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Fractional Trudinger–Moser Type Inequalities in One Dimension

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Abstract

We study the optimal fractional Trudinger–Moser inequalities on $\mathbb{R}$ when the integrands have the form $\left(e^{\pi u^2} - 1\right)|u|^{2a}$ for some $a \geq 0$. The equivalence of the subcritical and critical fractional Trudinger–Moser inequalities is set up in the spirit of Lam, Lu and Zhang. The existence of optimizers for the sharp subcritical fractional Trudinger–Moser inequalities is also investigated.

Keywords Trudinger–Moser inequality · Unbounded domains · Critical growth · Extremal function · Sharp constants · Fractional Laplacian

Mathematics Subject Classification Primary 35A23; Secondary 26D15 · 46E35

1 Introduction

The Trudinger–Moser inequalities are considered as the border line cases of the well-known Sobolev embeddings. They have been studied widely in the last 40 years, and there is a vast literature. Basically, the Trudinger–Moser inequalities provide that in the limiting case $p = N \geq 2$, the Sobolev space $W_0^{1,N}(\Omega)$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, can be embedded continuously into the Orlicz space $L_{\varphi_N}(\Omega)$ where $\varphi_N(t) = \exp\left(\alpha |t|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. This was first studied independently

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by Pohozaev [18], Trudinger [20] and Yudovich [21] without the optimal constant. This answers the question what the optimal target for the Sobolev embedding is when $p = N$. Indeed, it is clear that in this case, $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ continuously for all $q \geq 1$. However, we could construct counterexamples to show that $W_0^{1,N}(\Omega) \nsubseteq L^\infty(\Omega)$.

Motivated by the applications to the prescribed Gauss curvature problem, J. Moser optimized in [17] the above embedding. More precisely, using symmetrization arguments, he proved that

Theorem A *Let Ω be a domain with finite measure in Euclidean N-space $\mathbb{R}^N$, $N \geq 2$. Then, there exists a constant $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}} > 0$, where ω_{N-1} is the area of the surface of the unit N-ball, such that*

$$\sup_{\|\nabla u\|_N \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \mathrm{d}x < \infty \tag{1.1}$$

Moreover, the constant α_N is optimal in the sense that if we replace α_N by any number $\alpha > \alpha_N$, then

$$\sup_{\|\nabla u\|_N \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-1}}\right) \mathrm{d}x = \infty. \tag{1.2}$$

The fact that α_N is optimal was checked by Moser using the so-called Moser sequence:

$$u_n(x) = \begin{cases} \left(\frac{1}{\omega_{N-1}}\right)^{1/N} \left(\frac{n}{N}\right)^{\frac{N-1}{N}}, & 0 \leq |x| \leq e^{-\frac{n}{N}}, \\ \left(\frac{N}{\omega_{N-1} n}\right)^{1/N} \log\left(\frac{1}{|x|}\right), & e^{-\frac{n}{N}} < |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Moreover, from the Moser sequence, we can actually show that for any $a > 0$:

$$\sup_{\|\nabla u\|_N \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_N |u|^{\frac{N}{N-1}}\right) |u|^a \,\mathrm{d}x = \infty.$$

Hence, the inequality (1.1) is indeed sharp in this sense.

When $|\Omega| = \infty$, (1.1) is meaningless. Thus, it becomes interesting and nontrivial to extend such inequalities to domains of infinite volume. In this direction, we state the following three such results in the Euclidean spaces that could be found in [1,3,6, 13,15,16,19]:

Theorem B *Let $0 \leq \alpha < \alpha_N$. There hold*

$$\mathrm{STM}(\alpha) := \sup_{u \in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N \leq 1} \frac{1}{\|u\|_N^N} \int_{\mathbb{R}^N} \phi_N\left(\alpha |u|^{\frac{N}{N-1}}\right) \mathrm{d}x < \infty. \tag{1.3}$$

$$\mathrm{TM} := \sup_{u \in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N^N + \|u\|_N^N \leq 1} \int_{\mathbb{R}^N} \phi_N\left(\alpha_N |u|^{\frac{N}{N-1}}\right) \mathrm{d}x < \infty. \tag{1.4}$$

$$\mathrm{TME} := \sup_{u\in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N\le 1} \frac{1}{\|u\|_N^N}\int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N |u|^{\frac{N}{N-1}}\right)}{\left(1+|u|^{\frac{N}{N-1}}\right)}\mathrm{d}x < \infty. \tag{1.5}$$

Here,

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2}\frac{t^j}{j!}.$$

Moreover, the constant α_N is sharp.

