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# Infinitely Many Solutions for a Fourth-Order Semilinear Elliptic Equations Perturbed from Symmetry

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## Abstract

In this paper, we study the existence of multiple solutions for the following biharmonic problem

$$\begin{aligned} \Delta^2 u &= f(x,u) + g(x,u) \quad \text{in} \quad \Omega, \\ u &= \Delta u = 0 \qquad\qquad\quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$, $(N > 4)$ is a smooth bounded domain and $f(x,\xi)$ is odd in $\xi$, $g(x,\xi)$ is a perturbation term. By using the variant of Rabinowitz's perturbation method, under some growth conditions on $f$ and $g$, we show that there are infinitely many weak solutions to the problem.

**Keywords** Biharmonic · Boundary value problems · Critical points · Perturbation methods · Multiple solutions

**Mathematics Subject Classification** Primary 35J60; Secondary 35B33 · 35J25

## 1 Introduction

In the last decades, the biharmonic elliptic equation

$$\begin{aligned} \Delta^2 u &= f(x,u) \quad \text{in} \quad \Omega, \\ u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

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has been studied by many authors see [4,6–8,16–18] and the references therein.

In this paper, we study the existence of multiple weak solutions to the following problem

$$\begin{cases} \Delta^2 u = f(x,u) + g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $(N > 4)$ is a smooth bounded domain. To study the problem (1.2), we make the following assumptions:

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that

(A1) $f(x,\xi) = f_1(x,\xi) + f_2(x,\xi)$, $f_1, f_2 \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist constants $C_1 > 0$ and $1 < p < 2$ such that

$$|f_1(x,\xi)| \le C_1 |\xi|^{p-1}, \quad (x,\xi) \in \overline{\Omega} \times \mathbb{R}; \tag{1.3}$$

(A2) there exist constants $C_2 > 0$ and $1 < \mu < 2$ such that

$$f_1(x,\xi)\xi - \mu F_1(x,\xi) \le 0, \quad (x,\xi) \in \Omega \times \mathbb{R},$$

where $F_1(x,\xi) := \int_0^\xi f_1(x,\tau)\mathrm{d}\tau$;

(A3) there exist constants $C_2 > 0$, $1 < p_1 < 2$ and $2 < p_2 < 2_*$ such that

$$F_1(x,\xi) \ge C_2(|\xi|^{p_1} - |\xi|^{p_2}), \quad (x,\xi) \in \Omega_0 \times \mathbb{R},$$

where $2_* := \frac{2N}{N-4}$, $\Omega_0$ is a nonempty open and $\Omega_0 \subset \Omega$;

(A4) there exist constants $C_3 > 0$ and $2 < p_3 < 2_*$ such that

$$|f_2(x,\xi)| \le C_3 |\xi|^{p_3-1}, \quad (x,\xi) \in \overline{\Omega} \times \mathbb{R};$$

(A5) $f_i(x,\xi) = -f_i(x,-\xi)$, $i = 1, 2$, $(x,\xi) \in \Omega \times \mathbb{R}$.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that

(B) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist constants $C_4 > 0$ and $2 < \theta < 2_*$ such that

$$|g(x,\xi)| \le C_4 |\xi|^{\theta-1}, \quad (x,\xi) \in \overline{\Omega} \times \mathbb{R}.$$

Now, we formulate the main result of this paper.

**Theorem 1.1** *Assume that* (A1)–(A5), (B) *are satisfied and*

$$\frac{2p}{2-p} > \frac{N}{\theta - 2}. \tag{1.4}$$

*Then the problem* (1.2) *has a sequence of small negative energy solutions converging to zero.*

***Example 1.2*** Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^5$ and

$$f(x,\xi)=a(x)\,|\xi|^{-\frac{1}{2}}\,\xi\cos|\xi|^{\frac{3}{2}},\quad g(x,\xi)=\xi^{\theta-1},$$

where $a(x)\in C(\overline{\Omega},\mathbb{R})$ changes sign in $\Omega$, $\theta\in(\frac{7}{6},10)$. Set

$$f_1(x,\xi)=a(x)\,|\xi|^{-\frac{1}{2}}\,\xi,\quad f_2(x,\xi)=a(x)\,|\xi|^{-\frac{1}{2}}\,\xi\left(\cos|\xi|^{\frac{3}{2}}-1\right).$$

Thus, all conditions of Theorem 1.1 are satisfied with

$$2_*=10;\quad p=\mu=p_1=\frac{3}{2};\quad p_2=p_3=3.$$

By Theorem 1.1, the problem (1.2) has a sequence of small negative energy solutions converging to zero.

## 2 Proof of Theorem 1.1

Define the Euler–Lagrange functional associated with the problem (1.2) (see, e.g., [13,14]) as follows

$$I(u)=\frac{1}{2}\int_\Omega|\Delta u|^2\,dx-\int_\Omega F(x,u)\,dx-\int_\Omega G(x,u)\,dx,$$

where $F(x,u):=\int_0^u f(x,\xi)d\xi$ and $G(x,u):=\int_0^u g(x,\xi)d\xi$.

From (A1), (A4) and (B), we have $I$ is well defined on $H_0^2(\Omega)$ and $I\in C^1(H_0^2(\Omega),\mathbb{R})$ with

$$\langle I'(u),v\rangle=\int_\Omega\Delta u\Delta v dx-\int_\Omega f(x,u)\,v dx-\int_\Omega g(x,u)\,v dx \tag{2.1}$$

for all $v\in H_0^2(\Omega)$. One can also check that the critical points of $I$ are weak solutions of the problem (1.2).

Next, we introduce a cut-off function $\zeta\in C^\infty(\mathbb{R},\mathbb{R})$ satisfying

$$\begin{cases}\zeta(\xi)=1, & \xi\in(-\infty,1],\\ 0\le\zeta(\xi)\le 1, & \xi\in(1,2),\\ \zeta(\xi)=0, & \xi\in[2,+\infty),\\ \left|\zeta'(\xi)\right|\le 2 & \xi\in\mathbb{R}.\end{cases} \tag{2.2}$$

With the help of this cut-off function $\zeta$, define

$$\pi(u) := \zeta\left(\frac{\|u\|^2_{H_0^2(\Omega)}}{T_0}\right), \quad \forall u \in H_0^2(\Omega), \tag{2.3}$$

where $T_0$ is a small positive constant independent of $u$ given by (2.10) and (2.49).

**Lemma 2.1** *The functional $\pi$ defined by (2.3) is of class $C^1(H_0^2(\Omega), \mathbb{R})$ and*

$$\left|\langle \pi'(u), u\rangle\right| \le 8, \quad \forall u \in H_0^2(\Omega).$$

***Proof*** By direct computation, we get

$$\langle \pi'(u), v\rangle = 2\zeta'\left(\frac{\|u\|^2_{H_0^2(\Omega)}}{T_0}\right)\frac{(u, v)_{H_0^2(\Omega)}}{T_0}, \quad \forall u, v \in H_0^2(\Omega). \tag{2.4}$$

Assume that $u_n \to u_0$ in $H_0^2(\Omega)$. By the definition of $\zeta$ and (2.4), for any $v \in H_0^2(\Omega)$, we have that

$$\begin{aligned}
&\left|\langle \pi'(u_n) - \pi'(u_0), v\rangle\right| \\
&= 2\left|\zeta'\left(\frac{\|u_n\|^2_{H_0^2(\Omega)}}{T_0}\right)\frac{(u_n, v)_{H_0^2(\Omega)}}{T_0} - \zeta'\left(\frac{\|u_0\|^2_{H_0^2(\Omega)}}{T_0}\right)\frac{(u_0, v)_{H_0^2(\Omega)}}{T_0}\right| \\
&\le 2T_0^{-1}\|v\|_{H_0^2(\Omega)}\left[\left|\zeta'\left(\frac{\|u_n\|^2_{H_0^2(\Omega)}}{T_0}\right)\right|\|u_n - u_0\|_{H_0^2(\Omega)}\right. \\
&\quad \left.+\left|\zeta'\left(\frac{\|u_n\|^2_{H_0^2(\Omega)}}{T_0}\right) - \zeta'\left(\frac{\|u_0\|^2_{H_0^2(\Omega)}}{T_0}\right)\right|\|u_0\|_{H_0^2(\Omega)}\right],
\end{aligned}$$

which implies that $\left\|\pi'(u_n) - \pi'(u_0)\right\|_{(H_0^2(\Omega))^*} \to 0, n \to \infty$. So $\pi \in C^1(H_0^2(\Omega), \mathbb{R})$. By (2.2) and (2.4), we get

$$\left|\langle \pi'(u), u\rangle\right| \le 8, \quad \forall u \in H_0^2(\Omega).$$

