{x_n\}$ in X with $x_n \xrightarrow{\mathcal{I}} x$.*
(v) *A is a sequentially open set of X.*
Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v)$.

Proof (i) $\Rightarrow$ (ii). Suppose that A is an open set of X. If $x \in A$ and a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to x, then $\{n \in \mathbb{N} : x_n \notin A\} \in \mathcal{I}$ in X, i.e., the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in A; thus, A is an $\mathcal{I}$-sequential neighborhood of x. This shows that the set A is an $\mathcal{I}_{sn}$-open set of X.

(ii) $\Rightarrow$ (iii). If the set A is not an $\mathcal{I}$-open set of X, then the complement set $X\backslash A$ is not an $\mathcal{I}$-closed set, i.e., there exist a sequence $\{x_n\}$ in $X\backslash A$ and a point $x \in A$ with $x_n \xrightarrow{\mathcal{I}} x$; thus, $\{n \in \mathbb{N} : x_n \notin A\} = \mathbb{N} \notin \mathcal{I}$, i.e., the sequence $\{x_n\}$ is not $\mathcal{I}$-eventually in A. This implies the set A is not an $\mathcal{I}$-sequential neighborhood of the point x; therefore, A is not an $\mathcal{I}_{sn}$-open set.

(iii) $\Leftrightarrow$ (iv) has been proved in [32, Lemma 3.6].

(iii) $\Rightarrow$ (v). Since the usual convergence of sequences of the topological space X is always $\mathcal{I}$-convergence, every $\mathcal{I}$-closed set is sequentially closed and every $\mathcal{I}$-open set is sequentially open. □

In view of Lemma 2.1, given a topological space X, a point $x \in X$ and a neighborhood P of x, then P is also an $\mathcal{I}_{sn}$-neighborhood of x. Moreover, if P is an $\mathcal{I}_{sn}$-neighborhood of x, then P is both an $\mathcal{I}$-neighborhood of x and an $\mathcal{I}$-sequential neighborhood of x.

Lemma 2.2 *[32, Lemma 2.4] Let X be a topological space. If a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to a point $x \in X$, and $\{y_n\}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$, then the sequence $\{y_n\}$ is $\mathcal{I}$-convergent to $x \in X$.*

Lemma 2.3 *Let X be a topological space. The following statements hold.*

(i) *If $Y \subset X$ and A is an $\mathcal{I}_{sn}$-open (resp. $\mathcal{I}_{sn}$-closed, $\mathcal{I}$-open, or $\mathcal{I}$-closed) subset of X, then $A \cap Y$ is an $\mathcal{I}_{sn}$-open (resp. $\mathcal{I}_{sn}$-closed, $\mathcal{I}$-open, or $\mathcal{I}$-closed) subset of the subspace Y.*
(ii) *If Y is an $\mathcal{I}_{sn}$-open subset of X and A is an $\mathcal{I}$-open (resp. $\mathcal{I}_{sn}$-open) subset of the subspace Y, then A is an $\mathcal{I}$-open (resp. $\mathcal{I}_{sn}$-open) subset of X.*
(iii) *If Y is an $\mathcal{I}$-closed subset of X and A is an $\mathcal{I}$-closed subset of the subspace Y, then A is an $\mathcal{I}$-closed subset of X.*

Proof (i) We prove the cases for $\mathcal{I}_{sn}$-open and $\mathcal{I}$-open subsets, the cases for $\mathcal{I}_{sn}$-closed and $\mathcal{I}$-closed subsets can be showed by complement sets. Let $Y \subset X$. If $\{x_n\}$ is a sequence in Y with $x_n \xrightarrow{\mathcal{I}} x \in Y$ and U is a neighborhood of x in X, then $\{n \in \mathbb{N} : x_n \notin U\} = \{n \in \mathbb{N} : x_n \notin U \cap Y\} \in \mathcal{I}$. Thus, the sequence $x_n \xrightarrow{\mathcal{I}} x$ in X.

Suppose that A is an $\mathcal{I}_{sn}$-open subset of X. Let $\{x_n\}$ be a sequence in Y with $x_n \xrightarrow{\mathcal{I}} x \in A \cap Y$. Then the sequence $x_n \xrightarrow{\mathcal{I}} x$ in X. Since A is an $\mathcal{I}_{sn}$-open subset of X, the set $\{n \in \mathbb{N} : x_n \notin A \cap Y\} = \{n \in \mathbb{N} : x_n \notin A\} \in \mathcal{I}$, i.e., the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in $A \cap Y$. This shows that $A \cap Y$ is an $\mathcal{I}$-sequential neighborhood of x in Y. Therefore, $A \cap Y$ is an $\mathcal{I}_{sn}$-open subset of the subspace Y.

Suppose that A is an $\mathcal{I}$-open subset of X. Let $\{x_n\}$ be a sequence in Y with $x_n \xrightarrow{\mathcal{I}} x \in A \cap Y$. Then the sequence $x_n \xrightarrow{\mathcal{I}} x$ in X, and $\{n \in \mathbb{N} : x_n \in A \cap Y\} = \{n \in \mathbb{N} : x_n \in A\} \notin \mathcal{I}$ by part (iv) of Lemma 2.1. Thus, $A \cap Y$ is an $\mathcal{I}$-open subset of the subspace Y.

(ii) Suppose that Y is an $\mathcal{I}_{sn}$-open subset of X.

Let A be an $\mathcal{I}$-open subset of the subspace Y. If A is not an $\mathcal{I}$-open subset of X, then $X \backslash A$ is not an $\mathcal{I}$-closed subset of X; thus, there exist a sequence $\{x_n\}$ in $X \backslash A$ and a point $x \in A$ with $x_n \xrightarrow{\mathcal{I}} x$ in X. By $A \neq Y$, take a point $y \in Y \backslash A$, and define a sequence $\{y_n\}$ in Y as follows: $y_n = x_n$, if $x_n \in Y$; $y_n = y$, if $x_n \notin Y$. Since Y is an $\mathcal{I}$-sequential neighborhood of x, we have that $\{n \in \mathbb{N} : x_n \neq y_n\} = \{n \in \mathbb{N} : x_n \notin Y\} \in \mathcal{I}$. By Lemma 2.2, the sequence $\{y_n\}$ is $\mathcal{I}$-convergent to x in X; hence, $\{y_n\}$ is also $\mathcal{I}$-convergent to x in the subspace Y. It follows from part (iv) Lemma 2.1 that $\varnothing = \{n \in \mathbb{N} : y_n \in A\} \notin \mathcal{I}$, which is a contradiction. This shows that the set A is an $\mathcal{I}$-open subset of X.

Let A be an $\mathcal{I}_{sn}$-open subset of the subspace Y. If $x \in A$ and $\{x_n\}$ is a sequence with $x_n \xrightarrow{\mathcal{I}} x$ in X, then Y is an $\mathcal{I}$-sequential neighborhood of x in X, and $\{n \in \mathbb{N} : x_n \notin Y\} \in \mathcal{I}$. We can assume that $Y \neq A$ and take a point $y \in Y \backslash A$. Define a sequence $\{y_n\}$ in Y as follows: $y_n = x_n$, if $x_n \in Y$; $y_n = y$, if $x_n \notin Y$. Then $\{n \in \mathbb{N} : x_n \neq y_n\} = \{n \in \mathbb{N} : x_n \notin Y\} \in \mathcal{I}$. By Lemma 2.2, the sequence $\{y_n\}$ is $\mathcal{I}$-convergent to x in X; hence, $\{y_n\}$ is also $\mathcal{I}$-convergent to x in the subspace Y, thus $\{n \in \mathbb{N} : x_n \notin A\} = \{n \in \mathbb{N} : y_n \notin A\} \in \mathcal{I}$, i.e., the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in A, and so A is an $\mathcal{I}$-sequential neighborhood of x in X. Hence, A is an $\mathcal{I}_{sn}$-open subset of X.

(iii) Suppose that Y is an $\mathcal{I}$-closed subset of X and A is an $\mathcal{I}$-closed subset of the subspace Y. If a sequence $\{x_n\}$ in A is $\mathcal{I}$-convergent to a point x in X, then $x \in Y$, thus the sequence $\{x_n\}$ is $\mathcal{I}$-convergent to x in the subspace Y, so $x \in A$. Hence, the set A is an $\mathcal{I}$-closed subset of X.

□

Definition 2.4 A topological space X is called an *$\mathcal{I}$-FU-space* if for each $A \subset X$ and $x \in \overline{A}$, there exists a sequence $\{x_n\}$ in A with $x_n \xrightarrow{\mathcal{I}} x$ in X [30]. X is called an *$\mathcal{I}$-sequential space* if a subset A of X is open if and only if it is $\mathcal{I}$-open in X [29].

It is well known that every first-countable space is an $\mathcal{I}$-FU-space and every $\mathcal{I}$-FU-space is an $\mathcal{I}$-sequential space [26,30].

Let $\mathcal{I} = \mathcal{I}_{fin}$. The $\mathcal{I}$-sequential neighborhood of a point, $\mathcal{I}_{sn}$-open set, $\mathcal{I}_{sn}$-closed set, $\mathcal{I}$-open set, $\mathcal{I}$-closed set and $\mathcal{I}$-sequential space are called a *sequential neighborhood* of the point, *sn-open set*, *sn-closed set*, *sequentially open set*, *sequentially closed set* and *sequential space*, respectively.

Lemma 2.5 *The following statements hold.*

(i) *Sequentially open subsets and sn-open subsets coincide in a topological space.*
(ii) *Every sequential space is an $\mathcal{I}$-sequential space.*

Proof (i) By Lemma 2.1 every sn-open subset is sequentially open. If a subset A of a topological space X is not sn-open, then there exist a point $x \in A$ and a sequence $\{x_n\}$ in X converging to the point $x \in A$ such that $\{n \in \mathbb{N} : x_n \notin A\} \notin \mathcal{I}_{fin}$. Let $N = \{n \in \mathbb{N} : x_n \notin A\}$. Then N is infinite and the subsequence $\{x_n\}_{n \in N}$ still converges to x, thus the set $X \backslash A$ is not sequentially closed in X, and therefore, A is not sequentially open in X. This implies that sequentially open subsets and sn-open subsets coincide.

(ii) It follows from Lemma 2.1 that every $\mathcal{I}$-open subset of a topological space is sequentially open. Thus, every sequential space is an $\mathcal{I}$-sequential space.