Another interesting question is that whether we can prove the Trudinger–Moser inequalities in the one-dimensional case, namely, the fractional Trudinger–Moser inequality on $\mathbb{R}$. In [7], the authors established the following results:

Theorem C *Let $p > 1$ and*

$$\alpha_p = \frac{1}{2}\left[2\cos\left(\frac{\pi}{2p}\right)\Gamma\left(\frac{1}{p}\right)\right]^{\frac{p}{p-1}}.$$

Then, for any interval $I \Subset \mathbb{R}$:

$$\sup_{u\in \widetilde{H}^{\frac{1}{p},p}(I):\ \int_I \left|(-\Delta)^{\frac{1}{2p}}u\right|^p \mathrm{d}x\le 1} \frac{1}{|I|}\int_I \left(e^{\alpha_p|u|^{\frac{p}{p-1}}}-1\right)\mathrm{d}x < \infty;$$

$$\sup_{u\in \widetilde{H}^{\frac{1}{p},p}(I):\ \int_I \left|(-\Delta)^{\frac{1}{2p}}u\right|^p \mathrm{d}x\le 1} \frac{1}{|I|}\int_I \left(e^{\alpha_p|u|^{\frac{p}{p-1}}}-1\right)|u|^{2a}\,\mathrm{d}x = \infty \ \textit{for any } a > 0;$$

$$\sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}):\|u\|_2^2+\left\|(-\Delta)^{\frac{1}{4}}u\right\|_2^2\le 1} \int_{\mathbb{R}} \left(e^{\pi u^2}-1\right)\mathrm{d}x < \infty;$$

and

$$\sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}):\|u\|_2^2+\left\|(-\Delta)^{\frac{1}{4}}u\right\|_2^2\le 1} \int_{\mathbb{R}} \left(e^{\pi u^2}-1\right)|u|^{2a}\,\mathrm{d}x = \infty \ \textit{for any } a > 1. \tag{1.6}$$

The constants α_p and π are sharp.

Motivated by results in [7], the main purpose of this article is to study the Trudinger–Moser inequalities in the spirit of (1.6). Namely, we will investigate the problem where the integrand has the form $\left(e^{\pi u^2}-1\right)|u|^{2a}$ for some $a \geq 0$. Of course in this situation, some extra terms must be added in order to get the finiteness of the supremum. To find out the terms that we should use, we will first study the asymptotic behaviors of the sharp subcritical fractional Trudinger–Moser inequalities in the sense of [5,8,13]. More precisely, define

$$\mathrm{FSTM}_a(\alpha) = \sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}): \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \leq 1} \frac{1}{\|u\|_2^2} \int_{\mathbb{R}} \left(e^{\alpha u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x$$

$$\mathrm{FTM}_a = \sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}): \|u\|_2^2 + \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \leq 1} \|u\|_2^{2a} \int_{\mathbb{R}} \left(e^{\pi u^2}-1\right)\left(\sqrt{\pi}\,|u|\right)^{2a} \mathrm{d}x,$$

then we will prove that

Theorem 1.1 *For any* $a \geq 0$:

$$\mathrm{FSTM}_a(\alpha) \leq \frac{C(a)}{(\pi-\alpha)^{1+a}} \text{ for all } 0 \leq \alpha < \pi.$$

$$\mathrm{FTM}_a < \infty \text{ and } \mathrm{FTM}_a = \sup_{\alpha \in (0,\pi)} \frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}} \mathrm{FSTM}_a(\alpha).$$

As a consequence of Theorem 1.1, we obtain the following weighted Trudinger–Moser inequality that is somewhat related to versions of the Trudinger–Moser inequalities in [4,12]:

Theorem 1.2 *Let* $t \geq 0$ *and* $0 \leq \alpha < \pi$. *Then, there exists* $C(t,a) > 0$ *such that for all* u : $\int_{\mathbb{R}} |u|^2 |x|^t \,\mathrm{d}x < \infty$ *and* $\left\|(-\Delta)^{\frac{1}{4}} u\right\|_2 \leq 1$:

$$\int_{\mathbb{R}} \left(e^{\frac{\alpha}{(t+1)^2} u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} |x|^t \,\mathrm{d}x \leq \frac{C(t,a)}{(\pi-\alpha)^{1+a}} \int_{\mathbb{R}} |u|^2 |x|^t \,\mathrm{d}x$$

Motivated by the results in [9–11,14], our next aim is to study the attainability of the above Trudinger–Moser inequalities. Indeed, we will prove that

Theorem 1.3 *Let* $a \geq 0$. *Then,* $\mathrm{FSTM}_a(\alpha)$ *is attained for all* $0 < \alpha < \pi$. *Also, the mapping* $\mathrm{FSTM}_a(\cdot) : [0,\pi) \to \mathbb{R}^+$ *is continuous.*

2 Preliminary

Let $s \in (0,1)$. We consider $L_s(\mathbb{R})$ the spaces of functions $u \in L^1_{loc}(\mathbb{R})$ such that $\int_{\mathbb{R}} \frac{|u(x)|}{1+|x|^{1+2s}} \mathrm{d}x < \infty$. Then for a function $u \in L_s(\mathbb{R})$, we define $(-\Delta)^s u$ as a tempered distribution as follows:

$$\left\langle (-\Delta)^{s} u, \varphi \right\rangle = \int_{\mathbb{R}} u (-\Delta)^{s} \varphi \mathrm{d}x, \quad \varphi \in \mathcal{S}$$

