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Optimal Control for Time-Dependent Variational–Hemivariational Inequalities

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Abstract

The present work is intended to investigate optimal control for time-dependent variational–hemivariational inequalities in which the constraint set depends on time. Based on the existence, uniqueness and boundedness of the solution to the inequality, we deliver two continuous dependence results with respect to the time, and then, an existence result for an optimal control problem is presented. Finally, a semipermeability problem and a quasistatic frictional contact problem are given to illustrate our main results.

Keywords Time-dependent variational–hemivariational inequality · Optimal control · Continuous dependence · Constraint set · Semipermeability problem · Quasistatic frictional contact problem

Mathematics Subject Classification 47J20 · 49J40 · 49J45 · 74M10 · 74M15

1 Introduction

Let X be a reflexive Banach space and for every $t \in \mathbb{R}_+ := [0, +\infty)$ $K(t)$ be a nonempty, closed and convex subset of X. Let $A\colon \mathbb{R}_+ \times X \to X^*$ and $\varphi\colon X \times X \to \mathbb{R}$ be given maps to be specified later, $j\colon X \to \mathbb{R}$ be a locally Lipschitz function and $f\colon \mathbb{R}_+ \to X^*$ be fixed. The problem we will study in this paper is the following time-dependent variational–hemivariational inequality.

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Problem 1 *Find $u\colon \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, $u(t) \in K(t)$ and*

$$\begin{aligned} &\langle A(t, u(t)) - f(t), v - u(t)\rangle_X + \varphi(u(t), v) - \varphi(u(t), u(t)) \\ &\quad + j^0(u(t); v - u(t)) \geq 0, \qquad \forall v \in K(t). \end{aligned} \tag{1}$$

Hemivariational inequalities were introduced by Panagiotopoulos (see [14,16,17]) in the 1980s involving Clarke's generalized directional derivative of a locally Lipschitz function. Variational–hemivariational inequalities are the important generalization of variational inequalities and hemivariational inequalities which can be seen as a very effective mathematical tool which brings variational inequalities and hemivariational inequalities together since it involves both convex and nonconvex functions. They appear in a variety of mechanical problems, for example the unilateral contact problems in nonlinear elasticity, the problems describing the adhesive and friction effects, the nonconvex semipermeability problems, the masonry structures and the delamination problems in multilayered composites (see, e.g., [9,15]). For more about the existence, continuous dependence and convergence of solutions to several variational and hemivariational inequalities and related optimal control, we refer to [1,8,10,11,18,19,21–23] and the references therein.

The goal of this paper is to study the optimal control for time-dependent variational–hemivariational inequalities. Based on the existence, uniqueness and boundedness of the solution to the inequality (see [10,18]), we deliver some dependence results with respect to the time when the constraints have different forms in which the Mosco convergence is involved. Moreover, an existence result for an optimal control problem is presented. It is worth pointing out that there are several novelties of the present paper. On the one hand, we consider the constraint sets depending on the time in the inequality problem, which is first investigated, and this study develops the theory of variational–hemivariational inequalities. On the other hand, we obtain that if $t_n \to \bar{t}$ in $\mathbb{R}_+$, then $u(t_n) \to u(\bar{t})$ in X, when the constraint set $K(t)$ has different forms. Finally, we illustrate the abstract result to a semipermeability problem and a quasistatic frictional contact problem.

The rest of this paper is organized as follows: In Sect. 2, we review some of the standard facts which are used in the theory of variational and hemivariational inequalities. An existence and uniqueness result is given at the end of the section. Section 3 is first devoted to the proofs of continuous dependence results for the time-dependent variational–hemivariational inequality. An optimal control problem is then considered. In the last section, we apply the main results in Sect. 3 to a semipermeability problem and a quasistatic frictional contact problem.

2 Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by X^* its dual space and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. Let $C(\mathbb{R}_+; X)$ be the space of continuous functions defined on $\mathbb{R}_+$ with values in X.

Definition 2 A function $f: X \to \mathbb{R}$ is said to be lower semicontinuous (l.s.c.) at u, if for any sequence $\{u_n\}_{n\geq 1} \subset X$ with $u_n \to u$, we have $f(u) \leq \liminf f(u_n)$. A function f is said to be l.s.c. on X, if f is l.s.c. at every $u \in X$.

Definition 3 [2,4] Let $\varphi: X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and l.s.c. function. The mapping $\partial\varphi: X \to 2^{X^*}$ defined by

$$\partial\varphi(u) = \{\, u^* \in X^* \mid \langle u^*, v - u\rangle_X \leq \varphi(v) - \varphi(u) \;\text{ for all }\; v \in X \,\}$$

for $u \in X$ is called the subdifferential of φ. An element $u^* \in \partial\varphi(u)$ is called a subgradient of φ in u.

Definition 4 [2,4] Given a locally Lipschitz function $\varphi: X \to \mathbb{R}$, we denote by $\varphi^0(u; v)$ the (Clarke) generalized directional derivative of φ at the point $u \in X$ in the direction $v \in X$ defined by

$$\varphi^0(u; v) = \limsup_{\lambda \to 0^+,\, \zeta \to u} \frac{\varphi(\zeta + \lambda v) - \varphi(\zeta)}{\lambda}.$$

The generalized gradient of φ at $u \in X$, denoted by $\partial\varphi(u)$, is a subset of X^* given by

$$\partial\varphi(u) = \{\, u^* \in X^* \mid \varphi^0(u; v) \geq \langle u^*, v\rangle_X \;\text{ for all }\; v \in X \,\}.$$

Definition 5 [4,12] Let $K, K_n (n \in \mathbb{N}) \subset X$ be nonempty subsets. We say that K_n converge to K in the Mosco sense, as $n \to \infty$, denoted by $K_n \xrightarrow{M} K$ if and only if the two conditions hold:

(m1) for each $u \in K$, there exists $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in K_n$ and $u_n \to u$ in X as $n \to \infty$,
(m2) for each subsequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in K_n$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.

At the end of this section, we provide a result on existence, uniqueness and boundedness of solution to the variational–hemivariational inequality.