□

Let G be a method on a set X and $A \subset X$. The *G-hull* of the set A is defined as the set $\{G(\boldsymbol{x}) : \boldsymbol{x} \in s(A) \cap c_G(X)\}$, which is denoted by $[A]_G$; the *G-kernel* of A is defined as the set $\{l \in X :$ there is no $\boldsymbol{x} \in s(X \backslash A) \cap c_G(X)$ with $l = G(\boldsymbol{x})\}$, which is denoted by $(A)_G$ [26]. By means of $\mathcal{I}$-convergence, the following operators of a topological space X have been defined [26]. For each $A \subset X$, put

$$[A]_{\mathcal{I}_s} = \{x \in X : \text{ there exists a sequence } \{x_n\} \text{ in } A \text{ with } x_n \xrightarrow{\mathcal{I}} x\},$$
$$(A)_{\mathcal{I}_s} = \{x \in X : \text{ there exists no sequence } \{x_n\} \text{ in } X \backslash A \text{ with } x_n \xrightarrow{\mathcal{I}} x\},$$

which are called an $\mathcal{I}_s$*-hull* and $\mathcal{I}_s$*-kernel* of the set A in X, respectively. It is easy to check that a set A is an $\mathcal{I}$-closed subset in X if and only if $A = [A]_{\mathcal{I}_s}$, and a set A is an $\mathcal{I}$-open subset in X if and only if $A = (A)_{\mathcal{I}_s}$ [26]. For a topological space X and each $A \subset X$, the following operations have been discussed [27]:

$$[A]_{seq} = \{x \in X : \text{ there exists a sequence } \{x_n\} \text{ in } A \text{ with } x_n \to x\},$$
$$(A)_{seq} = \{x \in X : A \text{ is a sequential neighborhood of } x\}.$$

Therefore, we obtained some results of pseudo-open mappings and sequentially quotient mappings. The following new operators of a topological space X will be studied in this paper. For each $A \subset X$, put

$$[A]_{\mathcal{I}_{sn}} = \{x \in X : \text{if } U \text{ is an } \mathcal{I}\text{-sequential neighborhood of } x, \text{ then } U \cap A \neq \varnothing\},$$
$$(A)_{\mathcal{I}_{sn}} = \{x \in X : A \text{ is an } \mathcal{I}\text{-sequential neighborhood of } x\},$$

which are called an $\mathcal{I}_{sn}$*-closure* and an $\mathcal{I}_{sn}$*-interior* of the set A in X, respectively. It is easy to check that a set A is an $\mathcal{I}_{sn}$-closed subset in X if and only if $A = [A]_{\mathcal{I}_{sn}}$, and a set A is an $\mathcal{I}_{sn}$-open subset in X if and only if $A = (A)_{\mathcal{I}_{sn}}$.

Lemma 2.6 *Let X be a topological space. If $A, B \subset X$, then*

(i) $[A]_{\mathcal{I}_s} = X\backslash(X \setminus A)_{\mathcal{I}_s}$ *[26, Theorem 3.5];*
(ii) $[A]_{\mathcal{I}_{sn}} = X\backslash(X \setminus A)_{\mathcal{I}_{sn}}$*;*
(iii) $A^\circ \subset (A)_{\mathcal{I}_{sn}} \subset (A)_{\mathcal{I}_s} \subset A \subset [A]_{\mathcal{I}_s} \subset [A]_{\mathcal{I}_{sn}} \subset \overline{A}$*; and*
(iv) $(A \cap B)_{\mathcal{I}_{sn}} = (A)_{\mathcal{I}_{sn}} \cap (B)_{\mathcal{I}_{sn}}$*, and* $[A \cup B]_{\mathcal{I}_{sn}} = [A]_{\mathcal{I}_{sn}} \cup [B]_{\mathcal{I}_{sn}}$.

Proof (ii) If $x \in (X\backslash A)_{\mathcal{I}_{sn}}$, then $X\backslash A$ is an $\mathcal{I}$-sequential neighborhood of x and $(X\backslash A) \cap A = \varnothing$, and thus, $x \notin [A]_{\mathcal{I}_{sn}}$. This implies that $[A]_{\mathcal{I}_{sn}} \subset X\backslash(X \setminus A)_{\mathcal{I}_{sn}}$. If $x \notin [A]_{\mathcal{I}_{sn}}$, then there exists an $\mathcal{I}$-sequential neighborhood U of x with $U \cap A = \varnothing$, so $U \subset X\backslash A$, therefore $x \in (X\backslash A)_{\mathcal{I}_{sn}}$. Hence, $X\backslash(X \setminus A)_{\mathcal{I}_{sn}} \subset [A]_{\mathcal{I}_{sn}}$. This shows that $[A]_{\mathcal{I}_{sn}} = X\backslash(X \setminus A)_{\mathcal{I}_{sn}}$.

(iii) By Lemma 2.1, it results that $A^\circ \subset (A)_{\mathcal{I}_{sn}}$. If $x \in (A)_{\mathcal{I}_{sn}}\backslash(A)_{\mathcal{I}_s}$, then there exists a sequence $\{x_n\}$ in $X\backslash A$ with $x_n \xrightarrow{\mathcal{I}} x$, and thus, $\mathbb{N} = \{n \in \mathbb{N} : x_n \notin A\} \in \mathcal{I}$, which is a contradiction. This implies that $(A)_{\mathcal{I}_{sn}} \subset (A)_{\mathcal{I}_s}$. If $x \in X\backslash A$, since the constant sequence $x, x, \cdots$ is $\mathcal{I}$-convergent to x, $x \notin (A)_{\mathcal{I}_s}$. This shows that $(A)_{\mathcal{I}_s} \subset A$. By the above (i) and (ii), it results that $A \subset [A]_{\mathcal{I}_s} \subset [A]_{\mathcal{I}_{sn}} \subset \overline{A}$.

(iv) We only need prove that $(A \cap B)_{\mathcal{I}_{sn}} = (A)_{\mathcal{I}_{sn}} \cap (B)_{\mathcal{I}_{sn}}$. It is obvious that $(A \cap B)_{\mathcal{I}_{sn}} \subset (A)_{\mathcal{I}_{sn}} \cap (B)_{\mathcal{I}_{sn}}$. On the other hand, suppose that $x \in (A)_{\mathcal{I}_{sn}} \cap (B)_{\mathcal{I}_{sn}}$ and a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to the point x. Then A, B are all $\mathcal{I}$-sequential neighborhoods of x, so $\{n \in \mathbb{N} : x_n \notin A\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : x_n \notin B\} \in \mathcal{I}$. It follows that $\{n \in \mathbb{N} : x_n \notin A \cap B\} = \{n \in \mathbb{N} : x_n \notin A\} \cup \{n \in \mathbb{N} : x_n \notin B\} \in \mathcal{I}$. This implies that the set $A \cap B$ is an $\mathcal{I}$-sequential neighborhood of x, i.e., $x \in (A \cap B)_{\mathcal{I}_{sn}}$. Therefore, $(A \cap B)_{\mathcal{I}_{sn}} = (A)_{\mathcal{I}_{sn}} \cap (B)_{\mathcal{I}_{sn}}$. □

It is a question whether $(A \cap B)_{\mathcal{I}_s} = (A)_{\mathcal{I}_s} \cap (B)_{\mathcal{I}_s}$ and $[A \cup B]_{\mathcal{I}_s} = [A]_{\mathcal{I}_s} \cup [B]_{\mathcal{I}_s}$ for subsets A and B of a topological space X. There is an example showing that the union of two G-closed subsets in a topological space is not always G-closed [26, part (1) of Example 2.14], and thus, the intersection of two G-open subsets is not always G-open.

It is a classical topic in general topology to establish the relationship between spaces and mappings [4,28].

Definition 2.7 Let X, Y be topological spaces. Given a mapping $f : X \to Y$,

(i) f is called an $\mathcal{I}$*-continuous* provided U is an $\mathcal{I}$-open subset of Y then $f^{-1}(U)$ is an $\mathcal{I}$-open subset of X [32, Definition 4.1].
(ii) f is called an $\mathcal{I}_{sn}$*-continuous* provided U is an $\mathcal{I}_{sn}$-open subset of Y then $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X.
(iii) f is called *preserving* $\mathcal{I}$*-convergence* provided for each sequence $\{x_n\}$ in X with $x_n \xrightarrow{\mathcal{I}} x$, the sequence $\{(f(x_n)\}$ in Y is $\mathcal{I}$-convergent to $f(x)$ [24].

$\mathcal{I}$-continuous mappings and mappings preserving $\mathcal{I}$-convergence are most commonly used in ideal topological spaces. They have the following relations.

Lemma 2.8 *[32, Theorem 4.2] Every continuous mapping preserves $\mathcal{I}$-convergence, and every mapping preserving $\mathcal{I}$-convergence is $\mathcal{I}$-continuous.*

By $\mathcal{I}$-open subsets and $\mathcal{I}_{sn}$-open subsets in topological spaces, we will define and study certain topologies on a set and some mappings from a topological space onto another a set in the following sections. In order to facilitate reading and comparison, we will list some main terminology and notation to be discussed in this paper.

Remark 2.9 Terminology and notation.

Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$. Let (X, τ) be a topological space and $f : X \to Y$ be a surjective mapping.

(i) $\mathcal{I}$-topological space: the family of all $\mathcal{I}$-open subsets is closed under finite intersections, see Definition 5.2.

(ii) $\tau_{\mathcal{I}}$ ($\mathcal{I}$-open topology induced by τ): the topology of the set X generated by all $\mathcal{I}$-open subsets as a subbase, see Definition 5.2.

(iii) $X_{\mathcal{I}}$ ($\mathcal{I}$-open topological space induced by τ): $(X, \tau_{\mathcal{I}})$, see Definition 5.2.

(iv) $\tau_{\mathcal{I}_s}$ ($\mathcal{I}_s$-open topology induced by τ): the family of all $\mathcal{I}$-open subsets of (X, τ) where (X, τ) is an $\mathcal{I}$-topological space, see Definition 5.2.

(v) $X_{\mathcal{I}_s}$ ($\mathcal{I}_s$-open topological space induced by τ, or $\mathcal{I}_s$-coreflection of (X, τ)): $(X, \tau_{\mathcal{I}_s})$, see Definition 5.2.

(vi) $\tau_{f,\mathcal{I}}$ ($\mathcal{I}$-open topology induced by τ and f): the topology of the set Y generated by a family $\{U \subset Y : f^{-1}(U) \in \tau_{\mathcal{I}}\}$ as a subbase, see Lemma 5.7.

(vii) $\mathcal{I}$-neighborhood space: every $\mathcal{I}$-open subset is $\mathcal{I}_{sn}$-open, see Definition 3.1.

(viii) $\tau_{\mathcal{I}_{sn}}$ ($\mathcal{I}_{sn}$-open topology induced by τ): the family of all $\mathcal{I}_{sn}$-open subsets of (X, τ), see Definition 3.1.

(ix) $X_{\mathcal{I}_{sn}}$ ($\mathcal{I}_{sn}$-open topological space induced by τ, or $\mathcal{I}_{sn}$-coreflection of (X, τ)): $(X, \tau_{\mathcal{I}_{sn}})$, see Definition 3.1.

(x) $\tau_{f,\mathcal{I}_{sn}}$ ($\mathcal{I}_{sn}$-open topology induced by τ and f): the family

$$\{U \subset Y : f^{-1}(U) \text{ is an } \mathcal{I}_{sn}\text{-open subset of } X\},$$

see Remark 4.6 and Lemma 4.4.

3 $\mathcal{I}$-Neighborhood Spaces

In order to answer the question what a family related to $\mathcal{I}$-open subsets of a topological space generates a topology, in this section we introduce $\mathcal{I}$-neighborhood spaces, discuss their basic properties and give the equivalent conditions of the transformations among various neighborhoods defined by $\mathcal{I}$-convergence.

Any family of $\mathcal{I}_{sn}$-open subsets of a topological space is closed under arbitrary unions. In fact, let $\{A_\lambda\}_{\lambda\in 3}$ be a family of $\mathcal{I}_{sn}$-open subsets of a topological space X. Then $(\bigcup_{\lambda\in 3} A_\lambda)_{\mathcal{I}_{sn}} \subset \bigcup_{\lambda\in 3} A_\lambda = \bigcup_{\lambda\in 3}(A_\lambda)_{\mathcal{I}_{sn}} \subset (\bigcup_{\lambda\in 3} A_\lambda)_{\mathcal{I}_{sn}}$. Therefore, $\bigcup_{\lambda\in 3} A_\lambda = (\bigcup_{\lambda\in 3} A_\lambda)_{\mathcal{I}_{sn}}$, i.e., the set $\bigcup_{\lambda\in 3} A_\lambda$ is an $\mathcal{I}_{sn}$-open subset of X. Put

$$\tau_{\mathcal{I}_{sn}} = \{A \subset X : A = (A)_{\mathcal{I}_{sn}}\}.$$

By part (iv) of Lemma 2.6, $\tau_{\mathcal{I}_{sn}}$ is a topology for the set X.

