Bull. Malays. Math. Sci. Soc. (2021) 44:1837–1866
https://doi.org/10.1007/s40840-020-01035-1

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Stability Result for a New Viscoelastic–Thermoelastic Timoshenko System

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Received: 27 May 2020 / Revised: 27 September 2020 / Accepted: 6 October 2020 /
Published online: 17 October 2020

Abstract

In this work, we prove a general and optimal decay estimates for the solution energy of a new thermoelastic Timoshenko system with viscoelastic law acting on the transverse displacement. Therefore, exponential and polynomial decay rates are obtained as particular cases. The result is obtained under the assumption of equal speed of wave propagation.

Keywords General decay · Timoshenko system · Thermoelastic · Viscoelasticity

Mathematics Subject Classification 35D30 · 35D35 · 35B35 · 35L51 · 74D10 · 93D15

1 Introduction

In this paper, we investigate a thermoelastic Timoshenko system with a viscoelastic damping acting on the transverse displacement in the shear force equation and a thermoelastic dissipation effective on the shear force (1.11). The result obtained in this paper is general and optimal in the sense that it agrees with the decay rate of the relaxation function h (see conditions on h in Sect. 2). We indeed demonstrate this by giving some examples (see Sect. 4.1).

Timoshenko [1] in 1921 introduced a model which has been widely used to describe the vibration of a beam when the transverse shear strain is significant. Combining the

Communicated by Rosihan M. Ali.

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evolution equations

$$\rho_1 u_{tt} - S_x = 0, \quad \rho_2 v_{tt} - M_x + S = 0, \tag{1.1}$$

and the constitutive equations,

$$S = k(u_x + v), \quad M = bv_x, \tag{1.2}$$

(see [2,3] for detailed derivation), the following coupled hyperbolic system was derived

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x = 0, \\ \rho_2 v_{tt} - bv_{xx} + k(u_x + v) = 0, \end{cases} \tag{1.3}$$

where $u = u(x, t)$ represents the transverse displacement and $v = v(x, t)$ represents the rotation angle of the center of mass of a beam element. The positive parameters ρ_1, ρ_2, k and b are: mass density, moment of mass inertia, shear coefficient and flexural rigidity, respectively. System (1.3) coupled with different initial conditions, boundary conditions and various damping mechanisms has been extensively studied in the literature, see [4–7] and references therein.

When viscoelastic law acts on the rotation angle v, (1.3) takes the form

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x = 0, \\ \rho_2 v_{tt} - bv_{xx} + k(u_x + v) + \int_0^t g(t-s) v_{xx}(x, s) \mathrm{d}s = 0. \end{cases} \tag{1.4}$$

This has been discussed by many researchers and several stability results have been established (see [4,6,8–11]).

Concerning thermoelastic Timoshenko systems, when thermoelastic dissipation is effective on the bending moment, we have the evolution equation

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x = 0, \\ \rho_2 v_{tt} - bv_{xx} + k(u_x + v) + \gamma\theta + \int_0^t g(t-s) v_{xx}(x, s) \mathrm{d}s = 0, \\ \rho_3 \theta_t + q_x + \gamma v_{xt} = 0, \end{cases} \tag{1.5}$$

where $\theta = \theta(x, t)$ is the temperature difference and $q = -\beta\theta_x$ represents the heat flux, the positive constants ρ_3, β and γ are the capacity, diffusivity and adhesive stiffness, respectively. Rivera and Racke [12] studied (1.5) and proved that the system is exponentially stable if and only if the speeds of wave propagations are equal, that is,

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{1.6}$$

In the case when (1.6) does not hold, they only established a polynomial decay result.

Also, when thermoelastic dissipation is effective on the shear force equation, we have the evolution equations

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x + \gamma\theta_x = 0, \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v) - \gamma\theta + \int_0^t g(t-s) v_{xx}(x,s)\mathrm{d}s = 0, \\ \rho_3\theta_t + q_x + \gamma(u_x + v)_t = 0. \end{cases} \tag{1.7}$$

Apalara [13] discussed (1.7) and proved a general decay result for the solution energy, without imposing the equal-wave-speed condition (1.6). Almeida Júnior et al. [14] considered the system (1.7) in the absence of the memory term and showed that the system is exponentially stable if and only if (1.6) holds, and only polynomial decay is guaranteed otherwise. Interested readers may also see [4,8,9,13,15–20] for more results on thermoelastic Timoshenko systems.

Guesmia et al. [11] considered (1.5) with infinite memory acting on the rotation angle and established some general decay results depending on the speed of wave propagation. However, in their work, they raised a natural question, which is; how possible is it to have a viscoelastic dissipation effect on the transverse displacement u in the shear force equation instead of the rotation angle v as seen in (1.4). This question was answered later by Guesmia and Messaoudi [10].

Recently, Alves et al. [21] derived Timoshenko system (1.8) with a viscoelastic dissipation mechanism acting on the transverse displacement u in the shear force:

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x + k\int_0^t h(t-s)(u_x + v)_x(x,s)\mathrm{d}s = 0, \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v) - k\int_0^t h(t-s)(u_x + v)(x,s)\mathrm{d}s = 0, \end{cases} \tag{1.8}$$

coupled with Dirichlet–Neumann boundary condition and proved a uniform decay result under the condition that (1.6) holds. In the case of nonequal speed, i.e., $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$, they established only a polynomial decay (even if the relaxation function decays exponentially).

It is natural as well, to investigate the stability of (1.8) with an additional effect of a thermoelastic dissipation. This will be our goal in this paper.

Now, we consider instead of (1.1) and (1.2), the constitutive equations;

$$\begin{cases} S = k\left((u_x + v) - \int_0^t h(t-s)(u_x + v)(s)\mathrm{d}s\right) - \gamma\theta, \\ \\ M = b v_x, \quad q = -\beta\theta_x, \end{cases} \tag{1.9}$$

and the evolution equations

$$\begin{cases} \rho_1 u_{tt} - S_x = 0, \qquad \rho_2 v_{tt} - M_x + S = 0 \\ \rho_3 \theta_t + q_x + \gamma \left(u_x + v\right)_t = 0. \end{cases} \tag{1.10}$$

combining (1.9) and (1.10), we obtain the thermoelastic Timoshenko system

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x + \gamma\theta_x + k \displaystyle\int_0^t h(t-s)(u_x+v)_x(x,s)\mathrm{d}s = 0, \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v) - \gamma\theta - k \displaystyle\int_0^t h(t-s)(u_x+v)(x,s)\mathrm{d}s = 0, \\ \rho_3 \theta_t - \beta\theta_{xx} + \gamma\left(u_x + v\right)_t = 0, \end{cases} \tag{1.11}$$

where $x \in (0, 1)$, and $t > 0$. The relaxation function h is a given function which will be specified later. We endow the system (1.11) with the following mixed boundary conditions:

$$\begin{cases} u(0,t) = u_x(1,t) = 0, \ t \geq 0, \\ v_x(0,t) = v(1,t) = 0, \ t \geq 0, \\ \theta_x(0,t) = \theta(1,t) = 0, \ t \geq 0, \end{cases} \tag{1.12}$$

and initial data

$$\begin{cases} u(x,0) = u_0(x), \ v(x,0) = v_0(x), \ \theta(x,0) = \theta_0(x), \ x \in [0,1], \\ u_t(x,0) = u_1(x), \ v_t(x,0) = v_1(x), \quad x \in [0,1]. \end{cases} \tag{1.13}$$

The remaining part of this work is organized as follows: In Sect. 2, we present a few basic tools and state the assumptions on the relaxation function h. In Sect. 3, we prove some key technical lemmas that will be of help in obtaining our main result. In Sect. 4, we state and prove our general and optimal decay result.

2 Assumptions and Space Setting

Throughout this work, C and c_i denote positive constants that may change from one line to the other or within the same line. We denote by $\|.\|$ the usual norm in $L^2(0, 1)$. We consider the following assumptions on the function h

(A1) $h : [0, +\infty) \longrightarrow (0, +\infty)$ is a nonincreasing C^1-function such that

$$h(0) > 0, \quad 1 - \int_0^\infty h(\tau)\mathrm{d}\tau = l_0 > 0. \tag{2.1}$$

(A2) There exists a C^1-function $G : [0, +\infty) \to [0, +\infty)$ which is linear or it is strictly convex C^2-function on $(0, \mathrm{r}]$, $r \le h(t_0)$, for any $t_0 > 0$ with $\mathrm{G}(0) = \mathrm{G}'(0) = 0$ and a positive nonincreasing differentiable function $\xi : [0, +\infty) \to (0, +\infty)$, such that

$$h'(t) \le -\xi(t)G\left(h(t)\right), \quad t \ge 0. \tag{2.2}$$

Remark 2.1 As mentioned in [22], we have the following:

1. Conditions (A_1) and (A_2) implies that G is a strictly increasing convex C^2-function on $(0, r]$, with $G(0) = G'(0) = 0$, thus there exists an extension of G say

$$\bar{G} : [0, +\infty) \to (0, +\infty)$$

that is also a strictly increasing and a strictly convex C^2-function. For instance, for any $t > r$, we define $\bar{G}$ by

$$\bar{G}(s) = \frac{G''(r)}{2}s^2 + (G'(r) - G''(r)r)s + G(r) - G'(r)r + \frac{G''(r)}{2}r^2. \tag{2.3}$$

2. Also, since h is continuous, positive and $h(0) > 0$, then for any $t_0 > 0$ with $t \ge t_0$ we have

$$\int_0^t h(s)\mathrm{d}s \ge \int_0^{t_0} h(s)\mathrm{d}s = h_0 > 0. \tag{2.4}$$