For $s \in (0, 1)$ and $p \in [1, \infty]$, we define the Bessel-potential space

$$H^{s,p}(\mathbb{R}) = \left\{ u \in L^{p}(\mathbb{R}) : (-\Delta)^{\frac{s}{2}} u \in L^{p}(\mathbb{R}) \right\},$$

and with $I \Subset \mathbb{R}$ is a bounded interval:

$$\widetilde{H}^{s,p}(I) = \left\{ u \in L^{p}(\mathbb{R}) : (-\Delta)^{\frac{s}{2}} u \in L^{p}(\mathbb{R}), u = 0 \text{ on } \mathbb{R} \backslash I \right\}.$$

We also equip these spaces with the norm

$$\|u\|_{H^{s,p}} = \left[\|u\|_{p}^{p} + \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{p}^{p} \right]^{\frac{1}{p}}.$$

We note that this norm is equivalent to the smaller norm $\|u\|_{\widetilde{H}^{s,p}} = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{p}$ on $\widetilde{H}^{s,p}(I)$.

We also recall the following lemmata that can be found in [2]:

Lemma 2.1 *For* $s \in (0, 1)$, *there exists* $C_s > 0$ *such that*

$$\left\| (-\Delta)^{\frac{s}{2}} u \right\|_{2}^{2} = \frac{1}{C_s} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^{2}}{|x - y|^{1+2s}} \mathrm{d}x \mathrm{d}y.$$

Now, for a measurable function $u : \mathbb{R} \to \mathbb{R}$, we set $|u|^{*} : \mathbb{R} \to \mathbb{R}^{+}$ to be its non-increasing symmetric rearrangement. Then, we have the following properties:

Lemma 2.2 *Given a measurable function* $F : \mathbb{R} \to \mathbb{R}$ *and a measurable function* $u : \mathbb{R} \to \mathbb{R}$, *there holds*

$$\int_{\mathbb{R}} F(|u|) \mathrm{d}x = \int_{\mathbb{R}} F\left(|u|^{*}\right) \mathrm{d}x.$$

If $u \in H^{s,2}(\mathbb{R})$ *for* $0 < s < 1$:

$$\int_{\mathbb{R}} \left| (-\Delta)^{\frac{s}{2}} |u|^{*} \right|^{2} \mathrm{d}x \leq \int_{\mathbb{R}} \left| (-\Delta)^{\frac{s}{2}} u \right|^{2} \mathrm{d}x.$$

3 Proof of Theorem 1.1 and Theorem 1.2

Theorem 1.1 will be proved via the following lemmata:

Lemma 3.1 $\mathrm{FSTM}_a(\alpha) \le \frac{C_1(a)}{(\pi-\alpha)^{1+a}}$ *for all* $0 \le \alpha < \pi$ *for some* $C_1(a) > 0$ *depending only on* a.

Proof We just need to consider the case $\frac{\pi}{2} \le \alpha \lessapprox \pi$. We first recall that by Lemma 2.1, there is a universal constant $C_{\frac{1}{2}} > 0$ such that

$$\left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 = \frac{1}{C_{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y.$$

Now, for $u \in H^{\frac{1}{2},2}(\mathbb{R}) : \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \le 1$, we set $v(x) = u(\lambda x)$ where $\lambda = \|u\|_2^2$. Then,

$$\|v\|_2^2 = 1;$$
$$\left\|(-\Delta)^{\frac{1}{4}} v\right\|_2^2 = \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \le 1$$
$$\int_{\mathbb{R}} \left(e^{\alpha v^2} - 1\right)\left(\sqrt{\alpha}\,|v|\right)^{2a} \mathrm{d}x = \frac{1}{\|u\|_2^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

Hence,

$$\mathrm{FSTM}_a(\alpha) = \sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}) : \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \le 1;\ \|u\|_2^2 = 1} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

Moreover, by rearrangement arguments (Lemma 2.2), we can assume that $u \in H^{\frac{1}{2},2}(\mathbb{R}) : \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \le 1;\ \|u\|_2^2 = 1$ and u is even, non-increasing on $[0, \infty)$. Note that in this case

$$u^2(x) \le \frac{1}{2|x|} \int_{-|x|}^{|x|} u^2(y)\,\mathrm{d}y \le \frac{1}{2|x|}.$$

Now, we write for $R = \frac{\pi+\alpha}{\pi-\alpha}\frac{1}{2} \ge \frac{1}{2}$:

$$\int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x = \int_{-R}^{R} + \int_{\mathbb{R}\setminus(-R,R)} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

Then on $\mathbb{R}\setminus(-R,R)$, $|u(x)|\leq 1$ and

$$\int_{\mathbb{R}\setminus(-R,R)}\left(e^{\alpha u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a}\mathrm{d}x\leq\int_{\mathbb{R}\setminus(-R,R)}\alpha u^2e^{\alpha u^2}\left(\sqrt{\alpha}\,|u|\right)^{2a}\mathrm{d}x$$
$$\leq\int_{\mathbb{R}\setminus(-R,R)}\pi e^{\pi}\left(\sqrt{\pi}\right)^{2a}u^2\mathrm{d}x$$
$$\leq\pi e^{\pi}\left(\sqrt{\pi}\right)^{2a}\int_{\mathbb{R}}u^2\mathrm{d}x$$
$$=\pi e^{\pi}\left(\sqrt{\pi}\right)^{2a}.$$

