Bull. Malays. Math. Sci. Soc. (2021) 44:1727–1748
https://doi.org/10.1007/s40840-020-01029-z

**BULLETIN** *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

# Derivations on Banach algebras of connected multiplicative linear functionals

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Received: 19 April 2019 / Revised: 6 September 2020 / Accepted: 21 September 2020 /
Published online: 9 October 2020


**Abstract**
Let $A$ and $B$ be Banach algebras with $\sigma(B) \neq \emptyset$. Let $\theta, \phi, \gamma \in \sigma(B)$ and $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ be the set of all linear mappings $d : A \times B \to A \times B$ satisfying $d((a, b) \cdot_\theta (x, y)) = d(a, b) \cdot_\phi (x, y) + (a, b) \cdot_\gamma d(x, y)$ for all $a, x \in A$ and $b, y \in B$. In this paper, we characterize elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ in the case where $A$ has a right identity. We then investigate the concept of centralizing for elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ and determine dependent elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$. We also apply some results to group algebras.

**Keywords** Derivations · $\theta$-Lau products · Centralizing mappings · Dependent elements · Locally compact groups

**Mathematics Subject Classification** 47B47 · 43A15 · 43A20

## 1 Introduction

Throughout the paper, $\mathfrak{A}$ is a Banach algebra with Jacobson radical $\mathrm{rad}(\mathfrak{A})$, $A$ is a Banach algebra with a right identity $u$ and right annihilator

$$\mathrm{ran}(A) = \{z \in A : az = 0 \quad \text{forall} \quad a \in A\}$$

and $B$ is a Banach algebra with nonempty spectrum $\sigma(B)$. Let also $\theta, \phi$ and $\gamma$ be elements of $\sigma(B)$. In this paper, we endowed $\mathfrak{A}^{**}$, the second dual of $\mathfrak{A}$, with *the first*

Communicated by Pedro Tradacete.

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*Arens product* "$\Box$" defined by

$$\langle m\Box n, \tau\rangle = \langle m, n\tau\rangle,$$

where

$$\langle n\tau, a\rangle = \langle n, \tau a\rangle \quad \text{and} \quad \langle \tau a, x\rangle = \langle \tau, ax\rangle$$

for all $m, n \in \mathfrak{A}^{**}$, $\tau \in \mathfrak{A}^*$ and $a, x \in \mathfrak{A}$. Let us recall that the *topological center* of $\mathfrak{A}^{**}$ is denoted by $\mathrm{Z}_t(\mathfrak{A}^{**})$ and is defined by

$$\mathrm{Z}_t(\mathfrak{A}^{**}) = \{m \in \mathfrak{A}^{**} : \text{ themapping } n \mapsto m\Box n \text{ isweak}^* - \text{weak}^* \text{ continuouson } \mathfrak{A}^{**}\}.$$

Following [25], the $\theta$-Lau product $\mathfrak{A}$ and $B$ is denoted by $\mathfrak{A} \times_\theta B$ and it is the direct product $\mathfrak{A} \times B$ together with the component wise addition and the multiplication

$$(a, b) \cdot_\theta (x, y) = (ax + \theta(y)a + \theta(b)x, by).$$

We note that in the case where $B = \mathbb{C}$ and $\theta$ is the identity map on $\mathbb{C}$, the unitization $\mathfrak{A}$ will be obtained. We also note that if we permit $\theta = 0$, the $\theta$-Lau product $\mathfrak{A} \times_\theta B$ is the usual direct product of Banach algebras. Hence, we disregard the possibility that $\theta = 0$.

A linear mapping $\mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ is called *centralizing* if for every $a \in \mathfrak{A}$

$$[\mathfrak{D}(a), a] \in \mathrm{Z}(\mathfrak{A}),$$

where for each $a, x \in \mathfrak{A}$

$$[a, x] = ax - xa$$

and $\mathrm{Z}(\mathfrak{A})$ denotes the center of $\mathfrak{A}$. Also, $\mathfrak{D}$ is called a *derivation* if for every $a, x \in \mathfrak{A}$

$$\mathfrak{D}(ax) = \mathfrak{D}(a)x + a\mathfrak{D}(x).$$

The set of all derivations on $\mathfrak{A}$ is denoted by $\mathrm{Der}(\mathfrak{A})$. For any $x \in \mathfrak{A}$, the derivation $a \mapsto [a, x]$ on $\mathfrak{A}$ is called an *inner derivation* and it is denoted by $\mathrm{ad}_x$. Similarly, one can define derivations on rings.

The $\theta$-Lau products $\mathfrak{A} \times_\theta B$ were first introduced by Lau [17], for Banach algebras that are pre-duals of von Neumann algebras, and for which the identity of the dual is a multiplicative linear functional. Sanjani Monfared [25] extended this product to arbitrary Banach algebras $\mathfrak{A}$ and $B$. The $\theta$-Lau products have significance and utility due to the following reasons. First, the products can be regarded as a strongly splitting Banach algebra extension of $B$ by $\mathfrak{A}$; for the study of extensions of Banach algebras, see [1,9]. Second, many properties are not shared by arbitrary strongly splitting extensions, while the $\theta$-Lau products exhibit them; see [25]. Third, the $\theta$-Lau products can be used as a source of examples or counterexamples; see, for instance, [26]. These reasons caused that several authors studied various aspects of the products [7,15,26,27]. In this paper, we continue these investigations and study derivation-like maps of them. We also apply the results to Banach algebras that are important and useful in harmonic analysis on locally compact groups.

Derivations on rings were studied by several authors [4,6,11,14,24]. For example, Posner [24] showed that the product of two nonzero derivations on prime rings with characteristic different from two is not a derivation. He also proved that the zero map is the only centralizing derivation on a noncommutative prime ring. These results are known as the Posner's first and second theorems, respectively. Some authors studied the results obtained on derivations of prime rings for Banach algebras [5,8,22,23]. In the study of derivations on Banach algebras, we may assume that the algebra is unital, because otherwise we can replace the algebra $\mathfrak{A}$ by its unitization; i.e., $\mathfrak{A} \times \mathbb{C}$. On the hand, we know that the $\theta$-Lau products $\mathfrak{A} \times_\theta B$ are a generalization of the unitization of $\mathfrak{A}$. Therefore, it is natural to ask whether results concerning derivations on prime rings and Banach algebras hold for the $\theta$-Lau products $\mathfrak{A} \times_\theta B$? The other question comes to mind immediately: What happens to $\theta$ in these investigations? To answer these questions, we consider linear mappings $d : \mathfrak{A} \times B \to \mathfrak{A} \times B$ satisfying

$$d((a, b) \cdot_\theta (x, y)) = d(a, b) \cdot_\phi (x, y) + (a, b) \cdot_\gamma d(x, y)$$

for all $a, x \in \mathfrak{A}$, $b, y \in B$. We denote the set of all these mappings by $\mathrm{Der}(\mathfrak{A} \times_\theta^{\phi,\gamma} B)$. In this paper, we investigate the questions concerning derivations for elements of $\mathrm{Der}(\mathfrak{A} \times_\theta^{\phi,\gamma} B)$ and attention to group algebras.