Definition 3.1 Let (X, τ) be a topological space.

(i) X is called an *$\mathcal{I}$-neighborhood space* provided a subset A of X is $\mathcal{I}$-open if and only if $A = (A)_{\mathcal{I}_{sn}}$.

(ii) The family $\tau_{\mathcal{I}_{sn}}$ is called an *$\mathcal{I}_{sn}$-open topology* induced by the topology τ (and the ideal $\mathcal{I}$), and the topological space $(X, \tau_{\mathcal{I}_{sn}})$ is called an *$\mathcal{I}_{sn}$-open topological space* induced by the space (X, τ) or an *$\mathcal{I}_{sn}$-coreflection* of the space (X, τ), which is denoted by $X_{\mathcal{I}_{sn}}$.

It is well known that the *sequential coreflection* sX of a topological space X is the set X endowed with the topology consisting of sequentially open subsets of X [5]. By part (i) of Lemma 2.5, if X is a topological space, then the sequential coreflection sX is exactly the $\mathcal{I}_{sn}$-coreflection $X_{\mathcal{I}_{sn}}$ for the ideal $\mathcal{I}_{fin}$ on $\mathbb{N}$.

Lemma 3.2 *If a topology μ of a set X contains each $\mathcal{I}$-open subset of a topological space (X, τ), then both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-convergent sequences if and only if $\mu = \tau_{\mathcal{I}_{sn}}$, derived from this that (X, μ) is an $\mathcal{I}$-sequential space.*

Proof Let's first prove the following assertion.

Claim Both topological spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$ have the same $\mathcal{I}$-convergent sequences.

In fact, suppose that $x \in X$ and $\{x_n\}$ is a sequence in X. It follows from $\tau \subset \tau_{\mathcal{I}_{sn}}$ that if $x_n \xrightarrow{\mathcal{I}} x$ in $\tau_{\mathcal{I}_{sn}}$ then $x_n \xrightarrow{\mathcal{I}} x$ in τ. Conversely, suppose that $x_n \xrightarrow{\mathcal{I}} x$ in τ, and $x \in A \in \tau_{\mathcal{I}_{sn}}$. Then A is an $\mathcal{I}$-sequential neighborhood of x; thus, the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in A, i.e., $\{n \in \mathbb{N} : x_n \notin A\} \in \mathcal{I}$. This implies that $x_n \xrightarrow{\mathcal{I}} x$ in $\tau_{\mathcal{I}_{sn}}$. Claim is proved.

Suppose that both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-convergent sequences. Since the family μ contains each $\mathcal{I}$-open subset of (X, τ), by Lemma 2.1, $\tau_{\mathcal{I}_{sn}} \subset \mu$. On the other hand, if $A \in \mu$, then A is an $\mathcal{I}_{sn}$-open subset of (X, μ). Since both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-convergent sequences, they have the same $\mathcal{I}_{sn}$-open subsets, and thus, $A \in \tau_{\mathcal{I}_{sn}}$. This shows that $\mu = \tau_{\mathcal{I}_{sn}}$.

Now, suppose that $\mu = \tau_{\mathcal{I}_{sn}}$. Since both spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$ have the same $\mathcal{I}$-convergent sequences, both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-convergent sequences. If A is an $\mathcal{I}$-open subset of (X, μ), then A is an $\mathcal{I}$-open subset of (X, τ). And since μ contains each $\mathcal{I}$-open subset of (X, τ), we have that $A \in \mu$. This implies that (X, μ) is an $\mathcal{I}$-sequential space. □

Remark 3.3 (i) If (X, μ) is an $\mathcal{I}$-sequential space and the family μ contains each $\mathcal{I}$-open subset of (X, τ), then "$\mu = \tau_{\mathcal{I}_{sn}}$" is not necessarily true. For example, let (X, τ) be a non-discrete first-countable space, and let μ be the discrete topology of the set X. Then each $\mathcal{I}$-open subset of (X, τ) is also open in (X, τ) and $\tau = \tau_{\mathcal{I}_{sn}}$, thus μ contains each $\mathcal{I}$-open subset of (X, τ) and (X, μ) is an $\mathcal{I}$-sequential space, but $\mu \neq \tau_{\mathcal{I}_{sn}}$.

(ii) By Claim of the proof in Lemma 3.2, both topological spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$ have the same $\mathcal{I}$-closed subsets, $\mathcal{I}$-open subsets, $\mathcal{I}$-sequential neighborhoods of a point, $\mathcal{I}_{sn}$-closed subsets and $\mathcal{I}_{sn}$-open subsets; in addition, for each $A \subset X$ the operations $[A]_{\mathcal{I}_{sn}}$, $(A)_{\mathcal{I}_{sn}}$, $[A]_{\mathcal{I}_s}$ and $(A)_{\mathcal{I}_s}$ are consistent in topological spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$, respectively. Since the family of all $\mathcal{I}_{sn}$-open subsets in (X, τ) is $\tau_{\mathcal{I}_{sn}}$ and both spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$ have the same $\mathcal{I}_{sn}$-open subsets, we have that $(\tau_{\mathcal{I}_{sn}})_{\mathcal{I}_{sn}} = \tau_{\mathcal{I}_{sn}}$.

We discuss the relationship among various neighborhoods defined by $\mathcal{I}$-convergence in topological spaces.

Lemma 3.4 *Suppose that (X, τ) is a topological space. The following statements hold.*

(i) X is an $\mathcal{I}$-sequential space if and only if X is an $\mathcal{I}$-neighborhood space and $\tau = \tau_{\mathcal{I}_{sn}}$.

(ii) X is an $\mathcal{I}$-neighborhood space if and only if any $\mathcal{I}$-neighborhood of each point is an $\mathcal{I}_{sn}$-neighborhood of the point in X.

(iii) X is an $\mathcal{I}$-sequential space if and only if any $\mathcal{I}$-neighborhood of each point is a neighborhood of the point in X.

Proof We only need prove the necessity of part (i), the rest are easy to verify directly. Let X be an $\mathcal{I}$-sequential space. If A is an $\mathcal{I}$-open subset of X, then A is open in X, and by Lemma 2.1, A is $\mathcal{I}_{sn}$-open. Thus, X is an $\mathcal{I}$-neighborhood space. Obviously, we have that $\tau \subset \tau_{\mathcal{I}_{sn}}$. If $A \in \tau_{\mathcal{I}_{sn}}$, then $A = (A)_{\mathcal{I}_{sn}}$, and in view of Lemma 2.1, A is an $\mathcal{I}$-open subset of X, thus A is open in X, and therefore, $A \in \tau$. It follows that $\tau = \tau_{\mathcal{I}_{sn}}$. □

Theorem 3.5 *The following are equivalent for a topological space (X, τ).*

(i) Any $\mathcal{I}$-sequential neighborhood of each point is an $\mathcal{I}_{sn}$-neighborhood of the point in X.

(ii) For each $A \subset X$, $cl_{\tau_{\mathcal{I}_{sn}}}(A) = [A]_{\mathcal{I}_{sn}}$, and $int_{\tau_{\mathcal{I}_{sn}}}(A) = (A)_{\mathcal{I}_{sn}}$.

(iii) For each $A \subset X$, $[A]_{\mathcal{I}_{sn}}$ is $\mathcal{I}_{sn}$-closed, and $(A)_{\mathcal{I}_{sn}}$ is $\mathcal{I}_{sn}$-open in X.

Proof (i) $\Rightarrow$ (ii). By part (iii) of Lemma 2.6, $[A]_{\mathcal{I}_{sn}} \subset \mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A)$. If $x \in \mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A)$ and U is an $\mathcal{I}$-sequential neighborhood of x, by condition (i), then there exists an $\mathcal{I}_{sn}$-open subset V of X such that $x \in V \subset U$, thus $V \in \tau_{\mathcal{I}_{sn}}$, so $V \cap A \neq \varnothing$, therefore $U \cap A \neq \varnothing$, and hence, $x \in [A]_{\mathcal{I}_{sn}}$. This implies that $\mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A) \subset [A]_{\mathcal{I}_{sn}}$. It shows that $\mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A) = [A]_{\mathcal{I}_{sn}}$. By part (ii) of Lemma 2.6, $\mathrm{int}_{\tau_{\mathcal{I}_{sn}}}(A) = (A)_{\mathcal{I}_{sn}}$.

(ii) $\Rightarrow$ (iii). Since $\mathcal{I}$-sequential neighborhoods of each point in X and $X_{\mathcal{I}_{sn}}$ are same, by part (iii) of Lemma 2.6, we have that $[A]_{\mathcal{I}_{sn}} \subset [[A]_{\mathcal{I}_{sn}}]_{\mathcal{I}_{sn}} \subset \mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A)$. It follows from condition (ii) that $[[A]_{\mathcal{I}_{sn}}]_{\mathcal{I}_{sn}} = [A]_{\mathcal{I}_{sn}}$. Thus, $[A]_{\mathcal{I}_{sn}}$ is $\mathcal{I}_{sn}$-closed in X. By part (ii) of Lemma 2.6, $(A)_{\mathcal{I}_{sn}}$ is $\mathcal{I}_{sn}$-open in X.

(iii) $\Rightarrow$ (i). Suppose that $x \in X$ and A is an $\mathcal{I}$-sequential neighborhood of x in X. Then $x \in (A)_{\mathcal{I}_{sn}} \subset A$ and by condition (iii), $(A)_{\mathcal{I}_{sn}}$ is $\mathcal{I}_{sn}$-open in X, and thus, A is an $\mathcal{I}_{sn}$-neighborhood of x. □

Remark 3.6 (i) Since any family of $\mathcal{I}_{sn}$-open subsets of a topological space X is closed under arbitrary unions, a subset A of X is $\mathcal{I}_{sn}$-open if and only if A is an $\mathcal{I}_{sn}$-neighborhood of x for each $x \in A$. It is easy to check that the following are equivalent for a topological space (X, τ).

(a) $\tau = \tau_{\mathcal{I}_{sn}}$.
(b) Every $\mathcal{I}_{sn}$-open subset of X is open.
(c) Any $\mathcal{I}_{sn}$-neighborhood of each point is a neighborhood of the point in X.
(d) $\mathrm{int}_{\tau_{\mathcal{I}_{sn}}}(A) = \mathrm{int}_{\tau}(A)$, and $\mathrm{cl}_{\tau_{\mathcal{I}_{sn}}}(A) = \mathrm{cl}_{\tau}(A)$ for each $A \subset X$.

(ii) By Definition 2.4, a topological space X is an $\mathcal{I}$-FU-space if and only if for each $A \subset X$, we have that $[A]_{\mathcal{I}_s} = \overline{A}$, i.e., $(A)_{\mathcal{I}_s} = A^{\circ}$. In addition, for each $A \subset X$, $(A)_{\mathcal{I}_{sn}} = (A)_{\mathcal{I}_s}$ if and only if provided A is not an $\mathcal{I}$-sequential neighborhood of x in X then there is a sequence $\{x_n\}$ in $X\backslash A$ with $x_n \xrightarrow{\mathcal{I}} x$.

Corollary 3.7 *The following are equivalent for a topological space* (X, τ).