On $I=(-R,R)$, we define a new function

$$v(x)=\begin{cases}u(x)-u(R) & \text{if } |x|\leq R\\ 0 & \text{if } |x|>R\end{cases}$$

Remind that u is non-increasing, we have for a.e. $x\in I$:

$$\int_{\mathbb{R}}\frac{|v(x)-v(y)|^2}{|x-y|^2}\mathrm{d}y=\int_I\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}y+\int_{\mathbb{R}\setminus I}\frac{|u(x)-u(R)|^2}{|x-y|^2}\mathrm{d}y$$
$$\leq\int_{\mathbb{R}}\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}y$$

and for a.e. $x\in\mathbb{R}\setminus I$:

$$\int_{\mathbb{R}}\frac{|v(x)-v(y)|^2}{|x-y|^2}\mathrm{d}y=\int_I\frac{|u(R)-u(y)|^2}{|x-y|^2}\mathrm{d}y$$
$$\leq\int_{\mathbb{R}}\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}y.$$

Hence,

$$\left\|(-\Delta)^{\frac{1}{4}}v\right\|_2^2\leq\left\|(-\Delta)^{\frac{1}{4}}u\right\|_2^2\leq 1.$$

Actually, $v\in\widetilde{H}^{\frac{1}{2},2}(I)$. Also, on I with $\varepsilon=\frac{1}{2}\left(\frac{\pi}{\alpha}-1\right)$

$$u^2(x)\leq(v(x)+u(R))^2\leq(1+\varepsilon)v^2(x)+(1+\frac{1}{\varepsilon})u(R)^2$$
$$\leq\frac{\pi+\alpha}{2\alpha}v^2(x)+\frac{\pi+\alpha}{\pi-\alpha}\frac{1}{2R}=\frac{\pi+\alpha}{2\alpha}v^2(x)+1.$$

Hence,

$$\int_{-R}^{R}\left(e^{\alpha u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a}\mathrm{d}x$$
$$\le \alpha^a\int_{-R}^{R}\left(e^{\frac{\pi+\alpha}{2}v^2(x)+\alpha}-1\right)\left(\frac{\pi+\alpha}{2\alpha}v^2(x)+1\right)^a\mathrm{d}x$$
$$\le 2^{2a}\pi^a e^{\pi}\left(\int_{-R}^{R}e^{\frac{\pi+\alpha}{2}v^2(x)}v^{2a}(x)\,\mathrm{d}x+\int_{-R}^{R}e^{\frac{\pi+\alpha}{2}v^2(x)}\mathrm{d}x\right).$$

Noting that with

$$C(a)=\max_{t\in(0,\infty)}\frac{t^a}{e^t}$$

we have

$$v^{2a}(x)\le\frac{C(a)}{\left(\frac{\pi-\alpha}{2}\right)^a}\exp\left[\left(\frac{\pi-\alpha}{2}\right)|v(x)|^2\right].$$

Also, we note that since $v\in\widetilde{H}^{\frac{1}{2},2}(I)$, by the fractional Trudinger–Moser inequality on $\mathbb{R}$ (Theorem C), there exists an absolute constant $C>0$ such that $\int_{-R}^{R}e^{\pi v^2(x)}\le CR$ and $\int_{-R}^{R}e^{\frac{\pi+\alpha}{2}v^2(x)}\le CR$. Therefore,

$$\int_{-R}^{R}\left(e^{\alpha u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a}\mathrm{d}x$$
$$\le 2^{3a}\pi^a e^{\pi}\frac{C(a)}{(\pi-\alpha)^a}\int_{-R}^{R}e^{\pi v^2(x)}+2^{2a}\pi^a e^{\pi}\int_{-R}^{R}e^{\frac{\pi+\alpha}{2}v^2(x)}$$
$$\le\left(2^{3a}\pi^a e^{\pi}\frac{C(a)}{(\pi-\alpha)^a}+2^{2a}\pi^a e^{\pi}\right)CR$$
$$=\left(2^{3a}\pi^a e^{\pi}\frac{C(a)}{(\pi-\alpha)^a}+2^{2a}\pi^a e^{\pi}\right)C\frac{1}{2}\frac{\pi+\alpha}{\pi-\alpha}$$
$$\le\frac{C_1(a)}{(\pi-\alpha)^{a+1}}$$

for some $C_1(a) > 0$. □

Lemma 3.2 $\mathrm{FTM}_a < \infty$. *Moreover,*

$$\mathrm{FTM}_a = \sup_{\alpha\in(0,\pi)} \frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}} \mathrm{FSTM}_a(\alpha).$$

Proof Recall

$$\mathrm{FTM}_a = \sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}):\|u\|_2^2+\left\|(-\Delta)^{\frac{1}{4}}u\right\|_2^2\le 1} \|u\|_2^{2a}\int_{\mathbb{R}}\left(e^{\pi u^2}-1\right)\left(\sqrt{\pi}\,|u|\right)^{2a}\mathrm{d}x$$

Let $u \in H^{\frac{1}{2},2}(\mathbb{R}) : \|u\|_2^2 + \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \le 1$. Set $\theta = \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2$. Hence, $\|u\|_2^2 \le 1-\theta^2$.