Problem 6 *Find $u \in K$ such that*

$$\begin{aligned} &\langle Au, v - u\rangle_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) \\ &\geq \langle f, v - u\rangle_X, \qquad \forall v \in K. \end{aligned}$$

$$X \;\text{ is a reflexive Banach space.} \tag{2}$$

$$K \;\text{ is a nonempty, closed, convex subset of } X. \tag{3}$$

$$\begin{cases} A\colon X \to X^* \text{ is such that} \\ \text{(a) there exists } \alpha_A > 0 \text{ such that} \\ \qquad \langle Au_1 - Au_2, u_1 - u_2\rangle_X \geq \alpha_A \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \\ \text{(b) there exists } L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_{X^*} \leq L_A \|u_1 - u_2\|_X \\ \quad \text{for all } u_1, u_2 \in X. \end{cases} \tag{4}$$

$$\begin{cases} \varphi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a) } \varphi(u, \cdot)\colon X \to \mathbb{R} \text{ is convex and l.s.c. on } X, \text{ for all } u \in X. \\ \text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\ \quad \varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \\ \quad \leq \alpha_\varphi \|u_1 - u_2\|_X \|v_1 - v_2\|_X \\ \quad \text{for all } u_1, u_2, v_1, v_2 \in X. \\ \text{(c) } \varphi(u, \lambda v) = \lambda\varphi(u, v),\ \varphi(v, v) \geq 0 \text{ for all } u, v \in X, \lambda > 0. \end{cases} \tag{5}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) there exist } c_0, c_1 \geq 0 \text{ such that} \\ \qquad \|\partial j(u)\|_{X^*} \leq c_0 + c_1\|u\|_X \\ \quad \text{for all } u \in X. \\ \text{(c) there exists } \alpha_j > 0 \text{ such that} \\ \qquad j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \leq \alpha_j \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \end{cases} \tag{6}$$

Theorem 7 [10,18] *Assume that* (2)*,* (3)*,* (4)*,* (5)*,* (6) *hold and the following smallness condition is satisfied:*

$$\alpha_\varphi + \alpha_j < \alpha_A. \tag{7}$$

Then for any $f \in X^*$, Problem 6 *has a unique solution $u \in K$. Moreover, u satisfies the following estimate:*

$$\|u\|_X \leq \frac{1}{\alpha_A - \alpha_j}(\|A0_X\|_{X^*} + \|f\|_{X^*} + c_0). \tag{8}$$

3 Optimal Control

In this section we study the dependence of the solution to Problem 1 with respect to the time t and an optimal control problem. In the following, we will provide two dependence results.

At first, we consider the following hypotheses on the data of Problem 6:

$$K(t) \text{ is a nonempty, closed, convex subset of } X \text{ for every } t \in \mathbb{R}_+. \tag{9}$$

$$f \in C(\mathbb{R}_+; X^*). \tag{10}$$

$$\begin{cases} A\colon \mathbb{R}_+ \times X \to X^* \text{ is such that} \\ \text{(a) there exists } \alpha_{A1} > 0 \text{ such that} \\ \qquad \langle A(t,u_1) - A(t,u_2), u_1 - u_2\rangle_X \geq \alpha_{A1} \|u_1 - u_2\|_X^2 \\ \quad \text{for all } t \in \mathbb{R}_+, u_1, u_2 \in X. \\ \text{(b) there exists } L_{A1} > 0 \text{ such that} \\ \qquad \|A(t_1,u_1) - A(t_2,u_2)\|_X \leq L_{A1}(|t_1 - t_2| + \|u_1 - u_2\|_X) \\ \quad \text{for all } t_1, t_2 \in \mathbb{R}_+, u_1, u_2 \in X. \end{cases} \tag{11}$$

$$K(t_n) \xrightarrow{M} K(t) \qquad \text{as } t_n \to t. \tag{12}$$

$$\begin{cases} \text{For any } \{u_n\} \subset X \text{ with } u_n \rightharpoonup u \text{ in } X, \text{ and all } v \in X, \text{ we have} \\ \quad \limsup(\varphi(u_n, v) - \varphi(u_n, u_n)) \leq \varphi(u, v) - \varphi(u, u). \end{cases} \tag{13}$$

$$\begin{cases} \text{For any } \{u_n\} \subset X \text{ with } u_n \rightharpoonup u \text{ in } X, \text{ and all } v \in X, \text{ we have} \\ \quad \limsup j^0(u_n; v - u_n) \leq j^0(u; v - u). \end{cases} \tag{14}$$

The following result concerns the first dependence result to Problem 1.

Theorem 8 *Assume that* (2), (5), (6), (9), (10), (11) *hold and the following smallness condition is satisfied:*

$$\alpha_\varphi + \alpha_j < \alpha_{A1}. \tag{15}$$

Then, for every $t \in \mathbb{R}_+$, Problem 1 has a unique solution $u(t) \in K(t)$ and

$$\|u(t)\|_X \leq \frac{1}{\alpha_A - \alpha_j}(\|A(t, 0_X)\|_{X^*} + \|f(t)\|_{X^*} + c_0). \tag{16}$$

Moreover, suppose also that (12), (13), (14) *hold. If $t_n \to \bar{t}$ in $\mathbb{R}_+$, then $u(t_n) \to u(\bar{t})$ in X.*

Proof The existence and boundedness of solutions to Problem 1 can be deduced from Theorem 7.

Now, we prove the dependence result.

Let $t_n, \bar{t} \in \mathbb{R}_+$ and $t_n \to \bar{t}$ as $n \to +\infty$. Let $u_n = u(t_n) \in K(t_n)$ be the unique solution to Problem 1 corresponding to t_n, i.e.,

$$\begin{aligned} &\langle A(t_n, u_n) - f(t_n), v - u_n\rangle_X + \varphi(u_n, v) - \varphi(u_n, u_n) \\ &\quad + j^0(u_n; v - u_n) \geq 0, \qquad \forall v \in K(t_n). \end{aligned} \tag{17}$$

It follows from (16) that $\{u_n\}$ is a bounded sequence in X. Therefore, by the reflexivity of X, passing to a subsequence if necessary, $u_n \rightharpoonup \bar{u}$ in X as $n \to \infty$ for some $\bar{u} \in X$. We will show that $\bar{u} = u(\bar{t})$ is the solution to Problem 1.