For completeness purpose, we introduce the following spaces:

$$L^2_\star(0,1) = \{w \in L^2(0,1) \ : \int_0^1 w(s)\mathrm{d}s = 0\}, \quad H^1_\star = H^1(0,1) \cap L^2_\star(0,1),$$
$$H^2_\star(0,1) = \{w \in H^2(0,1) \ : w_x(0) = w_x(1) = 0\}.$$

Denote by $\mathcal{H}$ and $\mathcal{V}$ the following spaces:

$$\mathcal{H} := H^1_\star(0,1) \times L^2_\star(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_\star(0,1)$$

and

$$\mathcal{V} := \left(H^2_\star(0,1) \cap H^1_\star(0,1)\right) \times H^1_\star(0,1) \times \left(H^2(0,1) \cap H^1_0(0,1)\right)$$
$$\times H^1_0(0,1) \times \left(H^2_\star(0,1) \cap H^1_\star(0,1)\right).$$

We state without proof, the following existence and regularity result:

Theorem 2.1 *Let $W = (u, z, v, s, \theta)^T;\ z = u_t,\ s = v_t$, and assume that $(G1)$ holds. Then, for all $W_0 \in \mathcal{H}$, the systems (1.11)–(1.13) have a unique global (weak) solution*

$$u \in C\left(\mathbb{R}^+; H^1_\star(0,1)\right) \cap C^1\left(\mathbb{R}^+; L^2_\star(0,1)\right), \qquad \theta \in C\left(\mathbb{R}^+; H^1_\star(0,1)\right)$$

$$v \in C\left(\mathbb{R}^+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}^+; L^2(0,1)\right).$$

Moreover, if $W_0 \in \mathcal{V}$, then the solution satisfies,

$$\begin{aligned}
&u \in C\left(\mathbb{R}^+; H_\star^2(0,1) \cap H_\star^1(0,1)\right) \cap C^1\left(\mathbb{R}^+; H_\star^1(0,1)\right) \cap C^2\left(\mathbb{R}^+; L_\star^2(0,1)\right),\\
&v \in C\left(\mathbb{R}^+; H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}^+; H_0^1(0,1)\right) \cap C^2\left(\mathbb{R}^+; L^2(0,1)\right),\\
&\theta \in C\left(\mathbb{R}^+; H_\star^2(0,1)\right) \cap C^1\left(\mathbb{R}^+; H_\star^1(0,1)\right).
\end{aligned}$$

Remark 2.2 This result can be proved using the Faedo–Galerkin method and repeating the steps in [23].

Lemma 2.1 *Let $w \in L^2\big([0,\infty); L^2(0,1)\big)$ and $f, g \in L^2(0,1)$, we have*

$$\int_0^1 \left(\int_0^t h(t-s)\big[w(t)-w(s)\big]\mathrm{d}s\right)^2 \mathrm{d}x \le (1-l_0)(h \circ w)(t), \tag{2.5}$$

where

$$(h \circ w)(t) = \int_0^t h(t-s)\|w(t)-w(s)\|^2 \mathrm{d}s.$$

Proof Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t h(t-s)\big[w(t)-w(s)\big]\mathrm{d}s\right)^2 \mathrm{d}x\\
&= \int_0^1 \left(\int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}\big[w(t)-w(s)\big]\mathrm{d}s\right)^2 \mathrm{d}x\\
&\le \left(\int_0^{+\infty} h(s)\mathrm{d}s\right) \int_0^1 \int_0^t h(t-s)|w(t)-w(s)|^2 \mathrm{d}s\mathrm{d}x\\
&\le (1-l) \int_0^t h(t-s)\|w(t)-w(s)\|^2 \mathrm{d}s\\
&= (1-l)\,(h \circ w)\,(t).
\end{aligned} \tag{2.6}$$

□

As in [24], for any $0 < \alpha < 1$, let

$$g_\alpha(t) = \alpha h(t) - h'(t) \quad \text{and} \quad A_\alpha = \int_0^{+\infty} \frac{h^2(s)}{\alpha h(s) - h'(s)}\mathrm{d}s. \tag{2.7}$$

We have the following lemma.

Lemma 2.2 *Let* (u, v, θ) *be the solution of problem* (1.11)–(1.13). *Then, for any* $0 < \alpha < 1$ *we have*

$$\int_0^1 \left(\int_0^t h(t-s)\left((u_x+v)(t)-(u_x+v)(s)\right) \mathrm{d}s \right)^2 \mathrm{d}x \le A_\alpha \left(g \circ (u_x+v)\right)(t) \tag{2.8}$$

where

$$\left(g \circ (u_x+v)\right)(t) = \int_0^t g(t-s)\|(u_x+v)(t)-(u_x+v)(s)\|_2^2 \mathrm{d}s.$$

Proof Using Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\int_0^1 \left(\int_0^t h(t-s)\left((u_x+v)(t)-(u_x+v)(s)\right) \mathrm{d}s \right)^2 \mathrm{d}x \\ &= \int_0^1 \left(\int_0^t \frac{h(t-s)}{\sqrt{g(t-s)}}\sqrt{g(t-s)}\left((u_x+v)(t)-(u_x+v)(s)\right) \mathrm{d}s \right)^2 \mathrm{d}x \\ &\le \left(\int_0^{+\infty} \frac{h^2(s)}{g(s)} \mathrm{d}s \right) \int_0^1 \int_0^t g(t-s)\left((u_x+v)(t)-(u_x+v)(s)\right)^2 \mathrm{d}s\mathrm{d}x \\ &= A_\alpha \left(g \circ (u_x+v)\right)(t). \end{aligned} \tag{2.9}$$

□

Lemma 2.3 *Let* (u, v, θ) *be the solution of problem* (1.11)–(1.13). *Then, for any* $0 < \alpha < 1$ *we have*

(i)

$$\begin{aligned} \int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s \right)^2 \mathrm{d}x \le 2(\alpha + h(0))A_\alpha \int_0^1 (u_x+v)^2 \mathrm{d}x \\ + 2A_\alpha \left(g \circ (u_x+v)\right)(t), \end{aligned} \tag{2.10}$$

(ii)

$$\int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s \right)^2 dx \le 2\int_0^1 (u_x+v)^2 \mathrm{d}x + 2\left(h \circ (u_x+v)\right)(t). \tag{2.11}$$

Proof We apply Cauchy−Schwarz inequality.

(i)

$$\begin{aligned} &\int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s \right)^2 \mathrm{d}x \\ &= \int_0^1 \left(\int_0^t \frac{h(t-s)}{\sqrt{g(t-s)}}\sqrt{g(t-s)}\left[(u_x+v)(s)-(u_x+v)(t)+(u_x+v)(t)\right]\mathrm{d}s \right)^2 \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq 2A_\alpha \int_0^1 \int_0^t g(t-s)\big[\big((u_x+v)(s)-(u_x+v)(t)\big)^2+(u_x+v)^2(t)\big]\mathrm{d}s\mathrm{d}x \\
&= 2A_\alpha \left(\int_0^t g(s)\mathrm{d}s\right)\int_0^1 (u_x+v)^2\mathrm{d}x \\
&\quad + 2A_\alpha \int_0^1 \int_0^t g(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]^2\mathrm{d}x \\
&\leq 2(\alpha+h(0))A_\alpha \int_0^1 (u_x+v)^2\mathrm{d}x + 2A_\alpha\left(g\circ(u_x+v)\right)(t).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\right)^2 \mathrm{d}x \\
&= \int_0^1 \left(\int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}\big[(u_x+v)(s)-(u_x+v)(t)+(u_x+v)(t)\big]\mathrm{d}s\right)^2 \mathrm{d}x \\
&\leq 2\left(\int_0^t h(s)\mathrm{d}s\right)\int_0^1\int_0^t h(t-s)\big[\big((u_x+v)(s)-(u_x+v)(t)\big)^2+(u_x+v)^2(t)\big]\mathrm{d}s\mathrm{d}x \\
&= 2\left(\int_0^t h(s)\mathrm{d}s\right)^2\int_0^1 (u_x+v)^2\mathrm{d}x \\
&\quad + 2\left(\int_0^t h(s)\mathrm{d}s\right)\int_0^1\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]^2\mathrm{d}x \\
&\leq 2\int_0^1 (u_x+v)^2\mathrm{d}x + 2\left(h\circ(u_x+v)\right)(t).
\end{aligned}
$$

□

Lemma 2.4 *Let F be a convex function on the close interval $[a,b]$, and $|\Omega| \neq 0$. Let $f:\Omega\to[a,b]$ and j integrable function on Ω, such that $j(x)\geq 0$ and $\int_\Omega j(x)\mathrm{d}x = a > 0$. Then, we have the following Jensen inequality*

$$
\frac{1}{a}\int_\Omega F(f(y))j(y)dy \geq F\left(\frac{1}{a}\int_\Omega f(y)j(y)dy\right). \tag{2.12}
$$

3 Essential Lemmas

Lemma 3.1 *Let (u,v,θ) be the solution of problem (1.11)–(1.13). Therefore, the energy functional of system (1.11)–(1.13) defined by*

$$
\begin{aligned}
E(t) = &\frac{1}{2}\left(\rho_1\|u_t\|^2+\rho_2\|v_t\|^2+b\|v_x\|^2+\rho_3\|\theta\|^2+k\left(1-\int_0^t h(s)\mathrm{d}s\right)\|u_x+v\|^2\right) \\
&+\frac{k}{2}(h\circ(u_x+v))(t),
\end{aligned} \tag{3.1}
$$

satisfies

$$
E'(t) = -\beta\|\theta_x\|^2 - \frac{k}{2}h(t)\|u_x+v\|^2 + \frac{k}{2}(h'\circ(u_x+v))(t) \leq 0,\ \forall\, t\geq 0. \tag{3.2}
$$