□

With the help of this functional $\pi$, we define a new functional $\overline{I}$ on $H_0^2(\Omega)$ by

$$\overline{I}(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,\mathrm{d}x - \int_\Omega F_1(x, u)\,\mathrm{d}x - \pi(u)$$

$$\times\left(\int_{\Omega} F_2(x,u)\,\mathrm{d}x + \int_{\Omega} G(x,u)\,\mathrm{d}x\right), \tag{2.5}$$

where $F_2(x,u) = \int_0^u f_2(x,\tau)\mathrm{d}\tau$. From Lemma 2.1, hence $\overline{I} \in C^1(H_0^2(\Omega), \mathbb{R})$ and for any $u, v \in H_0^2(\Omega)$, we have

$$\begin{aligned}\langle \overline{I}'(u), v\rangle = &\int_{\Omega} \Delta u \Delta v \mathrm{d}x - \pi(u)\left(\int_{\Omega} f_2(x,u)\,v\mathrm{d}x + \int_{\Omega} g(x,u)\,v\mathrm{d}x\right)\\ &- \int_{\Omega} f_1(x,u)\,v\mathrm{d}x - \langle \pi'(u), v\rangle\left(\int_{\Omega} F_2(x,u)\,\mathrm{d}x + \int_{\Omega} G(x,u)\,\mathrm{d}x\right).\end{aligned} \tag{2.6}$$

**Lemma 2.2** *Assume that* (A1), (A2), (A4), (B) *are satisfied and u is a critical point of* $\overline{I}$. *Then*

$$\overline{I}(u) \le \frac{\mu-2}{4\mu}\|u\|^2_{H_0^2(\Omega)}. \tag{2.7}$$

***Proof*** Consider two cases.
*Case 1.* Let $u$ is a critical point of $\overline{I}$ and $\|u\|^2_{H_0^2(\Omega)} > 2T_0$, by the definition of $\zeta$, we have $\pi(u) = 0$ and $\pi'(u) = 0$. From (A1) and (2.6), we get that

$$\begin{aligned}\overline{I}(u) &= \overline{I}(u) - \mu^{-1}\langle \overline{I}'(u), u\rangle\\ &= \frac{\mu-2}{2\mu}\|u\|^2_{H_0^2(\Omega)} + \mu^{-1}\int_{\Omega}(f_1(x,u)u - \mu F_1(x,u))\,\mathrm{d}x\\ &\le \frac{\mu-2}{4\mu}\|u\|^2_{H_0^2(\Omega)}, \quad \text{by} \quad 1<\mu<2 \quad \text{and} \quad f_1(x,u)u - \mu F_1(x,u) \le 0.\end{aligned} \tag{2.8}$$

*Case 2.* Let $u$ is a critical point of $\overline{I}$ and $\|u\|^2_{H_0^2(\Omega)} \le 2T_0$. By applying embedding inequalities, Lemma 2.1, (A2), (A4) and (B), we get that

$$\begin{aligned}\overline{I}(u) &= \overline{I}(u) - \mu^{-1}\langle \overline{I}'(u), u\rangle\\ &\le \frac{\mu-2}{2\mu}\|u\|^2_{H_0^2(\Omega)} + \frac{\mu+9}{\mu}C_3\|u\|^{p_3}_{L^{p_3}(\Omega)} + \frac{\mu+9}{\mu}C_4\|u\|^{\theta}_{L^{\theta}(\Omega)}\\ &\le \frac{\mu-2}{2\mu}\|u\|^2_{H_0^2(\Omega)} + \frac{\mu+9}{\mu}C_3C_{p_3}^{p_3}\|u\|^{p_3}_{H_0^2(\Omega)}\\ &\quad + \frac{\mu+9}{\mu}C_4C_{\theta}^{\theta}\|u\|^{\theta}_{H_0^2(\Omega)},\end{aligned} \tag{2.9}$$

where $C_{p_3}$, $C_\theta$ are constants such that

$$\|u\|_{L^{p_3}(\Omega)} \leq C_{p_3} \|u\|_{H_0^2(\Omega)}, \|u\|_{L^\theta(\Omega)} \leq C_\theta \|u\|_{H_0^2(\Omega)}.$$

Since $p_3 > 2$, $p_4 > 2$, we can choose $T_0$ small enough such that if $\|u\|^2_{H_0^2(\Omega)} \leq 2T_0$

$$\frac{\mu+9}{\mu} C_3 C_{p_3}^{p_3} \|u\|^{p_3}_{H_0^2(\Omega)} + \frac{\mu+89}{\mu} C_4 C_\theta^\theta \|u\|^\theta_{H_0^2(\Omega)} \leq \frac{2-\mu}{4\mu} \|u\|^2_{H_0^2(\Omega)}, \tag{2.10}$$

By (2.8) and (2.10), we get the conclusion of the lemma. □

Next, we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \chi(\xi) = 1, & \xi \in (-\infty, \frac{A}{2}], \\ 0 \leq \chi(\xi) \leq 1, & \xi \in (\frac{A}{2}, \frac{A}{4}), \\ \chi(\xi) = 0, & \xi \in [\frac{A}{4}, +\infty), \\ \left|\chi'(\xi)\right| \leq -8A^{-1}, & \xi \in \mathbb{R}, \quad A := \frac{\mu-2}{4\mu}. \end{cases}$$

We put

$$\ell(u) := \chi\left(\|u\|^{-2}_{H_0^2(\Omega)} \overline{I}(u)\right), \quad \forall u \in H_0^2(\Omega)\backslash\{0\}, \tag{2.11}$$

$$\psi(u) := \begin{cases} \pi(u)\ell(u) \int_\Omega G(x,u)\mathrm{d}x, & \forall u \in H_0^2(\Omega)\backslash\{0\}, \\ 0, & u = 0, \end{cases} \tag{2.12}$$

and

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 \,\mathrm{d}x - \int_\Omega F_1(x,u)\,\mathrm{d}x - \pi(u)\int_\Omega F_2(x,u)\,\mathrm{d}x - \psi(u), \forall u \in H_0^2(\Omega). \tag{2.13}$$

From (A1), (A4) and (B), it is easy to verify that $\ell(u) \in C^1(H_0^2(\Omega)\backslash\{0\}, \mathbb{R})$. By direct computation, for $u \in H_0^2(\Omega)\backslash\{0\}$ and for any $v \in H_0^2(\Omega)$, we have

$$\langle \ell'(u), v\rangle = \chi'\left(\|u\|^{-2}_{H_0^2(\Omega)} \overline{I}(u)\right) \|u\|^{-4}_{H_0^2(\Omega)} \times \left(\|u\|^2_{H_0^2(\Omega)} \langle \overline{I}'(u), v\rangle - 2\overline{I}(u)(u,v)_{H_0^2(\Omega)}\right). \tag{2.14}$$

**Lemma 2.3** *Assume that* (A1), (A2), (A4), (B) *are satisfied. Then the functional* $\psi$ *defined by* (2.12) *is of class* $C^1(H_0^2(\Omega), \mathbb{R})$ *and*

$$\left|\langle \psi'(u), u\rangle\right| \leq 89 C_4 C_\theta^\theta \|u\|^\theta_{H_0^2(\Omega)}, \forall u \in H_0^2(\Omega). \tag{2.15}$$

***Proof*** For $u = 0$ for any $v \in H_0^2(\Omega)$, by (2.3), (2.12) and (B), we get

$$\left|\langle \psi'(0), v\rangle\right| = \lim_{t\to 0}\left|\frac{\psi(tv)-\psi(0)}{t}\right| \le C_4 \lim_{t\to 0}|t|^{\theta-1}\int_\Omega |v(x)|^\theta \,\mathrm{d}x = 0;$$

hence, $\psi'(0) = 0$. From (2.4), (2.12), (2.14) and (B) for $u \in H_0^2(\Omega)\backslash\{0\}$ and $v \in H_0^2(\Omega)$, we have that

$$\begin{aligned}\langle \psi'(u), v\rangle = \langle \pi'(u), v\rangle \ell(u)\int_\Omega G(x,u)\mathrm{d}x + \pi(u)\langle \ell'(u), v\rangle \int_\Omega G(x,u)\mathrm{d}x \\ + \pi(u)\ell(u)\int_\Omega g(x,u)v\mathrm{d}x. \end{aligned} \tag{2.16}$$

Next, we prove $\psi \in C^1(H_0^2(\Omega), \mathbb{R})$. Suppose that $u_n \to u_0$ in $H_0^2(\Omega)$. We consider two possible cases.
*Case 1.* $u_0 \neq 0$. By Lemma 2.1, (2.14) and (B), we have

$$\psi'(u_n) \to \psi'(u_0) \quad \text{as} \quad n \to \infty.$$

Case 2. $u_0 = 0$. Without loss of generality, we can assume $\|u_n\|^2_{H_0^2(\Omega)} < T_0$. Hence, by (2.2), (2.3) we get $\pi(u_n) = 1$ and $\pi'(u_n) = 0$; hence,

$$\langle \psi'(u_n), v\rangle = \langle \ell'(u_n), v\rangle \int_\Omega G(x,u_n)\mathrm{d}x + \ell(u_n)\int_\Omega g(x,u_n)v\mathrm{d}x. \tag{2.17}$$

On the other hand, by (2.14), we obtain

$$\begin{aligned}\langle \ell'(u_n), v\rangle \int_\Omega G(x,u_n)\mathrm{d}x = \chi'\left(\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-2}_{H_0^2(\Omega)}\langle \overline{I}'(u_n), v\rangle \int_\Omega G(x,u_n)\mathrm{d}x \\ - 2\chi'\left(\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-4}_{H_0^2(\Omega)}\overline{I}(u_n)(u_n, v)_{H_0^2(\Omega)} \\ \int_\Omega G(x,u_n)\mathrm{d}x.\end{aligned}$$