This paper is organized as follows. In Sect. 2, we give a characterization of elements of $\mathrm{Der}(\mathfrak{A} \times_\theta^{\phi,\gamma} B)$ in the case where $\mathfrak{A}$ has a right identity or a bounded approximate identity. In Sect. 3, we investigate the concept of centralizing for elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$. In Sect. 4, we study dependent elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ and show that if $(a, b)$ is dependent element on $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$, then $a = 0$ and $\eta(b) = 0$ for all $\eta \in \sigma(B)$. In Sect. 5, we give applications of results to group algebras. We show that if $\mathfrak{A}$ is $M(G)$ or $L^1(G)$ and $B$ is $M(G)$, $L^1(G)$ or $\mathbb{C}$, then elements of $\mathrm{Der}(\mathfrak{A} \times_\theta^{\phi,\gamma} B)$ are $(\eta, \eta)$-inner for all $\eta \in \sigma(B)$. We also prove that if $d$ is a nonzero $(\eta_1, \eta_2)$-centralizing elements of $\mathrm{Der}(L^\infty(G)^* \times_\theta^{\phi,\gamma} B)$, the $\theta = \phi = \gamma$ and $\eta_1 = \eta_2$. Finally, we prove that zero is the only dependent element on derivations of $A(G)$, when $G$ is a locally compact group.

## 2 The Characterization of Elements in $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$

In the following, let $\mathfrak{A}$ and $B$ be arbitrary Banach algebras and $A$ be a Banach algebra with a right identity $u$. Let also $\theta$, $\phi$ and $\gamma$ be elements of the spectrum of $B$. The main result of this section is the following theorem.

**Theorem 2.1** *Let $d \in \mathrm{Der}(A \times_{\theta}^{\phi,\gamma} B)$. Then, there exist unique derivations $d_A \in \mathrm{Der}(A)$ and $d_B \in \mathrm{Der}(B)$ such that*

$$d(a,b) = (d_A(a) + (\theta - \gamma)(b)d_A(u) - \phi(d_B(b))u, d_B(b))$$

*for all $a \in A$ and $b \in B$. Furthermore, either $\theta = \phi = \gamma$ or $d$ maps $A$ into $\mathrm{rad}(A)$.*

***Proof*** Let $a \in A$ and $d(a,0) = (x_0, y_0)$ for some $x_0 \in A$ and $y_0 \in B$. We have

$$(a,0) \cdot_{\theta} (u,0) = (a,0).$$

Thus,

$$\begin{aligned}(x_0, y_0) = d(a,0) &= d(a,0) \cdot_{\phi} (u,0) + (a,0) \cdot_{\gamma} d(u,0) \\ &= (x_0, y_0) \cdot_{\phi} (u,0) + (a,0) \cdot_{\gamma} (z_0, w_0) \\ &= (x_0 + \phi(y_0)u + az_0 + \gamma(w_0)a, 0),\end{aligned} \tag{1}$$

where $d(u,0) = (z_0, w_0)$ for some $z_0 \in A$ and $w_0 \in B$. Hence, $y_0 = 0$ and so

$$d(a,0) = (x_0, 0). \tag{2}$$

This implies that $w_0 = 0$. From this and (1), we infer that $az_0 = 0$ for all $a \in A$. Therefore,

$$z_0 \in \mathrm{ran}(A).$$

Let $b \in B$ and $d(0,b) = (x_1, y_1)$ for some $x_1 \in A$ and $y_1 \in B$. Then,

$$\begin{aligned}(\theta(b)z_0, 0) = d(\theta(b)u, 0) &= d((0,b) \cdot_{\theta} (u,0)) \\ &= d(0,b) \cdot_{\phi} (u,0) + (0,b) \cdot_{\gamma} d(u,0) \\ &= (x_1, y_1) \cdot_{\phi} (u,0) + (0,b) \cdot_{\gamma} (z_0, 0) \\ &= (x_1 + \phi(y_1)u + \gamma(b)z_0, 0).\end{aligned}$$

It follows that

$$x_1 = (\theta - \gamma)(b)z_0 - \phi(y_1)u.$$

Hence,

$$d(0,b) = ((\theta - \gamma)(b)z_0 - \phi(y_1)u, y_1).$$

From this and (2), one obtains that

$$d(a,b) = d(a,0) + d(0,b) = (x_0 + (\theta - \gamma)(b)z_0 - \phi(y_1)u, y_1).$$

Now, let $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ be the canonical projections. We define the functions $d_A : A \to A$ and $d_B : B \to B$ by the formulas

$$d_A(a) = \pi_A(d(a, 0)) \quad \text{and} \quad d_B(b) = \pi_B(d(0, b)).$$

It is easy to see that these functions are derivations and

$$d(a, b) = (d_A(a) + (\theta - \gamma)(b)d_A(u) - \phi(d_B(b))u, d_B(b))$$

for all $a \in A$ and $b \in B$. Since for every $a, x \in A$ and $b, y \in B$

$$d((a, b) \cdot_\theta (x, y)) = d(a, b) \cdot_\phi (x, y) + (a, b) \cdot_\gamma d(x, y),$$

we deduce that

$$\begin{aligned}
&\theta(b)d_A(x) + \theta(y)d_A(a) + (\theta - \gamma)(by)d_A(u) - \phi(d_B(by))u \\
&\quad = (\theta - \gamma)(b)d_A(u)x + \phi(d_B(b))(x - ux) + \phi(y)d_A(a) \\
&\qquad + \phi(y)(\theta - \gamma)(b)d_A(u) - \phi(y)\phi(d_B(b))u + (\gamma - \phi)(d_B(y))a \\
&\qquad + \gamma(b)d_A(x) + \gamma(b)(\theta - \gamma)(y)d_A(u) - \gamma(b)\phi(d_B(y))u
\end{aligned} \tag{3}$$

for all $a, x \in A$ and $b, y \in B$. Taking $a = x = 0$ in (3), we have

$$\begin{aligned}
&(\theta - \gamma)(by)d_A(u) - \phi(d_B(by))u \\
&\quad = \phi(y)(\theta - \gamma)(b)d_A(u) - \phi(y)\phi(d_B(b))u \\
&\qquad + \gamma(b)(\theta - \gamma)(y)d_A(u) - \gamma(b)\phi(d_B(y))u
\end{aligned} \tag{4}$$

for all $b, y \in B$. Subtracting (4) from (3), we arrive at

$$\begin{aligned}
&\theta(b)d_A(x) + \theta(y)d_A(a) \\
&\quad = (\theta - \gamma)(b)d_A(u)x + \phi(d_B(b))(x - ux) \\
&\qquad + \phi(y)d_A(a) + (\gamma - \phi)(d_B(y))a + \gamma(b)d_A(x).
\end{aligned} \tag{5}$$

Substituting $x = 0$ in (5), we obtain

$$(\theta - \phi)(y)d_A(a) = (\gamma - \phi)(d_B(y))a.$$

Hence,

$$\begin{aligned}
(\gamma - \phi)(d_B(y))u &= (\gamma - \phi)(d_B(y))uu \\
&= (\theta - \phi)(y)ud_A(u) \\
&= 0.
\end{aligned}$$

Thus,

$$(\gamma - \phi)(d_B(y))a = 0$$

and so

$$(\theta - \phi)(y)d_A(a) = 0 \tag{6}$$

for all $a \in A$ and $y \in B$. Hence, (5) reduces to

$$(\theta - \gamma)(b)(d_A(x) - d_A(u)x) = \phi(d_B(b))(x - ux) \tag{7}$$

for all $x \in A$ and $b \in B$. Now left multiplication of relation (4) by $u$ gives

$$(\phi - \gamma)(b)\phi(d_B(y))u = 0$$

for all $b, y \in B$. Note that if $\phi d_B \neq 0$, then $\phi = \gamma$. Thus, by (6) and (7), we have

$$(\theta - \gamma)(b)d_A(x - ux) = \phi(d_B(b))(x - ux) = 0.$$

This together with (6) shows that $\theta = \phi = \gamma$ or $d$ maps $A$ into $\mathrm{rad}(A)$. □

In the sequel, we list some consequences of the proof of Theorem 2.1, which we will frequently apply. Henceforth, we also suppose that $d_A$ and $d_B$ are derivations as in Theorem 2.1.