Proof We multiply (1.11)$_1$ by u_t, (1.11)$_2$ by v_t and (1.11)$_3$ by θ, then integrate over $(0, 1)$ and make use of the boundary conditions (1.12). Finally, addition of the resulting equations yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \big(\rho_1 u_t^2 + \rho_2 v_t^2 + b v_x^2 + \rho_3\theta^2 + k(u_x + v)^2\big)\mathrm{d}x\\ &\quad= -\beta\int_0^1 \theta_x^2 \mathrm{d}x + k\int_0^1 (u_x + v)_t(t)\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x.\end{aligned}\tag{3.3}$$

Now, we estimate the last integral in (3.3) as follows:

$$\begin{aligned}J =& k\int_0^1 (u_x+v)_t(t)\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\ =& k\int_0^1 (u_x+v)_t(t)\int_0^t h(t-s)\big[(u_x+v)(s)-(u_x+v)(t)+(u_x+v)(t)\big]\mathrm{d}s\mathrm{d}x\\ =& k\int_0^1 (u_x+v)_t(t)\int_0^t h(t-s)(u_x+v)(t)\mathrm{d}s\mathrm{d}x\\ &+ k\int_0^1\int_0^t h(t-s)\big[(u_x+v)(s)-(u_x+v)(t)\big](u_x+v)_t(t)\mathrm{d}s\mathrm{d}x\\ =& \frac{k}{2}\left(\int_0^t h(s)\mathrm{d}s\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_x+v)^2\mathrm{d}x\\ &- \frac{k}{2}\int_0^1\int_0^t h(t-s)\frac{\mathrm{d}}{\mathrm{d}t}\big[(u_x+v)(t)-(u_x+v)(s)\big]^2\mathrm{d}s\mathrm{d}x\\ =& \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\int_0^t h(s)\mathrm{d}s\right)\|u_x+v\|^2\right] - \frac{k}{2}h(t)\|u_x+v\|^2 - \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}(h\circ(u_x+v))(t)\\ &+ \frac{k}{2}(h'\circ(u_x+v))(t).\end{aligned}$$

Then, we substitute J into (3.3) and immediately deduce (3.2). □

According to (3.2), the energy is decreasing and $E(t) \le E(0)$ for all $t \ge 0$.

Lemma 3.2 *For any $\sigma > 0$, the functional I_1 defined along the solution of problem* (1.11)*, by*

$$I_1(t) = \rho_1\int_0^1 u_t\int_0^x \theta(y)\mathrm{d}y\mathrm{d}x + \frac{\gamma\rho_1}{\rho_3}\int_0^1 u_t\int_0^x v(y)\mathrm{d}y\mathrm{d}x - \rho_2\int_0^1 v v_t\mathrm{d}x,\tag{3.4}$$

satisfies

$$\begin{aligned}I_1'(t) \le& -\frac{\rho_1\gamma}{2\rho_3}\|u_t\|^2 - \rho_2\|v_t\|^2 + \sigma\|u_x+v\|^2 + C_\sigma\|v_x\|^2 + C_\sigma\|\theta_x\|^2\\ &+ CA_\alpha(g\circ(u_x+v))(t),\ \forall t\ge 0.\end{aligned}\tag{3.5}$$

Proof From (3.4), we set

$$\Psi_1(t) = \rho_1 \int_0^1 u_t \int_0^x \theta(y)\mathrm{d}y\mathrm{d}x,$$
$$\Psi_2(t) = \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x v(y)\mathrm{d}y\mathrm{d}x,$$
$$\Psi_3(t) = -\rho_2 \int_0^1 v v_t \mathrm{d}x,$$

thus differentiation of $I_1(t)$ gives

$$I_1'(t) = \Psi_1'(t) + \Psi_2'(t) + \Psi_3'(t). \tag{3.6}$$

Now, we evaluate $\Psi_1'(t)$, $\Psi_2'(t)$ and $\Psi_3'(t)$ as follows.

$$\Psi_1'(t) = \rho_1 \int_0^1 u_{tt} \int_0^x \theta(y)\mathrm{d}y\mathrm{d}x + \rho_1 \int_0^1 u_t \int_0^x \theta_t(y)\mathrm{d}y\mathrm{d}x.$$

Using $(1.11)_1$, $(1.11)_3$, integration by parts and the boundary conditions (1.12), we obtain

$$\begin{aligned}
\Psi_1'(t) = & -k \int_0^1 (u_x + v)\theta \mathrm{d}x + \gamma \int_0^1 \theta^2 \mathrm{d}x + k \int_0^1 \theta \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\
& + \frac{\beta\rho_1}{\rho_3} \int_0^1 u_t\theta_x \mathrm{d}x - \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x u_{yt}(y)\mathrm{d}y\mathrm{d}x \\
& - \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x v_t(y)\mathrm{d}y\mathrm{d}x \\
= & -k \int_0^1 (u_x + v)\theta \mathrm{d}x + \gamma \int_0^1 \theta^2 \mathrm{d}x + k \left(\int_0^t h(s)\mathrm{d}s \right) \int_0^1 (u_x+v)\theta \mathrm{d}x \\
& - k \int_0^1 \theta \int_0^t h(t-s)\big[(u_x+v)(t) - (u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x \\
& + \frac{\beta\rho_1}{\rho_3} \int_0^1 u_t\theta_x \mathrm{d}x - \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t^2 \mathrm{d}x - \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x v_t(y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Also, a direct differentiation yields

$$\Psi_2'(t) = \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x v_t(y)\mathrm{d}y\mathrm{d}x + \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_{tt} \int_0^x v(y)\mathrm{d}y\mathrm{d}x.$$

Using $(1.11)_1$, integration by parts and the boundary conditions (1.12), we get

$$\Psi_2'(t) = \frac{\gamma\rho_1}{\rho_3} \int_0^1 u_t \int_0^x v_t(y)\mathrm{d}y\mathrm{d}x - \frac{k\gamma}{\rho_3} \int_0^1 (u_x+v)v\mathrm{d}x + \frac{\gamma^2}{\rho_3} \int_0^1 \theta v \mathrm{d}x$$

$$
\begin{aligned}
&+\frac{k\gamma}{\rho_3}\int_0^1 v\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\
&=\frac{\gamma\rho_1}{\rho_3}\int_0^1 u_t\int_0^x v_t(y)\mathrm{d}y\mathrm{d}x-\frac{k\gamma}{\rho_3}\int_0^1(u_x+v)v\mathrm{d}x+\frac{\gamma^2}{\rho_3}\int_0^1\theta v\mathrm{d}x\\
&-\frac{k\gamma}{\rho_3}\int_0^1 v\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x\\
&+\frac{k\gamma}{\rho_3}\left(\int_0^t h(s)\mathrm{d}s\right)\int_0^1 v(u_x+v)\mathrm{d}x.
\end{aligned}
$$

Similarly, using $(1.11)_2$, integration by parts and the boundary conditions (1.12), we get

$$
\begin{aligned}
\Psi_3'(t)=&-\rho_2\int_0^1 v_t^2\mathrm{d}x-\rho_2\int_0^1 v_t v_{tt}\mathrm{d}x\\
=&-\rho_2\int_0^1 v_t^2\mathrm{d}x+v\int_0^1 v_x^2\mathrm{d}x+k\int_0^1 v(u_x+v)-\gamma\int_0^1 v\theta\mathrm{d}x\\
&-k\int_0^1 v\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\\
=&-\rho_2\|v_t\|^2+b\|v_x\|^2+k\int_0^1 v(u_x+v)-\gamma\int_0^1 v\theta\mathrm{d}x\\
&+k\int_0^1 v\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x\\
&-k\left(\int_0^t h(s)\mathrm{d}s\right)\int_0^1 v(u_x+v)\mathrm{d}x.
\end{aligned}
$$

Substituting $\Psi_1'(t)$, $\Psi_2'(t)$ and $\Psi_3'(t)$ into (3.6), we have

$$
\begin{aligned}
I_1'(t)=&-\frac{\gamma\rho_1}{\rho_3}\|u_t\|^2-\rho_2\|v_t\|^2+\gamma\|\theta\|^2+b\|v_x\|^2+\left(\frac{\gamma^2}{\rho_3}-\gamma\right)\int_0^1 v\theta\mathrm{d}x\\
&+\left[-k+k\left(\int_0^t h(s)\mathrm{d}s\right)\right]\int_0^1\theta(u_x+v)\mathrm{d}x+\frac{\beta\rho_1}{\rho_3}\int_0^1 u_t\theta_x\mathrm{d}x\\
&+\left(k-\frac{k\gamma}{\rho_3}\right)\int_0^1 v(u_x+v)\mathrm{d}x+k\left(\int_0^t h(s)\mathrm{d}s\right)\left(\frac{\gamma}{\rho_3}-1\right)\int_0^1 v(u_x+v)\mathrm{d}x\\
&-k\int_0^1\theta\int_0^t h(t-s)\big[(u_x+vt)-(u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x\\
&+k\left(1-\frac{\gamma}{\rho_3}\right)\int_0^1 v\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

Applying Young's and Cauchy−Schwarz's inequalities, as well as Lemma 2.2, we obtain

$$
\begin{aligned}
I_1'(t) \le & -\frac{\gamma\rho_1}{\rho_3}\|u_t\|^2 - \rho_2\|v_t\|^2 + b\|v_x\|^2 + 3\varepsilon_3\|u_x+v\|^2 \\
& + \left[1+\gamma+\frac{k^2}{4\varepsilon_3}+\frac{k^2}{4\varepsilon_3}\left(\int_0^t h(s)\mathrm{d}s\right)^2+\frac{1}{4}\left(\frac{\gamma^2}{\rho_3}-\gamma\right)^2\right]\|\theta\|^2 \\
& + \frac{1}{4}A_\alpha\big(g\circ(u_x+v)\big)(t) + \varepsilon_4\|u_t\|^2 + \frac{1}{4\varepsilon_4}\left(\frac{\beta\rho_1}{\rho_3}\right)^2\|\theta_x\|^2 \\
& + \left[2+\frac{1}{4\varepsilon_3}\left(k-\frac{k\gamma}{\rho_3}\right)^2+\frac{k^2}{4\varepsilon_3}\left(\int_0^t h(s)\mathrm{d}s\right)^2\left(\frac{\gamma}{\rho_3}-1\right)^2\right]\|v\|^2 \\
& + \frac{k^2}{4}\left(1-\frac{\gamma}{\rho_3}\right)^2 A_\alpha\big(g\circ(u_x+v)\big)(t).
\end{aligned}
\tag{3.7}
$$