If $\frac{1}{2} < \theta < 1$, then we set

$$v(x) = \frac{u(\lambda x)}{\theta}$$
$$\lambda = \frac{1-\theta^2}{\theta^2} > 0.$$

We get

$$\|v\|_2^2 = \frac{1}{\theta^2\lambda}\|u\|_2^2 \le \frac{1-\theta^2}{\theta^2\lambda} = 1.$$

Also, as above

$$\left\|(-\Delta)^{\frac{1}{4}} v\right\|_2^2 = \frac{1}{\theta^2}\left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 = 1.$$

Hence,

$$\begin{aligned}
&\|u\|_2^{2a}\int_{\mathbb{R}}\left(e^{\pi u^2}-1\right)\left(\sqrt{\pi}\,|u|\right)^{2a}\mathrm{d}x\\
&\le \left(1-\theta^2\right)^a\int_{\mathbb{R}}\left(e^{\pi u^2(\lambda x)}-1\right)\left(\sqrt{\pi}\,|u(\lambda x)|\right)^{2a}d(\lambda x)\\
&= \left(1-\theta^2\right)^a\lambda\int_{\mathbb{R}}\left(e^{\pi\theta^2 v^2(x)}-1\right)\left(\sqrt{\pi}\,|\theta v(x)|\right)^{2a}\mathrm{d}x\\
&= \left(1-\theta^2\right)^a\frac{1-\theta^2}{\theta^2}\int_{\mathbb{R}}\left(e^{\pi\theta^2 v^2(x)}-1\right)\left(\sqrt{\pi\theta^2}\,|v(x)|\right)^{2a}\mathrm{d}x
\end{aligned}$$

$$\leq \frac{\left(1-\theta^2\right)^{a+1}}{\theta^2}\mathrm{FSTM}_a\left(\pi\theta^2\right) \leq \frac{\left(1-\theta^2\right)^{a+1}}{\theta^2} C\left(a\right) \frac{1}{\left(\pi-\pi\theta^2\right)^{a+1}} \leq C\left(a\right).$$

It is also clear to see that when $0 < \theta \leq \frac{1}{2}$

$$\|u\|_2^{2a} \int_{\mathbb{R}} \left(e^{\pi u^2}-1\right)\left(\sqrt{\pi}\,|u|\right)^{2a} \mathrm{d}x \leq C\left(a\right).$$

Also, we can deduce that

$$\begin{aligned} \|u\|_2^{2a} \int_{\mathbb{R}} \left(e^{\pi u^2}-1\right)\left(\sqrt{\pi}\,|u|\right)^{2a} \mathrm{d}x &\leq \frac{\left(1-\theta^2\right)^{a+1}}{\theta^2}\mathrm{FSTM}_a\left(\pi\theta^2\right) \\ &\leq \sup_{\alpha\in(0,\pi)} \frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}}\mathrm{FSTM}_a\left(\alpha\right). \end{aligned}$$

Hence,

$$\mathrm{FTM}_a \leq \sup_{\alpha\in(0,\pi)} \frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}}\mathrm{FSTM}_a\left(\alpha\right).$$

Also, for any $u \in H^{\frac{1}{2},2}\left(\mathbb{R}\right) : \left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \leq 1;\ \ \|u\|_2^2 = 1$. We define

$$v\left(x\right) = \sqrt{\frac{\alpha}{\pi}} u\left(\lambda x\right);\ \lambda = \frac{\alpha}{\pi-\alpha}.$$

Then,

$$\begin{aligned} \left\|(-\Delta)^{\frac{1}{4}} v\right\|_2^2 &= \frac{\alpha}{\pi}\left\|(-\Delta)^{\frac{1}{4}} u\right\|_2^2 \\ \|v\|_2^2 &= \frac{\alpha}{\pi}\frac{1}{\lambda}\|u\|_2^2 = \frac{\alpha}{\pi}\frac{1}{\lambda} = \frac{\pi-\alpha}{\pi} \\ \left\|(-\Delta)^{\frac{1}{4}} v\right\|_2^2 + \|v\|_2^2 &\leq \frac{\alpha}{\pi} + \frac{\alpha}{\pi}\frac{1}{\lambda} = 1. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} \left(e^{\alpha u^2}-1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x &= \lambda \int_{\mathbb{R}} \left(e^{\alpha u^2(\lambda x)}-1\right)\left(\sqrt{\alpha}\,|u\left(\lambda x\right)|\right)^{2a} \mathrm{d}x \\ &= \frac{\alpha}{\pi-\alpha} \int_{\mathbb{R}} \left(e^{\pi v^2(x)}-1\right)\left(\sqrt{\pi}\,|v\left(x\right)|\right)^{2a} \mathrm{d}x \end{aligned}$$

$$\leq \frac{\alpha}{\pi-\alpha}\frac{1}{\|v\|_2^{2a}}\|v\|_2^{2a}\int_{\mathbb{R}}\left(e^{\pi v^2(x)}-1\right)\left(\sqrt{\pi}\,|v(x)|\right)^{2a}\mathrm{d}x$$
$$\leq \frac{\alpha}{\pi-\alpha}\left(\frac{\pi}{\pi-\alpha}\right)^{a}\mathrm{FTM}_a.$$

