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Uniqueness Theorems for Inverse Problems of Discontinuous Sturm–Liouville Operator

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Abstract
In this paper, inverse spectral problems of discontinuous Sturm–Liouville problems contained in the discontinuous coefficient and discontinuity conditions at an interior point of the finite interval are, respectively, studied according to

(i) the spectral data (the sequences of eigenvalues and normalized numbers) by using the Gelfand–Levitan–Marchenko method;
(ii) the Weyl function.

Keywords Inverse problems · Sturm–Liouville equation · Discontinuity conditions and discontinuous coefficient · Main equation · Weyl function

Mathematics Subject Classification 34B24 · 34A55 · 47E05

1 Introduction

This paper presents on inverse problems of discontinuous Sturm–Liouville operator which has both the Sturm–Liouville equation with the discontinuous coefficient and the discontinuity conditions (or transmission conditions) at an interior point of the finite interval. Inverse problems in spectral analysis consist in the reconstructing a linear operator from some of its spectral characteristics such as spectral data (the set of the eigenvalues and normalized numbers), spectra (for different boundary conditions), a spectral function, scattering data, Weyl function, nodal data, etc. In mathematical physics, inverse problem plays an important role in solving nonlinear evolution equations such as Korteweg–de Vries (KdV) equation, Zakharov and Shabat equation (see [1,11,17]); therefore, the theory of inverse problem has significantly developed and

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continues to evolve. Now, in this paper we consider the Sturm–Liouville equation

$$-u'' + q(x)u = \lambda^2 \mu(x)u, \quad 0 < x < \pi \tag{1}$$

with discontinuity conditions at the point $x = a \in (0, \pi)$

$$u(a+0) = \beta u(a-0), \quad u'(a+0) = \beta^{-1} u'(a-0) \tag{2}$$

and the boundary conditions

$$u(0) = u(\pi) = 0, \tag{3}$$

where $q(x) \in L_2(0, \pi)$ is a real-valued function, $\beta > 0$ is a real number, λ is a spectral parameter, $\mu(x)$ is the following piecewise-constant function:

$$\mu(x) = \begin{cases} 1, & 0 < x < a, \\ \alpha^2, & a < x < \pi, \end{cases}$$

$0 < \alpha \neq 1$ and assume that $a > \frac{\alpha\pi}{\alpha+1}$.

Discontinuous Sturm–Liouville problems arise in mathematics, physics, geophysics and other branches of natural sciences; for example, in geophysical models for oscillations of the Earth [18], in the theory of small transversal vibrations of a smooth inhomogeneous string damped by a pointwise force at the midpoint [12] and, etc. (see [16,25,30]). Therefore, the investigation of such problems is very attractive and there are many works on direct and inverse problems for discontinuous Sturm–Liouville problems examined in different aspects; in particular note that it was shown in [14] that if the potential is known over half the interval and if one boundary condition is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. In case of Sturm–Liouville Eq. (1) with discontinuity condition (2) and the boundary conditions $u'(0) - h_1 u(0) = u'(\pi) + h_2 u(\pi) = 0$, the spectral properties of this problem and the eigenfunction expansion are studied in [3]. When $\mu(x) \equiv 1$, we face Sturm–Liouville problem with discontinuity (or transmission) conditions at interior point of the finite interval. In this case, the direct problems are worked in [4,10,22,27] and the inverse problems are solved by different spectral characteristics in [8,9,13,15,21,26,28,29,31–34]. On the other hand, in case of $\beta = 1$, we encounter Sturm–Liouville problem with discontinuous coefficient. In this instance, the direct problems are investigated in [5,23] and the inverse problems are studied in [2,6,19] by Weyl function and in [7,24] by spectral data. Note that we give some works which is close to our topic of this paper.

The aim of this paper is to solve inverse problems for discontinuous Sturm–Liouville problem (1)–(3). The different inverse spectral problems appear in depending on the choice of the different spectral characteristics; in this way, we examine two different inverse problems stated as follows: to determine the potential function $q(x)$ from the spectral data (the set of eigenvalues and normalized numbers) and Weyl function, respectively. In the first inverse problem, we apply the Gelfand–Levitan–Marchenko method in which the transformation operator is used and a linear integral equation with respect to the kernel of the transformation operator plays the main role. However, the presence of discontinuities causes the solution of the Eq. (1) with discontinuity conditions (2) to be in the form of the integral representation (not transformation operator) given in [3]; for this reason, when applying the Gelfand–Levitan–Marchenko method, this integral representation is used. Then, we construct main equation (Gelfand–Levitan–Marchenko type integral equation) satisfied by the kernel of this integral representation and we obtain the algorithm for the reconstruction of potential function $q(x)$. In the second inverse problem, we define and examine the Weyl solution and the Weyl function. The uniqueness theorem for the solution of inverse problem according to the Weyl function is proved. Moreover, since the Weyl function is specified by the spectral data and two spectra, it is appeared that the particular cases of inverse problem of recovering Sturm–Liouville equation from the given Weyl function are inverse problems of recovering the Sturm–Liouville equation from the spectral data and from two spectra.