From the definition of $\chi$ and (B), applying embedding inequalities, we get that

$$\begin{aligned}\left|\chi'\left(\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-2}_{H_0^2(\Omega)}\langle \overline{I}'(u_n), v\rangle \int_\Omega G(x,u_n)\mathrm{d}x\right| \\ \le C_5\left\|\overline{I}'(u_n)\right\|_{(H_0^2(\Omega))^*}\|v\|_{H_0^2(\Omega)}\|u_n\|^{\theta-2}_{H_0^2(\Omega)},\end{aligned} \tag{2.18}$$

$$\left|2\chi'\left(\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-4}_{H_0^2(\Omega)}\overline{I}(u_n)(u_n,v)_{H_0^2(\Omega)}\int_\Omega G(x,u_n)\mathrm{d}x\right|$$
$$\leq C_6\|u_n\|^{\theta-1}_{H_0^2(\Omega)}\|v\|_{H_0^2(\Omega)}, \tag{2.19}$$

$$\left|\ell(u_n)\int_\Omega g(x,u_n)v\mathrm{d}x\right|\leq C_7\|u_n\|^{\theta-1}_{H_0^2(\Omega)}\|v\|_{H_0^2(\Omega)}, \tag{2.20}$$

where $C_j > 0$, $j = 5, 6, 7$, and $(H_0^2(\Omega))^*$ denotes the dual space of $H_0^2(\Omega)$. Since $\pi(u_n) = 1$, $\pi'(u_n) = 0$ and $u_n \to 0$, $n \to \infty$, we have that

$$\left\|\overline{I}'(u_n)\right\|_{(H_0^2(\Omega))^*}\to 0,\quad \text{as}\quad n\to\infty. \tag{2.21}$$

From (2.17)–(2.21), we see that

$$\|\psi'(u_n)-\psi'(0)\|_{(H_0^2(\Omega))^*}$$
$$=\sup_{\|v\|_{H_0^2(\Omega)}\leq 1}\left|\langle\ell'(u_n),v\rangle\int_\Omega G(x,u_n)\mathrm{d}x+\ell(u_n)\int_\Omega g(x,u_n)v\mathrm{d}x\right|\to 0,\quad \text{as}\quad n\to\infty;$$

hence, the continuity of $\psi'$ follows. So we have $\psi \in C^1(H_0^2(\Omega), \mathbb{R})$.

Next, we prove (2.15).

If $\|u\|^2_{H_0^2(\Omega)} > 2T_0$ or $u = 0$ then by (2.2), Lemma 2.1 and (2.16), we have $\langle\psi'(u), u\rangle = 0$. Otherwise, $\|u\|^2_{H_0^2(\Omega)} \leq 2T_0$ and $u \neq 0$. Arguing similarly as in (2.9), we obtain

$$\left|\overline{I}(u)-\mu^{-1}\langle\overline{I}'(u),u\rangle\right|$$
$$\leq 2|A|\|u\|^2_{H_0^2(\Omega)}+\frac{\mu+9}{\mu}C_3C_{p_3}^{p_3}\|u\|^{p_3}_{H_0^2(\Omega)}+\frac{\mu+9}{\mu}C_4C_\theta^\theta\|u\|^\theta_{H_0^2(\Omega)}. \tag{2.22}$$

From (2.10) and (2.22), we have that

$$\left|\langle\overline{I}'(u),u\rangle\right|\leq\mu\left(3|A|\|u\|^2_{H_0^2(\Omega)}+\left|\overline{I}(u)\right|\right). \tag{2.23}$$

By the definition of $\chi$, we have that if $\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u) \notin [\frac{A}{2}, \frac{A}{4}]$ then $\ell'(u) = 0$. Otherwise, if

$$\frac{A}{2}\leq\|u\|^{-2}_{H_0^2(\Omega)}\overline{I}(u)\leq\frac{A}{4}$$

then

$$\left|\overline{I}(u)\right|\leq|A|\|u\|^2_{H_0^2(\Omega)}. \tag{2.24}$$

In combination with (2.14), (2.23) and (2.24), we get

$$\left|\pi(u)\langle\ell'(u),u\rangle\int_{\Omega}G(x,u_n)\mathrm{d}x\right|\le 80C_4C_\theta^\theta\|u\|_{H_0^2(\Omega)}^\theta. \tag{2.25}$$

By Lemma 2.1 and (2.2), (2.11), we conclude that

$$\left|\langle\pi'(u),u\rangle\ell(u)\int_{\Omega}G(x,u)\mathrm{d}x\right.$$
$$\left.+\pi(u)\ell(u)\int_{\Omega}g(x,u)u\mathrm{d}x\right|\le 9C_4C_\theta^\theta\|u\|_{H_0^2(\Omega)}^\theta. \tag{2.26}$$

Combining with (2.16), (2.25) and (2.26), we get (2.15). The proof of Lemma 2.3 is complete. □

**Lemma 2.4** *Assume that* (A1)*,* (A2)*,* (A4)*,* (A5)*,* (B) *are satisfied. Then*

(K1) *The functional $J$ defined by* (2.13) *is of class $C^1(H_0^2(\Omega),\mathbb{R})$ and there exists a constants $C_8$ independent of $u$ such that*

$$|J(u)-J(-u)|\le C_8|J(u)|^{\frac{\theta}{2}},\quad \forall u\in H_0^2(\Omega). \tag{2.27}$$

(K2) *$J$ has no critical point with critical value on $H_0^2(\Omega)$ and $K_0=\{0\}$, where $K_0:=\{u\in H_0^2(\Omega):J(u)=0,J'(u)=0\}$.*

***Proof*** By Lemmas 2.1, 2.3, (A1) and (A4), we have $J\in C^1(H_0^2(\Omega),\mathbb{R})$ and

$$\langle J'(u),v\rangle=\int_{\Omega}\Delta u\Delta v\mathrm{d}x-\int_{\Omega}f_1(x,u)\,v\mathrm{d}x-\pi(u)\int_{\Omega}f_2(x,u)\,v\mathrm{d}x$$
$$-\langle\pi'(u),v\rangle\int_{\Omega}F_2(x,u)\,\mathrm{d}x-\langle\psi'(u),v\rangle,\quad \forall u,v\in H_0^2(\Omega). \tag{2.28}$$

Next, we prove (2.27). We consider two possible cases.

*Case 1*. If $\|u\|_{H_0^2(\Omega)}^2>2T_0$ or $\|u\|_{H_0^2(\Omega)}^{-2}\overline{I}(u)>\frac{A}{4}$, by the definition of $\zeta$ and $\chi$ we have $\psi(u)=0$. Then (2.27) holds by (A5) and (2.13).

*Case 2*. If $\|u\|_{H_0^2(\Omega)}^2\le 2T_0$ and $\|u\|_{H_0^2(\Omega)}^{-2}\overline{I}(u)\le\frac{A}{4}$,

$$\left|\overline{I}(u)\right|\ge\frac{|A|}{4}\|u\|_{H_0^2(\Omega)}^2. \tag{2.29}$$

From (B), (2.3), (2.10), (2.11) and (2.29), we get that

$$|J(u)| \geq \left|\overline{I}(u)\right| - 2\left|\int_{\Omega} G(x,u)\mathrm{d}x\right| \geq \frac{|A|}{4}\|u\|^2_{H_0^2(\Omega)} - 2C_4C_\theta^\theta\|u\|^\theta_{H_0^2(\Omega)}$$
$$\geq C_9\|u\|^2_{H_0^2(\Omega)};$$

hence,

$$|J(u)|^{\frac{\theta}{2}} \geq C_{10}\|u\|^\theta_{H_0^2(\Omega)}. \tag{2.30}$$

In view of (A5), (B), (2.3) and (2.11), we obtain

$$|J(u) - J(-u)| \leq 2\left|\int_{\Omega} G(x,u)\mathrm{d}x\right| \leq 2C_4C_\theta^\theta\|u\|^\theta_{H_0^2(\Omega)}. \tag{2.31}$$

It follows from (2.30) and (2.31) that (2.27) holds.