(*i*) *Any $\mathcal{I}$-sequential neighborhood of each point is a neighborhood of the point in X.*
(*ii*) *For each $A \subset X$, $A^{\circ} = (A)_{\mathcal{I}_{sn}}$, and $\overline{A} = [A]_{\mathcal{I}_{sn}}$.*
(*iii*) *$\tau = \tau_{\mathcal{I}_{sn}}$ and for each $A \subset X$, $cl_{\tau_{\mathcal{I}_{sn}}}(A) = [A]_{\mathcal{I}_{sn}}$, and $int_{\tau_{\mathcal{I}_{sn}}}(A) = (A)_{\mathcal{I}_{sn}}$.*
(*iv*) *For each $A \subset X$, $(A)_{\mathcal{I}_{sn}}$ is open, and $[A]_{\mathcal{I}_{sn}}$ is closed in X.*

Proof It follows from part (i) of Remark 3.6 and Theorem 3.5 that (i) ⇒ (iii). Moreover, the implications (iii) ⇒ (ii) ⇒ (iv) are obvious. Next, we prove that (iv) ⇒ (i). Suppose that $x \in X$ and A is an $\mathcal{I}$-sequential neighborhood of x. Then $x \in (A)_{\mathcal{I}_{sn}} \subset A$ and $(A)_{\mathcal{I}_{sn}}$ is open in X by part (iv), and thus, A is a neighborhood of x. □

Theorem 3.8 *$\mathcal{I}$-Neighborhood spaces have the following properties.*

(*i*) *Being hereditary with respect to $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) subspaces.*
(*ii*) *Being preserved by topological sums.*

Proof Let X be an $\mathcal{I}$-neighborhood space.

(i) Suppose that Y is an $\mathcal{I}$-open subset of X and A is an $\mathcal{I}$-open subset of Y. Then Y is an $\mathcal{I}_{sn}$-open subset in X. By part (ii) of Lemma 2.3, A is an $\mathcal{I}_{sn}$-open subset in X. By part (i) of Lemma 2.3, A is an $\mathcal{I}_{sn}$-open subset in Y. Hence, Y is an $\mathcal{I}$-neighborhood space.
Let Y be an $\mathcal{I}$-closed subset of X and F be an $\mathcal{I}$-closed subset of Y. By part (iii) of Lemma 2.3, F is $\mathcal{I}$-closed in X, thus F is $\mathcal{I}_{sn}$-closed in X, and so F is $\mathcal{I}_{sn}$-closed in Y. Therefore, Y is an $\mathcal{I}$-neighborhood space.

(ii) Let $\{X_\alpha\}_{\alpha\in\mathfrak{Z}}$ be a family of $\mathcal{I}$-neighborhood spaces. Put $X = \bigoplus_{\alpha\in\mathfrak{Z}} X_\alpha$ being the topological sum of $\{X_\alpha\}_{\alpha\in\mathfrak{Z}}$. We will show that the space X is an $\mathcal{I}$-neighborhood space. Let U be an $\mathcal{I}$-open subset in X. For each $\alpha \in \mathfrak{Z}$, by part (i) of Lemma 2.3, we see that $U \cap X_\alpha$ is $\mathcal{I}$-open in the subspace X_α. By the assumption, we have that $U \cap X_\alpha$ is $\mathcal{I}_{sn}$-open in X_α, and by part (ii) of Lemma 2.3, $U \cap X_\alpha$ is $\mathcal{I}_{sn}$-open in X. According to the definition of topological sums, we obtain that $U = \bigcup_{\alpha\in\mathfrak{Z}}(U \cap X_\alpha)$ is $\mathcal{I}_{sn}$-open in X. Thus, X is an $\mathcal{I}$-neighborhood space. □

Theorem 3.9 *The following are equivalent for a topological space (X, τ).*

(i) *X is an $\mathcal{I}$-neighborhood space.*
(ii) *$X_{\mathcal{I}_{sn}}$ is an $\mathcal{I}$-sequential space.*
(iii) *$X_{\mathcal{I}_{sn}}$ is an $\mathcal{I}$-neighborhood space.*

Proof (i) $\Rightarrow$ (ii). Suppose that A is an $\mathcal{I}$-open subset of $X_{\mathcal{I}_{sn}}$. By part (ii) of Remark 3.3, A is an $\mathcal{I}$-open subset of X. Since X is an $\mathcal{I}$-neighborhood space, we have that $A = (A)_{\mathcal{I}_{sn}}$, and A is an open subset of $X_{\mathcal{I}_{sn}}$. Therefore, $X_{\mathcal{I}_{sn}}$ is an $\mathcal{I}$-sequential space.

It follows from part (i) of Lemma 3.4 that (ii) $\Rightarrow$ (iii). Next, we prove that (iii) $\Rightarrow$ (i). Let A be an $\mathcal{I}$-open subset of X. Then A is an $\mathcal{I}$-open subset of $X_{\mathcal{I}_{sn}}$. Since $X_{\mathcal{I}_{sn}}$ is an $\mathcal{I}$-neighborhood space, we have that $A = (A)_{\mathcal{I}_{sn}}$; therefore, X is an $\mathcal{I}$-neighborhood space. □

By part (i) of Lemma 3.4 and Theorem 3.9 we have the following corollary.

Corollary 3.10 *A topological space (X, τ) is an $\mathcal{I}$-sequential space if and only if $X_{\mathcal{I}_{sn}}$ is an $\mathcal{I}$-sequential space and $\tau = \tau_{\mathcal{I}_{sn}}$.*

Example 3.11 If $\mathcal{I}$ is the minimal or a maximal ideal on $\mathbb{N}$, then each topological space is an $\mathcal{I}$-neighborhood space.

Proof It follows from part (i) of Lemma 2.5 that if $\mathcal{I}$ is the minimal ideal on $\mathbb{N}$ then each topological space is an $\mathcal{I}$-neighborhood space. Now, suppose that $\mathcal{I}$ is a maximal ideal on $\mathbb{N}$ and A is an $\mathcal{I}$-open subset of a topological space X. If $x \in A$ and a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to x, put $E = \{n \in \mathbb{N} : x_n \notin A\}$. By part (iv) of Lemma 2.1, $\mathbb{N} \setminus E \notin \mathcal{I}$. Since $\mathcal{I}$ is a maximal ideal on $\mathbb{N}$, we have that $E \in \mathcal{I}$. Therefore, the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in A. Hence, A is an $\mathcal{I}_{sn}$-open subset of X. So X is an $\mathcal{I}$-neighborhood space. □

We have not found an ideal $\mathcal{I}$ on $\mathbb{N}$ and a topological space X such that X is not an $\mathcal{I}$-neighborhood space.

Theorem 3.12 *Suppose that both X, Y are topological spaces. If $f : X \to Y$ is a mapping, then the following are equivalent.*

(i) *f preserves $\mathcal{I}$-convergence.*
(ii) *f is an $\mathcal{I}_{sn}$-continuous mapping.*
(iii) *If F is an $\mathcal{I}_{sn}$-closed subset of Y, then $f^{-1}(F)$ is an $\mathcal{I}_{sn}$-closed subset of X.*
(iv) *$f([A]_{\mathcal{I}_{sn}}) \subset [f(A)]_{\mathcal{I}_{sn}}$ for each $A \subset X$.*
(v) *If U is an $\mathcal{I}$-sequential neighborhood of a point y in Y and $x \in f^{-1}(y)$, then $f^{-1}(U)$ is an $\mathcal{I}$-sequential neighborhood of x in X.*

Proof (i) $\Rightarrow$ (v). Let U be an $\mathcal{I}$-sequential neighborhood of a point y in Y and $x \in f^{-1}(y)$. Suppose that a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to the point $x \in X$. Since the mapping f preserves $\mathcal{I}$-convergence, the sequence $\{f(x_n)\}$ in Y is $\mathcal{I}$-convergent to $f(x)$. Thus, the set $\{n \in \mathbb{N} : x_n \notin f^{-1}(U)\} = \{n \in \mathbb{N} : f(x_n) \notin U\} \in \mathcal{I}$, so the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in $f^{-1}(U)$. Hence, $f^{-1}(U)$ is an $\mathcal{I}$-sequential neighborhood of x.

(v) $\Rightarrow$ (iv). Let $A \subset X$. Suppose that $x \in [A]_{\mathcal{I}_{sn}} \subset X$. If U is an $\mathcal{I}$-sequential neighborhood of $f(x)$ in Y, by condition (v), then $f^{-1}(U)$ is an $\mathcal{I}$-sequential neighborhood of x in X, thus $f^{-1}(U) \cap A \neq \varnothing$, i.e., $U \cap f(A) \neq \varnothing$, and hence, $f(x) \in [f(A)]_{\mathcal{I}_{sn}}$. This implies that $f([A]_{\mathcal{I}_{sn}}) \subset [f(A)]_{\mathcal{I}_{sn}}$.

(iv) $\Rightarrow$ (iii). Let F be an $\mathcal{I}_{sn}$-closed subset of Y. It follows from condition (iv) that $f([f^{-1}(F)]_{\mathcal{I}_{sn}}) \subset [f(f^{-1}(F))]_{\mathcal{I}_{sn}} \subset [F]_{\mathcal{I}_{sn}} = F$, i.e., $[f^{-1}(F)]_{\mathcal{I}_{sn}} \subset f^{-1}(F)$. This shows that $f^{-1}(F)$ is an $\mathcal{I}_{sn}$-closed subset of X.

(iii) $\Rightarrow$ (ii). Let U be an $\mathcal{I}_{sn}$-open subset of Y. Then $Y \backslash U$ is an $\mathcal{I}_{sn}$-closed subset of Y. By condition (iii), $f^{-1}(Y \backslash U) = X \setminus f^{-1}(U)$ is an $\mathcal{I}_{sn}$-closed subset of X, and thus, $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X. By part (ii) of Definition 2.7, f is an $\mathcal{I}_{sn}$-continuous mapping.

(ii) $\Rightarrow$ (i). Suppose that a sequence $x_n \xrightarrow{\mathcal{I}} x$ in X and U is an open subset in Y with $f(x) \in U$. Since U is an $\mathcal{I}_{sn}$-open subset of Y, it follows from condition (ii) that $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X and $x \in f^{-1}(U)$. Thus, the sequence $\{x_n\}$ is $\mathcal{I}$-eventually in $f^{-1}(U)$, and therefore, $\{n \in \mathbb{N} : f(x_n) \notin U\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(U)\} \in \mathcal{I}$. It implies that the sequence $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ in Y. Hence, f preserves $\mathcal{I}$-convergence. □

By (i) $\Leftrightarrow$ (ii) in Theorem 3.12, we can rename a mapping preserving $\mathcal{I}$-convergence as an *$\mathcal{I}_{sn}$-continuous mapping*. Thus, if (X, τ) and (Y, μ) are topological spaces, then a mapping $f : (X, \tau) \to (Y, \mu)$ is $\mathcal{I}_{sn}$-continuous if and only if the mapping $f : (X, \tau_{\mathcal{I}_{sn}}) \to (Y, \mu_{\mathcal{I}_{sn}})$ is continuous. Recall the concept of sequential continuity. By Lemma 2.8, continuous mappings $\Rightarrow$ $\mathcal{I}_{sn}$-continuous mappings $\Rightarrow$ $\mathcal{I}$-continuous mappings. Suppose that both X, Y are topological spaces and $f : X \to Y$ is a mapping. The mapping f is said to be *sequentially continuous* provided U is sequentially open in Y then $f^{-1}(U)$ is sequentially open in X [7]; the mapping f is said to be *preserving convergence of sequences* if for each sequence $\{x_n\}$ with $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$ in Y. We have the following corollary by part (i) of Lemma 2.5 and Theorem 3.12.