2 Main Results

First of all, to achieve the main aim of this paper, we will give some properties of the spectral characteristics of the boundary value problem (1)–(3) and note that the detail investigation of these spectral characteristics is given in the work [3].

Denote $e(x, \lambda)$ by the solution of Eq. (1) with discontinuity conditions (2) satisfying the initial conditions

$$e(0, \lambda) = 1, \quad e'(0, \lambda) = i\lambda.$$

Theorem 1 *[3] The integral representation of the solution $e(x, \lambda)$ can be expressed as follows:*

$$e(x, \lambda) = e_0(x, \lambda) + \int_{-\sigma(x)}^{\sigma(x)} K(x, t)e^{i\lambda t} dt,$$

where

$$e_0(x, \lambda) = \begin{cases} e^{i\lambda x}, & 0 < x < a, \\ \kappa_1 e^{i\lambda(\alpha(x-a)+a)} + \kappa_2 e^{i\lambda(-\alpha(x-a)+a)}, & a < x < \pi, \end{cases}$$

with $\kappa_1 = \frac{1}{2}\left(\beta + \frac{1}{\alpha\beta}\right)$ and $\kappa_2 = \frac{1}{2}\left(\beta - \frac{1}{\alpha\beta}\right)$,

$$\sigma(x) = \begin{cases} x, & 0 < x < a, \\ \alpha(x - a) + a, & a < x < \pi, \end{cases}$$

the kernel $K(x, .) \in L_1(-\sigma(x), \sigma(x))$ for each fixed $x \in (0, \pi)$ and satisfies the inequality

$$\int_{-\sigma(x)}^{\sigma(x)} |K(x, t)| dt \leq e^{cp(x)} - 1$$

with

$$p(x)=\int_0^x (x-\xi)|q(\xi)|d\xi, \qquad c=(\alpha+4)|\kappa_1|+(\alpha+2)|\kappa_2|.$$

Remark 1 [3] The following properties of kernel function $K(x,t)$ are valid:

$$K(x,\sigma(x))=\begin{cases}\frac{1}{2}\int_0^x q(\xi)d\xi, & 0<a<x,\\ \frac{\kappa_1}{2}\int_0^x \frac{1}{\sqrt{\mu(\xi)}}q(\xi)d\xi, & a<x<\pi\end{cases}$$

$$K(x,t)|_{t=-\alpha(x-a)+a-0}^{-\alpha(x-a)+a+0}=-\frac{\kappa_2}{2}\left(\int_0^a q(\xi)d\xi-\frac{1}{\alpha}\int_a^x q(\xi)d\xi\right), \quad a<x<\pi,$$

$$K(x,-\sigma(x))=0.$$

Let $\vartheta(x,\lambda)$ be solution of Eq. (1) with discontinuity conditions (2) under initial conditions $\vartheta(0,\lambda)=0$ and $\vartheta'(0,\lambda)=1$. Then, the solution $\vartheta(x,\lambda)$ has the following integral representation

$$\vartheta(x,\lambda)=\vartheta_0(x,\lambda)+\int_0^{\sigma(x)} G(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{4}$$

where

$$\vartheta_0(x,\lambda)=\begin{cases}\frac{\sin\lambda x}{\lambda}, & 0<x<a,\\ \frac{\kappa_1\sin\lambda\upsilon^+(x)}{\lambda}+\frac{\kappa_2\sin\lambda\upsilon^-(x)}{\lambda}, & a<x<\pi,\end{cases} \tag{5}$$

with $\upsilon^{\pm}(x):=\pm\alpha(x-a)+a$, the kernel $G(x,t)$ satisfies the relations $G(x,t)=K(x,t)-K(x,-t)$ and

$$G(x,\sigma(x))=K(x,\sigma(x)). \tag{6}$$

Denote $\omega(x,\lambda)$ by a solution of equation (1) with discontinuity conditions (2) under the initial conditions $\omega(\pi,\lambda)=0$ and $\omega'(\pi,\lambda)=1$. The characteristic function of the problem (1)–(3) is in the form