So

$$\mathrm{FTM}_a \geq \sup_{\alpha\in(0,\pi)}\frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}}\mathrm{FSTM}_a(\alpha).$$

In conclusion:

$$\mathrm{FTM}_a = \sup_{\alpha\in(0,\pi)}\frac{\left(1-\frac{\alpha}{\pi}\right)^{a+1}}{\frac{\alpha}{\pi}}\mathrm{FSTM}_a(\alpha).$$

□

Proof of Theorem 1.2 Set

$$v(y)=\frac{1}{t+1}u\left(|y|^{\frac{1}{t+1}-1}\,y\right).$$

Then,

$$\int_{\mathbb{R}}\left(e^{\frac{\alpha}{(t+1)^2}u^2(x)}-1\right)\left(\sqrt{\alpha}\,|u(x)|\right)^{2a}|x|^t\,\mathrm{d}x$$
$$=\int_{\mathbb{R}}\left(e^{\frac{\alpha}{(t+1)^2}u^2\left(|y|^{\frac{1}{t+1}-1}y\right)}-1\right)\left(\sqrt{\alpha}\left|u\left(|y|^{\frac{1}{t+1}-1}y\right)\right|\right)^{2a}\left||y|^{\frac{1}{t+1}-1}y\right|^t d\left(|y|^{\frac{1}{t+1}-1}y\right)$$
$$=\int_{\mathbb{R}}\left(e^{\alpha v^2(y)}-1\right)\left[\sqrt{\alpha}\,(t+1)\,|v(y)|\right]^{2a}|y|^{\frac{t}{t+1}}\frac{1}{t+1}|y|^{\frac{1}{t+1}-1}\mathrm{d}y$$
$$=C(t,a)\int_{\mathbb{R}}\left(e^{\alpha v^2(y)}-1\right)\left(\sqrt{\alpha}\,|v(y)|\right)^{2a}\mathrm{d}y$$

and

$$\int_{\mathbb{R}}|u|^2|x|^t\,\mathrm{d}x=C(t,a)\int_{\mathbb{R}}|v(y)|^2\,\mathrm{d}y.$$

Also,

$$\left\|(-\Delta)^{\frac{1}{4}}v\right\|_2^2=\frac{1}{C_{\frac{1}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|v(x)-v(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{C_{\frac{1}{2}}} \frac{1}{(t+1)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left| u(|x|^{\frac{1}{t+1}-1} x) - u(|y|^{\frac{1}{t+1}-1} y) \right|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y$$

$$= \frac{1}{C_{\frac{1}{2}}} \frac{1}{(t+1)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(z) - u(w)|^2}{\left| |z|^t z - |w|^t w \right|^2} (t+1) |z|^t \mathrm{d}z \, (t+1) |w|^t \mathrm{d}w$$

$$= \frac{1}{C_{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(z) - u(w)|^2}{|z-w|^2} \frac{|z-w|^2 |z|^t |w|^t}{\left| |z|^t z - |w|^t w \right|^2} \mathrm{d}z \mathrm{d}w.$$

We note here that for all z, t :

$$|z-w|^2 |z|^t |w|^t \leq \left| |z|^t z - |w|^t w \right|^2 \tag{3.1}$$

Indeed, set $z = kw$, then (3.1) is equivalent to

$$|k-1|^2 |k|^t \leq \left| |k|^t k - 1 \right|^2 \Leftrightarrow \left(k^2 - 2k + 1 \right) |k|^t \leq |k|^{2t} k^2 - 2 |k|^t k + 1$$

$$\Leftrightarrow |k|^{2t+2} + 1 \geq |k|^{t+2} + |k|^t \Leftrightarrow \left(|k|^{t+2} - 1 \right) \left(|k|^t - 1 \right) \geq 0$$

Hence,

$$\left\| (-\Delta)^{\frac{1}{4}} v \right\|_2^2 \leq \frac{1}{C_{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(z) - u(w)|^2}{|z-w|^2} \mathrm{d}z \mathrm{d}w = \left\| (-\Delta)^{\frac{1}{4}} u \right\|_2^2 \leq 1.$$

□

Thus, by Theorem 1.1:

$$\int_{\mathbb{R}} \left(e^{\frac{\alpha}{(t+1)^2} u^2(x)} - 1 \right) \left(\sqrt{\alpha} |u(x)| \right)^{2a} |x|^t \mathrm{d}x = C(t, a) \int_{\mathbb{R}} \left(e^{\alpha v^2(y)} - 1 \right) \left(\sqrt{\alpha} |v(y)| \right)^{2a} \mathrm{d}y$$

$$\leq \frac{C(t,a)}{(\pi - \alpha)^{1+a}} \int_{\mathbb{R}} |v(y)|^2 \mathrm{d}y$$

$$\leq \frac{C(t,a)}{(\pi - \alpha)^{1+a}} \int_{\mathbb{R}} |u|^2 |x|^t \mathrm{d}x.$$