Next, we prove (K2) by contradiction. If $u_0$ is a critical point of $J$ with $J(u_0) > 0$, by (A1), (A4) and (B) we get $u_0 \neq 0$. We consider two possible cases.
*Case 1*. If $\|u_0\|^2_{H_0^2(\Omega)} > 2T_0$ then

$$\pi(u_0) = \pi'(u_0) = \psi'(u_0) = 0.$$

By (A2), (2.13) and (2.28), we have that

$$0 \leq J(u_0) = J(u_0) - \frac{1}{\mu}\langle J'(u_0), u_0\rangle$$
$$= \frac{\mu-2}{2\mu}\int_{\Omega}|\Delta u_0|^2\,\mathrm{d}x + \frac{1}{\mu}\int_{\Omega}\left(f_1(x,u_0)u_0 - \mu F_1(x,u_0)\right)\mathrm{d}x < 0,$$

which yields a contradiction.
*Case 2*. If $\|u_0\|^2_{H_0^2(\Omega)} \leq 2T_0$ then by Lemmas 2.1, 2.3, (2.10), (2.13) and (2.28), we obtain

$$0 \leq J(u_0) = J(u_0) - \frac{1}{\mu}\langle J'(u_0), u_0\rangle \leq \frac{\mu-2}{4\mu}\|u_0\|^2_{H_0^2(\Omega)} < 0,$$

which yields a contradiction. Moreover, by a similar proof and direct computation we obtain $K_0 = \{0\}$. □

**Lemma 2.5** *Assume that* (A1), (A4), (B) *are satisfied. Then the functional* $J$ *satisfies the Palais–Smale condition.*

***Proof*** Without loss of generality, assume $\|u\|^2_{H_0^2(\Omega)} > 2T_0$. Then, by the definition of $\pi$ and (A1) we obtain

$$J(u) \geq \frac{1}{2}\|u\|^2_{H_0^2(\Omega)} - C_{11}\|u\|^p_{H_0^2(\Omega)}. \tag{2.32}$$

Since $1 < p < 2$, (2.31) implies that

$$J(u) \to +\infty \text{ as } \|u\|_{H_0^2(\Omega)} \to +\infty. \tag{2.33}$$

Next, we show that $J(u)$ satisfies the Palais–Smale condition. Assume that $\{u_n\}_{n=1}^{\infty} \subset H_0^2(\Omega)$ is a Palais–Smale sequence, i.e., $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded and

$$J'(u_n) \to 0 \text{ as } n \to +\infty.$$

Since (2.33), $\{u_n\}_{n=1}^{\infty}$ is bounded in $H_0^2(\Omega)$. Therefore, we can (by passing to a subsequence, we can always suppose $u_n \neq 0$ for all $n$, otherwise, the thesis is obvious) suppose that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ weakly in } H_0^2(\Omega) \text{ as } n \to \infty\\ &u_n \to u_0 \text{ a.e., in } \Omega \text{ as } n \to \infty\\ &u_n \to u_0 \text{ strongly in } L^q(\Omega),\ 1 \leq q < 2_* \text{ as } n \to \infty. \end{aligned} \tag{2.34}$$

Since (2.34), by (A1), (A4), (B) and standard arguments we get

$$\int_\Omega f_i(x, u_n)(u_n - u_0)\mathrm{d}x \to 0 \text{ as } n \to \infty, i = 1, 2, \tag{2.35}$$

$$\int_\Omega g(x, u_n)(u_n - u_0)\mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{2.36}$$

From (2.34), we have

$$\lim_{n\to\infty} \left|\langle J'(u_n), u_n - u_0\rangle\right| = 0. \tag{2.37}$$

Next, we distinguish two cases.
*Case 1*. If $\|u_n\|^2_{H_0^2(\Omega)} > 2T_0$, from (2.2), (2.3), (2.10), (2.12), we get that

$$\pi(u_n) = 0, \pi'(u_n) = 0, \psi'(u_n) = 0.$$

By (2.28), we obtain

$$\langle J'(u_n) - J'(u_0), u_n - u_0\rangle = \|u_n - u_0\|^2_{H_0^2(\Omega)} - \int_\Omega (f_1(x, u_n) - f_1(x, u_0))(u_n - u_0)\mathrm{d}x. \tag{2.38}$$

*Case 2.* If $\|u_n\|^2_{H_0^2(\Omega)} \leq 2T_0$, from (2.28), we have

$$\begin{aligned}\langle J'(u_n), u_n - u_0\rangle &= \|u_n - u_0\|^2_{H_0^2(\Omega)} + \int_\Omega \Delta u_0 \Delta(u_n - u_0)\mathrm{d}x \\ &- \int_\Omega f_1(x, u_n)(u_n - u_0)\mathrm{d}x \\ &- \pi(u_n)\int_\Omega f_2(x, u_n)(u_n - u_0)\mathrm{d}x - \langle \pi'(u_n), u_n - u_0\rangle \int_\Omega F_2(x, u_n)\,\mathrm{d}x \\ &- \langle \psi'(u_n), u_n - u_0\rangle. \end{aligned} \tag{2.39}$$

By (A4), (2.2) and (2.4), we get that

$$\begin{aligned}&\left|\langle \pi'(u_n), u_n - u_0\rangle \int_\Omega F_2(x, u_n)\,\mathrm{d}x\right| \\ &\leq C_3 C_{p_3}^{p_3} \|u_n\|^{p_3}_{H_0^2(\Omega)} \left| 2\zeta'\left(\frac{\|u_n\|^2_{H_0^2(\Omega)}}{T_0}\right) \frac{(u_n, u_n - u_0)_{H_0^2(\Omega)}}{T_0}\right| \\ &\leq 2^{\frac{p_3+4}{2}} C_3 C_{p_3}^{p_3} T_0^{\frac{p_3-2}{2}} \left(\|u_n - u_0\|^2_{H_0^2(\Omega)} + (u_0, u_n - u_0)_{H_0^2(\Omega)}\right). \end{aligned} \tag{2.40}$$

On the other hand, from (2.16), we obtain

$$\begin{aligned}\langle \psi'(u_n), u_n - u_0\rangle &= \langle \pi'(u_n), u_n - u_0\rangle \ell(u_n) \int_\Omega G(x, u_n)\mathrm{d}x \\ &+ \pi(u_n)\langle \ell'(u_n), u_n - u_0\rangle \int_\Omega G(x, u_n)\mathrm{d}x \\ &+ \pi(u_n)\ell(u_n)\int_\Omega g(x, u_n)(u_n - u_0)\mathrm{d}x. \end{aligned} \tag{2.41}$$

Moreover, by (B), (2.4) and (2.11), we obtain

$$\left|\langle\pi'(u_n), u_n - u_0\rangle \ell(u_n) \int\limits_{\Omega} G(x, u_n)\mathrm{d}x\right| \leq 2^{\frac{\theta+4}{2}} C_4 C_\theta^\theta T_0^{\frac{\theta-2}{2}} \left(\|u_n - u_0\|^2_{H_0^2(\Omega)} + (u_0, u_n - u_0)_{H_0^2(\Omega)}\right). \tag{2.42}$$

From the defined of $\chi$, (B), (2.4) and (2.14), we have that

$$\pi(u_n)\langle\ell'(u_n), u_n - u_0\rangle \int\limits_{\Omega} G(x, u_n)\mathrm{d}x = \pi(u_n)\chi'\left(\|u_n\|^{-2}_{H_0^2(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-4}_{H_0^2(\Omega)} \left(\|u_n\|^2_{H_0^2(\Omega)}\langle\overline{I}'(u_n), u_n - u_0\rangle - 2\overline{I}(u_n)(u_n, u_n - u_0)_{H_0^2(\Omega)}\right) \times \int\limits_{\Omega} G(x, u_n)\mathrm{d}x. \tag{2.43}$$

From (2.6), we have

$$\langle\overline{I}'(u_n), u_n - u_0\rangle = \int\limits_{\Omega} \Delta u_n \Delta(u_n - u_0)\mathrm{d}x - \int\limits_{\Omega} f_1(x, u_n)(u_n - u_0)\mathrm{d}x - \pi(u_n)\left(\int\limits_{\Omega} f_2(x, u_n)(u_n - u_0)\mathrm{d}x + \int\limits_{\Omega} g(x, u_n)(u_n - u_0)\mathrm{d}x\right) - \langle\pi'(u_n), u_n - u_0\rangle\left(\int\limits_{\Omega} F_2(x, u_n)\,\mathrm{d}x + \int\limits_{\Omega} G(x, u_n)\,\mathrm{d}x\right). \tag{2.44}$$

By (A4), (B), (2.40), (2.42) and (2.44), we have

$$\left|\langle\overline{I}'(u_n), u_n - u_0\rangle\right| \leq \|u_n - u_0\|^2_{H_0^2(\Omega)} + \left|\langle\pi'(u_n), u_n - u_0\rangle \int\limits_{\Omega} F_2(x, u_n)\,\mathrm{d}x\right| + \left|\langle\pi'(u_n), u_n - u_0\rangle \int\limits_{\Omega} G(x, u_n)\,\mathrm{d}x\right| + o_n(1) \leq (1 + C_{12})\|u_n - u_0\|^2_{H_0^2(\Omega)} + o_n(1), \tag{2.45}$$

where $o_n(1) \to 0$ as $n \to \infty$ and

$$C_{12} = 2^{\frac{p_3+4}{2}} C_3 C_{p_3}^{p_3} T_0^{\frac{p_3-2}{2}} + 2^{\frac{\theta+4}{2}} C_4 C_\theta^\theta T_0^{\frac{\theta-2}{2}}.$$

Hence,

$$\left|\pi(u_n)\chi'\left(\|u_n\|^{-2}_{H^2_0(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-2}_{H^2_0(\Omega)}\langle\overline{I}'_{T_0}(u_n),u_n-u_0\rangle\int_\Omega G(x,u_n)\,\mathrm{d}x\right|$$
$$\le 2^{\frac{\theta+4}{2}}|A|^{-1}C_4C_\theta^\theta T_0^{\frac{\theta-2}{2}}(1+C_{12})\|u_n-u_0\|^2_{H^2_0(\Omega)}+o_n(1). \tag{2.46}$$

$$\left|\pi(u_n)\chi'\left(\|u_n\|^{-2}_{H^2_0(\Omega)}\overline{I}(u_n)\right)\|u_n\|^{-4}_{H^2_0(\Omega)}2\overline{I}(u_n)(u_n,u_n-u_0)_{H^2_0(\Omega)}\int_\Omega G(x,u_n)\,\mathrm{d}x\right|$$
$$\le 2^{\frac{\theta+4}{2}}C_4C_\theta^\theta T_0^{\frac{\theta-2}{2}}\|u_n-u_0\|^2_{H^2_0(\Omega)}+o_n(1). \tag{2.47}$$