$$\varphi(\lambda)=\vartheta(x,\lambda)\omega'(x,\lambda)-\vartheta'(x,\lambda)\omega(x,\lambda)$$

and clearly,

$$\varphi(\lambda)=-\omega(0,\lambda)=\vartheta(\pi,\lambda).$$

The function $\varphi(\lambda)$ is entire in λ, and then, it has an at most countable set of zeros $\{\lambda_n\}$ and the squares of the zeros $\{\lambda_n\}$ of the characteristic function coincide with the eigenvalues of the boundary value problem (1)–(3). The function $\vartheta(x,\lambda_n)$ and $\omega(x,\lambda_n)$ is eigenfunctions and

$$\omega(x,\lambda_n)=\rho_n\vartheta(x,\lambda_n), \qquad \rho_n\neq 0. \tag{7}$$

Denote

$$\gamma_n := \int_0^\pi \vartheta^2(x, \lambda_n)\mu(x)dx,$$

and the correlation is valid:

$$\dot{\varphi}(\lambda_n) = 2\lambda_n \rho_n \gamma_n. \tag{8}$$

The numbers γ_n are called the normalized numbers, and the numbers $\{\lambda_n^2, \gamma_n\}$ are called the spectral data of the boundary value problem (1)–(3).

Now, in the case of $q(x) \equiv 0$ at the boundary value problem (1)–(3), the characteristic function is as follows:

$$\varphi_0(\lambda) = \vartheta_0(\pi, \lambda) = \frac{\kappa_1 \sin \lambda \upsilon^+(\pi)}{\lambda} + \frac{\kappa_2 \sin \lambda \upsilon^-(\pi)}{\lambda}.$$

Then, the zeros $\{\lambda_n^0\}$ of this function have the form

$$\lambda_n^0 = \frac{n\pi}{\upsilon^+(\pi)} + s_n, \qquad \sup_n |s_n| = s < \infty.$$

Theorem 2 *[3] The boundary value problem (1)–(3) has a countable set of eigenvalues* $\{\lambda_n^2\}_{n\geq 1}$*:*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n},$$

where $\{d_n\} \in l_\infty$ *and* $\{k_n\} \in l_2$.

Theorem 3 ([3])

(i) The system of eigenfunctions $\{\vartheta(x, \lambda_n)\}_{n\geq 1}$ *of the boundary value problem (1)–(3) is complete in* $L_2(0, \pi; \mu)$.

(ii) The function $f(x) \in AC[0, a] \cap AC[a, \pi]$ *satisfying the discontinuity condition (2) and the boundary conditions (3) can be expanded into a uniformly convergent series of eigenfunctions of the problem (1)–(3):*

$$f(x) = \sum_{n=1}^{\infty} c_n \vartheta(x, \lambda_n), \quad c_n = \frac{1}{\gamma_n}\int_0^\pi \vartheta(x, \lambda_n) f(x)\mu(x)dx. \tag{9}$$

(iii) For $f(x) \in L_2(0, \pi; \mu)$*, the series (9) converges in* $L_2(0, \pi; \mu)$ *and the Parseval equality holds:*

$$\int_0^\pi |f(x)|^2 \mu(x)dx = \sum_{n=1}^{\infty} \gamma_n |c_n|^2.$$

2.1 Uniqueness of Inverse Problem by Spectral Data

Inverse Problem: *Given the spectral data* $\{\lambda_n^2, \gamma_n\}_{n\geq 1}$*; construct the potential* $q(x)$.

The aim of this subsection is to prove the uniqueness theorem for the solution of inverse problem by using the Gelfand–Levitan–Marchenko method.

Consider the spectral data $\{\lambda_n^2, \gamma_n\}_{n\geq 1}$ and the function

$$F(x,t) = \mu(t)\sum_{n=1}^{\infty}\left(\frac{\vartheta_0(x,\lambda_n)\,\vartheta_0(t,\lambda_n)}{\gamma_n} - \frac{\vartheta_0\left(x,\lambda_n^0\right)\vartheta_0\left(t,\lambda_n^0\right)}{\gamma_n^0}\right), \tag{10}$$

where the numbers γ_n^0 are normalized numbers of the problem (1)–(3) at $q(x)\equiv 0$.