(i) $d_A(u) \in \mathrm{ran}(A)$.
(ii) For every $a \in A$ and $b \in B$,

$$\begin{aligned}(\theta - \phi)(b)d_A(a) &= (\theta - \gamma)(b)(d_A(a) - d_A(u)a)\\ &= \phi(d_B(b))(a - ua)\\ &= 0.\end{aligned}$$

By (ii), the following statements hold.

(iii) If $d_A(u) = 0$, then $\theta = \phi = \gamma$ or $d_A$ is zero on $A$.
(iv) If $A$ has the identity, then $\theta = \phi = \gamma$ or $d_A$ is zero on $A$.
(v) If $A$ is a Banach algebra without identity, then $\phi d_B = 0$.

For a mapping $T$, we denote the range of $T$ by $\mathrm{Im}(T)$. We now give some corollaries of Theorem 2.1.

**Corollary 2.2** *Let* $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ *and* $\theta \neq \phi$. *Then, the additive group* $\mathrm{Im}(d)$ *is isomorphic to additive group* $\mathrm{Im}(d_B)$.

***Proof*** Let $\theta \neq \phi$. Then, $d_A = 0$ on $A$. So

$$\mathrm{Im}(d) = \{(-\phi(d_B(b))u, d_B(b)) : b \in B\}.$$

Define the function $\Phi : B \to \mathrm{Im}(d)$ by

$$\Phi(b) = (-\phi(d_B(b))u, d_B(b)).$$

It is obvious that $\Phi$ is a group epimorphism and $\ker(\Phi) = \ker(d_B)$. Therefore, the additive group $\mathrm{Im}(d)$ is isomorphic to the additive group

$$B/\ker(\Phi) = B/\ker(d_B),$$

which is isomorphic to $\mathrm{Im}(d_B)$. □

For $\eta_1, \eta_2 \in \sigma(B)$, an element $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ is called $(\eta_1, \eta_2)$-*inner* if there exist $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$

$$d(a, b) = (a, b) \cdot_{\eta_1} (a_0, b_0) - (a_0, b_0) \cdot_{\eta_2} (a, b).$$

Let us also recall that a Banach algebra $A$ is called *contractible* if, for every Banach $A$-bimodule $E$, every bounded derivation $d : A \to E$ is inner.

**Corollary 2.3** *Let $A$ and $B$ be contractible Banach algebras. If $d$ is a bounded element of* $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$*, then $d$ is $(\eta, \eta)$-inner for all $\eta \in \sigma(B)$.*

***Proof*** Let $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ be bounded. Then, $d_A$ and $d_B$ are bounded. If $A$ and $B$ are contractible, then $d_A = \mathrm{ad}_x$ and $d_B = \mathrm{ad}_y$ for some $x \in A$ and $y \in B$. Hence, $\phi d_B = 0$. Since $A$ is contractible, $u$ is the identity of $A$. Thus, $d_A(u) = 0$. It follows from Theorem 2.1 that

$$d(a, b) = (\mathrm{ad}_x(a), \mathrm{ad}_y(b))$$

for all $a \in A$ and $b \in B$. □

The following lemma is needed to prove our results.

**Lemma 2.4** *Let $\mathfrak{A}$ be any Banach algebra. If $\mathfrak{D} \in$* $\mathrm{Der}(\mathfrak{A})$*, then $\mathfrak{D}^{**} \in$* $\mathrm{Der}(\mathfrak{A}^{**})$.

***Proof*** Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{D} \in \mathrm{Der}(\mathfrak{A})$. Then,

$$\mathfrak{D}^*(\tau)a = \tau\mathfrak{D}(a) + \mathfrak{D}^*(\tau a)$$

and so

$$n\mathfrak{D}^*(\tau) = \mathfrak{D}^*(n\tau) + \mathfrak{D}^{**}(n)\tau$$

for all $n \in \mathfrak{A}^{**}$, $\tau \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$. Hence,

$$\begin{aligned} \langle \mathfrak{D}^{**}(m\Box n), \tau\rangle &= \langle m, \mathfrak{D}^*(n\tau) + \mathfrak{D}^{**}(n)\tau\rangle \\ &= \langle \mathfrak{D}^{**}(m), n\tau\rangle + \langle m\Box\mathfrak{D}^{**}(n), \tau\rangle \\ &= \langle \mathfrak{D}^{**}(m)\Box n + m\Box\mathfrak{D}^{**}(n), \tau\rangle \end{aligned}$$

for all $m, n \in \mathfrak{A}^{**}$ and $\tau \in \mathfrak{A}^*$. That is, $\mathfrak{D}^{**}$ is a derivation on $\mathfrak{A}^{**}$. □

Now, we investigate Theorem 2.1 in the case where $A$ has a bounded approximate identity instead of right identity.

**Proposition 2.5** *Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity and $d$ be bounded element of* $\mathrm{Der}(\mathfrak{A} \times_\theta^{\theta,\theta} B)$. *Then, there exist unique derivations $d_1 : \mathfrak{A} \to \mathrm{Z}_t(\mathfrak{A}^{**})$ and $d_2 : B \to \mathrm{Z}_t(B^{**})$ such that for every $a \in \mathfrak{A}$ and $b \in B$*

$$d(a, b) = (d_1(a) - \theta(d_2(b))u, d_2(b)),$$

*where $u$ is a right identity of $A^{**}$.*

***Proof*** Let $d \in \mathrm{Der}(\mathfrak{A} \times_\theta^{\theta,\theta} B)$ be bounded. By [25], $(\mathfrak{A} \times_\theta B)^{**}$ is isometrically isomorphic with $\mathfrak{A}^{**} \times_\theta B^{**}$. Thus, $d^{**} \in \mathrm{Der}(\mathfrak{A}^{**} \times_\theta^{\theta,\theta} B^{**})$. Since $\mathfrak{A}$ has a bounded approximate identity, $\mathfrak{A}^{**}$ has a right identity, say $u$. So there exist $d^{**}_{\mathfrak{A}^{**}} \in \mathrm{Der}(\mathfrak{A}^{**})$ and $d^{**}_{B^{**}} \in \mathrm{Der}(B^{**})$ such that

$$d^{**}(m, n) = (d^{**}_{\mathfrak{A}^{**}}(m) - \theta(d^{**}_{B^{**}}(n))u, d^{**}_{B^{**}}(n))$$

for all $m \in \mathfrak{A}^{**}$ and $n \in B^{**}$. By weak*–weak* continuity of $d^{**}$, we have $d^{**}_{\mathfrak{A}^{**}}$ is weak*–weak* continuous on $\mathfrak{A}^{**}$. Hence, for every $a \in \mathfrak{A}$, the mapping

$$n \longmapsto d^{**}_{\mathfrak{A}^{**}}(a)\Box n$$

is weak*–weak* continuous on $\mathfrak{A}^{**}$. Thus, $d^{**}_{\mathfrak{A}^{**}}(a) \in \mathrm{Z}_t(\mathfrak{A}^{**})$. That is, $d^{**}_{\mathfrak{A}^{**}}$ is a derivation from $\mathfrak{A}$ into $\mathrm{Z}_t(\mathfrak{A}^{**})$. Similarly, $d^{**}_{B^{**}}$ is a derivation from $B$ into $\mathrm{Z}_t(B^{**})$. The proof will be complete, if we only note that $d^{**}$ agree with $d$ on $\mathfrak{A}$. □

Let us recall that a Banach algebra $A$ is called *amenable* if every bounded derivation from $A$ into every dual Banach $A$-bimodule $E$ is inner.