Now, we apply Poincaré's inequality and assumption (A_1); thus, (3.7) yields

$$
\begin{aligned}
I_1'(t) \le & -\left(\frac{\gamma\rho_1}{\rho_3}-\varepsilon_4\right)\|u_t\|^2 - \rho_2\|v_t\|^2 + 3\varepsilon_3\|u_x+v\|^2 \\
& + \left[1+\gamma+\frac{k^2}{4\varepsilon_3}+\frac{k^2}{4\varepsilon_3}+\frac{1}{4}\left(\frac{\gamma^2}{\rho_3}-\gamma\right)^2+\frac{1}{4\varepsilon_4}\left(\frac{\beta\rho_1}{\rho_3}\right)^2\right]\|\theta_x\|^2 \\
& + \left[2+b+\frac{1}{4\varepsilon_3}\left(k-\frac{k\gamma}{\rho_3}\right)^2+\frac{k^2}{4\varepsilon_3}\left(\frac{\gamma}{\rho_3}-1\right)^2\right]\|v_x\|^2 \\
& + \frac{1}{4}A_\alpha\big(g\circ(u_x+v)\big)(t) + \frac{k^2}{4}\left(1-\frac{\gamma}{\rho_3}\right)^2 A_\alpha\big(g\circ(u_x+v)\big)(t).
\end{aligned}
$$

Lastly, choosing $\varepsilon_4 = \dfrac{\gamma\rho_1}{2\rho_3}$ and $\sigma = 3\varepsilon_3$, we arrive at (3.5). □

Lemma 3.3 *For any $\delta > 0$, the functional I_2 defined along the solution of problem (1.11), by*

$$
I_2(t) = -\frac{\rho_1 b}{k}\int_0^1 v_x u_t \mathrm{d}x - \rho_2\int_0^1 v_t u_x \mathrm{d}x + \rho_2\int_0^1 v_t\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x,
\tag{3.8}
$$

satisfies

$$
\begin{aligned}
I_2'(t) \le & -\frac{b}{2}\|v_x\|^2 + \delta\|v_t\|^2 + C_\delta\|u_x+v\|^2 + C_\delta\|\theta_x\|^2 \\
& + C_\delta(1+A_\alpha)(g\circ(u_x+v))(t) + \left(\rho_2-\frac{\rho_1 b}{k}\right)\int_0^1 v_{xt}u_t\mathrm{d}x, \ \forall t \ge 0.
\end{aligned}
\tag{3.9}
$$

Proof We differentiate $I_2(t)$ and find

$$I_2'(t) = -\frac{\rho_1 b}{k}\int_0^1 v_{xt}u_t\mathrm{d}x \underbrace{-\frac{\rho_1 b}{k}\int_0^1 v_x u_{tt}\mathrm{d}x}_{\Phi_1(t)} \underbrace{-\rho_2\int_0^1 v_{tt}u_x\mathrm{d}x}_{\Phi_2(t)} \underbrace{-\rho_2\int_0^1 v_t u_{xt}\mathrm{d}x}_{\Phi_3(t)}$$
$$+\underbrace{\rho_2\int_0^1 v_{tt}\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x + \rho_2 h(0)\int_0^1 v_t(u_x+v)\mathrm{d}x}_{\Phi_4(t)}$$
$$+\underbrace{\rho_2\int_0^1 v_t\int_0^t h'(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x.}_{\Phi_5(t)} \tag{3.10}$$

Using $(1.11)_1$ and $(1.11)_2$ as well as integration by parts and the boundary conditions (1.12), we estimate Φ_1, Φ_2, Φ_3, Φ_4 as follows.

$$\begin{aligned}\Phi_1(t) &= -\frac{\rho_1 b}{k}\int_0^1 v_x u_{tt}\mathrm{d}x\\ &= -b\int_0^1 u_{xx}v_x\mathrm{d}x - b\int_0^1 v_x^2\mathrm{d}x + \frac{\gamma b}{k}\int_0^1 \theta_x v_x\\ &\quad - b\int_0^1 v_{xx}\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Then, by Young's inequality, we have

$$\begin{aligned}\Phi_1(t) \le& -b\int_0^1 v_x^2\mathrm{d}x + \delta_1\int_0^1 v_x^2\mathrm{d}x + \frac{1}{4\delta_1}\left(\frac{\gamma b}{k}\right)^2\int_0^1 \theta_x^2 - b\int_0^1 u_{xx}v_x\mathrm{d}x\\ &- b\int_0^1 v_{xx}\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x,\end{aligned} \tag{3.11}$$

Also,

$$\begin{aligned}\Phi_2(t) =& -\rho_2\int_0^1 v_{tt}u_x\mathrm{d}x\\ =& \, b\int_0^1 v_x u_{xx}\mathrm{d}x + k\int_0^1 (u_x+v)u_x\mathrm{d}x - \gamma\int_0^1 \theta u_x\\ &- k\int_0^1 u_x\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\ =& \, b\int_0^1 v_x u_{xx}\mathrm{d}x + k\int_0^1 (u_x+v)^2\mathrm{d}x - k\int_0^1 (u_x+v)v\mathrm{d}x - \gamma\int_0^1 \theta(u_x+v)\mathrm{d}x\\ &+ \gamma\int_0^1 \theta v\mathrm{d}x - k\int_0^1 (u_x+v)\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\end{aligned}$$

$$+ k \int_0^1 v \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x.$$

Therefore, applying Young's inequality, we get

$$\Phi_2(t) \le b \int_0^1 v_x u_{xx}\mathrm{d}x + \left(k + \frac{k^2}{4\delta_1} + \frac{\gamma}{2} + \frac{k}{2}\right) \int_0^1 (u_x+v)^2\mathrm{d}x + 3\delta_1 \int_0^1 v^2\mathrm{d}x$$
$$+ \left(\frac{\gamma}{2} + \frac{\gamma}{4\delta_1}\right) \int_0^1 \theta^2\mathrm{d}x + \left(\frac{k}{2} + \frac{k^2}{4\delta_1}\right) \int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\right)^2 \mathrm{d}x, \tag{3.12}$$

we now make use of Lemma 2.3 to arrive at

$$\Phi_2(t) \le b \int_0^1 v_x u_{xx}\mathrm{d}x + \left(\frac{\gamma}{2} + \frac{\gamma}{4\delta_1}\right) \int_0^1 \theta^2\mathrm{d}x + 3\delta_1 \int_0^1 v^2\mathrm{d}x$$
$$+ \left[\frac{3k}{2} + \frac{k^2}{4\delta_1} + \frac{\gamma}{2} + (\alpha + h(0))A_\alpha \left(k + \frac{k^2}{2\delta_1}\right)\right] \int_0^1 (u_x+v)^2\mathrm{d}x$$
$$+ \left(k + \frac{k^2}{2\delta_1}\right) A_\alpha \big(g \circ (u_x+v)\big)(t). \tag{3.13}$$

Again,

$$\Phi_3(t) = -\rho_2 \int_0^1 v_t u_{xt}\mathrm{d}x = \rho_2 \int_0^1 v_{xt} u_t \mathrm{d}x. \tag{3.14}$$

We as well have

$$\begin{aligned}\Phi_4(t) + \Phi_5(t) = &- k \int_0^1 (u_x+v) \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\ &+ b \int_0^1 v_{xx} \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\ &+ \gamma \int_0^1 \theta \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\ &+ \int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\right)^2 \mathrm{d}x \\ &+ \rho_2 \int_0^1 v_t \int_0^t h'(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x + \rho_2 h(0) \int_0^1 v_t(u_x+v)\mathrm{d}x,\end{aligned}$$

therefore, recalling (2.7), we get

$$\begin{aligned}\Phi_4(t) + \Phi_5(t) = &- k \int_0^1 (u_x+v) \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\ &+ b \int_0^1 v_{xx} \int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\gamma\int_0^1\theta\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\
&+\rho_2\alpha\int_0^1 v_t\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\
&+\int_0^1\left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\right)^2\mathrm{d}x\\
&-\rho_2\int_0^1 v_t\int_0^t g(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x+\rho_2 h(0)\int_0^1 v_t(u_x+v)\mathrm{d}x.
\end{aligned}
$$

It follows from the application of Young's inequality that,

$$
\begin{aligned}
\Phi_4(t)+\Phi_5(t)\le{}& b\int_0^1 v_{xx}\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x\\
&+\left(\frac{k}{2}+\frac{(\rho_2 h(0))^2}{4\delta_2}\right)\int_0^1(u_x+v)^2\mathrm{d}x\\
&+\frac{\gamma}{2}\int_0^1\theta^2\mathrm{d}x+3\delta_2\int_0^1 v_t^2\mathrm{d}x\\
&+\frac{\rho_2^2}{4\delta_2}\int_0^1\left(\int_0^t g(t-s)(u_x+v)(s)\mathrm{d}s\right)^2\mathrm{d}x\\
&+\left(1+\frac{k}{2}+\frac{\gamma}{2}+\frac{(\rho_2\alpha)^2}{4\delta_2}\right)\int_0^1\left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\right)^2\mathrm{d}x.
\end{aligned}
$$