Now, let us write the integral representation (4) in the form of a transformation operator: Taking into account the expression (5), we find

$$\frac{\sin\lambda x}{\lambda} = \begin{cases} \vartheta_0(x,\lambda), & 0<x<a, \\ \frac{1}{\kappa_1}\vartheta_0\left(a+\frac{x-a}{\alpha},\lambda\right) - \frac{\kappa_2}{\kappa_1}\vartheta_0(2a-x,\lambda), & a<x<\pi. \end{cases}$$

Using this formula in the integral representation (4), we have

$$\vartheta(x,\lambda) = \vartheta_0(x,\lambda) + \int_0^x G(x,t)\vartheta_0(t,\lambda)dt, \quad 0<x<a$$

and

$$\vartheta(x,\lambda) = \vartheta_0(x,\lambda) + \int_0^a G(x,t)\vartheta_0(t,\lambda)dt + \frac{\alpha}{\kappa_1}\int_a^x G(x,\upsilon^+(t))\vartheta_0(t,\lambda)dt$$
$$-\frac{\kappa_2}{\kappa_1}\int_{\upsilon^-(x)}^a G(x,2a-t)\vartheta_0(t,\lambda)dt, \qquad a<x<\pi.$$

Thus, these equalities yield

$$\vartheta(x,\lambda) = \vartheta_0(x,\lambda) + \int_0^x \tilde{G}(x,t)\vartheta_0(t,\lambda)dt, \tag{11}$$

where

$$\tilde{G}(x,t) = \begin{cases} G(x,t), & 0<t<x<a \text{ and } 0<t<\upsilon^-(x),\ a<x<\pi, \\ G(x,t) - \frac{\kappa_2}{\kappa_1}G(x,2a-t), & \upsilon^-(x)<t<a,\ a<x<\pi, \\ \frac{\alpha}{\kappa_1}G(x,\upsilon^+(t)), & a<t<x<\pi. \end{cases} \tag{12}$$

Moreover, according to the transformation operator (11) as a Volterra integral equation with respect to $\vartheta_0(x,\lambda)$, we get

$$\vartheta_0(x,\lambda) = \vartheta(x,\lambda) + \int_0^x \tilde{H}(x,t)\vartheta(t,\lambda)dt, \tag{13}$$

where $\tilde{H}(x,t)$ is the kernel function of this operator.

Theorem 4 *For each fixed $x \in (0,\pi]$, the kernel $G(x,t)$ appearing in the integral representation (4) satisfies the linear integral equation*

$$\tilde{G}(x,t) + F(x,t) + \int_0^x \tilde{G}(x,\xi)F(\xi,t)d\xi = 0, \quad 0 < t < x. \tag{14}$$

Proof Taking into account (11) and (13), we obtain

$$\Psi_N(x,t) = \sum_{k=1}^{4} J_{N_k}(x,t), \tag{15}$$

where

$$\begin{aligned}
\Psi_N(x,t) &= \sum_{n=1}^{N}\left(\frac{\vartheta(x,\lambda_n)\vartheta(t,\lambda_n)}{\gamma_n} - \frac{\vartheta_0\left(x,\lambda_n^0\right)\vartheta_0(t,\lambda_n^0)}{\gamma_n^0}\right),\\
J_{N_1} &= \sum_{n=1}^{N}\left(\frac{\vartheta_0(x,\lambda_n)\vartheta_0(t,\lambda_n)}{\gamma_n} - \frac{\vartheta_0\left(x,\lambda_n^0\right)\vartheta_0\left(t,\lambda_n^0\right)}{\gamma_n^0}\right),\\
J_{N_2} &= \int_0^x \tilde{G}(x,\xi)\sum_{n=1}^{N}\left(\frac{\vartheta_0(\xi,\lambda_n)\vartheta_0(t,\lambda_n)}{\gamma_n} - \frac{\vartheta_0(\xi,\lambda_n^0)\vartheta_0(t,\lambda_n^0)}{\gamma_n^0}\right)d\xi,\\
J_{N_3} &= \int_0^x \tilde{G}(x,\xi)\sum_{n=1}^{N}\frac{\vartheta_0\left(\xi,\lambda_n^0\right)\vartheta_0\left(t,\lambda_n^0\right)}{\gamma_n^0}d\xi,\\
J_{N_4} &= -\int_0^t \tilde{H}(t,\xi)\sum_{n=1}^{N}\frac{\vartheta(x,\lambda_n)\vartheta(\xi,\lambda_n)}{\gamma_n}d\xi.
\end{aligned}$$