As $u_n \in K(t_n)$ and $K(t_n) \xrightarrow{M} K(\bar{t})$, we have $\bar{u} \in K(\bar{t})$. Moreover, we can find a sequence $\{u_n'\}$ such that $u_n' \in K(t_n)$ and $u_n \to \bar{u}$ in X, as $n \to \infty$. We set $v = u_n'$ in (17), and obtain

$$\begin{aligned} &\limsup\langle A(\bar{t}, u_n), u_n - \bar{u}\rangle_X \\ &\quad \leq \limsup\langle A(\bar{t}, u_n) - A(t_n, u_n), u_n - \bar{u}\rangle_X + \limsup\langle A(t_n, u_n), u_n - \bar{u}\rangle_X \end{aligned}$$

$$\leq \limsup L_{A1}|t_n - \bar{t}|\|u_n\|_X\|u_n - \bar{u}\|_X + \limsup\langle A(t_n, u_n), u_n - u_n'\rangle_X$$
$$+ \limsup\langle A(t_n, u_n), u_n' - \bar{u}\rangle_X$$
$$\leq \limsup\langle f(t_n), u_n - u_n'\rangle_X + \limsup(\varphi(u_n, u_n') - \varphi(u_n, u_n))$$
$$+ \limsup j^0(u_n; u_n' - u_n).$$

Using hypotheses (13), (14), we have

$$\limsup\langle A(\bar{t}, u_n), u_n - \bar{u}\rangle_X \leq 0.$$

It is well known that a monotone Lipschitz continuous operator is pseudomonotone, and hence, (11) implies that $A(t, \cdot)$ is pseudomonotone for all $t \in \mathbb{R}_+$. Therefore, we infer

$$\liminf\langle A(\bar{t}, u_n), u_n - v\rangle_X \geq \langle A(\bar{t}, \bar{u}), \bar{u} - v\rangle_X, \qquad \forall v \in X.$$

Subsequently, we will pass to the limit in (17). Let $w \in K(\bar{t})$. From $(m1)$ in Definition 5, we find a sequence $\{w_n\}$ such that $w_n \in K(t_n)$ and $w_n \to w$ in X, as $n \to \infty$. Setting $v = w_n$ in (1) we obtain

$$\langle A(t_n, u_n) - f(t_n), w_n - u_n\rangle_X + \varphi(u_n, w_n) - \varphi(u_n, u_n) + j^0(u_n; w_n - u_n) \geq 0.$$

Then, we have

$$\begin{aligned}
&\langle A(\bar{t}, \bar{u}), \bar{u} - w\rangle_X \leq \limsup\, \langle A(\bar{t}, u_n), u_n - w\rangle_X \\
&\quad\leq \limsup\, \langle A(\bar{t}, u_n), u_n - w_n\rangle_X + \limsup\, \langle A(\bar{t}, u_n), w_n - w\rangle_X \\
&\quad\leq \limsup\, \langle A(\bar{t}, u_n) - A(t_n, u_n), u_n - w_n\rangle_X + \limsup\, \langle A(t_n, u_n), u_n - w_n\rangle_X \\
&\quad\leq \limsup L_{A1}|t_n - \bar{t}|\|u_n\|_X\|u_n - w_n\|_X \\
&\qquad + \limsup\left(\langle f_n, u_n - w_n\rangle + \varphi(u_n, w_n) - \varphi(u_n, u_n) + j^0(u_n; w_n - u_n)\right) \\
&\quad\leq \langle f(\bar{t}), \bar{u} - w\rangle_X + \varphi(\bar{u}, w) - \varphi(\bar{u}, \bar{u}) + j^0(\bar{u}; w - \bar{u}).
\end{aligned}$$

Since $w \in K(\bar{t})$ is arbitrary, we obtain that

$$\langle A(\bar{t}, \bar{u}) - f(\bar{t}), w - \bar{u}\rangle_X + \varphi(\bar{u}, w) - \varphi(\bar{u}, \bar{u}) + j^0(\bar{u}; w - \bar{u}) \geq 0, \qquad \forall w \in K(\bar{t}),$$

which implies that $\bar{u} \in K(\bar{t})$ solves Problem 1. Since the solution of Problem 1 is unique, every subsequence $\{u_n\}$ converges weakly to the same limit, and hence, the whole original sequence $\{u_n\}$ converges weakly to $\bar{u} = u(\bar{t}) \in K(t)$.

Finally, we show that $u_n \to \bar{u}$, as $n \to \infty$. Since $K(t_n) \xrightarrow{M} K(\bar{t})$ as $n \to \infty$, we can find a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in K(t_n)$ such that $\tilde{u}_n \to \bar{u}$, as $\rho \to 0$. Choosing $v = \tilde{u}_n$ in (17), we have

$$\begin{aligned}
&\alpha_{A1}\|u_n - \tilde{u}_n\|_X^2 \leq \langle A(t_n, u_n) - A(t_n, \tilde{u}_n), u_n - \tilde{u}_n\rangle_X \\
&\quad= \langle Au_n, u_n - \tilde{u}_n\rangle_X + \langle A(t_n, \bar{u}) - A(t_n, \tilde{u}_n), u_n - \tilde{u}_n\rangle_X + \langle -A(t_n, \bar{u}), u_n - \tilde{u}_n\rangle_X
\end{aligned}$$

$$\leq \varphi(u_n, \tilde{u}_n) - \varphi(u_n, u_n) + j^0(u_n; \tilde{u}_n - u_n) + \langle f(t_n) - A(t_n, \overline{u}), u_n - \tilde{u}_n \rangle_X$$
$$+L_{A1} \|\tilde{u}_n - \overline{u}\|_X \|u_n - \tilde{u}_n\|_X.$$

Passing to the upper limit in the last inequality, as $n \to \infty$, and exploiting (11), (13), (14), we deduce $\limsup \|u_n - \tilde{u}_n\|_X^2 \leq 0$. Hence, we obtain $\|u_n - \tilde{u}_n\|_X \to 0$. Therefore, we have

$$0 \leq \lim \|u_n - \overline{u}\|_X \leq \lim \|u_n - \tilde{u}_n\|_X + \lim \|\tilde{u}_n - \overline{u}\|_X = 0,$$

which implies that $u_n \to \overline{u}$ in X, as $\rho \to 0$. This completes the proof. □

Next, we consider the constraint sets $K(t)$ and function φ which satisfy the following hypotheses:

$$\begin{cases} K(t) = c(t)K + d(t)\theta \text{ is such that} \\ \text{(a) } K \text{ is a nonempty, closed and convex subset of } X. \\ \text{(b) } 0_X \in K(t) \text{ and } \theta \text{ is a given element of } X. \\ \text{(c) } c, d \colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ are continuous.} \end{cases} \tag{18}$$

$$\begin{cases} \varphi \colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{there exists function } c_\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and a constant } N_k > 0 \\ \text{such that } c_\varphi(r) \leq N_k \text{ for all } r \in [0, k], \text{ and} \\ \quad \varphi(u, v_1) - \varphi(u, v_2) \leq c_\varphi(\|u\|_X)\|v_1 - v_2\|_X, \\ \text{for all } u, v_1, v_2 \in V. \end{cases} \tag{19}$$

Remark 9 We observe that if $K(t)$, for $t > 0$, is defined by (18), then $K(t_n) \to K(t)$ in the sense of Mosco, as $t_n \to t$, see [12].

Next, we give the second dependence result to Problem 1.