By (2.41), (2.42), (2.46) and (2.47), we have

$$\left|\langle\psi'(u_n),u_n-u_0\rangle\right|\le C_{13}\|u_n-u_0\|^2_{H^2_0(\Omega)}+o_n(1), \tag{2.48}$$

where

$$C_{13}=2^{\frac{\theta+4}{2}}C_4C_\theta^\theta T_0^{\frac{\theta-2}{2}}+2^{\frac{\theta+4}{2}}|A|^{-1}C_4C_\theta^\theta T_0^{\frac{\theta-2}{2}}(1+C_{12}).$$

Since $p_3>2$ and $\theta>2$, we can choose $T_0$ small enough such that

$$2^{\frac{p_3+4}{2}}C_3C_{p_3}^{p_3}T_0^{\frac{p_3-2}{2}}+C_{13}\le\frac{1}{2}. \tag{2.49}$$

By (2.39), (2.40), (2.48) and (2.49), we obtain

$$\left|\langle J'(u_n),u_n-u_0\rangle\right|\ge\left(1-2^{\frac{p_3+4}{2}}C_3C_{p_3}^{p_3}T_0^{\frac{p_3-2}{2}}-C_{13}\right)\|u_n-u_0\|^2_{H^2_0(\Omega)}+o_n(1)$$
$$\ge\frac{1}{2}\|u_n-u_0\|^2_{H^2_0(\Omega)}+o_n(1). \tag{2.50}$$

It follows from (2.37) and (2.50) that $u_n\to u_0$ as $n\to\infty$. The proof of Lemma 2.5 is complete. □

Now, we can show that $J$ has a sequence of critical values. For the problem

$$\begin{cases}\Delta^2u=\lambda u & \text{in}\quad\Omega,\\ u=\Delta u=0 & \text{on}\quad\partial\Omega,\end{cases} \tag{2.51}$$

we can show that the Dirichlet eigenvalue the problem (2.51) has a sequence of discrete eigenvalues $\{\lambda_k\}_{k=1}^\infty$ which satisfy

$$0<\lambda_1<\lambda_2\le\lambda_3\le\cdots\le\lambda_k\le\cdots\to\infty \text{ as } k\to\infty,$$

and $e_1, e_2, \ldots$ denote the corresponding eigenfunctions normalized such that $\|e_j\|_{H_0^2(\Omega)} = 1$, for all $j = 1, 2, \ldots$. For any $k > 0$, we put $\mathbb{V}_k = \text{span}\{e_j;\ j \le k\}$ in $H_0^2(\Omega)$, $\mathbb{V}_k^{\perp}$ be the orthogonal complement of $\mathbb{V}_k$ in $H_0^2(\Omega)$.

**Lemma 2.6** *There exists a normalized orthogonal sequence $\{\varphi_k\}_{k=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that supp $\varphi_k \subset \Omega_0, k \in \mathbb{N}$, where $\Omega_0$ is the nonempty open set given in* (A3).

***Proof*** By (A3), there exist $x_0 \in \Omega_0$ and $\delta_0 > 0$ such that $B(x_0, \delta_0) := \{x \in \mathbb{R}^N : |x - x_0| < \delta_0\} \subset \Omega_0$. Choose a strictly increasing sequence $\{\rho_k\}_{k=1}^{\infty}$ such that

$$0 < \rho_1 < \rho_2 < \cdots < \rho_k < \cdots \to \frac{\delta_0}{4}.$$

Define

$$O_k := B(x_0, \rho_{k+1}) \backslash \overline{B}(x_0, \rho_k), \quad k \in \mathbb{N}.$$

Let $x_k \in O_k$ and choose $r_k > 0$ such that

$$B(x_0, r_k) \subset O_k, \quad k \in \mathbb{N}. \tag{2.52}$$

Set

$$\varphi_0(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \ge 1. \end{cases} \tag{2.53}$$

By (2.53), define $\varphi_k$ as follows

$$\varphi_k(x) := \varphi_0\left((x - x_k)/r_k\right), \quad k \in \mathbb{N}. \tag{2.54}$$

By (2.53) and (2.54), we get

$$\varphi_k \in C_0^{\infty}(\Omega), \quad k \in \mathbb{N}.$$

Moreover, from (2.52)-(2.54), we have

$$\text{supp}\, \varphi_k \subset O_k \subset \Omega_0, \quad k \in \mathbb{N}.$$

Then the supports of $\varphi_k$ are disjoint to each other, which implies that $\{\varphi_k\}_{k=1}^{\infty}$ form a linearly independent sequence in $H_0^2(\Omega)$. By Gram–Schmidt orthogonalization process, there exists a normalized orthogonal sequence also denoted by $\{\varphi_k\}_{k=1}^{\infty}$ in $H_0^2(\Omega)$ and

$$\text{supp}\, \varphi_k \subset \Omega_0, k \in \mathbb{N}.$$

□

With the help of the normalized orthogonal sequence $\{\varphi_k\}_{k=1}^{\infty}$, define some subspaces as follows:

$$\mathbb{W}_k := \text{span}\{\varphi_j;\ j \le k\},$$
$$B_k := \{u \in \mathbb{W}_k : \|u\|_{H_0^2(\Omega)} \le 1\},\ S^k := \left\{u \in \mathbb{W}_k : \|u\|_{H_0^2(\Omega)} = 1\right\}$$

and

$$S_+^{k+1} := \left\{u = w + se_{k+1} : \|u\|_{H_0^2(\Omega)} = 1,\ w \in B_k,\ 0 \le s \le 1\right\}.$$

By these subspaces, we can introduce some continuous maps and minimax sequences of $J$ as follows

$$\Lambda_k := \left\{\varphi \in C(S^k, H_0^2(\Omega)) : \varphi \text{ is odd}\right\},$$
$$\Gamma_k := \left\{\varphi \in C(S_+^{k+1}, H_0^2(\Omega)) : \varphi\Big|_{S^k} \in \Lambda_k\right\}, \tag{2.55}$$

and

$$b_k := \inf_{\varphi \in \Lambda_k} \max_{u \in S^k} J(\varphi(u)), \quad c_k := \inf_{h \in \Gamma_k} \max_{u \in S_+^{k+1}} J(\varphi(u)), \quad k \in \mathbb{N}. \tag{2.56}$$

For any $\delta > 0$, put

$$\Gamma_k(\delta) := \left\{\varphi \in \Gamma_k : J(\varphi(u)) \le b_k + \delta,\ u \in S^k\right\}, \tag{2.57}$$
$$c_k(\delta) := \inf_{h \in \Gamma_k(\delta)} \max_{u \in S_+^{k+1}} J(\varphi(u)). \tag{2.58}$$

By (2.55)–(2.58), it is obvious that $b_k \le c_k \le c_k(\delta), k \in \mathbb{N}$. Next, we give some useful estimates for minimax values $b_k$ and $c_k(\delta)$.

**Lemma 2.7** *Assume that* (A1), (A3), (A4), (B) *are satisfied. Then for any* $k \in \mathbb{N}$, $b_k < 0$.

***Proof*** Since $\mathbb{W}_k$ is a finite-dimensional space, by (A3), (A4), (B), for any $u \in \mathbb{W}_k$ we get that

$$J(u) \le \frac{1}{2}\|u\|_{H_0^2(\Omega)}^2 + C_{14}\|u\|_{H_0^2(\Omega)}^{p_2} + C_{15}\|u\|_{H_0^2(\Omega)}^{p_3} + C_{16}\|u\|_{H_0^2(\Omega)}^{\theta} - C_{17}\|u\|_{H_0^2(\Omega)}^{p_1}.$$

Hence, there exist $\varepsilon(k) > 0$ and $\kappa(k) > 0$ such that $J(\kappa u) < -\varepsilon, u \in S^k$. Then we set $\varphi(u) = \kappa u, u \in S^k$. By (2.56), we obtain $b_k < 0$. □

**Lemma 2.8** *Assume that* (A1), (A3), (A4), (B) *are satisfied. Then for any* $k \in \mathbb{N}$ *and any* $\delta > 0$, *we have* $c_k(\delta) < 0$.

***Proof*** By (2.57) and (2.58), for fixed $k \in \mathbb{N}$, $0 < \delta < \delta'$, we have $\Gamma_k(\delta) \subset \Gamma_k(\delta')$ and $c_k(\delta) > c_k(\delta')$. Then we only need to prove $c_k(\delta) < 0$ for any $\delta \in (0, |b_k|)$. For any $\delta \in (0, |b_k|)$, from (2.56), there exists $\varphi_0 \in \Lambda_k$ such that $\max\limits_{u \in S^k} J(\varphi_0(u)) \leq b_n + \frac{\delta}{2}$. Since $\varphi_0(S^k)$ is a compact set in $H_0^2(\Omega)$, there exists a positive integer $m_0$ such that

$$\max_{u \in S^k} J((P_{m_0} \circ \varphi_0)(u)) \leq b_k + \delta, \tag{2.59}$$

where $P_{m_0}$ denotes the orthogonal projective operator from $H_0^2(\Omega)$ to $\mathbb{V}_{m_0}$.