Let $f(x) \in AC[0,a] \cap AC[a,\pi]$. According to Theorem 3, we have

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^\pi f(t)\Psi_N(x,t)\mu(t)dt\right| = 0 \tag{16}$$

and uniformly with respect to $x \in [0,\pi]$, we calculate

$$\lim_{N\to\infty}\int_0^\pi J_{N_1}(x,t)\mu(t)dt = \int_0^\pi f(t)F(x,t)dt, \tag{17}$$

$$\lim_{N\to\infty}\int_0^\pi J_{N_2}(x,t)\mu(t)dt = \int_0^\pi f(t)\left(\int_0^x \tilde{G}(x,\xi)F(\xi,t)d\xi\right)dt, \tag{18}$$

$$\lim_{N\to\infty}\int_0^\pi J_{N_3}(x,t)\mu(t)dt = \int_0^x \tilde{G}(x,\xi)f(\xi)d\xi, \tag{19}$$

$$\lim_{N\to\infty}\int_0^\pi J_{N_4}(x,t)\mu(t)dt = -\frac{1}{\mu(x)}\int_x^\pi f(t)\tilde{H}(t,x)\mu(t)dt. \tag{20}$$

Then, it follows from (15)–(20) that

$$\int_0^\pi f(t)F(x,t)dt + \int_0^\pi f(t)\left(\int_0^x \tilde{G}(x,\xi)F(\xi,t)d\xi\right)dt$$
$$+\int_0^x \tilde{G}(x,\xi)f(\xi)d\xi - \frac{1}{\mu(x)}\int_x^\pi f(t)\tilde{H}(t,x)\mu(t)dt = 0.$$

Considering the properties $\tilde{G}(x,t) = \tilde{H}(x,t) = 0$ for $x < t$ and in view of the arbitrariness of $f(x)$, we derive

$$F(x,t) + \tilde{G}(x,t) + \int_0^x \tilde{G}(x,\xi)F(\xi,t)d\xi - \frac{\mu(t)}{\mu(x)}\tilde{H}(t,x) = 0.$$

As a result, this equality yields (14) for $t < x$. □

Definition 1 Eq. (14) is called main equation or Gelfand–Levitan–Marchenko type equation of the problem (1)–(3).

Theorem 5 *The Gelfand–Levitan–Marchenko type Eq. (14) has a unique solution $\tilde{G}(x,.)$ in $L_2(0,x)$ for each fixed $x \in (0,\pi]$.*

Proof Since (14) is a Fredholm equation, it is sufficient to prove that the homogeneous equation

$$g(t) + \int_0^x F(s,t)g(s)ds = 0 \tag{21}$$

has only the trivial solution $g(t) = 0$. Assume that $g(t)$ is a solution of Eq. (21) and $g(t) = 0$ for $t \in (x,\pi)$. Then,

$$\int_0^x g^2(t)\mu(t)dt + \int_0^x\int_0^x F(s,t)g(s)g(t)\mu(t)dsdt = 0$$

and this yields

$$\int_0^x g^2(t)\mu(t)dt + \sum_{n=1}^\infty \frac{1}{\gamma_n}\left(\int_0^x \vartheta_0(t,\lambda_n)g(t)\mu(t)dt\right)^2$$
$$-\sum_{n=1}^\infty \frac{1}{\gamma_n^0}\left(\int_0^x \vartheta_0(t,\lambda_n^0)g(t)\mu(t)dt\right)^2 = 0.$$

Using Parseval equality

$$\int_0^x g^2(t)\mu(t)dt = \sum_{n=1}^\infty \frac{1}{\gamma_n^0}\left(\int_0^x \vartheta_0\left(t,\lambda_n^0\right)g(t)\mu(t)dt\right)^2,$$

we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\gamma_n} \left(\int_0^x \vartheta_0(t, \lambda_n) g(t) \mu(t) dt \right)^2 = 0.$$

Consequently, since $\gamma_n > 0$ and and the system $\{\vartheta_0(t, \lambda_n)\}$ is complete in $L_2(0, \pi; \mu)$ from Theorem 3, we find $g(t) = 0$. □

From Theorems 4 and 5, we obtain the following theorem:

Theorem 6 *The spectral data* $\{\lambda_n^2, \gamma_n\}_{n\geq 1}$ *uniquely determine the boundary value problem (1)-(3).*

The potential function $q(x)$ can be constructed according to the the following algorithm:

- From the given numbers $\{\lambda_n^2, \gamma_n\}_{n\geq 1}$ construct the function $F(x, t)$ by (10).
- Find the function $\tilde{G}(x, t)$ by solving the main equation (14).
- Calculate $q(x)$ by the following formula obtained from (6) and (12)

$$\tilde{G}(x, x) = \frac{1}{2} \int_0^x q(\xi) d\xi.$$

2.2 Uniqueness of Inverse Problem by Weyl Function

Inverse Problem: *Given the Weyl function* $m(\lambda)$; *construct the potential* $q(x)$.

The goal of this subsection is to prove the uniqueness theorem for the solution of inverse problem by Weyl function. To carry out this, firstly we define and examine the Weyl function.