4 Proof of Theorem 1.3

We first recall that

$$\mathrm{FSTM}_a(\alpha) = \sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}): \left\| (-\Delta)^{\frac{1}{4}} u \right\|_2^2 \leq 1} \frac{1}{\|u\|_2^2} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) \left(\sqrt{\alpha} |u| \right)^{2a} \mathrm{d}x.$$

Let fix $\alpha \in (0, \pi)$. We then can choose a sequence $(u_n) \in H^{\frac{1}{2},2}(\mathbb{R}) : \left\| (-\Delta)^{\frac{1}{4}} u_n \right\|_2^2 \leq 1$ such that

$$\frac{1}{\|u_n\|_2^2} \int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 \right) \left(\sqrt{\alpha} |u_n| \right)^{2a} \mathrm{d}x \uparrow \mathrm{FSTM}_a(\alpha).$$

Moreover, by rearrangement arguments, we can also assume that u_n is even, non-increasing on $[0, \infty)$. Also, as in Lemma 3.1, we will assume $\|u_n\|_2^2 = 1$. Hence, we can assume that

$$u_n \rightharpoonup u \text{ weakly in } H^{\frac{1}{2},2}(\mathbb{R}).$$

As a consequence,

$$\left\| (-\Delta)^{\frac{1}{4}} u \right\|_2^2 \leq 1 \text{ and } \|u\|_2^2 \leq 1.$$

Noting that

$$u_n^2(x) \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} u_n^2(y)\, \mathrm{d}y \leq \frac{1}{2|x|},$$

we have that for any $\varepsilon > 0$:

$$|u_n(x)| \leq \varepsilon \text{ and } |u(x)| \leq \varepsilon \text{ when } |x| \geq \frac{1}{2\varepsilon}.$$

Now, we distinguish two cases:

Case $a > 0$: Then, we write

$$\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 \right) \left(\sqrt{\alpha} |u_n| \right)^{2a} \mathrm{d}x = \int_{\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} + \int_{\mathbb{R} \setminus \left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - 1 \right) \left(\sqrt{\alpha} |u_n| \right)^{2a} \mathrm{d}x.$$

First,

$$I_\varepsilon = \int_{\mathbb{R} \setminus \left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - 1 \right) \left(\sqrt{\alpha} |u_n| \right)^{2a} \mathrm{d}x \leq \left(\sqrt{\alpha} \varepsilon \right)^{2a} \mathrm{FSTM}_0(\alpha) \downarrow 0 \text{ as } \varepsilon \downarrow 0.$$

On the interval $\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)$, we have that

$$\left(e^{\alpha u_n^2} - 1 \right) \left(\sqrt{\alpha} |u_n| \right)^{2a} \to \left(e^{\alpha u^2} - 1 \right) \left(\sqrt{\alpha} |u| \right)^{2a} \text{ a.e.,}$$

and for some $q \gtrapprox 1$:

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\left(e^{\alpha u_n^2} - 1\right)\left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right]^q \mathrm{d}x \\
&\lesssim \int_{\mathbb{R}} \left[\left(e^{\alpha q u_n^2} - 1\right)\left(\sqrt{\alpha}\,|u_n|\right)^{2aq}\right] \mathrm{d}x \lesssim \mathrm{FSTM}_{aq}(\alpha q).
\end{aligned}$$

Hence,

$$\int_{\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - 1\right)\left(\sqrt{\alpha}\,|u_n|\right)^{2a} \mathrm{d}x \to \int_{\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

Letting $n \to \infty$ and then $\varepsilon \downarrow 0$, we get

$$\mathrm{FSTM}_a(\alpha) \le \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

In other words, $u \neq 0$ and

$$\mathrm{FSTM}_a(\alpha) = \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x \le \frac{1}{\|u\|_2^2}\int_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right)\left(\sqrt{\alpha}\,|u|\right)^{2a} \mathrm{d}x.$$

Hence, $\|u\|_2^2 = 1$ and u is a maximizer for $\mathrm{FSTM}_a(\alpha)$.

Case $a = 0$. Then, we write

$$\begin{aligned}
\int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x &= \int_{\mathbb{R}} \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1\right) \mathrm{d}x + \alpha \\
&= \int_{\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} + \int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1\right) \mathrm{d}x + \alpha.
\end{aligned}$$

In this case,

$$\begin{aligned}
\int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1\right) \mathrm{d}x &\le \int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \alpha u_n^2 \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x \\
&\le \alpha \varepsilon^2 \mathrm{FSTM}_0(\alpha) \to 0 \text{ as } \varepsilon \downarrow 0.
\end{aligned}$$

With the same arguments as in the first case, we get

$$\int\limits_{\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1\right) \mathrm{d}x \to \int\limits_{\left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)} \left(e^{\alpha u^2} - \alpha u^2 - 1\right) \mathrm{d}x$$
$$\leq \int\limits_{\mathbb{R}} \left(e^{\alpha u^2} - \alpha u^2 - 1\right) \mathrm{d}x.$$

In conclusion, we have after letting $n \to \infty$ and then $\varepsilon \downarrow 0$:

$$\mathrm{FSTM}_0(\alpha) \leq \int\limits_{\mathbb{R}} \left(e^{\alpha u^2} - \alpha u^2 - 1\right) \mathrm{d}x + \alpha.$$

Again, it means that $u \neq 0$. Hence,

$$\mathrm{FSTM}_0(\alpha) \leq \frac{1}{\|u\|_2^2} \int\limits_{\mathbb{R}} \left(e^{\alpha u^2} - \alpha u^2 - 1\right) \mathrm{d}x + \alpha$$
$$= \frac{1}{\|u\|_2^2} \int\limits_{\mathbb{R}} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x.$$

Equivalently, $\|u\|_2^2 = 1$ and u is an optimizer for $\mathrm{FSTM}_0(\alpha)$.