For any $c \in \mathbb{R}$, let $J^c := \{u \in H_0^2(\Omega) : J(u) \leq c\}$. Choose $\overline{\varepsilon} = -\frac{b_k + \delta}{2} > 0$. By (A1), (A4), (B) and (2.13), there exists a positive constant $\rho_0$ such that if $u \in \overline{B}(0, \rho_0)$, $J(u) \leq \varepsilon$, where $B(x_0, \rho)$ denotes the open ball of radius $\rho$ centered at $u_0$ in $H_0^2(\Omega)$ and $\overline{B}$ denotes the closure in $H_0^2(\Omega)$. From (2.13) and $J(0) = 0$, hence $\mathrm{dist}(0, J^{-\overline{\varepsilon}}) > 0$. Setting

$$\rho_0' := \min\{\rho_0, \mathrm{dist}(0, J^{-\overline{\varepsilon}})\},$$

then $\rho_0' > 0$. By deformation theorem in [2] (or see deformation theorem in [9]), we have there exist $\varepsilon \in (0, \overline{\varepsilon})$ and a continuous map $\eta \in C([0, 1] \times H_0^2(\Omega), H_0^2(\Omega))$ such that

$$\eta(1, u) = u, \quad \text{if} \quad J(u) \notin [-\overline{\varepsilon}, \overline{\varepsilon}], \tag{2.60}$$

and

$$\eta(1, J^{\varepsilon} \backslash B(0, \rho_0')) \subset J^{-\varepsilon}, \tag{2.61}$$

where $B(0, \rho_0')$ is a neighborhood of $K_0$.

From (2.55), we obtain $P_{m_0} \circ \varphi_0 \in C(S^k, \mathbb{V}_{m_0})$. Since $\mathbb{V}_{k+1}$ is a metric space with the norm $\|\cdot\|_{H_0^2(\Omega)}$ and $S^k$ is a closed subset in $\mathbb{V}_{k+1}$, by Dugundji extension theorem (see Theorem 4.1 in [5]), we have there exists an extension

$$\widetilde{P_{m_0} \circ \varphi_0} : \mathbb{W}_{k+1} \to \mathbb{V}_{m_0};$$

furthermore,

$$\left((\widetilde{P_{m_0} \circ \varphi_0})\mathbb{W}_{k+1}\right) \subset \mathrm{co}\left((P_{m_0} \circ \varphi_0)S^k\right), \tag{2.62}$$

where the symbol co denotes the convex hull. Since $(P_{m_0} \circ \varphi_0)S^k$ is a compact set in $\mathbb{V}_{m_0}$, by the definition of convex hull, $\mathrm{co}\left((P_{m_0} \circ \varphi_0)S^k\right)$ is a bounded set in $\mathbb{V}_{m_0}$. Then there exists a constant $\nu$ such that

$$J(u) \leq \nu, \quad u \in \mathrm{co}\left((P_{m_0} \circ \varphi_0)S^k\right).$$

By (2.62), we have

$$J\left(\widetilde{(P_{m_0}\circ\varphi_0)}u\right)\le\nu,\quad \forall u\in\mathbb{W}_{k+1}. \tag{2.63}$$

Next, we distinguish two cases.
*Case 1.* $\nu\le\varepsilon$. Since $\widetilde{P_{m_0}\circ\varphi_0}\in C(\mathbb{W}_{k+1},\mathbb{V}_{m_0})$, by (2.63), we get that

$$\widetilde{(P_{m_0}\circ\varphi_0)}u\in J^{\varepsilon}_{m_0},\quad u\in\mathbb{W}_{k+1}, \tag{2.64}$$

where $J^{\varepsilon}_{m_0}:=\{u\in\mathbb{V}_{m_0}:J(u)\le\varepsilon\}$. Define a map $\Theta$ as follows:

$$\Theta(u)=\begin{cases}u, & u\notin\overline{B}(0,\rho'_0)\cap\mathbb{V}_{m_0}\\ u+\left(\rho'^2_0-\|u\|^2_{H^2_0(\Omega)}\right)^{\frac{1}{2}}e_{m_0+1}, & u\in\overline{B}(0,\rho'_0)\cap\mathbb{V}_{m_0}.\end{cases} \tag{2.65}$$

It is clear that $\Theta\in C(\mathbb{V}_{m_0},\mathbb{V}_{m_0+1})$.

On the other hand, if $u\in\mathbb{W}_{k+1}$ and $\left\|\left(\widetilde{P_{m_0}\circ\varphi_0}\right)u\right\|_{H^2_0(\Omega)}>\rho'_0$ then by (2.64) and (2.65), we have

$$\begin{aligned}&\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u=\left(\widetilde{P_{m_0}\circ\varphi_0}\right)u\in J^{\varepsilon}_{m_0},\\ &\left\|\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u\right\|_{H^2_0(\Omega)}>\rho'_0.\end{aligned} \tag{2.66}$$

Otherwise, $u\in\mathbb{W}_{k+1}$ and $\left\|\left(\widetilde{P_{m_0}\circ\varphi_0}\right)u\right\|_{H^2_0(\Omega)}\le\rho'_0$ from (2.65) we have

$$\left\|\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u\right\|_{H^2_0(\Omega)}=\rho'_0. \tag{2.67}$$

Combining the definition of $\rho'_0$, (2.66) and (2.67), we obtain

$$\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u\notin B(0,\rho'_0),\quad u\in\mathbb{W}_{k+1}, \tag{2.68}$$

and

$$\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u\in J^{\varepsilon},\quad \forall u\in\mathbb{W}_{k+1}. \tag{2.69}$$

Define a map

$$\begin{aligned}\Theta_{m_0}:\mathbb{W}_{k+1}&\longrightarrow H^2_0(\Omega)\\ u&\longmapsto\Theta_{m_0}(u)=\eta\left(1,\left(\Theta\circ\left(\widetilde{P_{m_0}\circ\varphi_0}\right)\right)u\right).\end{aligned} \tag{2.70}$$

We need to prove $\Theta_{m_0} \in \Gamma_k(\delta)$ and $\max_{u \in S_+^{k+1}} J(\Theta_{m_0}(u)) < 0$. First, it is obvious that $\Theta_{m_0} \in C(S_+^{k+1}, H_0^2(\Omega))$. Next, we prove $\Theta_{m_0}\Big|_{S^k} \in \Lambda_k$. By Dugundji extension theorem, we get

$$\left(\widetilde{P_{m_0} \circ \varphi_0}\right) u = \left(P_{m_0} \circ \varphi_0\right) u, \quad \forall u \in S^k. \tag{2.71}$$

From (2.59), hence $\left(P_{m_0} \circ \varphi_0\right) u \in J^{-2\varepsilon}$, $u \in S^k$. By the definition of $\rho_0'$ and $J^{-2\varepsilon} \subset J^{-\varepsilon}$ implies that

$$\left\| \left(P_{m_0} \circ \varphi_0\right) u \right\|_{H_0^2(\Omega)} \geq \rho_0', \quad \forall u \in S^k. \tag{2.72}$$

From (2.65), (2.71) and (2.72), we have that

$$\left(\Theta \circ \left(\widetilde{P_{m_0} \circ \varphi_0}\right)\right) u = \Theta\left(\left(P_{m_0} \circ \varphi_0\right) u\right) = \left(P_{m_0} \circ \varphi_0\right) u, \quad \forall u \in S^k. \tag{2.73}$$

Since $\left(P_{m_0} \circ \varphi_0\right) u \in J^{-2\varepsilon}$, $\forall u \in S^k$, by (2.59), (2.60), (2.70) and (2.73), we have

$$\Theta_{m_0}(u) = \eta\left(1, \left(\Theta \circ \left(\widetilde{P_{m_0} \circ \varphi_0}\right)\right) u\right) = \left(P_{m_0} \circ \varphi_0\right) u, \quad \forall u \in S^k, \tag{2.74}$$

which implies that $\Theta_{m_0}\Big|_{S^k} \in \Lambda_k$. Moreover, from (2.57), (2.59) and (2.74), we have $\Theta_{m_0} \in \Gamma_k(\delta)$. Since $S^{k+1} \subset \mathbb{W}_{k+1}$, by (2.68) and (2.69), we obtain

$$\left(\Theta \circ \left(\widetilde{P_{m_0} \circ \varphi_0}\right)\right) u \notin B(0, \rho_0') \quad \text{and} \quad \left(\Theta \circ \left(\widetilde{P_{m_0} \circ \varphi_0}\right)\right) u \in J^{\varepsilon}, \quad \forall u \in S^{k+1}.$$

From (2.61) and (2.70), we get $\max_{u \in S_+^{k+1}} J\left(\Theta_{m_0}(u)\right) \leq -\varepsilon < 0$ which implies that $c_k(\delta) < 0$.