**Corollary 2.6** *Let $\mathfrak{A}$ and $B$ be amenable Banach algebras. If $d$ is a bounded element of* $\mathrm{Der}(\mathfrak{A} \times_\theta^{\theta,\theta} B)$*, then there exist $m \in \mathfrak{A}^{**}$ and $n \in B^{**}$ such that*

$$d(a, b) = (\mathrm{ad}_m(a), \mathrm{ad}_n(b))$$

*for all $a \in \mathfrak{A}$ and $b \in B$.*

***Proof*** Let $\mathfrak{A}$ and $B$ be amenable Banach algebras. Then, $\mathfrak{A}$ and $B$ both have a bounded approximate identity. By Proposition 2.5, there exist derivations $d_1 : \mathfrak{A} \to \mathfrak{A}^{**}$ and $d_2 : B \to B^{**}$ with

$$d(a, b) = (d_1(a) - \theta(d_2(b))u, d_2(b))$$

for all $a \in \mathfrak{A}$ and $b \in B$. Since $\mathfrak{A}$ and $B$ are amenable, $d_1$ and $d_2$ are inner. Hence, $d_1 = \mathrm{ad}_m$ and $d_2 = \mathrm{ad}_n$ for some $m \in A^{**}$ and $n \in B^{**}$. This shows that $\theta d_2 = 0$. Thus,

$$d(a, b) = (\mathrm{ad}_m(a), \mathrm{ad}_n(b)),$$

as desired. □

## 3 Analogues of Posner's First and Second Theorems

We commence this section with an analogue of Posner's first theorem [24]; see also [8].

**Theorem 3.1** *Let $B$ be prime, $d_1 \in \mathrm{Der}(A \times_{\theta}^{\phi,\phi} B)$ and $d_2 \in \mathrm{Der}(A \times_{\gamma}^{\theta,\theta} B)$. If $d_1d_2$ is a nonzero element of $\mathrm{Der}(A \times_{\gamma}^{\phi,\phi} B)$, then $\theta = \phi = \gamma$ and $d_1d_2$ maps $A \times B$ into $\mathrm{rad}(A)$.*

***Proof*** By Theorem 2.1, there exist $d_{1A}, d_{2A} \in \mathrm{Der}(A)$ and $d_{1B}, d_{2B} \in \mathrm{Der}(B)$ such that

$$d_1(a,b) = (d_{1A}(a) - \phi(d_{1B}(b))u, d_{1B}(b)) \quad \text{and} \quad d_2(a,b) = (d_{2A}(a) - \theta(d_{2B}(b))u, d_{2B}(b))$$

for all $a \in A$ and $b \in B$. Hence,

$$\begin{aligned} d_1d_2(a,b) &= d_1(d_{2A}(a) - \theta(d_{2B}(b))u, d_{2B}(b)) \\ &= (d_{1A}d_{2A}(a) - \theta(d_{2B}(b))d_{1A}(u) - \phi(d_{1B}d_{2B}(b))u, d_{1B}d_{2B}(b)) \end{aligned}$$

for all $a \in A$ and $b \in B$. If $d_1d_2 \in \mathrm{Der}(A \times_{\gamma}^{\phi,\phi} B)$, then there exist $d_A \in \mathrm{Der}(A)$ and $d_B \in \mathrm{Der}(B)$ such that

$$d_1d_2(a,b) = (d_A(a) - \phi(d_B(b))u, d_B(b))$$

for all $a \in A$ and $b \in B$. This shows that $d_B(b) = d_{1B}d_{2B}(b)$ for all $b \in B$. Thus, $d_{1B}d_{2B} \in \mathrm{Der}(B)$. It follows from [24] that $d_B = d_{1B}d_{2B} = 0$ on $B$. Whence

$$d_1d_2(a,b) = (d_A(a), 0) = (d_{1A}d_{2A}(a) - \theta(d_{2B}(b))d_{1A}(u), 0)$$

and hence $d_A(a) = d_{1A}d_{2A}(a)$ for all $a \in A$ and $b \in B$. Hence,

$$d_1d_2(a,b) = (d_A(a), 0) = (d_{1A}d_{2A}(a), 0) \tag{8}$$

for all $a \in A$ and $b \in B$. Now, if $\theta \neq \phi$ or $\theta \neq \gamma$, then $d_{1A} = 0$ or $d_{2A} = 0$. In either case, $d_1d_2 = 0$ by (8). To complete the proof, note that the last assertion of the theorem follows from (8) and Theorem 2 in [8]. □

Let $\eta_1, \eta_2 \in \sigma(B)$. A mapping $T : A \times B \to A \times B$ is called $(\eta_1, \eta_2)$-*centralizing* if for every $a \in A$ and $b \in B$,

$$[T(a,b), (a,b)]_{\eta_1,\eta_2} := T(a,b) \cdot_{\eta_1} (a,b) - (a,b) \cdot_{\eta_2} T(a,b) \in \mathrm{Z}(A) \times \mathrm{Z}(B).$$

In a special case when

$$[T(a,b),(a,b)]_{\eta_1,\eta_2}=0,$$

$d$ is called $(\eta_1,\eta_2)$-*commuting*. In the following, we present some analogues of Posner's second theorem.

**Theorem 3.2** *Let $\eta_1,\eta_2\in\sigma(B)$, $A$ be an algebra without identity and $d\in \mathrm{Der}(A\times_\theta^{\phi,\gamma} B)$. Then, the following statements hold.*

(i) *If the mapping $(a,b)\mapsto[d(a,b),(a,b)]_{\eta_1,\eta_2}$ is $(\eta_1,\eta_2)$-centralizing, then either $\eta_1=\eta_2$ and $\theta=\phi=\gamma$ or $d_A$ maps $A$ into $\mathrm{Z}(A)$.*
(ii) *If the mapping $(a,b)\mapsto[d(a,b),(a,b)]_{\eta_1,\eta_2}$ is $(\eta_1,\eta_2)$-commuting, then either $\eta_1=\eta_2$ and $\theta=\phi=\gamma$ or $d_A$ is zero on $A$*

***Proof*** Let the mapping $(a,b)\mapsto[d(a,b),(a,b)]_{\eta_1,\eta_2}$ be $(\eta_1,\eta_2)$-centralizing. Then, the mapping $a\mapsto[d_A(a),a]$ is centralizing. Thus,

$$d_A(u)=[[d_A(u),u],u]\in\mathrm{Z}(A).$$

This implies that

$$d_A(u)=d_A(u)u=ud_A(u)=0.$$

Hence, $\theta=\phi=\gamma$ or $d_A=0$ on $A$.