Now, let $\varepsilon_n \downarrow 0$ be an arbitrary sequence. We will now show that

$$\mathrm{FSTM}_a(\alpha + \varepsilon_n) - \mathrm{FSTM}_a(\alpha) \downarrow 0$$

and

$$\mathrm{FSTM}_a(\alpha) - \mathrm{FSTM}_a(\alpha - \varepsilon_n) \downarrow 0$$

as $n \to \infty$. Then, we can conclude that $\mathrm{FSTM}_a(\cdot)$ is continuous. Indeed, using the fact the $\mathrm{FSTM}_a(x)$ is attained for all $x \in (0, \pi)$, we can find a sequence $(u_n) \in H^{\frac{1}{2},2}(\mathbb{R}) : \left\|(-\Delta)^{\frac{1}{4}} u_n\right\|_2^2 \leq 1$ and $\|u_n\|_2^2 = 1$ (and so we will assume $u_n \rightharpoonup u$ weakly in $H^{\frac{1}{2},2}(\mathbb{R})$) such that

$$\mathrm{FSTM}_a(\alpha + \varepsilon_n) = \int\limits_{\mathbb{R}} \left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha + \varepsilon_n}\, |u_n|\right)^{2a} \mathrm{d}x.$$

Moreover, again we will assume that u_n is even, non-increasing on $[0, \infty)$. Then, it is clear that

$$0 \leq \mathrm{FSTM}_a(\alpha + \varepsilon_n) - \mathrm{FSTM}_a(\alpha)$$

$$\leq \int_{\mathbb{R}} \left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} \mathrm{d}x - \int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a} \mathrm{d}x$$

$$= \int_{\mathbb{R}} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right] \mathrm{d}x.$$

We will again consider two cases:

Case $a > 0$. We will use the same method as above and write the above integral as

$$\int_{\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} + \int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right] \mathrm{d}x.$$

Then, we have

$$\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a} \downarrow 0 \text{ a.e.}$$

and it is easy to see that there is some $q \gtrapprox 1$ such that

$$\int_{\mathbb{R}} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right]^q \mathrm{d}x \lesssim C(a, \alpha, q).$$

So

$$\int_{\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right] \mathrm{d}x \downarrow 0 \text{ as } n \to \infty.$$

Also, by the Radial Lemma, we have for large n :

$$\int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) \left(\sqrt{\alpha+\varepsilon_n}\,|u_n|\right)^{2a} - \left(e^{\alpha u_n^2} - 1\right) \left(\sqrt{\alpha}\,|u_n|\right)^{2a}\right] \mathrm{d}x$$

$$\leq \left(\sqrt{\alpha+1}\varepsilon\right)^{2a} \mathrm{FSTM}_0\left(\frac{\alpha+\pi}{2}\right) + \left(\sqrt{\alpha}\varepsilon\right)^{2a} \mathrm{FSTM}_0(\alpha) \to 0 \text{ as } \varepsilon \to 0.$$

Hence, in this case

$$\mathrm{FSTM}_a(\alpha+\varepsilon_n) - \mathrm{FSTM}_a(\alpha) \downarrow 0.$$

Case $a = 0$. Then, we will write

$$\int_{\mathbb{R}} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - 1\right) - \left(e^{\alpha u_n^2} - 1\right)\right] \mathrm{d}x$$

$$= \int_{\mathbb{R}} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - (\alpha+\varepsilon_n)\, u_n^2 - 1 \right) - \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1 \right) \right] \mathrm{d}x + \varepsilon_n$$

$$= \int_{\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} + \int_{\mathbb{R}\setminus\left(-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)} \left[\left(e^{(\alpha+\varepsilon_n)u_n^2} - (\alpha+\varepsilon_n)\, u_n^2 - 1 \right) - \left(e^{\alpha u_n^2} - \alpha u_n^2 - 1 \right) \right] \mathrm{d}x + \varepsilon_n.$$

Then, by the same arguments as above, we can also deduce that

$$\mathrm{FSTM}_a\,(\alpha+\varepsilon_n) - \mathrm{FSTM}_a\,(\alpha) \downarrow 0.$$

The fact that

$$\mathrm{FSTM}_a\,(\alpha) - \mathrm{FSTM}_a\,(\alpha-\varepsilon_n) \downarrow 0$$

can be proved similarly.

Hence, we now can conclude that the mapping $\mathrm{FSTM}_a\,(\cdot)$ is continuous.

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