*Case 2.* $\nu > \varepsilon$. By a similar proof as in Lemmas 2.3 and 2.5, we can prove that $J\Big|_{\mathbb{V}_{m_0}} \in C^1(\mathbb{V}_{m_0}, \mathbb{R})$ and satisfies Palais–Smale condition. Moreover, $J\Big|_{\mathbb{V}_{m_0}}$ has no critical points with positive critical values on $\mathbb{V}_{m_0}$. By noncritical interval theorem (see Theorem 5.1.6 in [3]), we see that $J_{m_0}^{\varepsilon}$ is a strong deformation retract of $J_{m_0}^{\nu}$. So there exists a map $\psi$ such that $\psi \in C(J_{m_0}^{\nu}, J_{m_0}^{\varepsilon})$ and $\psi(u) = u$, if $u \in J_{m_0}^{\varepsilon}$. Define a map $\Psi$ as follows:

$$\begin{aligned} \Psi : \mathbb{W}_{k+1} &\longrightarrow H_0^2(\Omega) \\ u &\longmapsto \Psi(u) := \psi\left(1, \left(\Theta \circ \left(\widetilde{P_{m_0} \circ \varphi_0}\right)\right)(u)\right). \end{aligned}$$

By a similar proof as in Case 1, we get $\Psi \in \Gamma_k(\delta)$ and $\max_{u \in S_+^{k+1}} J(\Psi(u)) \leq -\varepsilon < 0$ which implies that $c_k(\delta) < 0$. The proof of Lemma 2.8 is complete. □

**Lemma 2.9** *Suppose that $f$ satisfies* (A1), (A3), (A4) *and $g$ satisfies* (B). *Then there exists a positive constant $C_{18}$ independent of $k$ such that for all $k$ large enough*

$$b_k \geq -C_{18} k^{\frac{-4p}{N(2-p)}}. \tag{2.75}$$

***Proof*** For any $\varphi \in \Lambda_k (k \geq 2)$ when $0 \notin \varphi(S^k)$, then the genus $\gamma(\varphi(S^k))$ is well defined and $\gamma(\varphi(S^k)) \geq \varphi(S^k) = k$. By Proposition 7.8 in [12], hence $\varphi(S^k) \cap \mathbb{V}_{k-1}^{\perp} \neq \emptyset$. Otherwise, if $0 \in \varphi(S^k)$ then $0 \in \varphi(S^k) \cap \mathbb{V}_{k-1}^{\perp}$. So for any $\varphi \in \Lambda_k (k \geq 2)$ we have $\varphi(S^k) \cap \mathbb{V}_{k-1}^{\perp} \neq \emptyset$. Therefore, for any $\varphi \in \Lambda_k (k \geq 2)$, we obtain

$$\max_{u \in S^k} J(\varphi(u)) \geq \inf_{u \in \mathbb{V}_{k-1}^{\perp}} J(u). \tag{2.76}$$

From (A1), (A4), (B), (2.10) and (2.13), we get that

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|u\|_{H_0^2(\Omega)}^2 - C_{19} \|u\|_{L^p(\Omega)}^p - \pi(u) \int_\Omega F_2(x,u) \mathrm{d}x - \psi(u) \\ &\geq \frac{1}{4} \|u\|_{H_0^2(\Omega)}^2 - C_{19} \|u\|_{L^2(\Omega)}^p, \forall u \in H_0^2(\Omega). \end{aligned} \tag{2.77}$$

Moreover, by $u \in \mathbb{V}_{k-1}^{\perp}$, hence

$$\|u\|_{L^2(\Omega)} \leq \lambda_k^{\frac{-1}{2}} \|u\|_{H_0^2(\Omega))}. \tag{2.78}$$

Combining (2.56), (2.76), (2.77) and (2.78), for any $k \geq 2$, we have

$$b_k \geq \inf_{t \geq 0} \left( \frac{1}{4} t^2 - C_{19} \lambda_k^{\frac{-p}{2}} t^p \right) = -C_{20} \lambda_k^{\frac{-p}{2-p}}, \tag{2.79}$$

where $C_{20}$ is a positive constant independent of $k$ and $\lambda_k$. On the other hand, it follows from Agmon's generalization [1] of Weyl's formula [15], which in fact is an extension of earlier work of Pleijel [10] for $N = 2$, we have

$$\lambda_k \geq C_{21} k^{\frac{4}{N}}. \tag{2.80}$$

Combining (2.79) and (2.80), we arrive at the conclusion of the lemma. □

**Lemma 2.10** *Suppose that $c_k = b_k$ for $k \geq k_0$, where $k_0 \in \mathbb{N}$. Then there exists a positive integer $k_1$ such that*

$$|b_k| \geq C_{22} k^{\frac{2}{2-\theta}}, \quad k \geq k_1, \tag{2.81}$$

*where $C_{22}$ is a positive constant independent of $k$.*

***Proof*** For any $k \geq k_0$ and any $\varepsilon \in (0, |b_k|)$, by (2.56) there exists a map $\varphi_0 \in \Gamma_k$ such that

$$\max_{u \in S_+^{k+1}} J(\varphi_0(u)) < c_k + \varepsilon = b_k + \varepsilon. \tag{2.82}$$

From $S^{k+1} = S_+^{k+1} \cup (-S_+^{k+1})$, $\varphi_0$ can be continuously extended to $S^{k+1}$ as an odd function, also denoted by $\varphi_0$, then $\varphi_0 \in \Lambda_{k+1}$. From (2.56), we have

$$b_{k+1} \leq \max_{u \in S^{k+1}} J(\varphi_0(u)) = J(\varphi_0(u_0)) \tag{2.83}$$

for some $u_0 \in S^{k+1}$. If $u_0 \in S_+^{k+1}$, in combination with (2.27), (2.82) and (2.83), we have

$$b_{k+1} < b_k + \varepsilon + C_8 \left| b_{k+1} \right|^{\frac{\theta}{2}}. \tag{2.84}$$

Otherwise, $u_0 \in -S_+^{k+1}$, from (2.27) and (2.82), we get that

$$\begin{aligned} J(\varphi_0(u_0)) &\leq J(\varphi_0(-u_0)) + C_8 \left| J(\varphi_0(u_0)) \right|^{\frac{\theta}{2}} \\ &\leq b_k + \varepsilon + C_8 \left| J(\varphi_0(u_0)) \right|^{\frac{\theta}{2}}. \end{aligned} \tag{2.85}$$

Next, we consider two possible cases.
*Case 1.* $J(\varphi_0(u_0)) \leq |b_{k+1}|$, from (2.83) and (2.84), we obtain

$$b_{k+1} < b_k + \varepsilon + C_8 \left| b_{k+1} \right|^{\frac{\theta}{2}}. \tag{2.86}$$

*Case 2.* $J(\varphi_0(u_0)) > |b_{k+1}|$. By (2.82), there exists $u_1 \in S_+^{k+1}$ such that

$$J(\varphi_0(u_1)) < b_k + \varepsilon < 0. \tag{2.87}$$

Since $J \circ \varphi_0 \in C(S^{k+1}, \mathbb{R})$ and $S^{k+1}$ is a connected space with the norm $\|\cdot\|_{H_0^2(\Omega))}$, by the intermediate value theorem, there exists $u_2 \in S^{k+1}$ such that $J(\varphi_0(u_2)) = \frac{|b_{k+1}|}{2}$. By (2.82), hence $u_2 \in -S^{k+1}$. From (2.27) and (2.82), we get

$$\begin{aligned} J(\varphi_0(u_2)) &\leq J(\varphi_0(-u_2)) + C_8 \left| J(\varphi_0(u_2)) \right|^{\frac{\theta}{2}} \\ &\leq b_k + C_8 \left| J(\varphi_0(u_2)) \right|^{\frac{\theta}{2}}, \end{aligned}$$

which implies that

$$b_{k+1} \leq b_k + \varepsilon + C_8 \left| b_{k+1} \right|^{\frac{\theta}{2}}. \tag{2.88}$$

By Lemma 2.7, (2.84), (2.86) and (2.88), it is easy to see that

$$|b_k| \le |b_{k+1}| + C_8 \, |b_{k+1}|^{\frac{\theta}{2}}, \quad k \ge k_0. \tag{2.89}$$

Next, we show that (2.89) implies (2.81). The proof will be done by induction. First, we introduce a useful inequality as follows:

$$(1+t)^\alpha \ge 1 + \frac{\alpha t}{2}, \quad t \in [0, \beta], \tag{2.90}$$

where $\alpha, \beta$ are positive constants and $\beta$ depends on $\alpha$. Set $\alpha = 2(\theta-2)^{-1}$. In view of (2.90), there exists $\widetilde{k_0} \in \mathbb{N}$ such that

$$\left(1+\frac{1}{k}\right)^{\frac{2}{\theta-2}} \ge 1 + \frac{1}{(\theta-2)k}, \quad k \ge \widetilde{k_0}. \tag{2.91}$$