Since $A$ is an algebra without identity, $\phi d_B=0$ on $B$. So our hypothesis gives

$$2(\eta_1-\eta_2)(b)[d_A(a),a]+(\eta_1-\eta_2)(b)(\eta_1-\eta_2)(d_B(b))a \\ +(\eta_1-\eta_2)(b)^2d_A(a)\in\mathrm{Z}(A). \tag{9}$$

Let us substitute $a=u$ in (9). Then,

$$(\eta_1-\eta_2)(b)(\eta_1-\eta_2)(d_B(b))u\in\mathrm{Z}(A),$$

which implies that

$$(\eta_1-\eta_2)(b)(\eta_1-\eta_2)(d_B(b))=0.$$

The substitution $-b$ for $b$ in (9) leads to

$$-2(\eta_1-\eta_2)(b)[d_A(a),a]+(\eta_1-\eta_2)(b)^2d_A(a)\in\mathrm{Z}(A). \tag{10}$$

Subtracting (9) from (10), we arrive at

$$2(\eta_1-\eta_2)(b)[d_A(a),a]\in\mathrm{Z}(A).$$

This together with (10) shows that

$$(\eta_1 - \eta_2)(b)^2 d_A(a) \in \mathrm{Z}(A).$$

So either $\eta_1 = \eta_2$ or $d_A$ maps $A$ into $\mathrm{Z}(A)$. Therefore, (i) holds. A similar argument proves (ii). ◻

As an immediate consequence of Theorem 3.2, we have the following result.

**Corollary 3.3** *Let $\eta_1, \eta_2 \in \sigma(B)$, $A$ be a Banach algebra without identity and $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$. Then, the following statements hold.*

(i) *If $d$ is $(\eta_1, \eta_2)$-centralizing, then either $\eta_1 = \eta_2$ and $\theta = \phi = \gamma$ or $d_A$ maps $A$ into $\mathrm{Z}(A)$.*
(ii) *If $d$ is $(\eta_1, \eta_2)$-commuting, then either $\eta_1 = \eta_2$ and $\theta = \phi = \gamma$ or $d_A$ is zero on $A$.*

**Theorem 3.4** *Let $\eta_1, \eta_2 \in \sigma(B)$ and $A$ be a Banach algebra without identity. If $d_1$ and $d_2$ are elements of $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ satisfying $d_1(a, b) \cdot_{\eta_1} (a, b) - (a, b) \cdot_{\eta_2} d_2(a, b) \in \mathrm{Z}(A) \times \mathrm{Z}(B)$ for every $a \in A$ and $b \in B$, then either $\eta_1 = \eta_2$ and $\theta = \phi = \gamma$ or $d_1$ and $d_2$ map $A$ into $\mathrm{Z}(A)$.*

***Proof*** In view of Theorem 2.1, there exist $d_{1A}, d_{2A} \in \mathrm{Der}(A)$ and $d_{1B}, d_{2B} \in \mathrm{Der}(B)$ such that

$$d_1(a, b) = (d_{1A}(a) + (\theta - \gamma)(b) d_{1A}(u) - \phi(d_{1B}(b))u, d_{1B}(b))$$

and

$$d_2(a, b) = (d_{2A}(a) + (\theta - \gamma)(b) d_{2A}(u) - \phi(d_{2B}(b))u, d_{2B}(b))$$

for all $a \in A$ and $b \in B$. Hence,

$$\begin{aligned} d_{1A}(a)a - a d_{2A}(a) &+ (\theta - \gamma)(b) d_{1A}(u)a - \phi(d_{1B}(b))ua \\ &+ \phi(d_{2B}(b))a + \eta_1(d_{1B}(b))a - \eta_2(d_{2B}(b))a + \eta_1(b) d_{1A}(a) \\ &- \eta_2(b) d_{2A}(a) + \eta_1(b)(\theta - \gamma)(b) d_{1A}(u) - \eta_1(b)\phi(d_{1B}(b))u \\ &- \eta_2(b)(\theta - \gamma)(b) d_{2A}(u) + \eta_2(b)\phi(d_{2B}(b))u \\ &\in \mathrm{Z}(A). \end{aligned} \tag{11}$$

From this, we infer that

$$d_{1A}(a)a - a d_{2A}(a) \in \mathrm{Z}(A). \tag{12}$$

So

$$d_{1A}(u) = 0.$$

Hence, $\theta = \phi = \gamma$ or $d_{1A}$ is zero on $A$. Setting $a = 0$ in (11), the following relation obtains.

$$\begin{aligned} & -\phi(d_{1B}(b))ua + \phi(d_{2B}(b))a + \eta_1(d_{1B}(b))a \\ & \quad - \eta_2(d_{2B}(b))a + \eta_1(b)d_{1A}(a) \\ & \quad - \eta_2(b)d_{2A}(a) \\ & \in Z(A). \end{aligned} \tag{13}$$

Substituting $a = u$ in (13), we get

$$\begin{aligned} & -\phi(d_{1B}(b))u + \phi(d_{2B}(b))u + \eta_1(d_{1B}(b))u \\ & \quad - \eta_2(d_{2B}(b))u - \eta_2(b)d_{2A}(u) \\ & \in Z(A). \end{aligned}$$

It follows that

$$\begin{aligned} & -\phi(d_{1B}(b))u + \phi(d_{2B}(b))u + \eta_1(d_{1B}(b))u \\ & \quad - \eta_2(d_{2B}(b))u - \eta_2(b)d_{2A}(u) \\ & = -\phi(d_{1B}(b))u + \phi(d_{2B}(b))u \\ & \quad + \eta_1(d_{1B}(b))u - \eta_2(d_{2B}(b))u \\ & \quad - \eta_2(b)ud_{2A}(u). \end{aligned}$$

This shows that

$$d_{2A}(u) = 0.$$

Thus,

$$\begin{aligned} & -\phi(d_{1B}(b))u + \phi(d_{2B}(b))u \\ & \quad + \eta_1(d_{1B}(b))u - \eta_2(d_{2B}(b))u \\ & \in Z(A). \end{aligned}$$

Hence,

$$\phi(d_{1B}(b)) - \phi(d_{2B}(b)) = \eta_1(d_{1B}(b)) - \eta_2(d_{2B}(b)).$$

So relation (13) becomes

$$\eta_1(b)d_{1A}(a) - \eta_2(b)d_{2A}(a) \in Z(A). \tag{14}$$

If we replace $a$ by $a + u$ in (12), then

$$d_{1A}(a) - ud_{2A}(a) \in Z(A).$$

It follows that

$$-\eta_1(b)d_{1A}(a)+\eta_1(b)ud_{2A}(a)\in \mathrm{Z}(A) \tag{15}$$

and

$$\eta_2(b)d_{1A}(a)-\eta_2(b)ud_{2A}(a)\in \mathrm{Z}(A). \tag{16}$$

From (14) and (15), we have

$$\eta_1(b)ud_{2A}(a)-\eta_2(b)d_{2A}(a)\in \mathrm{Z}(A). \tag{17}$$

This implies that

$$\eta_2(b)d_{2A}(a)=\eta_2(b)ud_{2A}(a).$$

Hence,

$$d_{2A}(a)=ud_{2A}(a). \tag{18}$$

This together with (17) shows that

$$(\eta_1-\eta_2)(b)d_{2A}(a)\in \mathrm{Z}(A).$$

So, $\eta_1=\eta_2$ or $d_{2A}(a)\in \mathrm{Z}(A)$. On the hand, from (16) and (18), we obtain

$$\eta_2(b)d_{1A}(a)-\eta_2(b)d_{2A}(a)\in \mathrm{Z}(A). \tag{19}$$

Regarding (14) and (19), we have

$$(\eta_1-\eta_2)(b)d_{1A}(a)\in \mathrm{Z}(A)$$

for all $a\in A$ and $b\in B$. So $\eta_1=\eta_2$ or $d_{1A}(a)\in \mathrm{Z}(A)$. Therefore, the proof is complete. □