Corollary 3.13 *[7, Theorem 3.1] Sequential continuous mappings and the mappings preserving convergence of sequences coincide.*

Theorem 3.14 *A topological space X is an $\mathcal{I}$-neighborhood space if and only if every $\mathcal{I}$-continuous mapping on the space X is an $\mathcal{I}_{sn}$-continuous mapping.*

Proof Suppose that X is an $\mathcal{I}$-neighborhood space and a mapping $f : X \to Y$ is $\mathcal{I}$-continuous. Let U be an $\mathcal{I}_{sn}$-open subset of Y. Since U is an $\mathcal{I}$-open subset of Y and f is $\mathcal{I}$-continuous, $f^{-1}(U)$ is an $\mathcal{I}$-open subset of X. It follows from that X is an $\mathcal{I}$-neighborhood space that $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X. Therefore, f is an $\mathcal{I}_{sn}$-continuous mapping.

Conversely, suppose that X is not an $\mathcal{I}$-neighborhood space. Then there exists an $\mathcal{I}$-open subset O of X such that it is not $\mathcal{I}_{sn}$-open. Let $Y = \{0, 1\}$ and the set Y be endowed with the following topology: the sets $\varnothing$, $\{0\}$ and Y are open in Y, and the set $\{1\}$ is open in Y if and only if the set O is $\mathcal{I}$-closed in X. Define a mapping $f : X \to Y$

as $f(x) = 0$, if $x \in O$; and $f(x) = 1$, if $x \in X \backslash O$. First, we prove that the mapping f is $\mathcal{I}$-continuous. Let U be an $\mathcal{I}$-open subset of Y. We can assume that $U = \{1\}$, because O is an $\mathcal{I}$-open subset of X. If $\{1\}$ is not an open subset of Y, we define a sequence $\{y_n\}$ in Y with each $y_n = 0$, then the sequence $y_n \to 1$, and thus, $y_n \xrightarrow{\mathcal{I}} 1$. Since U is an $\mathcal{I}$-open subset, by part (iv) of Lemma 2.1, $\varnothing = \{n \in \mathbb{N} : y_n \in U\} \notin \mathcal{I}$, which is a contradiction. Therefore, $\{1\}$ is open in Y, and O is $\mathcal{I}$-closed in X, i.e., $f^{-1}(U)$ is $\mathcal{I}$-open in X. This shows that f is an $\mathcal{I}$-continuous mapping. Since $\{0\}$ is open in Y and $f^{-1}(\{0\}) = O$ is not $\mathcal{I}_{sn}$-open in X, the mapping f is not $\mathcal{I}_{sn}$-continuous. □

Another form of Theorem 3.14 is described in part (iii) of Remark 4.2.

Example 3.15 There exists an $\mathcal{I}$-sequential space (X, τ) such that property "$\tau = \tau_{\mathcal{I}_{sn}}$" is not hereditary.

Let $\mathcal{I}$ be the minimal ideal on $\mathbb{N}$, and let (X, τ) be Arens's space S_2 [28, Example 1.8.6]. Then X is a sequential space. Thus, X is an $\mathcal{I}$-sequential space, and by part (i) of Lemma 3.4, $\tau = \tau_{\mathcal{I}_{sn}}$. Denote $X = \{a_n : n \in \omega\} \cup \{a_{n,m} : n, m \in \mathbb{N}\}$, where the sequence $a_n \to a_0$ and the sequence $a_{n,m} \to a_n$ for each $n \in \mathbb{N}$. Put $Y = X \backslash \{a_n : n \in \mathbb{N}\}$. Then there exists no non-trivial convergent sequence in Y [28], and hence, each subset of Y is an $\mathcal{I}_{sn}$-open subset in Y. Since Y is not a discrete subspace of X, the topology $\mu = \tau|_Y$ of Y is not of property "$\mu = \mu_{\mathcal{I}_{sn}}$".

Let $A = \{a_n : n \in \omega\}$. Then $(A)_{\mathcal{I}_{sn}} = \{a_0\}$ and $((A)_{\mathcal{I}_{sn}})_{\mathcal{I}_{sn}} = \varnothing$. Thus, the set $(A)_{\mathcal{I}_{sn}}$ is not always an $\mathcal{I}_{sn}$-open subset. Similarly, the set $[A]_{\mathcal{I}_{sn}}$ is not always an $\mathcal{I}_{sn}$-closed subset for each $A \subset X$.

Question 3.16 *Is the property of $\mathcal{I}$-neighborhood spaces hereditary with respect to subspaces?*

Example 3.17 Define a topology τ on the set $\mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$, as follows.

(a) Each point $n \in \mathbb{N}$ is isolated.
(b) Each open neighborhood U of ∞ is of the form $(\mathbb{N} \backslash I) \cup \{\infty\}$, for each $I \in \mathcal{I}$.
We denote the set $\mathbb{N} \cup \{\infty\}$ equipped with this topology by $\Sigma(\mathcal{I})$. Since $\{n \in \mathbb{N} : n \notin (\mathbb{N} \backslash I) \cup \{\infty\}\} = I \in \mathcal{I}$ for each $I \in \mathcal{I}$, the sequence $n \xrightarrow{\mathcal{I}} \infty$ in $\Sigma(\mathcal{I})$.

Proof It is showed that the space $\Sigma(\mathcal{I})$ is a compact space if and only if $\mathcal{I}$ is the minimal ideal on $\mathbb{N}$ [32, Example 2.7], where the $\mathcal{I}$-convergence is exactly the usual convergence. In [22, Exercise 4M], it was mentioned that for a maximal ideal $\mathcal{I}$ on $\mathbb{N}$, the space $\Sigma(\mathcal{I})$ is extremally disconnected, and hence, there is no non-trivial convergent sequence in it.

(i) $\tau = \tau_{\mathcal{I}_{sn}}$. It is obvious that $\tau \subset \tau_{\mathcal{I}_{sn}}$. Let $A \in \tau_{\mathcal{I}_{sn}}$, and we can assume $\infty \in A$. Then A is an $\mathcal{I}$-sequential neighborhood of the point ∞. By the sequence $n \xrightarrow{\mathcal{I}} \infty$, we have that $\{n \in \mathbb{N} : n \notin A\} \in \mathcal{I}$. Put $I = \{n \in \mathbb{N} : n \notin A\}$, then $I \in \mathcal{I}$ and $A = (\mathbb{N} \backslash I) \cup \{\infty\} \in \tau$. This shows that $\tau = \tau_{\mathcal{I}_{sn}}$.

(ii) $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-neighborhood space if and only if $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-FU-space.
It is obvious that every $\mathcal{I}$-FU-space is an $\mathcal{I}$-neighborhood space. Suppose that $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-neighborhood space. It follows from part (i) of Lemma 3.4 that $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-sequential space. Since the point ∞ is a unique accumulation point

of $\Sigma(\mathcal{I})$, each subset of $\Sigma(\mathcal{I})$ is open or closed. By part (i) of Theorem 3.8, each subspace of $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-sequential space, and therefore, $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-FU-space by [32, Example 6.5].

By Example 3.11, $\Sigma(\mathcal{I})$ is an $\mathcal{I}$-FU-space if $\mathcal{I}$ is a maximal ideal on $\mathbb{N}$. □

4 $\mathcal{I}_{sn}$-Quotient Spaces

With regard to Question 1.4, in this section we will answer the following question: How to characterize the finest topology of the range that makes the mapping from a topological space onto a set preserve $\mathcal{I}$-convergence (be $\mathcal{I}_{sn}$-continuous)? We introduce $\mathcal{I}_{sn}$-quotient mappings and discuss some topological properties of $\mathcal{I}_{sn}$-quotient spaces.

Definition 4.1 Suppose that both X, Y are topological spaces and a mapping $f : X \to Y$ is surjective.

(i) f is called a *quotient* (resp. an $\mathcal{I}$-*quotient*) *mapping* [32, Definition 5.1] if for each $U \subset Y$, the set $f^{-1}(U)$ is open (resp. $\mathcal{I}$-open) in X if and only if U is open (resp. $\mathcal{I}$-open) in Y, where the space Y is called a *quotient* (resp. an $\mathcal{I}$-*quotient*) *space* induced by the mapping f (and the ideal $\mathcal{I}$).

(ii) f is called an $\mathcal{I}_{sn}$-*quotient mapping* if for each $U \subset Y$, the set $f^{-1}(U)$ is $\mathcal{I}_{sn}$-open in X if and only if U is $\mathcal{I}_{sn}$-open in Y, where the space Y is called an $\mathcal{I}_{sn}$-*quotient space* induced by the mapping f and the ideal $\mathcal{I}$.

(iii) f is called an $\mathcal{I}$-*covering mapping* [32, Definition 5.1] if whenever L is an $\mathcal{I}$-convergent sequence in Y, there exists an $\mathcal{I}$-convergent sequence S in X with $f(S) = L$.

Remark 4.2 (i) It is obvious that every quotient (resp. $\mathcal{I}$-quotient) mapping is a continuous (resp. an $\mathcal{I}$-continuous) mapping, and by Theorem 3.12, every $\mathcal{I}_{sn}$-quotient mapping is an $\mathcal{I}_{sn}$-continuous mapping.

(ii) A mapping $f : (X, \tau) \to (Y, \mu)$ is an $\mathcal{I}_{sn}$-quotient mapping if and only if the mapping $f : (X, \tau_{\mathcal{I}_{sn}}) \to (Y, \mu_{\mathcal{I}_{sn}})$ is a quotient mapping.

(iii) Since the mapping $f : X \to Y$ in the proof of the sufficiency of Theorem 3.14 is an $\mathcal{I}$-quotient mapping, we may restate Theorem 3.14 in the following form: A topological space X is an $\mathcal{I}$-neighborhood space if and only if every $\mathcal{I}$-quotient mapping on the space X is an $\mathcal{I}_{sn}$-continuous mapping.

(iv) Let $\mathcal{I}$ be the minimal ideal on $\mathbb{N}$. The $\mathcal{I}$-quotient mapping is called a *sequentially quotient mapping* [7], and the $\mathcal{I}_{sn}$-quotient mapping is called an *sn-quotient mapping*. By part (i) of Lemma 2.5, sequentially quotient mappings and *sn*-quotient mappings coincide in topological spaces.