Theorem 10 *Assume that* (2), (5), (6), (10), (11), (15), (18), (19) *are satisfied. If $t_n \to \overline{t}$ in $\mathbb{R}_+$, then $u(t_n) \to u(\overline{t})$ in X.*

Proof Let $t, t_n \in \mathbb{R}_+$ and $t_n \to t$ as $n \to +\infty$. We have

$$\langle A(t, u(t)) - f(t), v - u(t)\rangle_X + \varphi(u(t), v) - \varphi(u(t), u(t))$$
$$+j^0(u(t); v - u(t)) \geq 0, \qquad \forall v \in K(t), \tag{20}$$
$$\langle A(t_n, u(t_n)) - f(t_n), v_n - u(t_n)\rangle_X + \varphi(u(t_n), v_n) - \varphi(u(t_n), u(t_n))$$
$$+j^0(u(t_n); v_n - u(t_n)) \geq 0, \qquad \forall v_n \in K(t_n). \tag{21}$$

By the definition of $K(t_n)$, we get $\frac{u(t_n)-d(t_n)\theta}{c(t_n)} \in K$. Let $c_n = \frac{c(t)}{c(t_n)}$. Taking $v = c_n(u(t_n) - d(t_n)\theta) + d(t)\theta \in K(t)$ in (20) we obtain

$$\langle A(t, u(t)) - f(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta - u(t)\rangle_X$$
$$+\varphi(u(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t), u(t))$$
$$+j^0(u(t); c_n(u(t_n) - d(t_n)\theta) + d(t)\theta - u(t)) \geq 0.$$

Taking $v_n = \frac{1}{c_n}(u(t) - d(t)\theta) + d(t_n)\theta \in K(t_n)$ in (21) and multiplying by c_n, we obtain

$$\begin{aligned}
&\langle A(t_n, u(t_n)) - f(t_n), u(t) - d(t)\theta - c_n(u(t_n) - d(t_n)\theta)\rangle_X \\
&\quad +\varphi(u(t_n), u(t) - d(t)\theta + c_n d(t_n)\theta) - \varphi(u(t_n), c_n u(t_n)) \\
&\quad + j^0(u(t_n); u(t) - d(t)\theta - c_n(u(t_n) - d(t_n)\theta)) \geq 0.
\end{aligned}$$

Adding the above two inequalities, we deduce that

$$\begin{aligned}
&\langle A(t, u(t)) - A(t, u(t_n)), u(t) - u(t_n)\rangle_X \\
&\leq \langle A(t, u(t_n)) - A(t, u(t)), -d(t)\theta - (c_n - 1)u(t_n) + c_n d(t_n)\theta\rangle_X \\
&\quad +\langle A(t_n, u(t_n)) - A(t, u(t_n)), u(t) - d(t)\theta - c_n(u(t_n) - d(t_n)\theta)\rangle_X \\
&\quad +\varphi(u(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t), u(t)) \\
&\quad +\varphi(u(t_n), u(t) - d(t)\theta + c_n d(t_n)\theta) - \varphi(u(t_n), c_n u(t_n)) \\
&\quad + j^0(u(t); c_n(u(t_n) - d(t_n)\theta) + d(t)\theta - u(t)) \\
&\quad + j^0(u(t_n); u(t) - d(t)\theta - c_n(u(t_n) - d(t_n)\theta)) \\
&\quad +\langle f(t_n) - f(t), u(t) - d(t)\theta - c_n(u(t_n) - d(t_n)\theta)\rangle_X.
\end{aligned}$$

From (5)(c) and (d), we have

$$\begin{aligned}
&\varphi(u(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t), u(t)) \\
&\quad +\varphi(u(t_n), u(t) - d(t)\theta + c_n d(t_n)\theta) - \varphi(u(t_n), c_n u(t_n)) \\
&= \varphi(u(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t), u(t) - d(t)\theta + c_n d(t_n)\theta) \\
&\quad +\varphi(u(t_n), u(t) - d(t)\theta + c_n d(t_n)\theta) - \varphi(u(t_n), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) \\
&\quad +\varphi(u(t), u(t) - d(t)\theta - c_n d(t_n)\theta) - \varphi(u(t), u(t)) \\
&\quad +\varphi(u(t_n), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t_n), c_n u(t_n)) \\
&\leq \alpha_\varphi \|u(t_n) - u(t)\|_X \|c_n u(t_n) - u(t) + 2d(t)\theta - 2c_n d(t_n)\theta\|_X \\
&\quad +\|c_n d(t_n)\theta - d(t)\theta\|(c_\varphi(\|u(t_n)\|_X) + c_\varphi(\|u(t)\|_X)).
\end{aligned}$$

By using the identity

$$c_n u(t_n) - u(t) = u(t_n) - u(t) + (c_n - 1)u(t_n),$$

we obtain

$$\begin{aligned}
&\varphi(u(t), c_n(u(t_n) - d(t_n)\theta) + d(t)\theta) - \varphi(u(t), u(t)) \\
&\quad +\varphi(u(t_n), u(t) - d(t)\theta - c_n d(t_n)\theta) - \varphi(u(t_n), c_n u(t_n)) \\
&\leq \alpha_\varphi \|u(t_n) - u(t)\|_X \|c_n u(t_n) - u(t) + 2d(t)\theta - 2c_n d(t_n)\theta\|_X \\
&\quad +\|c_n d(t_n)\theta - d(t)\theta\|_X (c_\varphi(\|u(t_n)\|_X) + c_\varphi(\|u(t)\|_X)) \\
&\leq \alpha_\varphi \|u(t_n) - u(t)\|_X^2 + \alpha_\varphi \|u(t_n) - u(t)\|_X \|(c_n - 1)u(t_n) + 2d(t)\theta - 2c_n d(t_n)\theta\|_X \\
&\quad +\|c_n d(t_n)\theta - d(t)\theta\|_X (c_\varphi(\|u(t_n)\|_X) + c_\varphi(\|u(t)\|_X)).
\end{aligned}$$