For a ring $R$, a map $T: R\to R$ is called *strong commutativity preserving* if

$$[T(r),T(s)]=[r,s]$$

for all $r,s\in R$. Derivation as well as strong commutativity preserving mappings have been studied by several authors; see, for example, [3,4]. In the next result, we investigate this concept for elements of $\mathrm{Der}(A\times_\theta^{\phi,\gamma} B)$.

**Theorem 3.5** *Let $\eta_i,\rho_i\in\sigma(B)$ for $i=1,2$ and $A$ be a Banach algebra without identity. If $d$ is an element of $\mathrm{Der}(A\times_\theta^{\phi,\gamma} B)$ satisfying $[d(a,b),d(x,y)]_{\eta_1,\eta_2}=[(a,b),(x,y)]_{\rho_1,\rho_2}$ for all $a,x\in A$ and $b,y\in B$, then $\theta=\phi$, $\rho_1=\rho_2$ and $\eta_1 d_B=\eta_2 d_B$.*

***Proof*** By hypothesis, we have

$$\begin{aligned}&[d_A(a), d_A(x)] - (\theta - \gamma)(y)d_A(u)d_A(a) + (\theta - \gamma)(b)d_A(u)d_A(x)\\&\quad + (\eta_1 - \eta_2)(d_B(b))d_A(x) + (\eta_1 - \eta_2)(d_B(y))d_A(a)\\&\quad + (\eta_1 - \eta_2)(d_B(b))(\theta - \gamma)(y)d_A(u)\\&\quad + (\eta_1 - \eta_2)(d_B(y))(\theta - \gamma)(b)d_A(u)\\&= [a, x] + (\rho_1 - \rho_2)(b)x + (\rho_1 - \rho_2)(y)a.\end{aligned}$$

Substitute $a = x = 0$ in the above relation. Then, it reduces to

$$\begin{aligned}&[d_A(a), d_A(x)] + (\eta_1 - \eta_2)(d_B(b))d_A(x) + (\eta_1 - \eta_2)(d_B(y))d_A(a)\\&= [a, x] + (\rho_1 - \rho_2)(b)x + (\rho_1 - \rho_2)(y)a.\end{aligned} \tag{20}$$

If we set $a = x = u$ in (20), then

$$(\eta_1 - \eta_2)(d_B(b))d_A(u) + (\eta_1 - \eta_2)(d_B(y))d_A(u) = (\rho_1 - \rho_2)(b)u + (\rho_1 - \rho_2)(y)u.$$

Since $d_A(u) \in \mathrm{ran}(A)$, we obtain $\rho_1 = \rho_2$. Hence, (20) becomes

$$\begin{aligned}&[d_A(a), d_A(x)] + (\eta_1 - \eta_2)(d_B(b))d_A(x)\\&\quad + (\eta_1 - \eta_2)(d_B(y))d_A(a)\\&= [a, x].\end{aligned} \tag{21}$$

Taking $a = 0$ in (21), we get

$$(\eta_1 - \eta_2)(d_B(b))d_A(x) = 0. \tag{22}$$

If $d_A$ is zero, then by (21)

$$[a, x] = [d_A(a), d_A(x)] = 0$$

for all $a, x \in A$. Hence, $A$ is commutative and so $A$ has the identity element. This contradiction shows that $d_A$ is nonzero. Thus, $\theta = \phi$ and by (22), $\eta_1 d_B = \eta_2 d_B$. □

It is easy to see that if $R$ is a prime ring, $rs, s \in \mathrm{Z}(R)$ and $s$ is nonzero, then $r \in \mathrm{Z}(R)$.

**Proposition 3.6** *Let $\eta \in \sigma(B)$ and $B$ be prime. If $d$ is an element of* $\mathrm{Der}(A \times_\theta^{\phi,\gamma} B)$ *satisfying $d((a, b) \cdot_\theta (x, y)) - (a, b) \cdot_\eta (x, y) \in \mathrm{Z}(A) \times \mathrm{Z}(B)$, then either $\theta = \eta$ or $A \times_\theta B$ is commutative. In both cases, $d_A$ maps $A$ into* $\mathrm{rad}(A)$.

***Proof*** For every $a, x \in A$ and $b, y \in B$, we have

$$\begin{aligned}&d_A(ax) + \theta(b)d_A(x) + \theta(y)d_A(a)\\&\quad + (\theta - \gamma)(by)d_A(u) - \phi(d_B(by))u\\&\quad - (ax + \eta(b)x + \eta(y)a) \in \mathrm{Z}(A)\end{aligned} \tag{23}$$

and

$$d_B(by) - by \in Z(B). \tag{24}$$

Suppose that $Z(B) = \{0\}$. Then,

$$d_B([b, y]) = [b, y] \tag{25}$$

for all $b, y \in B$. Replacing $y$ by $yb$ in (25), we have

$$[b, y]d_B(b) = 0.$$

The substitution $d_B(b)y$ for $y$ in the above relation leads to

$$[b, d_B(b)]yd_B(b) = 0,$$

which yields $[b, d_B(b)] = 0$ for all $b \in B$ by primeness of $B$. Hence, $d_B$ is centralizing on $B$. So either $d_B = 0$ or $B$ is commutative [24]. This and (25) imply that $B$ is commutative. Thus,

$$B = Z(B) = \{0\}.$$

This contradictions shows that $Z(B) \neq \{0\}$. Choose a nonzero element in $Z(B)$, say $b_0$. It follows from (24) that

$$d_B(bb_0) - bb_0 \in Z(B) \tag{26}$$

and

$$d_B(bb_0^2) - bb_0^2 \in Z(B) \tag{27}$$

for all $b \in B$. Since

$$d_B(bb_0^2) = d_B(bb_0)b_0 + bb_0d_B(b_0),$$

from (27) we have

$$(d_B(bb_0) - bb_0)b_0 + bb_0d_B(b_0) \in Z(B).$$

This together with (26) follows that $bd_B(b_0)b_0 \in Z(B)$ and so $bd_B(b_0) \in Z(B)$ for all $b \in B$. That is,

$$d_B(bb_0) - d_B(b)b_0 \in Z(B)$$

for all $b \in B$. From this and (26), we infer that $(d_B(b) - b)b_0 \in Z(B)$. Thus,

$$d_B(b) - b \in Z(B). \tag{28}$$

This implies that $d_B$ is commuting. Hence, $d_B = 0$ or $B$ is commutative. In every case, it follows from (28) that $B$ is commutative.