Define

$$C_{22} := \min\left\{ k_1^{\frac{2}{\theta-2}} \left|b_{k_1}\right|, \left(\frac{1}{C_8(\theta-2)}\right)^{\frac{2}{\theta-2}} \right\}, \tag{2.92}$$

where $k_1 = \max\{k_0, \widetilde{k_0}\}$. Then we claim (2.81) holds. By (2.92), we have

$$\left|b_{k_1}\right| \ge C_{22} k_1^{\frac{2}{2-\theta}}. \tag{2.93}$$

Suppose that (2.81) holds for $k \ge k_1$. Then we only need to prove (2.81) also holds for $k+1$. If not, we get that

$$|b_{k+1}| \le C_{22}(k+1)^{\frac{2}{2-\theta}}. \tag{2.94}$$

Since (2.81) holds for $k$, by (2.27), (2.89) and (2.94), we obtain

$$\begin{aligned} C_{22} k_1^{\frac{2}{2-\theta}} \le |b_k| \le |b_{k+1}| + C_8 \, |b_{k+1}|^{\frac{\theta}{2}} &\le C_{22}(k+1)^{\frac{2}{2-\theta}} \\ &+ C_8 C_{22}^{\frac{\theta}{2}} (k+1)^{\frac{\theta}{2-\theta}}. \end{aligned} \tag{2.95}$$

When we divide (2.95) by $C_{22}(k+1)^{\frac{2}{2-\theta}}$ on both sides, in view of (2.92), we get that

$$\left(1+\frac{1}{k}\right)^{\frac{2}{\theta-2}} < 1 + C_8 C_{22}^{\frac{\theta-2}{2}} \frac{1}{k+1} < 1 + C_8 C_{22}^{\frac{\theta-2}{2}} \frac{1}{k} \le 1 + \frac{1}{(\theta-2)k},$$

which contradicts (2.91). So (2.81) holds. The proof of Lemma 2.10 is complete. □

**Lemma 2.11** *Suppose that $c_k > b_k$. Then for any $\delta \in (0, c_k - b_k)$, $c_k(\delta)$ given by (2.57) is a critical value of $J$.*

***Proof*** By using deformation theorem in [2], the proof of this lemma is similar to the one of Lemma 1.57 in [11]. We omit the details. □

Proof of Theorem 1.1 From (1.4), Lemmas 2.7, 2.9 and 2.10, it is impossible that $c_k = b_k$ for all large $k$, we can choose subsequence $\{k_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $c_{k_j} > b_{k_j}$. By Lemmas 2.8, 2.9 and 2.11, there exists a sequence of critical points $\{u_{k_j}\}_{j=1}^{\infty}$ of $J$ such that

$$-C_{18}k_j^{\frac{-2p}{N(2-p)}} \le b_{k_j} < c_{k_j} \le c_{k_j}(\delta_j) = J(u_{k_j}) < 0, \tag{2.96}$$

where $\delta_j \in (0, c_{k_j} - b_{k_j})$. It is obvious that $u_{k_j} \neq 0$, $j \in \mathbb{N}$. Next, we consider the following two possible cases.
*Case 1.* $\|u_{k_j}\|^2_{H_0^2(\Omega))} > 2T_0$. From (2.2), (2.3) and (2.16), hence

$$\pi(u_{k_j}) = 1 \quad \text{and} \quad \psi'(u_{k_j}) = 0.$$

By (A2), (2.5) and (2.28), we get that

$$\begin{aligned}\overline{I}(u_{k_j}) &= \overline{I}(u_{k_j}) - \mu^{-1}\langle \widetilde{I}'(u_{k_j}), u_{k_j}\rangle \\ &= 2A\left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} + \int\limits_{\Omega}\left(\mu^{-1}f_1(x, u_{n_j})u_{n_j} - F_1(x, u_{n_j})\right)\mathrm{d}x \\ &\le A\left\|u_{k_j}\right\|^2_{H_0^2(\Omega))}. \end{aligned} \tag{2.97}$$

*Case 2.* $\|u_{k_j}\|^2_{H_0^2(\Omega))} \le 2T_0$. By Lemmas 2.1, 2.3, (A2), (A4), (B) (2.5) and (2.28), we get that

$$\overline{I}(u_{k_j}) \le \frac{1}{2}\left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} - \int\limits_{\Omega} F_1(x, u_{k_j})\mathrm{d}x + C_3C_{p_3}^{p_3}\left\|u_{k_j}\right\|^{p_3}_{H_0^2(\Omega))}$$
$$+ C_4C_\theta^\theta\left\|u_{k_j}\right\|^\theta_{H_0^2(\Omega))},$$

and

$$\langle \widetilde{I}'(u_{k_j}), u_{k_j}\rangle \ge \left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} - \int\limits_{\Omega} f_1(x, u_{k_j})u_{k_j}\mathrm{d}x - 9C_3C_{p_3}^{p_3}\left\|u_{k_j}\right\|^{p_3}_{H_0^2(\Omega))}$$
$$- 89C_4C_\theta^\theta\left\|u_{k_j}\right\|^\theta_{H_0^2(\Omega))}.$$

Hence,

$$
\begin{aligned}
\overline{I}(u_{k_j}) &= \overline{I}(u_{k_j}) - \mu^{-1}\langle \widetilde{I}'(u_{k_j}), u_{k_j}\rangle \\
&\le 2A \left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} + \frac{\mu+9}{\mu} C_3 C_{p_3}^{p_3} \left\|u_{k_j}\right\|^{p_3}_{H_0^2(\Omega))} + \frac{\mu+89}{\mu} C_4 C_\theta^\theta \left\|u_{k_j}\right\|^{\theta}_{H_0^2(\Omega))} \\
&\le A \left\|u_{k_j}\right\|^2_{H_0^2(\Omega))}. \qquad (2.98)
\end{aligned}
$$

In both cases, by (2.11), (2.97) or (2.98), we get $\ell(u_{k_j}) = 0$ and $\ell'(u_{k_j}) = 0$. Hence,

$$J(u_{k_j}) = \overline{I}(u_{k_j}) \le A \left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} < 0.$$

By (2.96), it is easy to see that

$$\left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} \to 0 \text{ as } j \to \infty.$$

So there exists $j_0 \in \mathbb{N}$ such that $\left\|u_{k_j}\right\|^2_{H_0^2(\Omega))} < T_0$ for all $j \ge j_0$. By (2.3) and (2.16), hence

$$\pi(u_{k_j}) = 1, \quad \pi'(u_{k_j}) = 0 \quad \text{for all } j \ge j_0.$$

In combination with (2.52), (2.16) and (2.28), when $j$ is large enough, we conclude that $u_{k_j}$ are weak solutions of the problem (1.2). □

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## References

1. Agmon, S.: On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems. Commun. Pure Appl. Math. **18**, 627–663 (1965)
2. Bartolo, P., Benci, V., Fortunato, D.: Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity. Nonlinear Anal. **7**(9), 981–1012 (1983)
3. Chang, K.C.: Methods in Nonlinear Analysis. Springer Monographs in Mathematics, p. x+439. Springer, Berlin (2005)
4. Chipot, M.: Chipot, variational inequalities and flow in porous media. In: Applied Mathematical Sciences, vol. **52**, pp. 7–118. Springer, New York (1984)
5. Dugundji, J.: An extension of Tietze's theorem. Pac. J. Math. **1**, 353–367 (1951)
6. Lazer, A.C., Mckenna, P.J.: Large amplitude periodic oscillation in suspension bridges: some new connections with nonlinear analysis. SIAM Rev. **32**, 537–578 (1990)
7. Liu, X.Q., Huang, Y.S.: One sign-changing solution for a fourth-order asymptotically linear elliptic problem. Nonlinear Anal. **72**, 2271–2276 (2010)
8. Micheletti, A.M., Pistoia, A.: Multiplicity solutions for a fourth order semilinear elliptic problems. Nonlinear Anal. **31**, 895–908 (1998)
9. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear Analysis-Theory and Methods. Springer Monographs in Mathematics, p. xi+577. Springer, Cham (2019)

10. Pleijel, Å.: On the eigenvalues and eigenfunctions of elastic plates. Commun. Pure Appl. Math. **3**, 1–10 (1950)
11. Rabinowitz, P.H.: Multiple critical points of perturbed symmetric functionals. Trans. Am. Math. Soc. **272**(2), 753–769 (1982)
12. Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations. In: CBMS Regional Conference Series in Mathematics, vol. **65**, pp. 8–100. Published for the Conference Board of the Mathematical Sciences. By the American Mathematical Society, Providence, RI (1986)
13. Ragusa, M.A., Tachikawa, A.: Regularity of minimizers of some variational integrals with discontinuity. Z. Anal. Anwend. **27**(4), 469–482 (2008)
14. Ragusa, M.A., Tachikawa, A.: Regularity for minimizers for functionals of double phase with variable exponents. Adv. Nonlinear Anal. **9**(1), 710–728 (2020)
15. Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). Math. Ann. **71**, 441–479 (1912)
16. Zhang, J.H., Li, S.J.: Multiple nontrivial solutions for some fourth-order semilinear elliptic problems. Nonlinear Anal. **60**, 221–230 (2005)
17. Zhang, J., Wei, Z.L.: Infinitely many nontrivial solutions for a class of biharmonic equations via variant fountain theorems. Nonlinear Anal. **74**, 7474–7485 (2011)
18. Zhou, J.W., Wu, X.: Sign-changing solutions for some fourth-order nonlinear elliptic problems. J. Math. Anal. Appl. **342**, 542–558 (2008)