Moreover, it follows from (6)(b) that

$$
\begin{aligned}
& j^0(u(t); c_n(u(t_n)-d(t_n)\theta)+d(t)\theta-u(t)) \\
&\quad + j^0(u(t_n); u(t)-d(t)\theta-c_n(u(t_n)-d(t_n)\theta)) \\
&\le \alpha_j\|u(t_n)-u(t)\|_X\|c_n(u(t_n)-d(t_n)\theta)+d(t)\theta-u(t)\|_X \\
&\le \alpha_j\|u(t_n)-u(t)\|_X\|u(t_n)-u(t)+(c_n-1)u(t_n)-c_nd(t_n)\theta+d(t)\theta\|_X \\
&\le \alpha_j\|u(t_n)-u(t)\|_X^2+\alpha_j\|u(t_n)-u(t)\|_X\|c_n-1)u(t_n)-c_nd(t_n)\theta+d(t)\theta\|_X.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& (\alpha_{A1}-\alpha_\varphi-\alpha_j)\|u(t_n)-u(t)\|_X^2 \\
&\quad \le L_{A1}\|-d(t)\theta-(c_n-1)u(t_n)+c_nd(t_n)\theta\|_X\|u(t)-u(t)\|_X \\
&\qquad + L_{A1}|t_n-t|\|u(t)-d(t)\theta-c_n(u(t_n)-d(t_n)\theta)\|_X \\
&\qquad + \alpha_\varphi\|u(t_n)-u(t)\|_X\|(c_n-1)u(t_n)+2d(t)\theta-2c_nd(t_n)\theta\|_X \\
&\qquad + \|c_nd(t_n)\theta-d(t)\theta\|_X(c_\varphi(\|u(t_n)\|_X)+c_\varphi(\|u(t)\|_X)) \\
&\qquad + \alpha_j\|u(t_n)-u(t)\|_X\|(c_n-1)u(t_n)-c_nd(t_n)\theta+d(t)\theta\|_X \\
&\qquad + \|f(t_n)-f(t)\|_{X^*}\|u(t_n)-u(t)\|_X \\
&\qquad + \|f(t_n)-f(t)\|_{X^*}\|-d(t)\theta-(c_n-1)u(t_n)+c_nd(t_n)\theta)\|_X \\
&\quad \le \Bigg(L_{A1}(|c_n-1|\|u(t_n)\|_X+|c_nd(t_n)\theta-d(t)|\|\theta\|_X) \\
&\qquad + \alpha_\varphi(|c_n-1|\|u(t_n)\|_X+|2d(t)-2c_nd(t_n)|\|\theta\|_X) \\
&\qquad + \alpha_j(|c_n-1|\|u(t_n)\|_X+|c_nd(t_n)-d(t)|\|\theta\|_X) \\
&\qquad + \|f(t_n)-f(t)\|_{X^*}\Bigg)\|u(t_n)-u(t)\|_X \\
&\qquad + L_{A1}|t_n-t|(\|u(t)\|_X+|d(t)|\|\theta\|_X+|c_n|(\|u(t_n)\|_X+|d(t_n)|\|\theta\|_X)) \\
&\qquad + |c_nd(t_n)-d(t)|\|\theta\|_X(c_\varphi(\|u(t_n)\|_X)+c_\varphi(\|u(t)\|_X)) \\
&\qquad + \|f(t_n)-f(t)\|_{X^*}(|c_n-1|\|u(t_n)\|_X+|c_nd(t_n)\theta-d(t)|\|\theta\|_X).
\end{aligned}
$$

From (16), it follows that there exists a constant $k>0$ such that $\|u(t_n)\|_X\le k$ and $\|u(t)\|_X\le k$ for sufficiently large n. Then, there exists a constant $N_k>0$ such that

$$
c_\varphi(\|u(t_n)\|_X)\le N_k,\ \ c_\varphi(\|u(t)\|_X)\le N_k. \tag{22}
$$

Then, we have

$$
\begin{aligned}
& (\alpha_{A1}-\alpha_\varphi-\alpha_j)\|u(t_n)-u(t)\|_X^2 \\
&\quad \le \Bigg(L_{A1}(|c_n-1|k+|c_nd(t_n)\theta-d(t)|\|\theta\|_X) \\
&\qquad + \alpha_\varphi(|c_n-1|k+|2d(t)-2c_nd(t_n)|\|\theta\|_X)+\|f(t_n)-f(t)\|_{X^*}\Bigg)\|u(t_n)-u(t)\|_X \\
&\qquad + L_{A1}|t_n-t|(k+|d(t)|\|\theta\|_X+|c_n|(k+|d(t_n)|\|\theta\|_X))+2N_k|c_nd(t_n)-d(t)|\|\theta\|_X
\end{aligned}
$$

$$+\|f(t_n)-f(t)\|_{X^*}(|c_n-1|k+|c_nd(t_n)\theta-d(t)|\|\theta\|_X).$$

From the following fact

$$x,a,b\geq 0 \text{ and } x^2\leq ax+b \qquad\Rightarrow\qquad x\leq a+\sqrt{b},$$

it follows that

$$\begin{aligned}
&\|u(t_n)-u(t)\|_X\\
&\leq \frac{1}{\alpha_{A1}-\alpha_\varphi-\alpha_j}\Big(L_{A1}(|c_n-1|k+|c_nd(t_n)\theta-d(t)|\|\theta\|_X)\\
&\quad+\alpha_\varphi(|c_n-1|k+|2d(t)-2c_nd(t_n)|\|\theta\|_X)\\
&\quad+\alpha_j(|c_n-1|k+|c_nd(t_n)-d(t)|\|\theta\|_X)+\|f(t_n)-f(t)\|_{X^*}\Big)\\
&\quad+\sqrt{\frac{1}{\alpha_{A1}-\alpha_\varphi-\alpha_j}\Big(L_{A1}|t_n-t|(k+|d(t)|\|\theta\|_X+|c_n|(k+|d(t_n)|\|\theta\|_X))}\\
&\quad\overline{+2N_k|c_nd(t_n)-d(t)|\|\theta\|_X}\\
&\quad\overline{+\|f(t_n)-f(t)\|_{X^*}(c_n-1|k+|c_nd(t_n)\theta-d(t)|\|\theta\|_X)\Big)}.
\end{aligned}$$

Since $c(t_n)\to c(t)$, $d(t_n)\to d(t)$ and $f(t_n)\to f(t)$ as $n\to+\infty$, we deduce that the right hand of the above inequality tends to 0 as $n\to+\infty$, and hence, $u(t_n)\to u(t)$ as $n\to+\infty$. Therefore, we obtain that $u\in C(\mathbb{R}_+;X)$. ◻

Remark 11 We observe that if $d(t)=0$ for all $t\in\mathbb{R}_+$ in (18), then from the proof of the above theorem, we can omit the condition (13).

We consider the following special case.

Problem 12 *Find $u:\mathbb{R}_+=[0,+\infty)\to X$ such that, for all $t\in\mathbb{R}_+$, $u(t)\in K$ and*

$$\begin{aligned}
&\langle A(t,u(t))-f(t),v-u(t)\rangle_X+\varphi(u(t),v)-\varphi(u(t),u(t))\\
&\quad+j^0(u(t);v-u(t))\geq 0,\qquad \forall v\in K.
\end{aligned}$$

The following result is a consequence of Theorem 10.