Let $\psi(x, \lambda)$ be a solution of Eq. (1) with discontinuity conditions (2) satisfying the conditions $\psi(0, \lambda) = 1$ and $\psi(\pi, \lambda) = 0$ and denote $\phi(x, \lambda)$ be a solution of Eq. (1) with discontinuity conditions (2) under the initial conditions $\phi(0, \lambda) = 1$ and $\phi'(0, \lambda) = 0$. Then, the solution $\omega(x, \lambda)$ can be represented as follows:

$$-\frac{\omega(x, \lambda)}{\varphi(\lambda)} = \phi(x, \lambda) - \frac{\omega'(0, \lambda)}{\varphi(\lambda)} \vartheta(x, \lambda).$$

Denote

$$m(\lambda) := -\frac{\omega'(0, \lambda)}{\varphi(\lambda)}. \tag{22}$$

It is clear that

$$\psi(x, \lambda) = -\frac{\omega(x, \lambda)}{\varphi(\lambda)} = \phi(x, \lambda) + m(\lambda)\vartheta(x, \lambda). \tag{23}$$

The function $\psi(x, \lambda)$ is called the *Weyl solution*, and the function $m(\lambda) = \psi'(0, \lambda)$ is called the *Weyl function* of the boundary value problem (1)–(3). Weyl solution and Weyl function are meromorphic functions with simple poles at the points $\lambda = \lambda_n$, $n \geq 1$, and the squares of these points $\{\lambda_n\}_{n\geq 1}$ are eigenvalues of the boundary value problem (1)–(3).

Remark 2 The characteristic function of boundary value problem generated by Eq. (1) with discontinuity conditions (2) and the boundary conditions $y'(0) = y(\pi) = 0$ has the following form

$$\tilde{\varphi}(\lambda) = \omega'(0, \lambda) = \phi(\pi, \lambda).$$

Then, we can express the Weyl function (22) as

$$m(\lambda) = -\frac{\phi(\pi, \lambda)}{\varphi(\lambda)} = -\frac{\tilde{\varphi}(\lambda)}{\varphi(\lambda)}. \tag{24}$$

Note that the integral representation of solution $\phi(x, \lambda)$ can be represented by

$$\phi(x, \lambda) = \phi_0(x, \lambda) + \int_0^{\sigma(x)} A(x, t) \cos \lambda t dt,$$

where $A(x, t) = K(x, t) + K(x, -t)$ and

$$\phi_0(x, \lambda) = \begin{cases} \cos \lambda x, & 0 < x < a, \\ \kappa_1 \cos \lambda \upsilon^+(x) + \kappa_2 \cos \lambda \upsilon^-(x), & a < x < \pi. \end{cases}$$

Lemma 1 *The following representation holds*

$$m(\lambda) = m(0) + \sum_{n=1}^{\infty} \frac{\lambda^2}{\gamma_n \lambda_n^2 (\lambda_n^2 - \lambda^2)}. \tag{25}$$

Proof Consider the contour integral

$$I_N(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{m(\xi) - m_0(\xi)}{\xi(\xi - \lambda)} d\xi, \quad \lambda \in int\Gamma_N,$$

where $\Gamma_N = \{\lambda : |\lambda| = |\lambda_N^0| + \frac{p}{2}\}$ with $p = \inf_{n \neq k} |\lambda_n^0 - \lambda_k^0| > 0$ is a contour of counterclockwise circuit (see [3]) and

$$m_0(\lambda) = -\frac{\phi_0(\pi, \lambda)}{\vartheta_0(\pi, \lambda)}$$

is the Weyl function of the problem (1)–(3) at $q(x) \equiv 0$. Now, let us show that

$$\lim_{N \to \infty} I_N(\lambda) = 0.$$

Assume that

$$f_1(\lambda) = \phi(\pi, \lambda) - \phi_0(\pi, \lambda) \quad and \quad f_2(\lambda) = \vartheta(\pi, \lambda) - \vartheta_0(\pi, \lambda)$$

and according to Lemma 1.3.1 in [20], we have

$$\lim_{|\lambda|\to\infty} e^{-|Im\lambda|\upsilon^{+}(\pi)}|f_i(\lambda)| = 0, \quad i = 1, 2. \tag{26}$$

Then, we can write

$$\frac{m(\lambda) - m_0(\lambda)}{\lambda} = \frac{-1}{\lambda}\left(\frac{f_1(\lambda)}{\varphi(\lambda)} + \frac{f_2(\lambda)m_0(\lambda)}{\varphi(\lambda)}\right).$$

Using the inequality $|\varphi(\lambda)| \geq C_\delta \frac{1}{|\lambda|} e^{|Im\lambda|\upsilon^{+}(\pi)}$, $\lambda \in G_\delta := \{\lambda : |\lambda_n - \lambda_n^0| \geq \delta\}$, where $\delta << \frac{p}{2}$ is a sufficiently positive number and the relation (26), we find

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \left|\frac{m(\lambda) - m_0(\lambda)}{\lambda}\right| = 0.$$

Consequently, this yields $\lim\limits_{N\to\infty} I_N(\lambda) = 0$.