On the hand, by (23) we find that

$$d_A(ax) - ax \in Z(A) \tag{29}$$

for all $a, x \in A$. Thus,

$$d_A(u) - u \in Z(A).$$

This implies that

$$d_A(u) = 0,$$

and so $u$ is the identity of $A$. So (23) reduces to

$$\theta(b)d_A(x) + \theta(y)d_A(a) - (\eta(b)x + \eta(y)a) \in Z(A). \tag{30}$$

Let us substitute in (30), $y = 0$. Then,

$$\theta(b)d_A(x) - \eta(b)x \in Z(A). \tag{31}$$

In view of (29), we have

$$d_A(x) - x \in Z(A) \tag{32}$$

for all $x \in A$. Thus, $d_A$ is centralizing and so $d_A$ maps $A$ into $\mathrm{rad}(A)$. From (32), we also have

$$\theta(b)d_A(x) - \theta(b)x \in Z(A)$$

for all $x \in A$ and $b \in B$. By this and (31),

$$(\eta - \theta)(b)x \in Z(A).$$

Therefore, $\eta = \theta$ or $A$ is commutative. □

## 4 Dependent Elements of Der$(A \times_{\theta}^{\phi,\gamma} B)$

An element $(a, b) \in A \times B$ is said to be $(\eta_1, \eta_2)$-*dependent* on $d \in Der(A \times_{\theta}^{\phi,\gamma} B)$ if

$$d(x, y) \cdot_{\eta_1} (a, b) = (a, b) \cdot_{\eta_2} (x, y)$$

for all $x \in A$ and $y \in B$. For the study of dependent elements on derivations of rings, see [6,14,16].

**Theorem 4.1** *Let $\eta_1, \eta_2 \in \sigma(b)$ and $d \in Der(A \times_\theta^{\phi,\gamma} B)$. If $(a, b)$ is $(\eta_1, \eta_2)$-dependent on $d$, then $a = 0$ and $\eta(b) = 0$ for all $\eta \in \sigma(B)$.*

***Proof*** Let $(a, b)$ be $(\eta_1, \eta_2)$-dependent on $d$. Then, $b$ is dependent on $d_B$. So for every $y \in B$, we have

$$d_B(y)b = by. \tag{33}$$

Suppose that $\eta \in \sigma(B)$ and $\eta(b) \neq 0$. Then, by (33) we get

$$\eta(d_B(y)) = \eta(y)$$

for all $y \in B$. Choose $y_0 \in B$ with $\eta(y_0) \neq 0$. Then,

$$\begin{aligned}\eta(y_0)^2 &= \eta(y_0^2) = \eta(d_B(y_0^2)) \\ &= \eta(d_B(y_0))\eta(y_0) \\ &\quad + \eta(y_0)\eta(d_B(y_0)) \\ &= 2\eta(y_0)^2,\end{aligned}$$

a contradiction. Hence, $\eta(b) = 0$. Since $(a, b)$ is dependent on $d$, it follows that $a$ is dependent on $d_A$. Thus,

$$d_A(x)a = ax$$

for all $x \in A$. Thus,

$$a = au = d_A(u)a.$$

Hence,

$$a = d_A(u)d_A(u)a = 0,$$

as claimed. □

The following is an immediate consequence of Theorem 4.1.

**Corollary 4.2** *Let $d \in \mathrm{Der}(A \times_\theta^{\phi,\gamma} \mathbb{C})$. Then, zero is the only dependent element on $d$.*

## 5 Applications to Group Algebras

Let $G$ denote a locally compact group with a fixed left Haar measure $\lambda$ on $G$. Let $L^\infty(G)$ be the usual Lebesgue space as defined in [13] equipped with the essential supremum norm $\|\cdot\|_\infty$ and $L^1(G)$ be the Banach space of integrable functions with respect to $\lambda$. Then, with the norm $\|\cdot\|_1$ and the convolution product "$*$," $L^1(G)$ is a Banach algebra with a bounded approximate identity. Let us remark that $L^\infty(G)$ is the dual of $L^1(G)$ under the usual duality. This let us endow $L^\infty(G)^*$ with the first Arens product. Hence, $L^\infty(G)^*$ with this product is a Banach algebra with a right identity; see [12]. Let $M(G)$ denote the measure algebra of $G$ as defined in [13] endowed with the convolution product "$*$" and the total norm $\|\cdot\|$. Then, $M(G)$ is a Banach algebra with identity element $\delta_e$, the Dirac measure at the identity element $e$ of $G$. Also, $M(G)$ is the dual of $C_0(G)$, the space of all continuous functions on $G$ vanishing at infinity.

**Proposition 5.1** *Let $\mathfrak{A}$ be $M(G)$ or $L^1(G)$ and $B$ be either $M(G)$, $L^1(G)$ or $\mathbb{C}$. If $d \in \mathrm{Der}(\mathfrak{A} \times_\theta^{\phi,\gamma} B)$, then $d$ is $(\eta, \eta)$-inner for all $\eta \in \sigma(B)$.*

***Proof*** First, we assume that $d \in \mathrm{Der}(M(G) \times_\theta^{\phi,\gamma} B)$. It follows from Theorem 2.1 that

$$\begin{aligned} d(\nu_1, \nu_2) &= (d_{M(G)}(\nu_1) + (\theta - \gamma)(\nu_2) d_{M(G)}(\delta_e) \\ &\quad - \phi(d_B(\nu_2))\delta_e, d_B(\nu_2)) \\ &= (d_{M(G)}(\nu_1) - \phi(d_B(\nu_2))\delta_e, d_B(\nu_2)) \end{aligned}$$

for all $\nu_1, \nu_2 \in M(G)$. Since

$$d_{M(G)} \in \mathrm{Der}(M(G)),$$

we have $d_{M(G)} : L^1(G) \to M(G)$ is a derivation. From [2,20], we infer that there exists $\mu_1 \in M(G)$ such that $d_{M(G)} = \mathrm{ad}_{\mu_1}$ on $L^1(G)$. Apply Theorems 2.9.53 and 3.3.40 of [9] to $C_0(G)$. Then, $d_{M(G)} = \mathrm{ad}_{\mu_1}$ on $M(G)$. Similarly, $d_B = \mathrm{ad}_{\mu_2}$ on $B$. So $\phi d_B = 0$. Hence,

$$d(\nu_1, \nu_2) = (\mathrm{ad}_{\mu_1}(\nu_1), \mathrm{ad}_{\mu_2}(\nu_2))$$

for all $\nu_1, \nu_2 \in M(G)$. Therefore, $d$ is $(\eta, \eta)$-inner for all $\eta \in \sigma(B)$.

Now, let $d \in \mathrm{Der}(L^1(G) \times_\theta^{\theta,\theta} B)$. Then, by Proposition 2.5, there are derivations $d_1 : L^1(G) \to \mathrm{Z}_t(L^\infty(G)^*)$ and $d_2 : B \to \mathrm{Z}_t(B^{**})$ such that for every $f \in L^1(G)$ and $g \in B$

$$d(f, g) = (d_1(f) - \theta(d_2(g))u, d_2(g)),$$

where $u$ is a right identity of $L^\infty(G)^*$. It is well known from [18,21] that

$$\mathrm{Z}_t(B^{**}) = B.$$

Hence, $d_1$ and $d_2$ are derivations on $L^1(G)$ and $B$, respectively. So $d_1 = \mathrm{ad}_{\mu_1}$ and $d_2 = \mathrm{ad}_{\mu_2}$ for some $\mu_1, \mu_2 \in M(G)$. That is, $d$ is $(\eta, \eta)$-inner for all $\eta \in \sigma(B)$. □

Let $A(G)$ be the Fourier algebra of $G$ as defined in [10] and recall that

$$\sigma(A(G)) = \{\varepsilon_t : t \in G\},$$

where $\varepsilon_t$ denotes the evaluation functional at $t$.