Theorem 13 *Assume that (2), (5), (6), (10), (11), (15), (19) are satisfied. If $t_n\to\bar{t}$ in $\mathbb{R}_+$, then $u(t_n)\to u(\bar{t})$ in K.*

Finally, we provide an existence result for an optimal control problem governed by Problem 1.

Consider a closed interval $[a,b]\subset\mathbb{R}_+$ and a cost functional $F:[a,b]\times X\to\mathbb{R}$, and find a solution $t^*\in[a,b]$ to the following problem:

$$F(t^*,u(t^*))=\min_{t\in[a,b]}F(t,u(t)),\tag{23}$$

where $u = u(t) \in K(t)$ denotes the unique solution of Problem 1 corresponding to the time t.

We are now in a position to state the main result on the existence of solutions to problem (23). We admit the following hypothesis:

$$F \text{ is l.s.c. on } [a, b] \times X. \tag{24}$$

Theorem 14 *Assume all the hypotheses of Theorem 8 or Theorem 10 are satisfied. If (24) holds, then the problem (23) has at least one solution.*

Proof Let $\{(t_n, u_n)\} \subset [a, b] \times X$ be a minimizing sequence of the functional F, i.e.,

$$\lim F(t_n, u_n) = \inf_{t\in[a,b]} F(t, u(t)),$$

where $t_n \in [a, b]$ and $u_n \in K(t_n)$ is the unique solution of Problem 1 that corresponds to t_n, i.e., $u_n = u(t_n)$. It is clear that $\{t_n\}$ is bounded. Then, there is a subsequence of $\{t_n\}$, denoted in the same way, such that $t_n \to \bar{t}$ for some $\bar{t} \in [a, b]$. From Theorem 8 or Theorem 10, we infer that the sequence $\{u_n\} \subset K(t_n)$ converges weakly in X to the unique solution $u(\bar{t}) \in K(\bar{t})$ of Problem 1. Finally, from (24), we have

$$F(\bar{t}, u(\bar{t})) \leq \liminf F(t_n, u_n) = \inf_{t\in[a,b]} F(t, u(t)),$$

which shows that $\bar{t}$ is a solution of the problem (23). This completes the proof. □

4 Applications

4.1 Semipermeability Problem

In this part we consider a semipermeability problem (see [3,14–16]) to illustrate our main results of Sect. 3.

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega = \Gamma$ which consists of two disjoint measurable parts Γ_1 and Γ_2 such that $m(\Gamma_1) > 0$. Consider the following semipermeability problem.

Problem 15 *Find a temperature* $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$-\operatorname{div} a(\boldsymbol{x}, \nabla u) = \tilde{f}(t, u) \ \ in \ \ \Omega \times \mathbb{R}_+, \tag{25}$$

$$\tilde{f}(t, u) = f_1(t) + f_2(u), \ \ -f_2(u) \in \partial h(\boldsymbol{x}, u) \ \ in \ \ \Omega \times \mathbb{R}_+, \tag{26}$$

$$u(t) \in U(t) \ \ for \ \ t \in \mathbb{R}_+, \tag{27}$$

$$u = 0 \ \ on \ \ \Gamma_1 \times \mathbb{R}_+, \tag{28}$$

$$-\frac{\partial u}{\partial \nu_a} \in k(u)\partial g_c(\boldsymbol{x}, u) \ \ on \ \ \Gamma_2 \times \mathbb{R}_+. \tag{29}$$

The description of Problem 15 can be found in [23] when U is independent of t.

We introduce the following spaces:

$$V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \}, \quad H = L^2(\Omega). \tag{30}$$

Since $m(\Gamma_1) > 0$, on V we can consider the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega)^d}$ for $v \in V$ which is equivalent on V to the $H^1(\Omega)$ norm. By $\gamma\colon V \to L^2(\Gamma)$ we denote the trace operator which is known to be linear, bounded and compact. Moreover, by γv we denote the trace of an element $v \in H^1(\Omega)$.

We need the following hypotheses study Problem 15:

$$\left\{\begin{array}{l} a\colon \Omega\times\mathbb{R}^d \to \mathbb{R}^d \text{ is such that} \\ \quad \text{(a) } a(\cdot,\boldsymbol{\xi}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^d, \\ \qquad \text{and } a(\boldsymbol{x},0)=0 \text{ for a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(b) } a(\boldsymbol{x},\cdot) \text{ is continuous on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(c) } \|a(\boldsymbol{x},\boldsymbol{\xi})\| \le m_a\,(1+\|\boldsymbol{\xi}\|) \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega \\ \qquad \text{with } m_a>0. \\ \quad \text{(d) } (a(\boldsymbol{x},\boldsymbol{\xi}_1)-a(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \ge \alpha_a\,\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2 \\ \qquad \text{for all } \boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega \text{ with } \alpha_a>0. \end{array}\right. \tag{31}$$

$$\left\{\begin{array}{l} h\colon \Omega\times\mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } h(\cdot,r) \text{ is measurable on } \Omega \text{ for all } r\in\mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e}\in L^2(\Omega) \text{ such that } h(\cdot,\overline{e}(\cdot))\in L^1(\Omega). \\ \quad \text{(b) } h(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(c) there exist } \overline{c}_0,\ \overline{c}_1\ge 0 \text{ such that} \\ \qquad |\partial h(\boldsymbol{x},r)| \le \overline{c}_0+\overline{c}_1|r| \text{ for all } r\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(d) there exists } \alpha_h\ge 0 \text{ such that} \\ \qquad h^0(\boldsymbol{x},r_1;r_2-r_1)+h^0(\boldsymbol{x},r_2;r_1-r_2) \le \alpha_h|r_1-r_2|^2 \\ \qquad \text{for all } r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Omega. \end{array}\right. \tag{32}$$

$$\left\{\begin{array}{l} g\colon \Gamma_2\times\mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } g(\cdot,r) \text{ is measurable on } \Gamma_2 \text{ for all } r\in\mathbb{R}. \\ \quad \text{(b) } g(\boldsymbol{x},\cdot) \text{ is convex on } \Gamma_2, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \quad \text{(c) there exists } L_g>0 \text{ such that} \\ \qquad |g(\boldsymbol{x},r_1)-g(\boldsymbol{x},r_2)| \le L_g|r_1-r_2| \\ \qquad \text{for all } r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_2. \end{array}\right. \tag{33}$$

$$\left\{\begin{array}{l} k\colon \Gamma_2\times\mathbb{R} \to \mathbb{R}_+ \text{ is such that} \\ \quad \text{(a) } k(\cdot,r) \text{ is measurable on } \Gamma_2 \text{ for all } r\in\mathbb{R}. \\ \quad \text{(b) there exists } L_k>0 \text{ such that} \\ \qquad |k(\boldsymbol{x},r_1)-k(\boldsymbol{x},r_2)| \le L_k|r_1-r_2| \\ \qquad \text{for all } r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_2. \\ \quad \text{(c) } k(\boldsymbol{x},0)=0 \text{ for a.e. } \boldsymbol{x}\in\Omega. \end{array}\right. \tag{34}$$

$$\text{For every} t\in\mathbb{R}_+\ U(t) \text{ is a closed, convex subset of } V, \quad f_1\in C(\mathbb{R}_+;H). \tag{35}$$

By standard procedure, we obtain the variational formulation of Problem 15 with the following form.