Next, due to Lemma 2.3, we obtain

$$
\begin{aligned}
&\Phi_4(t)+\Phi_5(t)\\
&\le b\int_0^1 v_{xx}\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x+\frac{\gamma}{2}\int_0^1\theta^2\mathrm{d}x+3\delta_2\int_0^1 v_t^2\mathrm{d}x\\
&+\left[\frac{k}{2}+\frac{(\rho_2 h(0))^2}{4\delta_2}+\frac{\rho_2^2}{2\delta_2}+(\alpha+h(0))\left(2+k+\gamma+\frac{(\rho_2\alpha)^2}{2\delta_2}\right)\right]\int_0^1(u_x+v)^2\mathrm{d}x\\
&+\frac{\rho_2^2}{2\delta_2}\big(g\circ(u_x+v)\big)(t)+2A_\alpha\left(1+\frac{k}{2}+\frac{\gamma}{2}+\frac{(\rho_2\alpha)^2}{4\delta_2}\right)\big(g\circ(u_x+v)\big)(t).
\end{aligned}
\tag{3.15}
$$

Now, we substitute (3.11)–(3.15) into (3.10) and apply the Poincaré's inequality to get

$$
\begin{aligned}
I_2'(t)\le{}&-(b-3\delta_1)\|v_x\|^2+3\delta_2\|v_t\|^2+c_1(\delta_1,\delta_2)\|u_x+v\|^2+c_2(\delta_1,\delta_2)\|\theta_x\|^2\\
&+c_2(\delta_1,\delta_2)(1+A_\alpha)\big(g\circ(u_x+v)\big)(t)+\left(\rho_2-\frac{\rho_1 b}{k}\right)\int_0^1 v_{xt}u_t\mathrm{d}x.
\end{aligned}
$$

Finally, choosing $\delta_1=\dfrac{b}{6}$ and setting $\delta:=3\delta_2$, we arrive at (3.9). □

Lemma 3.4 *For any $\delta > 0$, the functional I_3 defined along the solution of problem (1.11), by*

$$I_3(t) = \rho_1 \int_0^1 u_t \int_0^x (u_y + v)(y)\mathrm{d}y\mathrm{d}x + \frac{\rho_1\rho_3}{\gamma}\int_0^1 u_t \int_0^x \theta(y)\mathrm{d}y\mathrm{d}x, \quad (3.16)$$

satisfies

$$\begin{aligned} I_3'(t) \leq &- \frac{kl_0}{2}\|u_x + v\|^2 + \varepsilon\|u_t\|^2 + C\left(1 + \frac{1}{4\varepsilon} + \frac{6}{kl_0}\right)\|\theta_x\|^2 \\ &+ C\left(1 + \frac{6}{kl_0}\right)A_\alpha(g\circ(u_x + v))(t), \ \forall t \geq 0. \end{aligned} \quad (3.17)$$

Proof Differentiation of I_3 yields

$$\begin{aligned} I_3'(t) = &\rho_1 \int_0^1 u_t \int_0^x (u_y + v)_t(y)\mathrm{d}y\mathrm{d}x + \underbrace{\rho_1 \int_0^1 u_{tt}\int_0^x (u_y + v)(y)\mathrm{d}y\mathrm{d}x}_{\Upsilon_1(t)} \\ &+ \underbrace{\frac{\rho_1\rho_3}{\gamma}\int_0^1 u_{tt}\int_0^x \theta(y)\mathrm{d}y\mathrm{d}x}_{\Upsilon_2(t)} + \underbrace{\frac{\rho_1\rho_3}{\gamma}\int_0^1 u_t \int_0^x \theta_t(y)\mathrm{d}y\mathrm{d}x}_{\Upsilon_3(t)}. \end{aligned} \quad (3.18)$$

Now, we estimate $\Upsilon_1(t)$, $\Upsilon_2(t)$ and $\Upsilon_3(t)$ as follows.

Using $(1.11)_1$, integration by parts and the boundary conditions (1.12), we have

$$\begin{aligned} \Upsilon_1(t) = &- k\int_0^1 (u_x + v)^2\mathrm{d}x + k\int_0^1 (u_x + v)\int_0^t h(t-s)(u_x + v)(s)\mathrm{d}s\mathrm{d}x \\ &+ \gamma\int_0^1 \theta(u_x + v)\mathrm{d}x \\ = &- k\left(1 - \int_0^t h(s)\mathrm{d}s\right)\int_0^1 (u_x + v)^2\mathrm{d}x + \gamma\int_0^1 \theta(u_x + v)\mathrm{d}x \\ &- k\int_0^1 (u_x + v)\int_0^t h(t-s)\big[(u_x + v)(t) - (u_x + v)(s)\big]\mathrm{d}s\mathrm{d}x. \end{aligned}$$

Applying Young's and Poincaré's inequalities, we get

$$\begin{aligned} \Upsilon_1(t) \leq &- k\left(1 - \int_0^t h(s)\mathrm{d}s\right)\int_0^1 (u_x + v)^2\mathrm{d}x + 2\varepsilon_2\int_0^1 (u_x + v)^2\mathrm{d}x \\ &+ \frac{\gamma^2}{4\varepsilon_2}\int_0^1 \theta^2\mathrm{d}x + \frac{k^2}{4\varepsilon_2}\int_0^1 \left(\int_0^t h(t-s)\big[(u_x + v)(t) - (u_x + v)(s)\big]\mathrm{d}s\right)^2\mathrm{d}x, \end{aligned}$$

then using Poincaré's inequality, Lemma 2.2 and assumption (A_1), we have

$$\begin{aligned}\Upsilon_1(t) \leq & -k\left(1-\int_0^t h(s)\mathrm{d}s\right)\|u_x+v\|^2 + 2\varepsilon_2\|u_x+v\|^2 + \frac{\gamma^2}{4\varepsilon_2}\|\theta_x\|^2 \\ & + \frac{k^2}{4\varepsilon_2}A_\alpha(g\circ(u_x+v))(t) \\ \leq & -kl_0\|u_x+v\|^2 + 2\varepsilon_2\|u_x+v\|^2 + \frac{\gamma^2}{4\varepsilon_2}\|\theta_x\|^2 + \frac{k^2}{4\varepsilon_2}A_\alpha(g\circ(u_x+v))(t).\end{aligned} \tag{3.19}$$

Next, using $(1.11)_1$, integration by parts and the boundary conditions (1.12), we have

$$\begin{aligned}\Upsilon_2(t) = & -\frac{k\rho_3}{\gamma}\int_0^1 (u_x+v)\theta\mathrm{d}x + \frac{k\rho_3}{\gamma}\int_0^1 \theta\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s\mathrm{d}x \\ & + \rho_3\int_0^1 \theta^2\mathrm{d}x \\ = & -\frac{k\rho_3}{\gamma}\left(1-\int_0^t h(s)\mathrm{d}s\right)\int_0^1 \theta(u_x+v)\mathrm{d}x + \rho_3\int_0^1\theta^2\mathrm{d}x \\ & - \frac{k\rho_3}{\gamma}\int_0^1\theta\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Young's inequality leads to

$$\begin{aligned}\Upsilon_2(t) \leq & \ \varepsilon_2\int_0^1 (u_x+v)^2\mathrm{d}x + \frac{1}{4\varepsilon_2}\left(\frac{k\rho_3}{\gamma}\right)^2\left(1-\int_0^t h(s)\mathrm{d}s\right)^2\int_0^1\theta^2\mathrm{d}x \\ & + \int_0^1\left(\int_0^t h(t-s)\big[(u_x+v)(t)-(u_x+v)(s)\big]\mathrm{d}s\right)^2\mathrm{d}x \\ & + \left[\rho_3 + \frac{1}{4}\left(\frac{k\rho_3}{\gamma}\right)^2\right]\int_0^1\theta^2\mathrm{d}x,\end{aligned}$$

then applying Poincaré's inequality, Lemma 2.2 and assumption (A_1), we get

$$\Upsilon_2(t) \leq \varepsilon_2\|u_x+v\|^2 + \left[\rho_3 + \frac{1}{4}\left(\frac{k\rho_3}{\gamma}\right)^2\left(1+\frac{1}{\varepsilon_2}\right)\right]\|\theta_x\|^2 + A_\alpha(g\circ(u_x+v))(t). \tag{3.20}$$

Also, using $(1.11)_3$, integration by parts and the boundary conditions (1.12), as well as Young's inequality, we obtain

$$\begin{aligned}\Upsilon_3(t) = & -\rho_1\int_0^1 u_t\int_0^x (u_y+v)_t(y)\mathrm{d}y\mathrm{d}x + \frac{\rho_1\beta}{\gamma}\int_0^1 u_t\theta_x\mathrm{d}x \\ \leq & -\rho_1\int_0^1 u_t\int_0^x (u_y+v)_t(y)\mathrm{d}y\mathrm{d}x + \varepsilon\|u_t\|^2 + \frac{1}{4\varepsilon}\left(\frac{\rho_1\beta}{\gamma}\right)^2\|\theta_x\|^2.\end{aligned} \tag{3.21}$$

Substitution of (3.19)–(3.21) into (3.18), we get

$$I_3'(t) \leq -(kl_0 - 3\varepsilon_2)\|u_x + v\|^2 + \varepsilon\|u_t\|^2 + C\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon_2}\right)\|\theta_x\|^2 + C\left(1 + \frac{1}{\varepsilon_2}\right)A_\alpha(g \circ (u_x + v))(t).$$