Lemma 4.3 *The following are hold.*

(i) *For each topological space X, the identity id : $X_{\mathcal{I}_{sn}} \to X$ is a continuous and $\mathcal{I}$-covering mapping.*

(ii) *Every $\mathcal{I}$-covering mapping is an $\mathcal{I}$-quotient (resp. $\mathcal{I}_{sn}$-quotient) mapping if and only if it is an $\mathcal{I}$-continuous (resp. $\mathcal{I}_{sn}$-continuous) mapping.*

Proof Part (i) and the necessity of part (ii) follow by Lemma 3.2 and part (i) of Remark 4.2, respectively. Next, we prove that the sufficiency of part (ii) holds. Suppose that $f : X \to Y$ is an $\mathcal{I}$-covering and $\mathcal{I}$-continuous (resp. $\mathcal{I}_{sn}$-continuous) mapping. If $U \subset Y$, $\{y_n\}$ is a sequence in Y with $y_n \xrightarrow{\mathcal{I}} y \in U$ and $f^{-1}(U)$ is an $\mathcal{I}$-open (resp. $\mathcal{I}_{sn}$-open) subset of X, then there exists a sequence $\{x_n\}$ in X with $x_n \xrightarrow{\mathcal{I}} x$ such that each $f(x_n) = y_n$ and $f(x) = y$, thus $x \in f^{-1}(U)$ and $\{n \in \mathbb{N} : x_n \in f^{-1}(U)\} \notin \mathcal{I}$ (resp. $\{n \in \mathbb{N} : x_n \notin f^{-1}(U)\} \in \mathcal{I}$), therefore $\{n \in \mathbb{N} : y_n \in U\} \notin \mathcal{I}$ (resp. $\{n \in \mathbb{N} : y_n \notin U\} \in \mathcal{I}$), and hence, U is an $\mathcal{I}$-open (resp. $\mathcal{I}_{sn}$-open) subset of Y. This shows that f is an $\mathcal{I}$-quotient (resp. $\mathcal{I}_{sn}$-quotient) mapping. □

Lemma 4.4 *Suppose that (X, τ) is a topological space, Y is a set and $f : X \to Y$ is a surjective mapping. Put $\tau_{f,\mathcal{I}_{sn}} = \{U \subset Y : f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of $X\}$. Then*

- *(i)* *$\tau_{f,\mathcal{I}_{sn}}$ is a topology of Y.*
- *(ii)* *$\tau_{f,\mathcal{I}_{sn}}$ is the topology of Y which is the finest topology that makes f preserve $\mathcal{I}$-convergence.*
- *(iii)* *$\tau_{f,\mathcal{I}_{sn}} = (\tau_{f,\mathcal{I}_{sn}})_{\mathcal{I}_{sn}}$.*
- *(iv)* *$f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}_{sn}})$ is an $\mathcal{I}_{sn}$-quotient mapping.*

Proof For the convenience of narration, let $\mu = \tau_{f,\mathcal{I}_{sn}}$.

- (i) Since any family of $\mathcal{I}_{sn}$-open subsets of a topological space is closed under arbitrary unions or finite intersections, it is easy to see that the family μ is a topology of the set Y.
- (ii) First, we prove that the mapping $f : (X, \tau) \to (Y, \mu)$ preserves $\mathcal{I}$-convergence. Suppose that a sequence $\{x_n\}$ in X is $\mathcal{I}$-convergent to a point $x \in X$ and $f(x) \in U \in \mu$. Then $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X and $x \in f^{-1}(U)$. We have that $\{n \in \mathbb{N} : f(x_n) \notin U\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(U)\} \in \mathcal{I}$, and thus, the sequence $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ in (Y, μ). This implies that the mapping $f : (X, \tau) \to (Y, \mu)$ preserves $\mathcal{I}$-convergence. On the other hand, suppose that ν is a topology of Y and $f : (X, \tau) \to (Y, \nu)$ preserves $\mathcal{I}$-convergence. If $V \in \nu$, i.e., V is an open subset of (Y, ν), then V is an $\mathcal{I}_{sn}$-open subset of (Y, ν). It follows from Theorem 3.12 that $f^{-1}(V)$ is an $\mathcal{I}_{sn}$-open subset of X. Hence, $V \in \mu$. Therefore $\nu \subset \mu$. The proof of (ii) is completed.
- (iii) It is obvious that $\mu \subset \mu_{\mathcal{I}_{sn}}$. Suppose that $U \in \mu_{\mathcal{I}_{sn}}$. Then U is an $\mathcal{I}_{sn}$-open subset of the space (Y, μ). By Theorem 3.12 and the above-mentioned part (ii), $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X, so $U \in \mu$. Hence, $\mu = \mu_{\mathcal{I}_{sn}}$.
- (iv) By the above-mentioned condition (ii), $f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}_{sn}})$ preserves $\mathcal{I}$-convergence. Suppose that $U \subset Y$ and $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X. Then $U \in \mu$, and thus, U is an $\mathcal{I}_{sn}$-open subset of Y. Therefore, the mapping f is an $\mathcal{I}_{sn}$-quotient mapping.

□

The following is the main result in this section.

Theorem 4.5 *Suppose that both X, Y are topological spaces and $f : (X, \tau) \to (Y, \mu)$ is a surjective mapping. Then the following are equivalent.*

(*i*) *The topology μ of the space Y is the finest topology that makes f preserve $\mathcal{I}$-convergence.*

(*ii*) $\mu = \tau_{f,\mathcal{I}_{sn}}$.

(*iii*) *The mapping f is an $\mathcal{I}_{sn}$-quotient mapping and $\mu = \mu_{\mathcal{I}_{sn}}$.*

Proof By Lemma 4.4, (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Suppose that $f : X \to (Y, \mu)$ is an $\mathcal{I}_{sn}$-quotient mapping and $\mu = \mu_{\mathcal{I}_{sn}}$. Let $U \in \mu$. Since f preserves $\mathcal{I}$-convergence, $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X. On the other hand, if $V \subset Y$ and $f^{-1}(V)$ is an $\mathcal{I}_{sn}$-open subset of X, since f is an $\mathcal{I}_{sn}$-quotient mapping, V is an $\mathcal{I}_{sn}$-open subset of Y, therefore $V \in \mu_{\mathcal{I}_{sn}} = \mu$. This shows that $\mu = \{U \subset Y : f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of $X\} = \tau_{f,\mathcal{I}_{sn}}$. □

Remark 4.6 The topology $\tau_{f,\mathcal{I}_{sn}}$ of Y in Lemma 4.4 is called an $\mathcal{I}_{sn}$-*open topology* induced by the topology τ and the mapping f. Theorem 4.5 shows that the $\mathcal{I}_{sn}$-open topology induced by the topology τ and the mapping f is exactly the finest topology of Y that makes f preserve $\mathcal{I}$-convergence (be $\mathcal{I}_{sn}$-continuous).

The following example will show that two conditions "$\mathcal{I}_{sn}$-quotient mapping" and "$\mu = \mu_{\mathcal{I}_{sn}}$" in part (iii) of Theorem 4.5 are independent. Thus, the topology of an $\mathcal{I}_{sn}$-quotient space induced by a mapping f is not always an $\mathcal{I}_{sn}$-open topology induced by the topology τ and the mapping f.

Example 4.7 (*i*) There exist an ideal $\mathcal{I}$ on $\mathbb{N}$ and an $\mathcal{I}_{sn}$-quotient (resp. $\mathcal{I}$-quotient) mapping $f : X \to Y$ such that the topology μ of Y is not the finest topology of Y that makes f preserve $\mathcal{I}$-convergence (resp. be $\mathcal{I}$-continuous).

In fact, take $Y = \mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N}\backslash\mathbb{N}$, and the set Y is endowed with the subspace topology μ of the Čech-Stone compactification $\beta\mathbb{N}$. Since the space Y has no non-trivial convergent sequence [19, Corollary 3.6.15], every subset of Y is sn-open in Y. Let X be the set Y endowed with the discrete topology and let $f : X \to Y$ be the identity. Then f is continuous. Let $\mathcal{I}$ be the minimal ideal on $\mathbb{N}$. Then f is an $\mathcal{I}_{sn}$-quotient (resp. $\mathcal{I}$-quotient) mapping. Since each $\mathcal{I}$-convergent sequence of X is trivial (resp. X is a discrete space), the mapping $f : X \to (Y, \nu)$ preserves $\mathcal{I}$-convergence (resp. is $\mathcal{I}$-continuous) for any topology ν of Y, and thus, the finest topology of Y that makes f preserve $\mathcal{I}$-convergence (resp. be $\mathcal{I}$-continuous) is the discrete topology, which is different with μ.

(*ii*) There exist an ideal $\mathcal{I}$ on $\mathbb{N}$ and a continuous and non-$\mathcal{I}_{sn}$-quotient (resp. non-$\mathcal{I}$-quotient) mapping $f : X \to (Y, \mu)$ with $\mu = \mu_{\mathcal{I}_{sn}}$.

Let $\mathcal{I}$ be the minimal ideal on $\mathbb{N}$, and let (Y, μ) be Arens's space S_2, see Example 3.15. Then Y is an $\mathcal{I}$-sequential space, and thus, $\mu = \mu_{\mathcal{I}_{sn}}$. Let X be the set Y endowed with the discrete topology and let $f : X \to Y$ be the identity. Then f is continuous. Since X is a discrete space and Y is not a discrete space, the mapping f is neither an $\mathcal{I}_{sn}$-quotient mapping nor an $\mathcal{I}$-quotient mapping. By Theorems 4.5 and 3.14, the topology μ of Y is not the finest topology of Y that makes f preserve $\mathcal{I}$-convergence or be $\mathcal{I}$-continuous.

Lemma 4.8 *Suppose that both X and Y are topological spaces and $f : X \to Y$ is a mapping. Then the following are hold.*

(*i*) *Let X be an $\mathcal{I}$-neighborhood space. If f is an $\mathcal{I}_{sn}$-quotient mapping, then f is an $\mathcal{I}$-quotient mapping and Y is an $\mathcal{I}$-neighborhood space.*

(*ii*) *Let Y be an $\mathcal{I}$-neighborhood space. If f is an $\mathcal{I}_{sn}$-continuous and $\mathcal{I}$-quotient mapping, then f is an $\mathcal{I}_{sn}$-quotient mapping.*

Proof (i) Let X be an $\mathcal{I}$-neighborhood space. Suppose that $f : X \to Y$ is an $\mathcal{I}_{sn}$-quotient mapping. Then f is an $\mathcal{I}_{sn}$-continuous mapping, and thus, f is an $\mathcal{I}$-continuous mapping. If $U \subset Y$ and $f^{-1}(U)$ is an $\mathcal{I}$-open subset of X, since X is an $\mathcal{I}$-neighborhood space, $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X, thus U is an $\mathcal{I}_{sn}$-open subset of Y, and therefore, U is an $\mathcal{I}$-open subset of Y. This shows that f is an $\mathcal{I}$-quotient mapping. Let V be an $\mathcal{I}$-open subset of Y. Then $f^{-1}(V)$ is an $\mathcal{I}$-open subset of X, thus $f^{-1}(V)$ is an $\mathcal{I}_{sn}$-open subset of X, and hence, V is an $\mathcal{I}_{sn}$-open subset of Y. This implies that Y is an $\mathcal{I}$-neighborhood space.

(ii) Let Y be an $\mathcal{I}$-neighborhood space. Suppose that $f : X \to Y$ is an $\mathcal{I}_{sn}$-continuous and $\mathcal{I}$-quotient mapping. If $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of X, then $f^{-1}(U)$ is an $\mathcal{I}$-open subset of X, thus U is an $\mathcal{I}$-open subset of Y, and therefore, U is an $\mathcal{I}_{sn}$-open subset of Y. It implies that the mapping f is an $\mathcal{I}_{sn}$-quotient mapping. □

Lemma 4.9 *Suppose that (X, τ) is a topological space. Then every continuous $\mathcal{I}_{sn}$-quotient mapping onto the space X is a quotient mapping if and only if $\tau = \tau_{\mathcal{I}_{sn}}$.*

Proof Suppose that every continuous $\mathcal{I}_{sn}$-quotient mapping onto the space X is a quotient mapping. It is obvious that $\tau \subset \tau_{\mathcal{I}_{sn}}$. Let id : $X_{\mathcal{I}_{sn}} \to X$ be the identity. Then id is a continuous and $\mathcal{I}_{sn}$-quotient mapping, and thus, id is a quotient mapping. This shows that the mapping id is a homeomorphism, and $\tau = \tau_{\mathcal{I}_{sn}}$.