Problem 16 *Find* $u\colon \mathbb{R}_+ \to V$ *such that for all* $t \in \mathbb{R}_+$, $u(t) \in U(t)$ *and*

$$\int_\Omega a(\boldsymbol{x}, \nabla u(t)) \cdot \nabla(v - u(t))\,\mathrm{d}x + \int_{\Gamma_2} \big(k(u(t))g(\boldsymbol{x}, v) - k(u(t))g(\boldsymbol{x}, u(t))\big)\,\mathrm{d}\Gamma$$
$$+ \int_\Omega h^0(\boldsymbol{x}, u(t); v - u(t))\,\mathrm{d}x \geq \int_\Omega f_1(t)(v - u(t))\,\mathrm{d}x$$

for all $v \in U(t)$.

Theorem 17 *Assume that* (31)–(35) *hold and the following smallness condition is satisfied:*

$$L_k L_g \|\gamma\|^2 + \alpha_h < \alpha_a. \tag{36}$$

Then, Problem 16 *has a unique solution* $u \in C(\mathbb{R}_+; V)$.

Proof We can apply Theorem 7 in the following functional framework: $X = V$, $K = U$, $f(t) = f_1(t)$ for all $t \in \mathbb{R}_+$ and

$$A\colon V \to V^*,\ \ \langle Au, v\rangle_V = \int_\Omega a(\boldsymbol{x}, \nabla u) \cdot \nabla v\,\mathrm{d}x \quad \text{for } u, v \in V, \tag{37}$$

$$\varphi\colon V \times V \to \mathbb{R},\ \ \varphi(u, v) = \int_{\Gamma_2} k(u)g(v)\,\mathrm{d}\Gamma \quad \text{for } u, v \in V, \tag{38}$$

$$j\colon V \to \mathbb{R},\quad j(v) = \int_\Omega h(v)\,\mathrm{d}x \quad \text{for } v \in V. \tag{39}$$

From the proof of [23, Theorem 32], the operator A and functions φ and j satisfy hypotheses (11), (5) and (6) with $\alpha_{A1} = \alpha_a$, $\alpha_\varphi = L_k L_g \|\gamma\|^2$ and $\alpha_j = \alpha_h$, respectively. □

We conclude this part with the following example.

Example 18 Hypothesis (12) is satisfied for the following constraint sets for a bilateral obstacle problem:

$$U(t) = U(\psi_1(t), \psi_2(t)) = \{\, v \in V \mid \psi_1(t) \leq v \leq \psi_2(t) \ \text{ a.e. in } \ \Omega \,\},$$

where $\psi_1, \psi_2 \in C(\mathbb{R}_+; V)$. It is clear that for every $t \in \mathbb{R}_+$, $U(t)$ is closed convex subset of V. We will show that

$$U(t_n) \xrightarrow{M} U(t) \qquad \text{as } t_n \to t. \tag{40}$$

In fact, let $v_n \in U(t_n)$ be such that $v_n \rightharpoonup v$ in V, as $n \to \infty$. Since

$$U(t_n) = \{\, z \in V \mid z \geq \psi_1(t_n) \ \text{ a.e. in } \ \Omega \,\} \cap \{\, z \in V \mid z \leq \psi_2(t_n) \ \text{ a.e. in } \ \Omega \,\},$$

we obtain $v_n - \psi_1(t_n) \in \{\, z \in V \mid z \geq 0 \ \text{ a.e. in } \ \Omega \,\}$ and $v_n - \psi_2(t_n) \in \{\, z \in V \mid z \leq 0 \ \text{ a.e. in } \ \Omega \,\}$. Moreover, since the sets $\{\, z \in V \mid z \geq 0 \ \text{ a.e. in } \ \Omega \,\}$ and

$\{z \in V \mid z \le 0 \text{ a.e. in } \Omega\}$ are weakly closed by Mazur's theorem, we deduce that $v - \psi_1(t) \in \{z \in V \mid z \ge 0 \text{ a.e. in } \Omega\}$ and $v - \psi_2(t) \in \{z \in V \mid z \le 0 \text{ a.e. in } \Omega\}$, and hence, $v \in U(t)$.

On the other hand, for any $v \in U(t)$, there exist $v_1 \in \{z \in V \mid z \ge 0 \text{ a.e. in } \Omega\}$ and $v_2 \in \{z \in V \mid z \le 0 \text{ a.e. in } \Omega\}$ such that $v = v_1 + \psi_1(t) = v_2 + \psi_2(t)$.

Since $\psi_1, \psi_2 \in C(\mathbb{R}_+; V)$, it is clear that $(\psi_1(t_n), \psi_2(t_n)) \to (\psi_1(t), \psi_2(t))$ in $V \times V$. Put $v_n = v_1 + \psi_1(t_n)$. Then, for n large enough, we get $v_n \in U(t_n)$. Hence, $v_n = v_1 + \psi_1(t_n) \to v_1 + \psi_1(t) = v_2 + \psi_2(t) = v$ in V. Therefore, the convergence (40) holds.

4.2 Quasistatic Frictional Contact Problem

In this part, we consider a quasistatic frictional contact problem which variational formulation is a time-dependent variational–hemivariational inequality. For more frictional contact problems, we refer to [5–7,13].