Finally, choosing $\varepsilon_2 = \frac{kl_0}{6}$, we get (3.17). □

Lemma 3.5 *The functional I_4, defined by*

$$I_4(t) = \int_0^1 \int_0^t J(t-s)(u_x+v)^2(x,s)\mathrm{d}s\mathrm{d}x, \quad \text{where } J(t) = \int_t^{+\infty} h(s)\mathrm{d}s,$$

satisfies along the solution of problem (1.11), the estimate

$$I_4'(t) \leq -\frac{1}{2}\left(h \circ (u_x + v)\right)(t) + 3(1-l)\|u_x + v\|_2^2, \quad \forall\, t \geq 0. \tag{3.22}$$

Proof We differentiate I_4 and recall that $J(t) = J(0) - \int_0^t h(s)\mathrm{d}s$ and $J'(t) = -h(t)$; thus, we have

$$\begin{aligned}
I_4'(t) &= \int_0^1 \int_0^t J'(t-s)(u_x+v)^2(x,s)\mathrm{d}s\mathrm{d}x + J(0)\int_0^1 (u_x+v)^2(x,s)\mathrm{d}x \\
&= -\int_0^1 \int_0^t h(t-s)(u_x+v)^2(x,s)\mathrm{d}s\mathrm{d}x + J(t)\int_0^1 (u_x+v)^2(x,s)\mathrm{d}x \\
&\quad + \int_0^1 \int_0^t h(t-s)(u_x+v)^2(x,s)\mathrm{d}s\mathrm{d}x \\
&= J(t)\|u_x+v\|_2^2 - \int_0^1 h(t-s)\big[(u_x+v)(x,t) - (u_x+v)(x,s)\big]^2\mathrm{d}s\mathrm{d}x \\
&\quad + 2\int_0^1 (u_x+v)\int_0^t h(t-s)\big[(u_x+v)(x,t) - (u_x+v)(x,s)\big]\mathrm{d}s\mathrm{d}x \\
&\leq -\left(h \circ (u_x+v)\right)(t) + J(t)\|u_x+v\|_2^2 + 2(1-l)\|u_x+v\|_2^2 \\
&\quad + \frac{\left(\int_0^t h(s)\mathrm{d}s\right)}{2(1-l)}\left(h \circ (u_x+v)\right)(t) \\
&\leq -\frac{1}{2}\left(h \circ (u_x+v)\right)(t) + 2(1-l)\|u_x+v\|_2^2 + J(t)\|u_x+v\|_2^2.
\end{aligned} \tag{3.23}$$

By recalling that h is decreasing and positive, hence $J(t) \leq J(0) = (1-l)$, the result (3.22) follows. □

4 General and Optimal Decay Result

In this section, we state and prove our main general and optimal decay result. In order to do this, we define a Lyapunov functional $\mathcal{L}$ as

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t), \tag{4.1}$$

for N, N_1, N_2, N_3 to be fixed later. In the lemma that follows, we show the equivalence of $\mathcal{L}$ and the energy functional E.

Lemma 4.1 *There exist positive constants α_1, α_2 such that for N large enough, the functional $\mathcal{L}$ satisfies*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0. \tag{4.2}$$

Moreover, if $\rho_2 = \dfrac{\rho_1 b}{k}$ then

$$\begin{aligned}\mathcal{L}(t) \leq & - \lambda \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|v_x\|_2^2 + \|u_x + v\|_2^2 + \|\theta_x\|_2^2 \right) \\ & + \frac{1}{4} \left(h \circ (\varphi_x + \psi) \right)(t), \ \forall\, t \geq 0, \end{aligned} \tag{4.3}$$

for some positive constants N, N_1, N_2, N_3 to be later chosen appropriately.

Proof We have

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| \leq & \, N_1 |I_1(t)| + N_2 |I_2(t)| + N_3 |I_3(t)| \\ \leq & \, \rho_1 N_1 \left| \int_0^1 u_t \int_0^x \theta(y) \mathrm{d}y \mathrm{d}x \right| + \frac{\gamma \rho_1}{\rho_3} N_1 \left| \int_0^1 u_t \int_0^x v(y) \mathrm{d}y \mathrm{d}x \right| \\ & + \rho_2 N_1 \left| \int_0^1 v v_t \mathrm{d}x \right| + \frac{\rho_1 b}{k} N_2 \left| \int_0^1 v_x u_t \mathrm{d}x \right| + \rho_2 N_2 \left| \int_0^1 v_t u_x \mathrm{d}x \right| \\ & + \rho_2 N_2 \left| \int_0^1 v_t \int_0^t h(t-s)(u_x + v)(s) \mathrm{d}s \mathrm{d}x \right| \\ & + \rho_1 N_3 \left| \int_0^1 u_t \int_0^x (u_y + v)(y) \mathrm{d}y \mathrm{d}x \right| \\ & + \frac{\rho_1 \rho_3}{\gamma} N_3 \left| \int_0^1 u_t \int_0^x \theta(y) \mathrm{d}y \mathrm{d}x \right|. \end{aligned}$$

Applying Young's inequality and Lemma 2.1, we get

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| \leq & \, C \big(\rho_1 \|u_t\|^2 + \rho_2 \|v_t\|^2 + \rho_3 \|\theta\|^2 + b \|v_x\|^2 + k \|u_x + v\|^2 \big) \\ & + C \int_0^1 \left(\int_0^t h(t-s)(u_x + v)(s) \mathrm{d}s \right)^2 \mathrm{d}x. \end{aligned}$$

Using Cauchy−Schwarz's inequality, the last integral above can be estimated in like manner as in Lemma 2.3, we get

$$\int_0^1 \left(\int_0^t h(t-s)(u_x+v)(s)\mathrm{d}s \right)^2 \mathrm{d}x \le 2\|u_x+v\|^2 + 2\big(h \circ (u_x+v)\big)(t).$$

Hence, there exists a constant $C > 0$ such that

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| \le & \frac{C}{2}\Bigg[\rho_1\|u_t\|^2 + \rho_2\|v_t\|^2 + \rho_3\|\theta\|^2 + b\|v_x\|^2 \\ & + k\left(1 - \int_0^t h(s)\mathrm{d}s\right)\|u_x+v\|^2\Bigg] + \frac{Ck}{2}(h \circ (u_x+v))(t) \\ \le & CE(t). \end{aligned}$$

It follows that

$$|\mathcal{L}(t) - NE(t)| \le CE(t) \Longleftrightarrow (N-C)E(t) \le \mathcal{L}(t) \le (N+C)E(t).$$

Now, we choose N large enough so that

$$N - C > 0. \tag{4.4}$$

Therefore, there exist positive constants α_1, α_2 such that (4.2) holds. Hence, $\mathcal{L} \equiv E$.

Now, we differentiate (4.1), bearing in mind (3.2), (3.5), (3.9) and (3.17) and the fact that $g = \alpha h - h'$, we get for any $t \ge 0$,

$$\begin{aligned} \mathcal{L}'(t) \le & -\left[\frac{\gamma\rho_1}{2\rho_3}N_1 - \varepsilon N_3\right]\|u_t\|^2 - \left[\rho_2 N_1 - \delta N_2\right]\|v_t\|^2 - \left[\frac{b}{2}N_2 - C_\sigma N_1\right]\|v_x\|^2 \\ & - \left[\frac{kl_0}{2}N_3 - \sigma N_1 - C_\delta N_2\right]\|u_x+v\|^2 \\ & - \left[\beta N - C_\sigma N_1 - C_\delta N_2 + C\left(1 + \frac{6}{kl_0}\right)N_3\right]\|\theta_x\|^2 \\ & + \left[\frac{k}{2}N - (1+A_\alpha)\left(CN_1 + C_\delta N_2\left(1 + \frac{6}{kl_0}\right)N_3\right)\right](g \circ (u_x+v))(t) \\ & + \frac{\alpha k}{2}N\big(h \circ (u_x+v)\big)(t) + N_2\left(\rho_2 - \frac{\rho_1 b}{k}\right)\int_0^1 v_{xt}u_t \mathrm{d}x. \end{aligned}$$

Setting $N_1 = 1$, $\varepsilon = \dfrac{\gamma\rho_1 N_1}{4\rho_3 N_3}$, $\sigma = \dfrac{kl_0 N_3}{4N_1}$, $\delta = \dfrac{\rho_2 N_1}{2N_2}$ and recalling that $\rho_2 = \dfrac{\rho_1 b}{k}$, we find

$$\mathcal{L}'(t) \leq -\frac{\gamma\rho_1}{4\rho_3}\|u_t\|^2 - \frac{\rho_2}{2}\|v_t\|^2 - \left[\frac{kl_0}{4}N_3 - c_2N_2\right]\|u_x + v\|^2 - \left[\frac{b}{2}N_2 - c_1\right]\|v_x\|^2$$
$$- [\beta N - C(1 + N_2 + N_3)]\,\|\theta_x\|^2 + \frac{\alpha k}{2}N\big(h \circ (u_x + v)\big)(t)$$
$$- \left[\frac{k}{2}N - C(1 + A_\alpha)(1 + N_2 + N_3)\right](g \circ (u_x + v))(t).$$

Choosing $N_2 = \dfrac{kl_0N_3}{8c_2}$, we get

$$\mathcal{L}'(t) \leq -\frac{\gamma\rho_1}{4\rho_3}\|u_t\|^2 - \frac{\rho_2}{2}\|v_t\|^2 - \frac{kl_0N_3}{8}\|u_x + v\|^2 - \left[\frac{bkl_0N_3}{16c_2} - c_1\right]\|v_x\|^2$$
$$- [\beta N - C(1 + N_2 + N_3)]\,\|\theta_x\|^2 + \frac{\alpha k}{2}N\big(h \circ (u_x + v)\big)(t)$$
$$- \left[\frac{k}{2}N - C(1 + A_\alpha)(1 + N_2 + N_3)\right](g \circ (u_x + v))(t). \tag{4.5}$$

Now, we pick N_3 large such that

$$\frac{bkl_0N_3}{16c_2} - c_1 > 0. \tag{4.6}$$

Observe that $\dfrac{\alpha h^2(s)}{g(s)} = \dfrac{\alpha h^2(s)}{\alpha h(s) - h'(s)} < h(s)$; therefore, application of the dominated convergence theorem yields

$$\alpha A_\alpha = \int_0^{+\infty} \frac{\alpha h^2(s)}{\alpha h(s) - h'(s)}\mathrm{d}s \to 0 \;\; as \;\; \alpha \to 0. \tag{4.7}$$

Hence, there exists $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0 < 1$, we have

$$\alpha A_\alpha < \frac{1}{8C\,(1 + N_2 + N_3)}. \tag{4.8}$$

Finally, we choose N large enough and set $\alpha = \dfrac{1}{2Nk}$ so that (4.2) remains valid and

$$\frac{Nk}{3} - CA_\alpha\,(1 + N_2 + N_3) > 0 \tag{4.9}$$
$$\frac{Nk}{6} - C\,(1 + N_2 + N_3) > 0. \tag{4.10}$$

as well as

$$\beta N - C(1 + N_2 + N_3) > 0. \tag{4.11}$$

Combining (4.5)–(4.11), we obtain (4.3). This completes the proof. □

Now we will state and prove our main decay result.