**Proposition 5.2** *Let G be a locally compact group. Then, the following statements hold.*

(i) *The only dependent element on derivations of $A(G)$ is zero.*
(ii) *If G is abelian, the statement* (i) *holds for $M(G)$ and $L^1(G)$ instead of $A(G)$.*

***Proof*** Let $B$ be either $A(G)$, $M(G)$ or $L^1(G)$, and let $\mathfrak{D} : B \to B$ be a derivation. We define $d : \{0\} \times B \to \{0\} \times B$ by

$$d(0, b) = (0, \mathfrak{D}(b)).$$

It is easy to see that $d \in Der(\{0\} \times_{\theta}^{\theta,\theta} B)$. If $\vartheta \in B$ is dependent on $\mathfrak{D}$, then $(0, \vartheta)$ is dependent on $d$. By Theorem 4.1, $\eta(\vartheta) = 0$ for all $\eta \in \sigma(B)$. It follows that $\vartheta = 0$ in the case where $B = A(G)$. So (i) holds. For (ii), suppose that $\vartheta$ is a nonzero measure in $M(G)$. From Theorem 23.11 of [13], we infer that

$$\int_G \bar{\chi} \, \mathrm{d}\vartheta \neq 0$$

for some $\chi \in \widehat{G}$, the character group of $G$. We define the mapping $\eta_0 : M(G) \to \mathbb{C}$ by

$$\eta_0(\mu) = \int_G \bar{\chi} \, \mathrm{d}\mu.$$

Then, $\eta_0 \in \sigma(M(G))$ and $\eta_0(\vartheta) \neq 0$; see Theorem 23.4 of [13]. Therefore, zero is the only dependent element on derivations of $M(G)$. The result is proved for $L^1(G)$ similarly. □

Before, we give some applications of our results to the Banach algebra $L^\infty(G)^*$, let us recall that $\mathrm{LUC}(G)$ denotes the Banach space of all bounded continuous functions $f$ on $G$ such that the mapping $t \mapsto f_t$ from $G$ into $C(G)$ is continuous, where $f_t(s) = f(ts)$ for all $t, s \in G$.

**Theorem 5.3** *Let $\eta_1, \eta_2 \in \sigma(B)$ and $d$ be an $(\eta_1, \eta_2)$-centralizing element of* $\mathrm{Der}(L^\infty(G)^* \times_{\theta}^{\phi,\gamma} B)$*. Then, either $\eta_1 = \eta_2$ and $\theta = \phi = \gamma$ or $d_{L^\infty(G)^*}$ is zero on $L^\infty(G)^*$.*

***Proof*** Let $d$ be an $(\eta_1, \eta_2)$-centralizing element of $\mathrm{Der}(L^\infty(G)^* \times_\theta^{\phi,\gamma} B)$. Then, $d_{L^\infty(G)^*}$ is centralizing on $L^\infty(G)^*$. Hence,

$$d_{L^\infty(G)^*}(m)\Box m - m\Box d_{L^\infty(G)^*}(m) \in \mathrm{Z}(L^\infty(G)^*).$$

Replacing $m$ by $m + u$ in the above relation, we have

$$d_{L^\infty(G)^*}(m) - u\Box d_{L^\infty(G)^*}(m) \in \mathrm{Z}(L^\infty(G)^*)$$

for all $m \in L^\infty(G)^*$. Thus,

$$d_{L^\infty(G)^*}(m) - u\Box d_{L^\infty(G)^*}(m) = u\Box d_{L^\infty(G)^*}(m) - u\Box d_{L^\infty(G)^*}(m).$$

So

$$d_{L^\infty(G)^*}(m) = u\Box d_{L^\infty(G)^*}(m)$$

for all $m \in L^\infty(G)^*$. This together with [23] yields that $d_{L^\infty(G)^*}$ maps $L^\infty(G)^*$ into $\mathrm{rad}(u\Box L^\infty(G)^*)$. On the hand, by [12,19], the Banach algebra $u\Box L^\infty(G)^*$ is isometrically isomorphic to the Banach algebra $\mathrm{LUC}(G)^*$ and

$$\mathrm{LUC}(G)^* = M(G) \oplus C_0(G)^\perp,$$

where

$$C_0(G)^\perp := \{H \in \mathrm{LUC}(G)^* : H|_{C_0(G)} = 0\}.$$

Thus, $\mathrm{LUC}(G)^*/C_0(G)^\perp$ is isomorphic to the semisimple Banach algebra $M(G)$. Hence,

$$\mathrm{rad}(\mathrm{LUC}(G)^*) \subseteq C_0(G)^\perp.$$

It follows that $d_{L^\infty(G)^*}$ maps $L^\infty(G)^*$ into $C_0(G)^\perp$.

Now, let $d_{L^\infty(G)^*}$ be nonzero on $L^\infty(G)^*$. Then, $\theta = \phi = \gamma$, because

$$d_{L^\infty(G)^*}(u) = 0.$$

Suppose that $\eta_1 \neq \eta_2$. For every $m \in L^\infty(G)^*$, we have

$$d_{L^\infty(G)^*}(m) \in \mathrm{Z}(L^\infty(G)^*)$$

by Corollary 3.3. Thus,

$$d_{L^\infty(G)^*}(m) \in \mathrm{Z}_t(L^\infty(G)^*) = L^1(G).$$

So

$$d_{L^\infty(G)^*}(m) \in L^1(G) \cap C_0(G)^\perp = \{0\}.$$

Therefore, $d_{L^\infty(G)^*} = 0$ on $L^\infty(G)^*$, a contradiction. □

Let $j$ be a fixed integer. The additive group of integers modulo $j$ is denoted by $\mathbb{Z}_j$.

**Proposition 5.4** *Let $G$ be a compact group and $A = L^\infty(G)^*$ in Proposition 3.6. Then, $\theta = \eta$ or $G$ is isomorphic with $\mathbb{Z}_{j_1} \bigoplus \cdots \bigoplus \mathbb{Z}_{j_\ell}$ for some positive integer $j_1, \ldots, j_\ell$.*

***Proof*** Let $L^\infty(G)^* \times B$ be commutative. Then, $L^\infty(G)^*$ is commutative. Hence, $L^1(G)$ is commutative and

$$L^\infty(G)^* = L^\infty(G)^* \Box u = u \Box L^\infty(G)^* = LUC(G)^*,$$

where $u$ is a right identity of $L^\infty(G)^*$. So $G$ is abelian and discrete. Since $G$ is also compact, it is a finite abelian group. Hence, $G$ is isomorphic with $\mathbb{Z}_{j_1} \bigoplus \cdots \bigoplus \mathbb{Z}_{j_\ell}$ for some positive integer $j_1, \ldots, j_\ell$. This fact and Proposition 3.6 prove the result. □

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