On the other hand, in case of the boundary value problem (1)–(3) at $q(x) \equiv 0$ the following relations are valid:

$$\dot{\varphi}_0(\lambda_n^0) = 2\lambda_n^0 \rho_n^0 \gamma_n^0, \qquad \rho_n^0 = \tilde{\varphi}_0(\lambda_n^0). \tag{27}$$

According to (7), (8) and (27), we calculate

$$\underset{\lambda=\lambda_n}{Res} \frac{m(\lambda)}{\lambda} = -\frac{1}{2\lambda_n^2\gamma_n} \quad and \quad \underset{\lambda=\lambda_n^0}{Res} \frac{m_0(\lambda)}{\lambda} = -\frac{1}{2(\lambda_n^0)^2\gamma_n^0}.$$

Then, applying the residue theorem, we obtain

$$\begin{aligned} I_N(\lambda) = {} & \frac{m(\lambda) - m_0(\lambda)}{\lambda} + \sum_{n=1}^{N} \underset{\xi=\lambda_n}{Res} \frac{m(\xi)}{\xi}\left(\frac{1}{(\lambda_n - \lambda)} - \frac{1}{(\lambda_n + \lambda)}\right) \\ & - \sum_{n=1}^{N} \underset{\xi=\lambda_n^0}{Res} \frac{m_0(\xi)}{\xi}\left(\frac{1}{(\lambda_n^0 - \lambda)} - \frac{1}{(\lambda_n^0 + \lambda)}\right) + \underset{\xi=0}{Res} \frac{m(\xi) - m_0(\xi)}{\xi(\xi - \lambda)} \\ = {} & \frac{m(\lambda) - m_0(\lambda)}{\lambda} - \frac{m(0) - m_0(0)}{\lambda} \\ & - \sum_{n=1}^{N} \frac{\lambda}{\gamma_n\lambda_n^2(\lambda_n^2 - \lambda^2)} + \sum_{n=1}^{N} \frac{\lambda}{\gamma_n^0(\lambda_n^0)^2((\lambda_n^0)^2 - \lambda^2)}. \end{aligned}$$

Since $\lim_{N\to\infty} I_N(\lambda) = 0$ and

$$\frac{m_0(\lambda)}{\lambda} = \frac{m_0(0)}{\lambda} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \underset{\lambda=\lambda_n^0}{Res} \frac{m_0(\lambda)}{\lambda}\left(\frac{1}{(\lambda - \lambda_n^0)} + \frac{1}{\lambda_n^0}\right)$$

$$= \frac{m_0(0)}{\lambda} + \sum_{n=1}^{\infty} \frac{\lambda}{\gamma_n^0 (\lambda_n^0)^2 ((\lambda_n^0)^2 - \lambda^2)},$$

we have as $N \to \infty$

$$m(\lambda) = m(0) + \sum_{n=1}^{\infty} \frac{\lambda^2}{\gamma_n \lambda_n^2 (\lambda_n^2 - \lambda^2)}.$$

□

Now, we agree that together with the boundary value problem (1)–(3) we consider a boundary value problem which is the same form but with different potential function $\hat{q}(x)$. If a certain symbol τ denotes an object related to the problem (1)–(3), then the corresponding symbol $\hat{\tau}$ denotes an object related to the problem with $\hat{q}(x)$.

Theorem 7 *The boundary value problem (1)–(3) is uniquely determined by the Weyl function.*

Proof To prove the theorem, we show that if $m(\lambda) = \hat{m}(\lambda)$, then $q(x) = \hat{q}(x)$. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda) \begin{pmatrix} \hat{\vartheta}(x,\lambda) & \hat{\psi}(x,\lambda) \\ \hat{\vartheta}'(x,\lambda) & \hat{\psi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \vartheta(x,\lambda) & \psi(x,\lambda) \\ \vartheta'(x,\lambda) & \psi'(x,\lambda) \end{pmatrix}. \tag{28}$$