Conversely, suppose that $\tau = \tau_{\mathcal{I}_{sn}}$ and $f : Z \to X$ is a continuous $\mathcal{I}_{sn}$-quotient mapping. If $U \subset X$ and $f^{-1}(U)$ is open in Z, then $f^{-1}(U)$ is an $\mathcal{I}_{sn}$-open subset of Z, thus U is an $\mathcal{I}_{sn}$-open subset of X, and therefore, U is an open subset of X. This shows that the mapping f is quotient. □

Theorem 4.10 *The following are equivalent for a topological space X.*

(*i*) *X is an $\mathcal{I}$-sequential space.*

(*ii*) *Every quotient mapping on the space X is $\mathcal{I}$-quotient.*

(*iii*) *X is an $\mathcal{I}$-neighborhood space and every continuous $\mathcal{I}$-covering (resp. $\mathcal{I}$-quotient, or $\mathcal{I}_{sn}$-quotient) mapping onto the space X is quotient.*

Proof (i) ⇔ (ii) has been proved in [32, Theorem 5.8].

(i) ⇒ (iii). Suppose that (X, τ) is an $\mathcal{I}$-sequential space. By part (i) of Lemma 3.4, X is an $\mathcal{I}$-neighborhood space and $\tau = \tau_{\mathcal{I}_{sn}}$. By parts (ii) of Lemma 4.3 and (ii) of Lemma 4.8, we can assume the mapping $f : Z \to X$ is continuous and $\mathcal{I}_{sn}$-quotient. It follows from Lemma 4.9 that the mapping f is quotient.

(iii) ⇒ (i). Suppose that (X, τ) is an $\mathcal{I}$-neighborhood space and every continuous $\mathcal{I}$-covering (resp. $\mathcal{I}$-quotient, or $\mathcal{I}_{sn}$-quotient) mapping onto X is quotient. By Lemma 4.3, the identity id : $X_{\mathcal{I}_{sn}} \to X$ is a continuous $\mathcal{I}$-covering (resp. $\mathcal{I}$-quotient, or $\mathcal{I}_{sn}$-quotient) mapping, thus it is a quotient mapping, and hence, $\tau_{\mathcal{I}_{sn}} = \tau$. It follows from part (i) of Lemma 3.4 that X is an $\mathcal{I}$-sequential space. □

By Theorem 4.10 and Example 3.11, we have the following corollary [32, Theorem 5.6]: Let $\mathcal{I}$ be a maximal ideal on $\mathbb{N}$ and X be a topological space. Then X is an $\mathcal{I}$-sequential space if and only if every $\mathcal{I}$-quotient mapping onto the space X is quotient.

Example 4.11 Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The set $X = [0, \omega_1]$ is endowed with the following topology: the only non-isolated point ω_1 has the neighborhoods of the usual ordered topology. Then

(i) There is not any non-trivial convergent sequence in X.
(ii) A sequence $x_n \xrightarrow{\mathcal{I}} x \in X$ if and only if the set $\{n \in \mathbb{N} : x_n \neq x\} \in \mathcal{I}$.
(iii) Each subset X is an $\mathcal{I}_{sn}$-subset.

(i) and (ii) were showed in [32, Example 2.2]. Next, we show that (iii) holds. If $x \in A \subset X$, then A is an $\mathcal{I}$-sequential neighborhood of x. In fact, suppose that a sequence $\{x_n\}$ is $\mathcal{I}$-convergent to the point x in X. Then $\{n \in \mathbb{N} : x_n \notin A\} \subset \{n \in \mathbb{N} : x_n \neq x\} \in \mathcal{I}$. Thus, each subset of X is an $\mathcal{I}_{sn}$-open subset.

This implies that X is an $\mathcal{I}$-neighborhood space, $\tau_{\mathcal{I}_{sn}}$ is the discrete topology of the set X and $\tau \neq \tau_{\mathcal{I}_{sn}}$. This also shows that $(X, \tau_{\mathcal{I}_{sn}})$ is an $\mathcal{I}$-FU-space, but X is even not an $\mathcal{I}$-sequential space. Since the identity id : $X_{\mathcal{I}_{sn}} \to X$ is an $\mathcal{I}$-covering mapping, an $\mathcal{I}$-covering mapping does not preserve an $\mathcal{I}$-sequential space.

Question 4.12 *Is a space X satisfying one of the following conditions an $\mathcal{I}$-neighborhood space?*

(i) *Every $\mathcal{I}_{sn}$-continuous and $\mathcal{I}$-quotient mapping onto the space X is an $\mathcal{I}_{sn}$-quotient mapping.*
(ii) *Every $\mathcal{I}_{sn}$-quotient mapping on the space X is an $\mathcal{I}$-quotient mapping.*

5 $\mathcal{I}$-Topological Spaces and $\mathcal{I}$-Quotient Spaces

Quotient, sequentially quotient and sequence-covering mappings are one of the most powerful tools in studying sequential spaces [28]. We introduced and studied $\mathcal{I}$-quotient mappings and $\mathcal{I}$-covering mappings [32]. When we discuss $\mathcal{I}$-continuous mappings and $\mathcal{I}$-quotient mappings, we will encounter the following natural question, see Question 1.4.

Question 5.1 *Suppose that (X, τ) is a topological space and $f : X \to Y$ is a surjective mapping. How to characterize a topology μ of the set Y such that it is the finest topology that makes $f : (X, \tau) \to (Y, \mu)$ be $\mathcal{I}$-continuous?*

We answer the above question to the $\mathcal{I}_{sn}$-continuous mapping f in Theorem 4.5. From the angle of $\mathcal{I}$-convergence, $\mathcal{I}$-open sets are more simpler and more natural than $\mathcal{I}_{sn}$-open sets in ideal topological spaces. In comparison with Lemma 4.4 and Theorem 4.5, this question involves the topology generated by all $\mathcal{I}$-open sets in topological spaces. We introduce the following concepts.

Definition 5.2 Let (X, τ) be a topological space.

(i) The topology of the set X generated by all $\mathcal{I}$-open subsets as a subbase is called an *$\mathcal{I}$-open topology* induced by the topology τ (and the ideal $\mathcal{I}$), which is denoted by $\tau_{\mathcal{I}}$.

(ii) The topological space $(X, \tau_{\mathcal{I}})$ is called an *$\mathcal{I}$-open topological space* induced by the topological space (X, τ), which is denoted by $X_{\mathcal{I}}$.

(iii) (X, τ) is called an *$\mathcal{I}$-topological space* if the family of all $\mathcal{I}$-open subsets of X is closed under finite intersections. Therefore, the $\mathcal{I}$-open topology

$$\tau_{\mathcal{I}} = \{A \subset X : A = (A)_{\mathcal{I}_s}\}$$

is called an *$\mathcal{I}_s$-open topology* induced by the topology τ (and the ideal $\mathcal{I}$) [26, Definition 6.1], which is denoted by $\tau_{\mathcal{I}_s}$; and the $\mathcal{I}$-open topological space $X_{\mathcal{I}}$ is called an *$\mathcal{I}_s$-open topological space* induced by the topological space (X, τ), or an *$\mathcal{I}_s$-coreflection* of the topological space (X, τ), which is denoted by $X_{\mathcal{I}_s}$.

By Lemma 2.1, we have that $\tau \subset \tau_{\mathcal{I}_{sn}} \subset \tau_{\mathcal{I}}$. Obviously, a topological space (X, τ) is an $\mathcal{I}$-sequential space if and only if $\tau = \tau_{\mathcal{I}}$, and X is an $\mathcal{I}$-neighborhood space if and only if $\tau_{\mathcal{I}_{sn}} = \tau_{\mathcal{I}}$. Thus, each $\mathcal{I}$-neighborhood space is an $\mathcal{I}$-topological space. Around Question 5.1, in this section we will discuss the relationship among $\mathcal{I}$-continuous mappings, $\mathcal{I}$-quotient mappings and $\mathcal{I}_{sn}$-quotient mappings in $\mathcal{I}$-topological spaces.

Lemma 5.3 *$\mathcal{I}$-topological spaces are preserved by $\mathcal{I}$-quotient mappings.*

Proof Suppose that $f : X \to Y$ is an $\mathcal{I}$-quotient mapping, where X is an $\mathcal{I}$-topological space. If U and V are $\mathcal{I}$-open subsets of the space Y, then $f^{-1}(U)$ and $f^{-1}(V)$ are $\mathcal{I}$-open subsets of X. Since X is an $\mathcal{I}$-topological space, we obtain that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ is an $\mathcal{I}$-open subset of X. And since f is $\mathcal{I}$-quotient, the intersection $U \cap V$ is an $\mathcal{I}$-open subset of Y. Hence, Y is an $\mathcal{I}$-topological space. □

The following lemma is a continuation of Lemma 3.2 and part (ii) of Remark 3.3.

Lemma 5.4 *The following are equivalent for a topological space X.*

(i) Both spaces X and $X_{\mathcal{I}}$ have the same $\mathcal{I}$-convergent sequences.

(ii) The identity $id : X_{\mathcal{I}} \to X$ is a continuous and $\mathcal{I}$-covering mapping.

(iii) The identity $id : X_{\mathcal{I}} \to X$ is an $\mathcal{I}_{sn}$-quotient mapping.

(iv) The identity $id : X \to X_{\mathcal{I}}$ is an $\mathcal{I}_{sn}$-continuous mapping.

(v) X is an $\mathcal{I}$-neighborhood space.

Proof (i) ⇔ (v). By Lemma 3.2, both spaces (X, τ) and $(X, \tau_{\mathcal{I}})$ have the same $\mathcal{I}$-convergent sequences if and only if $\tau_{\mathcal{I}_{sn}} = \tau_{\mathcal{I}}$, i.e., (X, τ) is an $\mathcal{I}$-neighborhood space.

(i) ⇒ (ii) is obvious. By part (ii) of Lemma 4.3 we have that (ii) ⇒ (iii). And (iii) ⇒ (iv) is obvious.

(iv) ⇒ (v). If A is an $\mathcal{I}$-open subset of the space (X, τ), then A is open in the space $X_{\mathcal{I}}$, and thus, A is $\mathcal{I}_{sn}$-open in $X_{\mathcal{I}}$. Since the mapping $id : X \to X_{\mathcal{I}}$ is an $\mathcal{I}_{sn}$-continuous mapping, the set A is $\mathcal{I}_{sn}$-open in X. This implies that X is an $\mathcal{I}$-neighborhood space. □

By Lemma 3.2, $\tau_{\mathcal{I}_{sn}}$ is the finest topology of the set X which contains each $\mathcal{I}$-open subset of the space (X, τ) and has the same $\mathcal{I}$-convergent sequences as the space (X, τ). Next, we discuss the following question involved in $\mathcal{I}$-open subsets: How to determine the finest topology of the set X which contains each $\mathcal{I}$-open subsets of the space (X, τ) and has the same $\mathcal{I}$-open subsets as the space (X, τ)?

Lemma 5.5 *The following are equivalent for a topological space X.*

(i) *Both spaces X and $X_{\mathcal{I}}$ have the same $\mathcal{I}$-open subsets.*
(ii) *The identity $id : X_{\mathcal{I}} \to X$ is an $\mathcal{I}$-quotient mapping.*
(iii) *The identity $id : X \to X_{\mathcal{I}}$ is an $\mathcal{I}$-continuous mapping.*
(iv) *X is an $\mathcal{I}$-topological space and $X_{\mathcal{I}}$ is an $\mathcal{I}$-sequential space.*

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). Suppose that the mapping id : $X \to X_{\mathcal{I}}$ is an $\mathcal{I}$-continuous mapping. If A and B are $\mathcal{I}$-open subsets of X, then A and B are open in the space $X_{\mathcal{I}}$, thus $A \cap B$ is open in $X_{\mathcal{I}}$, and therefore, $A \cap B$ is $\mathcal{I}$-open in $X_{\mathcal{I}}$. Since id : $X \to X_{\mathcal{I}}$ is $\mathcal{I}$-continuous, the set $A \cap B$ is $\mathcal{I}$-open in the space X. Hence, the family of $\mathcal{I}$-open subsets of X is closed under finite intersections, and it implies that X is an $\mathcal{I}$-topological space. If V is an $\mathcal{I}$-open subset of the space $X_{\mathcal{I}}$, then V is $\mathcal{I}$-open in X, and thus, $V \in \tau_{\mathcal{I}}$, i.e., V is open in $X_{\mathcal{I}}$. This shows that $X_{\mathcal{I}}$ is an $\mathcal{I}$-sequential space.