Theorem 4.1 *Let* (u, v, θ) *be the solution of problem* (1.11)–(1.13). *Assume* (A_1)) *and* (A_2) *hold and that* $\rho_2 = \frac{\rho_1 b}{k}$. *Then, there exist* $\omega_1, \omega_2 > 0$ *such that the energy functional* E *of problem* (1.11)–(1.13) *satisfies*

$$E(t) \leq \omega_2 G_1^{-1}\left(\omega_1 \int_{t_0}^{t} \xi(s)\mathrm{d}s\right), \quad \text{where } G_1(t) = \int_t^r \frac{1}{sG'(s)}\mathrm{d}s \tag{4.12}$$

and G_1 *is a decreasing and strictly convex function on* $(0, r]$, *with* $r = h(t_0) > 0$ *and* $\lim_{t\to 0} G_1(t) = +\infty$.

Proof According to (A_1) and (A_2), we have that ξ and h are continuous, decreasing and positive. In addition, G is continuous and positive. Thus, $\forall\, t \in [0, t_0]$, we have

$$0 < h(t_0) \leq h(t) \leq h(0), \quad 0 < \xi(t_0) \leq \xi(t) \leq \xi(0)$$

then we can find some positive constants a_1 and a_2

$$a_1 \leq \xi(t)G(h(t)) \leq a_2.$$

Therefore, it follows that

$$h'(t) \leq -\xi(t)G(h(t)) \leq -\frac{a_1}{h(0)}h(0) \leq -\frac{a_1}{h(0)}h(t), \quad \forall\, t \in [0, t_0]. \tag{4.13}$$

From (3.1) and (4.13), we have

$$\begin{aligned}
&\int_0^{t_0} h(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2\mathrm{d}s \\
&\leq -\frac{h(0)}{a_1}\int_0^{t_0} h'(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2\mathrm{d}s \\
&\leq -CE'(t), \quad \forall\, t \in [0, t_0].
\end{aligned} \tag{4.14}$$

Using (4.3) and (4.14), we get

$$\begin{aligned}
\mathcal{L}'(t) &\leq -\lambda E(t) + \frac{1}{4}(h \diamond (\varphi_x + \psi)(t) \\
&= -\lambda E(t) + \frac{1}{4}\int_0^{t_0} h(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2\mathrm{d}s \\
&\quad + \frac{1}{4}\int_{t_0}^{t} h(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2\mathrm{d}s
\end{aligned}$$

$$\leq -\lambda E(t) - CE'(t) + \frac{1}{4}\int_{t_0}^{t} h(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 \mathrm{d}s.$$

Therefore, for all $t \geq t_0$, we get

$$\mathcal{L}_1'(t) \leq -\lambda E(t) + \frac{1}{4}\int_{t_0}^{t} h(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 \mathrm{d}s, \quad (4.15)$$

where $\mathcal{L}_1(t) = \mathcal{L}(t) + CE(t)$ and with (4.2) in mind, we deduce that $\mathcal{L}_1$ is equivalent to E.

To complete the proof of Theorem 4.1, we discuss two cases:

Case 1 $G(t)$ is linear.

For this case, we multiply (4.15) by $\xi(t)$, then with (3.1) and (A_2) in mind, we have

$$\begin{aligned}\xi(t)\mathcal{L}_1'(t) &\leq -\lambda\xi(t)E(t) + \frac{1}{4}\xi(t)\int_{t_0}^{t} h(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 \mathrm{d}s \\ &\leq -\lambda\xi(t)E(t) + \frac{1}{2}\int_{t_0}^{t} \xi(s)h(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 \mathrm{d}s \\ &\leq -\lambda\xi(t)E(t) - \frac{1}{2}\int_{t_0}^{t} h'(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 \mathrm{d}s \\ &\leq -\lambda\xi(t)E(t) - CE'(t). \end{aligned} \quad (4.16)$$

Knowing that ξ is decreasing, we have

$$(\xi\mathcal{L}_1 + CE)'(t) \leq -\lambda\xi(t)E(t), \quad \forall\, t \geq t_0 \quad (4.17)$$

and since $\mathcal{L}_1 \sim E$, we obtain

$$\xi\mathcal{L}_1 + CE \sim E. \quad (4.18)$$

Hence, setting $\mathcal{L}_2(t) = \xi(t)\mathcal{L}_1(t) + CE(t)$, then we can find some positive constant ω such that

$$\mathcal{L}_2'(t) \leq -\lambda\xi(t)E(t) \leq -\omega\xi(t)\mathcal{L}_2(t), \quad \forall\, t \geq t_0. \quad (4.19)$$

Integration of (4.19) over (t_0, t) and recalling (4.18) yield

$$E(t) \leq \omega e^{-\widetilde{\omega}\int_{t_0}^{t}\xi(s)\mathrm{d}s} = \omega G_1^{-1}\left(\widetilde{\omega}\int_{t_0}^{t}\xi(s)\mathrm{d}s\right),$$

where $\widetilde{\omega}$ is a positive constant.

Case 2 $G(t)$ is nonlinear.

Firstly, for this case, we set $\mathcal{F}(t) = \mathcal{L}(t) + I_4(t)$. Thus, from Lemma 3.5 and (4.5), we have

$$\begin{aligned}\mathcal{F}'(t) \leq &-\frac{\gamma\rho_1}{4\rho_3}\|u_t\|^2 - \frac{\rho_2}{2}\|v_t\|^2 - \left[\frac{kl_0N_3}{8} - 3(1-l_0)\right]\|u_x+v\|^2\\ &-\left[\frac{bkl_0N_3}{16c_2} - c_1\right]\|v_x\|^2 - [\beta N - C(1+N_2+N_3)]\,\|\theta_x\|^2\\ &-\left[\frac{k}{2}N - C(1+A_\alpha)(1+N_2+N_3)\right](g\circ(u_x+v))(t)\\ &+\left[\frac{\alpha Nk}{2} - \frac{1}{2}\right]\big(h\circ(u_x+v)\big)(t). \end{aligned}\tag{4.20}$$

Next, we choose N_3 large so that

$$\frac{kl_0N_3}{8} - 3(1-l_0) > 0 \text{ and } \frac{bkl_0N_3}{16c_2} - c_1 > 0,$$

then choose N large enough so that

$$\beta N - C(1+N_2+N_3) > 0 \text{ and } \frac{k}{2}N - C(1+A_\alpha)(1+N_2+N_3) > 0.$$

Now, we take $\alpha = \dfrac{1}{2Nk}$; thus, there exists a constant $\widetilde{\alpha} > 0$ such that

$$\mathcal{F}'(t) \leq -\widetilde{\alpha}\,E(t),\ \forall t \geq t_0. \tag{4.21}$$

From (4.21), it follows that

$$\widetilde{\alpha}\int_{t_0}^{t} E(s)\mathrm{d}s \leq \mathcal{F}(t_0) - \mathcal{F}(t) \leq \mathcal{F}(t_0).$$

Therefore, we have

$$\int_0^{+\infty} E(s)\mathrm{d}s < \infty. \tag{4.22}$$

We next define the functional φ as

$$\varphi(t) := \alpha\int_{t_0}^{t} \|(u_x+v)(t) - (u_x+v)(t-s)\|_2^2\mathrm{d}s.$$

Due to (4.22), we can select $0 < \alpha < 1$ such that

$$\varphi(t) < 1,\ \ \forall\, t \geq t_0. \tag{4.23}$$

Henceforth, we can assume without loss of generality that $\varphi(t) > 0, \ \forall\, t \geq t_0$, otherwise, we obtain immediately an exponential stability result, from (4.15). We as well define the function ψ as

$$\psi(t) = -\int_{t_0}^{t} h'(s)\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s$$

and notice that

$$\psi(t) \leq -CE'(t). \tag{4.24}$$

We can see from assumption (A_2) that G is strictly convex on $(0, r], \ r = h(t_0)$ and $G(0) = G'(0) = 0$. Therefore, we have

$$G(\epsilon s) \leq \epsilon G(s), \quad 0 \leq \epsilon \leq 1, \quad s \in (0, r]. \tag{4.25}$$