An elastic body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 1$, 2, 3. The boundary of Ω is denoted by $\Gamma = \partial\Omega$ and it is assumed to be Lipschitz continuous. We also suppose that Γ consists of three mutually disjoint and measurable parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ such that meas $(\Gamma_1) > 0$. Let $\boldsymbol{\nu} = (\nu_i)$ be the outward unit normal at Γ and let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$. For a vector field, notation u_ν and $\boldsymbol{u}_\tau$ represent the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$. Also, σ_ν and $\boldsymbol{\sigma}_\tau$ represent the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary, i.e., $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

The classical model for the quasistatic frictional contact problem is as follows:

Problem 19 *Find a displacement field* $\boldsymbol{u} \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} \colon \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *and an interface force* $\eta \colon \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \;\; in \;\; \Omega \times \mathbb{R}_+, \tag{41}$$

$$\mathrm{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \;\; in \;\; \Omega \times \mathbb{R}_+, \tag{42}$$

$$\boldsymbol{u} = \boldsymbol{0} \;\; on \;\; \Gamma_1 \times \mathbb{R}_+, \tag{43}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \;\; on \;\; \Gamma_2 \times \mathbb{R}_+, \tag{44}$$

$$u_\nu \le g, \;\; \sigma_\nu + \eta \le 0, \;\; (u_\nu - g)(\sigma_\nu + \eta) = 0, \;\; \eta \in \partial j_\nu(u_\nu) \;\; on \;\; \Gamma_3 \times \mathbb{R}_+, \tag{45}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \;\; if \;\; \|\boldsymbol{u}_\tau\| \ne 0 \;\; on \;\; \Gamma_3 \times \mathbb{R}_+. \tag{46}$$

The description of Problem 19 can be found in [9,10,20] for fixed $t \in \mathbb{R}_+$.

We will use the spaces V and $\mathcal{H}$ defined by

$$V = \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d).$$

The space $\mathcal{H}$ will be endowed with the Hilbertian structure given by the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\,\mathrm{d}x,$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. On the space V we consider the inner product and the corresponding norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \;\; \|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \;\text{ for all }\; \boldsymbol{u}, \boldsymbol{v} \in V.$$

Let $\gamma\colon V \to L^2(\Gamma; \mathbb{R}^d)$ be the trace operator. By the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)} \le \|\gamma\|\|\boldsymbol{v}\|_V \;\text{ for all }\; \boldsymbol{v} \in V. \tag{47}$$

We need the following hypotheses on the data of Problem 19:

$$\left\{\begin{array}{l} \mathcal{A}\colon \Omega \times \mathbb{R}_+ \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \;\text{(a) } \mathcal{A}(\cdot, t, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } t \in \mathbb{R}_+, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \;\text{(b) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \qquad\quad \|\mathcal{A}(\boldsymbol{x}, t_1, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, t_2, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}}(|t_1 - t_2| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \qquad \text{for all } t_1, t_2 \in \mathbb{R}_+, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \;\text{(c) there exists } \alpha_{\mathcal{A}} > 0 \text{ such that} \\ \qquad\quad (\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge \alpha_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad \text{for all } t \in \mathbb{R}_+, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \;\text{(d) } \mathcal{A}(\boldsymbol{x}, t, \boldsymbol{0}) = \boldsymbol{0} \text{ for all } t \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{48}$$

$$\left\{\begin{array}{l} F_b\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \;\text{(a) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \;\text{(b) there exists } L_{F_b} > 0 \text{ such that} \\ \qquad\quad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b}|r_1 - r_2| \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \;\text{(c) } F_b(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, F_b(\boldsymbol{x}, r) \ge 0 \text{ for all } r \ge 0 \\ \qquad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{49}$$

$$\left\{\begin{array}{l} j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \;\text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3). \\ \;\text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \;\text{(c) there exist } \overline{c}_0 \ge 0 \text{ and } \overline{c}_1 \ge 0 \text{ such that} \\ \qquad |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1|r| \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \;\text{(d) there exist } \alpha_{j_\nu} > 0 \text{ such that} \\ \qquad j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu}|r_1 - r_2|^2 \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{50}$$

We also assume that the densities of body forces and surface tractions satisfy

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^d)), \;\; \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2; \mathbb{R}^d)) \;\text{ and }\; g \in C(\mathbb{R}_+; \mathbb{R}_+).$$

We introduce the set of admissible displacement fields $U(t)$ for $t \in \mathbb{R}_+$ defined by

$$U(t) = \{ \boldsymbol{v} \in V \mid v_\nu \le g(t) \text{ on } \Gamma_3 \}$$

and define an element $\boldsymbol{f}: \mathbb{R}_+ \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle_V = \langle \boldsymbol{f}_0(t), \boldsymbol{v} \rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \boldsymbol{f}_2(t), \boldsymbol{v} \rangle_{L^2(\Gamma_2;\mathbb{R}^d)} \tag{51}$$

for all $\boldsymbol{v} \in V$, $t \in \mathbb{R}_+$.

The variational formulation of Problem 19 has the following form.

Problem 20 *Find* $\boldsymbol{u}: \mathbb{R}_+ \to V$ *such that, for all* $t \in \mathbb{R}_+$, $u(t) \in U(t)$ *and*

$$\langle \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \rangle_{\mathcal{H}} + \int_{\Gamma_3} F_b(u_\nu(t))(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|) \, \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) \, \mathrm{d}\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_V, \qquad \forall \boldsymbol{v} \in U(t).$$

Theorem 21 *Assume that* (48), (49), (50) *hold and the following smallness condition is satisfied:*

$$(L_{F_b} + \alpha_{j_\nu}) \|\gamma\|^2 < \alpha_{\mathcal{A}}. \tag{52}$$

Then, Problem 20 *has a unique solution* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$.

Proof We apply Theorem 10 in the following functional framework: $X = V$, $K(t) = U(t)$ and

$$A: \mathbb{R}_+ \times V \to V^*, \quad \langle A(t, \boldsymbol{u}), \boldsymbol{v} \rangle = \langle \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{53}$$
$$\varphi: V \times V \to \mathbb{R}, \quad \varphi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F_b(u_\nu) \|\boldsymbol{v}_\tau\| \, \mathrm{d}\Gamma \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{54}$$
$$j: V \to \mathbb{R}, \quad j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu) \, \mathrm{d}\Gamma \quad \text{for } \boldsymbol{v} \in V. \tag{55}$$

For any $g_0 > 0$, let

$$K = \{ \boldsymbol{v} \in V \mid v_\nu \le g_0 \text{ on } \Gamma_3 \}.$$

Then, $K = \frac{g_0}{g(t)} K(t)$ and (18) is obvious with $c_1(t) = c_2(t) = \frac{g_0}{g(t)}$ and $d_1(t) = d_2(t) = 0$.

From the proof of [10, Theorem 32], the operator A and functions φ and j satisfy hypotheses (11), (5) and (6) with $\alpha_{A1} = \alpha_{\mathcal{A}}$, $\alpha_\varphi = L_{F_b} \|\gamma\|^2$ and $\alpha_j = \alpha_{j_\nu} \|\gamma\|^2$, respectively. From Remark 11, we complete the proof. □

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Compliance with Ethical Standards

Conflict of interests The author declares to have no competing interests.

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