Taking into account the relation

$$\langle \vartheta(x,\lambda), \psi(x,\lambda) \rangle = -1 \tag{29}$$

and the formula (28), we obtain

$$\begin{aligned} \vartheta(x,\lambda) &= P_{11}(x,\lambda)\hat{\vartheta}(x,\lambda) + P_{12}(x,\lambda)\hat{\vartheta}'(x,\lambda), \\ \psi(x,\lambda) &= P_{11}(x,\lambda)\hat{\psi}(x,\lambda) + P_{12}(x,\lambda)\hat{\psi}'(x,\lambda), \end{aligned} \tag{30}$$

$$\begin{aligned} P_{j1}(x,\lambda) &= \psi^{(j-1)}(x,\lambda)\hat{\vartheta}'(x,\lambda) - \vartheta^{(j-1)}(x,\lambda)\hat{\psi}'(x,\lambda), \\ P_{j2}(x,\lambda) &= \vartheta^{(j-1)}(x,\lambda)\hat{\psi}(x,\lambda) - \psi^{(j-1)}(x,\lambda)\hat{\vartheta}(x,\lambda), \quad j = 1, 2. \end{aligned} \tag{31}$$

It follows from (23), (29) and (31) that

$$P_{11}(x,\lambda) = 1 - \frac{\omega(x,\lambda)}{\varphi(\lambda)}(\hat{\vartheta}'(x,\lambda) - \vartheta'(x,\lambda)) + \vartheta(x,\lambda)\left(\frac{\hat{\omega}'(x,\lambda)}{\hat{\varphi}(\lambda)} - \frac{\omega'(x,\lambda)}{\varphi(\lambda)}\right),$$

$$P_{12}(x,\lambda) = \frac{\omega(x,\lambda)}{\varphi(\lambda)}\hat{\vartheta}(x,\lambda) - \frac{\hat{\omega}(x,\lambda)}{\hat{\varphi}(\lambda)}\vartheta(x,\lambda).$$

Since

$$\vartheta(x,\lambda) = O\left(\frac{e^{|Im\lambda|\sigma(x)}}{|\lambda|}\right), \quad \omega(x,\lambda) = O\left(\frac{e^{|Im\lambda|(\sigma(\pi)-\sigma(x))}}{|\lambda|}\right), \quad |\lambda| \to \infty$$

and $|\Delta(\lambda)| \geq C_\delta \frac{1}{|\lambda|} e^{|Im\lambda|\upsilon^+(\pi)}$, $\lambda \in G_\delta$, we calculate

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{11}(x,\lambda) - 1| = \lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{12}(x,\lambda)| = 0. \tag{32}$$

Using (23) and (31), we get

$$P_{11}(x,\lambda) = \phi(x,\lambda)\hat{\vartheta}'(x,\lambda) - \vartheta(x,\lambda)\hat{\phi}'(x,\lambda) + \vartheta(x,\lambda)\hat{\vartheta}'(x,\lambda)(m(\lambda) - \hat{m}(\lambda)),$$
$$P_{12}(x,\lambda) = \vartheta(x,\lambda)\hat{\phi}(x,\lambda) - \hat{\vartheta}(x,\lambda)\phi(x,\lambda) + \vartheta(x,\lambda)\hat{\vartheta}(x,\lambda)(\hat{m}(\lambda) - m(\lambda)).$$

If $m(\lambda) = \hat{m}(\lambda)$, then the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ and according to (32) this yields $P_{11}(x,\lambda) \equiv 1$ and $P_{12}(x,\lambda) \equiv 0$. Thus, substituting into (30), we find $\vartheta(x,\lambda) \equiv \hat{\vartheta}(x,\lambda)$ and $\psi(x,\lambda) \equiv \hat{\psi}(x,\lambda)$. Consequently, we obtain $q(x) = \hat{q}(x)$. □

Remark 3 Considering the expansion (25) of the Weyl function, this function is specified by the spectral data $\{\lambda_n^2, \gamma_n\}_{n\ge1}$. Then, we can state that the spectral data $\{\lambda_n^2, \gamma_n\}_{n\ge1}$ uniquely determine the boundary value problem (1)–(3). Moreover, according to (24) the poles and zeros of the Weyl function $m(\lambda)$ coincide with the zeros λ_n and $\tilde{\lambda}_n$ of the characteristic functions $\varphi(\lambda)$ and $\tilde{\varphi}(\lambda)$, respectively. Therefore, the Weyl function $m(\lambda)$ is specified by two spectra $\{\lambda_n^2\}$ and $\{\tilde{\lambda}_n^2\}$ and the boundary value problem is uniquely determined by two spectra. As a result, inverse problem of the boundary value problem (1)–(3) according to spectral data and two spectra is particular cases of inverse problem according to Weyl function in this subsection.

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