(iv) $\Rightarrow$ (i). Suppose that X is an $\mathcal{I}$-topological space and $X_{\mathcal{I}}$ is an $\mathcal{I}$-sequential space. Obviously, every $\mathcal{I}$-open subset of X is an $\mathcal{I}$-open subset of $X_{\mathcal{I}}$. If U is an $\mathcal{I}$-open subset of $X_{\mathcal{I}}$, then U is open in $X_{\mathcal{I}}$, and thus, $U \in \tau_{\mathcal{I}}$, i.e., U is $\mathcal{I}$-open in $X_{\mathcal{I}}$. Therefore, both spaces X and $X_{\mathcal{I}}$ have the same $\mathcal{I}$-open subsets. □

Theorem 5.6 *Let (X, τ) be a topological space. If a topology μ of the set X contains each $\mathcal{I}$-open subset of (X, τ), then both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-open subsets if and only if $\mu = \tau_{\mathcal{I}}$, (X, τ) is an $\mathcal{I}$-topological space and (X, μ) is an $\mathcal{I}$-sequential space.*

Proof The sufficiency is true by Lemma 5.5. Next, we will show the necessary. Suppose that both spaces (X, τ) and (X, μ) have the same $\mathcal{I}$-open subsets. Since the family μ contains each $\mathcal{I}$-open subset of (X, τ), we have that $\tau_{\mathcal{I}} \subset \mu$. O the other hand, if $A \in \mu$, then A is $\mathcal{I}$-open in the space (X, μ), and thus, $A \in \tau_{\mathcal{I}}$. This implies that $\mu = \tau_{\mathcal{I}}$. By Lemma 5.5, the space (X, τ) is an $\mathcal{I}$-topological space and the space (X, μ) is an $\mathcal{I}$-sequential space. □

Next, we will further consider extending the identity in Lemma 5.5 to a mapping from a topological space onto another topological space. Suppose that (X, τ) is a topological space and $f : X \to Y$ is a surjective mapping. The topology of the set Y generated by a family $\{U \subset Y : f^{-1}(U) \in \tau_{\mathcal{I}}\}$ as a subbase is called an *$\mathcal{I}$-open topology* induced by the topology τ and the mapping f, which is denoted by $\tau_{f,\mathcal{I}}$. Obviously, if (X, τ) is an $\mathcal{I}$-topological space, then $\tau_{f,\mathcal{I}} = \{U \subset Y : f^{-1}(U)$ is $\mathcal{I}$-open in $X\}$; if (X, τ) is an $\mathcal{I}$-neighborhood space, then $\tau_{f,\mathcal{I}} = \tau_{f,\mathcal{I}_{sn}}$.

Lemma 5.7 *Suppose that $f : X \to Y$ is a surjective mapping. The following statements hold.*

(*i*) *If* (X, τ) *is an* $\mathcal{I}$*-topological space and* $(Y, \tau_{f,\mathcal{I}})$ *is an* $\mathcal{I}$*-sequential space, then* $f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}})$ *is* $\mathcal{I}$*-continuous.*

(*ii*) *If* $f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}})$ *is* $\mathcal{I}$*-continuous, then*

(a) $(Y, \tau_{f,\mathcal{I}})$ *is an* $\mathcal{I}$*-sequential space and* f *is an* $\mathcal{I}$*-quotient mapping; and*

(b) *a topology* μ *of the set* Y *is the finest topology that makes* $f : (X, \tau) \to (Y, \mu)$ *be* $\mathcal{I}$*-continuous if and only if* $\mu = \tau_{f,\mathcal{I}}$.

Proof (i) Suppose that (X, τ) is an $\mathcal{I}$-topological space and $(Y, \tau_{f,\mathcal{I}})$ is an $\mathcal{I}$-sequential space. Let U be an $\mathcal{I}$-open subset of the space Y. By the property of $\mathcal{I}$-sequential spaces of Y, the set U is open in Y, i.e., $U \in \tau_{f,\mathcal{I}}$. Since X is an $\mathcal{I}$-topological space, we have that $f^{-1}(U)$ is $\mathcal{I}$-open in the space X. Hence, f is $\mathcal{I}$-continuous.

(ii) Suppose that $f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}})$ is an $\mathcal{I}$-continuous mapping.

(a) If V is an $\mathcal{I}$-open subset of the space $(Y, \tau_{f,\mathcal{I}})$, then $f^{-1}(V)$ is an $\mathcal{I}$-open subset of the space (X, τ), and thus, $V \in \tau_{f,\mathcal{I}}$, i.e., V is open in Y. Therefore, Y is an $\mathcal{I}$-sequential space. Let U be a subset of Y such that $f^{-1}(U)$ is $\mathcal{I}$-open in X. Then $U \in \tau_{f,\mathcal{I}}$, and thus, U is $\mathcal{I}$-open in Y. Therefore, f is an $\mathcal{I}$-quotient mapping.

(b) Suppose that a topology μ of the set Y is the finest topology that makes $f : (X, \tau) \to (Y, \mu)$ be $\mathcal{I}$-continuous. Since the mapping $f : (X, \tau) \to (Y, \tau_{f,\mathcal{I}})$ is $\mathcal{I}$-continuous, we have that $\tau_{f,\mathcal{I}} \subset \mu$. Let $U \in \mu$. Since the mapping $f : (X, \tau) \to (Y, \mu)$ is $\mathcal{I}$-continuous, we have that $U \in \tau_{f,\mathcal{I}}$, and thus, $\mu \subset \tau_{f,\mathcal{I}}$. Therefore, $\mu = \tau_{f,\mathcal{I}}$.
Conversely, suppose that $\mu = \tau_{f,\mathcal{I}}$. Then the mapping $f : (X, \tau) \to (Y, \mu)$ is $\mathcal{I}$-continuous. Let ν be a topology of the set Y such that $f : (X, \tau) \to (Y, \nu)$ is an $\mathcal{I}$-continuous mapping. Then $\nu \subset \tau_{f,\mathcal{I}} = \mu$, and thus, μ is the finest topology that makes $f : (X, \tau) \to (Y, \mu)$ be $\mathcal{I}$-continuous.

□

The following is the main result in this section.

Theorem 5.8 *Let* (X, τ) *be an* $\mathcal{I}$*-topological space,* $f : X \to Y$ *be a surjective mapping and* $(Y, \tau_{f,\mathcal{I}})$ *be an* $\mathcal{I}$*-sequential space. Then the following are equivalent.*

(*i*) *A topology* μ *of the set* Y *is the finest topology that makes* $f : (X, \tau) \to (Y, \mu)$ *be* $\mathcal{I}$*-continuous.*

(*ii*) $\mu = \tau_{f,\mathcal{I}}$.

(*iii*) $f : (X, \tau) \to (Y, \mu)$ *is an* $\mathcal{I}$*-quotient mapping and* (Y, μ) *is an* $\mathcal{I}$*-sequential space.*

Proof By Lemma 5.7, we have that (i) ⇔ (ii) ⇒ (iii). Next, we will show that (iii) ⇒ (i). Suppose that $f : (X, \tau) \to (Y, \mu)$ is an $\mathcal{I}$-quotient mapping and (Y, μ) is an $\mathcal{I}$-sequential space. And suppose that ν is a topology of the set Y and $f : (X, \tau) \to (Y, \nu)$ is $\mathcal{I}$-continuous. If $V \in \nu$, then V is $\mathcal{I}$-open in the space (Y, ν), and thus, $f^{-1}(V)$ is $\mathcal{I}$-open in the space (X, τ). Since $f : (X, \tau) \to (Y, \mu)$ is an $\mathcal{I}$-quotient mapping, we have that the set V is $\mathcal{I}$-open in (Y, μ). And since (Y, μ) is an $\mathcal{I}$-sequential space, we obtain that $V \in \mu$. This implies that $\nu \subset \mu$. It follows from that the mapping $f : (X, \tau) \to (Y, \mu)$ is $\mathcal{I}$-continuous that the family μ is the finest topology that makes $f : (X, \tau) \to (Y, \mu)$ be $\mathcal{I}$-continuous. □

Part (i) of Example 4.7 shows that the condition "$\mathcal{I}$-sequential space" in part (iii) of Theorem 5.8 cannot be omitted. Part (ii) of Example 4.7 shows that the condition "$(Y, \tau_{f,\mathcal{I}})$ be an $\mathcal{I}$-sequential space" cannot be replaced by "(Y, μ) be an $\mathcal{I}$-sequence space" in Theorem 5.8.

Corollary 5.9 *Let X be an $\mathcal{I}$-neighborhood space. Then the following are equivalent for a surjective mapping $f : X \to (Y, \mu)$.*

(i) *μ is the finest topology that makes f be $\mathcal{I}$-continuous.*
(ii) $\mu = \tau_{f,\mathcal{I}}$.
(iii) *f is an $\mathcal{I}$-quotient mapping and Y is an $\mathcal{I}$-sequential space.*
(iv) *f is an $\mathcal{I}_{sn}$-quotient mapping and $\mu = \mu_{\mathcal{I}_{sn}}$.*

Proof Since X is an $\mathcal{I}$-neighborhood space, it follows from Lemmas 4.4 and part (ii) of 5.7 that $(Y, \tau_{f,\mathcal{I}})$ is an $\mathcal{I}$-sequential space. By Theorem 5.8, we have that (i) ⇔ (ii) ⇔ (iii). By Theorems 3.14 and 4.5, we have that (i) ⇔ (iv). □

Applying Corollary 5.9 and Example 3.11 to the usual convergence of sequences in topological spaces, we have the following corollary, which is a partial answer to Question 1.4, and a new and interesting result for sequentially quotient mappings.

Corollary 5.10 *Suppose that both X, Y are topological spaces and $f : X \to Y$ is a surjective mapping. Then the following are equivalent.*

(i) *The topology of Y is the finest topology that makes f be sequentially continuous.*
(ii) *A subset U of Y is open in Y if and only if $f^{-1}(U)$ is sequentially open in X.*
(iii) *The mapping f is a sequentially quotient mapping and Y is a sequential space.*

Acknowledgements Questions 1.1–1.4 discussed in this paper come from the academic seminar on general topology in Ningde Normal University. In the seminar, we studied some properties of $\mathcal{I}$-quotient mappings for a maximal ideal and related topics for $\mathcal{I}$-convergence. I would like to thank Professor Xiangeng Zhou for participating in the discussion on this article and proposing valuable comments. One of the main tools of this article is the $\mathcal{I}_{sn}$-interior operator $(A)_{\mathcal{I}_{sn}}$ defined for each subset A of a topological space and an ideal $\mathcal{I}$ on $\mathbb{N}$, which was first introduced by Professor Zhou in topological spaces for a maximal ideal with different terminologies and notations. I would like to express my gratitude to thank the reviewers for reviewing the manuscript and offering many valuable comments. For example, the reviewers give a list of references in a broad perspective of studying ideals to address the interested reader toward this prolific research area.

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