It thus follows from assumption (A_2), (4.23) and Jensen's inequality (2.12) that

$$\begin{aligned}
\psi(t) &= \frac{1}{\alpha\varphi(t)} \int_{t_0}^{t} \varphi(t)(-h'(s))\alpha\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s \\
&\geq \frac{1}{\alpha\varphi(t)} \int_{t_0}^{t} \varphi(t)\xi(s)G(h(s))\alpha\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s \\
&\geq \frac{\xi(t)}{\alpha\varphi(t)} \int_{t_0}^{t} G(\varphi(t)h(s))\alpha\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s \\
&\geq \frac{\xi(t)}{\alpha} G\left(\frac{1}{\varphi(t)} \int_{t_0}^{t} \varphi(t)h(s)\alpha\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s\right) \\
&= \frac{\xi(t)}{\alpha} G\left(\alpha \int_{t_0}^{t} h(s)\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s\right) \\
&= \frac{\xi(t)}{\alpha} \widetilde{G}\left(\alpha \int_{t_0}^{t} h(s)\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s\right),
\end{aligned} \tag{4.26}$$

where $\widetilde{G}$ is the extension of G on $(0, +\infty)$ in (2.3). From (4.26), we get

$$\int_{t_0}^{t} h(s)\|(u_x + v)(t) - (u_x + v)(t - s)\|_2^2 \mathrm{d}s \leq \frac{1}{\alpha} \widetilde{G}^{-1}\left(\frac{\alpha\psi(t)}{\xi(t)}\right).$$

Thus, from (4.15), we get

$$\mathcal{L}_1'(t) \leq -\lambda E(t) + C\widetilde{G}^{-1}\left(\frac{\alpha\psi(t)}{\xi(t)}\right), \quad \forall\, t \geq t_0. \tag{4.27}$$

Now, for $r_0 < r$ to be chosen later, we define the functional

$$M_2(t) := \widetilde{G}'\left(r_0 \frac{E(t)}{E(0)}\right) \mathcal{L}_1(t) + E(t)$$

which is equivalent to E since $\mathcal{L}_1 \sim E$. Using (4.27) and due to $E'(t) \leq 0,\ \widetilde{G}'(t) > 0,\ \widetilde{G}''(t) > 0$, we have

$$\begin{aligned} M_2'(t) =& r_0 \frac{E'(t)}{E(0)} \widetilde{G}'' \left(r_0 \frac{E(t)}{E(0)} \right) \mathcal{L}_1(t) + \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) \mathcal{L}_1'(t) + E'(t) \\ \leq& -\lambda E(t) \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) + C \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) \widetilde{G}^{-1} \left(\alpha \frac{\psi(t)}{\xi(t)} \right) + E'(t). \quad (4.28) \end{aligned}$$

In what follows, we consider the convex conjugate $\widetilde{G}^*$ of $\widetilde{G}$ in the sense of Young (see [25] page 61–64) defined by

$$\widetilde{G}^*(s) = s(\widetilde{G}')^{-1}(s) - \widetilde{G}\left[(\widetilde{G}')\right], \quad (4.29)$$

with $\widetilde{G}^*$ satisfying the generalized Young inequality

$$U_1 U_2 \leq \widetilde{G}^*(U_1) + \widetilde{G}(U_2). \quad (4.30)$$

Let us set $U_1 = \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right)$ and $U_2 = \widetilde{G}^{-1} \left(\alpha \frac{\psi(t)}{\xi(t)} \right)$. It follows from Lemma 3.1 and (4.28)–(4.30), that for all $t \geq t_0$

$$\begin{aligned} M_2'(t) \leq& -\lambda E(t) \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) + C \widetilde{G}^* \left(\widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) \right) + C\alpha \frac{\psi(t)}{\xi(t)} + E'(t) \\ \leq& -\lambda E(t) \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) + C r_0 \frac{E(t)}{E(0)} \widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) + C\alpha \frac{\psi(t)}{\xi(t)} + E'(t). \end{aligned} \quad (4.31)$$

We multiply (4.31) by $\xi(t)$, observe that $r_0 \frac{E(t)}{E(0)} < r$ and

$$\widetilde{G}' \left(r_0 \frac{E(t)}{E(0)} \right) = G' \left(r_0 \frac{E(t)}{E(0)} \right),$$

then making use of (3.2) and (4.24), we obtain

$$\begin{aligned} \xi(t) M_2'(t) \leq& -\lambda \xi(t) E(t) G' \left(r_0 \frac{E(t)}{E(0)} \right) + C r_0 \frac{E(t)}{E(0)} \xi(t) G' \left(r_0 \frac{E(t)}{E(0)} \right) \\ &+ C\alpha\psi(t) + \xi(t) E'(t) \\ \leq& -\lambda \xi(t) E(t) G' \left(r_0 \frac{E(t)}{E(0)} \right) + C r_0 \frac{E(t)}{E(0)} \xi(t) G' \left(r_0 \frac{E(t)}{E(0)} \right) \\ &- C E'(t), \ \forall\ t \geq t_0. \end{aligned} \quad (4.32)$$

Let $M_3(t) = \xi(t)M_2(t) + CE(t)$, then $M_3 \sim E$ since $M_2 \sim E$, i.e., there exist constants $a_0, a_1 > 0$, such that M_3 satisfies

$$a_0 M_3(t) \leq E(t) \leq a_1 M_3(t). \tag{4.33}$$

Thus, from (4.32), we have

$$M_3'(t) \leq -(\lambda E(0) - Cr_0)\xi(t)\frac{E(t)}{E(0)}\xi(t)G'\left(r_0\frac{E(t)}{E(0)}\right), \ \forall t \geq t_0.$$

Next, we choose $r_0 < r$ small enough such that $\lambda E(0) - Cr_0 > 0$, then for some positive constant $\omega > 0$, we have

$$\begin{aligned} M_3'(t) \leq & -\omega\xi(t)\frac{E(t)}{E(0)}\xi(t)G'\left(r_0\frac{E(t)}{E(0)}\right) \\ = & -\omega\xi(t)G_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_0, \end{aligned} \tag{4.34}$$

where

$$G_2(s) = sG'(r_0 s).$$

We therefore remark that

$$G_2'(s) = G'(r_0 s) + r_0 s G''(r_0 s),$$

thus the strict convexity of G on $(0, r]$ implies $G_2(s) > 0, \ G_2'(s) > 0$ on $(0, r]$.

Now, let $M(t) = a_0\frac{M_3(t)}{E(0)}$, then due to (4.33) and (4.34), we have

$$M(t) \sim E(t) \tag{4.35}$$

and

$$M'(t) = a_0\frac{M_3'(t)}{E(0)} \leq -\omega_1\xi(t)G_2(M(t)), \ \forall t \geq t_0. \tag{4.36}$$

We now integrate (4.36) over (t_0, t), to get

$$\omega_1\int_{t_0}^{t}\xi(s)\mathrm{d}s \leq -\int_{t_0}^{t}\frac{M'(s)}{G_2(M(s))}\mathrm{d}s = \frac{1}{r_0}\int_{r_0 M(t)}^{r_0 M(t_0)}\frac{1}{sG'(s)}\mathrm{d}s$$

which yields

$$M(t) \leq \frac{1}{r_0}G_1^{-1}\left(\widetilde{\omega}_1\int_{t_0}^{t}\xi(s)\mathrm{d}s\right), \tag{4.37}$$

where

$$G_1(t) = \int_t^r \frac{1}{sG'(s)}\mathrm{d}s.$$

It follows from (A_2), that G_1 is a strictly decreasing function on $(0, r]$. Furthermore,

$$\lim_{t \longrightarrow 0} G_1(t) = +\infty.$$

Finally, making use of (4.35) and (4.37), we obtain decay estimate (4.12). Hence, Theorem 4.1 is completely proved. □

4.1 Examples

(1). Let $h_1(t) = \tau_0 \mathrm{e}^{-2\tau_0 t}, \ t \geq 0,$ where $\tau_0 > 0$; thus, (A_1) holds with $l_0 = \frac{1}{2}$. Therefore,

$$h_1'(t) = -2\tau_0^2 \mathrm{e}^{-2\tau_0 t} = -2\tau_0 G(h_1(t)), \quad \text{where} \quad G(s) = s.$$

In this case, the solution energy $(3.1)_1$ satisfies

$$E(t) \leq \Theta \mathrm{e}^{-\nu t}, \quad \forall\, t \geq 0,$$

for some constants $\Theta > 0$, $\nu > 0$.

(2). Let $h_2(t) = \tau_0 \mathrm{e}^{-(1+t)^{\tau_1}}, \ t \geq 0,$ where $\tau_0 > 0,$ and $0 < a_1 < 1$ with τ_0 chosen such that (A_1) holds. Therefore,

$$h_2'(t) = -\tau_0 \tau_1 (1+t)^{\tau_1 - 1} \mathrm{e}^{-(1+t)^{\tau_1}} = -\xi(t) G(h_2(t)),$$

where

$$\xi(t) = \tau_1 (1+t)^{\tau_1 - 1}, \quad G(s) = s.$$

In this case, the solution energy $(3.1)_1$ satisfies

$$E(t) \leq \Theta \mathrm{e}^{-\nu (1+t)^{\tau_1}}, \quad \forall\, t \geq 0,$$

for some constants $\Theta > 0$, $\nu > 0$.

(3). Let $h_3(t) = \frac{\tau_0}{(1+t)^{\tau_1}}, \ t \geq 0,$ where $\tau_0 > 0,$ and $\tau_1 > 1$ with τ_0 chosen such that (A_1) holds. Therefore,

$$h_3'(t) = \frac{-\tau_0 \tau_1}{(1+t)^{\tau_1 + 1}} = -\tau_1 G(h_3(t)), \quad \text{where} \quad G(s) = s^q, \ \ q = \frac{\tau_1 + 1}{\tau_1}$$

with q satisfying $1 < q < 2$. Hence, the solution energy (3.1) satisfies

$$E(t) \leq \frac{\Theta}{(1+t)^{\tau_1}}$$

for some constant $\Theta > 0$.

Acknowledgements The first author would like to appreciate the continuous support of University of Hafr Al Batin. Baowei Feng has been supported by the National Natural Science Foundation of China, Grant No. 11701465.

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