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The Effect of Points Fattening on del Pezzo Surfaces

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Abstract

In this paper, we study the fattening effect of points over the complex numbers for del Pezzo surfaces $\mathbb{S}_r$ arising by blowing-up of $\mathbb{P}^2$ at r general points, with $r \in \{1, \ldots, 8\}$. Basic questions when studying the problem of points fattening on an arbitrary variety are what is the minimal growth of the initial sequence and how are the sets on which this minimal growth happens characterized geometrically. We provide a complete answer for del Pezzo surfaces.

Keywords Initial degree · Initial sequence · Blow-up · Alpha problem · Chudnovsky-type results

Mathematics Subject Classification 52C30 · 14N20 · 05B30

1 Introduction

In this paper, we follow the approach to fat point schemes initiated by Bocci and Chiantini in [3]. The initial degree $\alpha(I)$ of a homogeneous ideal $I \subset \mathbb{C}[\mathbb{P}^n]$ is the least degree t such that the homogeneous component I_t in degree t is nonzero. Although this notion was known since 1981 (see [4]), Bocci and Chiantini used this invariant for the first time in order to study fat point subschemes in the projective plane.

This definition can be extended to symbolic powers $I^{(m)}$ of I; namely, $\alpha(I^{(m)})$ is the least degree t such that the homogeneous component $(I^{(m)})_t$ in degree t is nonzero.

Let $Z \subset \mathbb{P}^2(\mathbb{C})$ be a set of points and I be its radical ideal. By Nagata–Zariski theorem ([7], Theorem 3.14), the ideal of scheme mZ is the mth symbolic power of I. Bocci and Chiantini proved, among others, that sets of points Z in $\mathbb{P}^2(\mathbb{C})$ such that

$$\alpha(I^{(2)}) - \alpha(I) = 1,$$

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are either contained in a single line or form the so-called star configuration.

Results of Bocci and Chiantini have been generalized in [6] by Dumnicki, Szemberg and Tutaj-Gasińska. They were studying configurations of points in $\mathbb{P}^2(\mathbb{C})$ with

$$\alpha(I^{(m+1)}) - \alpha(I^{(m)}) = 1$$

for some $m \geq 2$ and obtained their full characterization (see [6], Theorem 3.4).

These considerations were extended for another types of spaces. Except for spaces $\mathbb{P}^n$, the problem of points fattening was considered among others by me in [1] for the space $\mathbb{P}^1 \times \mathbb{P}^1$ and by Di Rocco, Lundman and Szemberg in [5] for Hirzebruch surfaces (with appropriately modified definition of the initial degree).

The aim of this paper is to make similar classification with respect to points fattening on del Pezzo surfaces. In this paper, a del Pezzo surface is a smooth complex surface X with the ample anticanonical bundle $-K_X$.

In fact, considerations on points fattening effect were initiated on del Pezzo surface $\mathbb{P}^2(\mathbb{C})$ and this path of research was continued to another one, namely $\mathbb{P}^1 \times \mathbb{P}^1$. Over $\mathbb{C}$, there are exactly ten del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and eight surfaces $\mathbb{S}_r$ arising by blowing-up of $\mathbb{P}^2$ in r general points, where $1 \leq r \leq 8$. In this paper, we complete the picture for the last eight del Pezzo surfaces. More precisely, for each of the surfaces $\mathbb{S}_r$ we establish the maximal integer m, such that

$$\alpha(I) = \alpha(I^{(2)}) = \cdots = \alpha(I^{(m)}) = 1$$

holds and we describe subschemes realizing this sequence of equalities. We focus mainly on the smallest possible value of $\alpha(I)$, namely 1. In the case of the surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$, we additionally give characterization of subschemes satisfying a more general condition, namely

$$\alpha(I^{(m)}) = \alpha(I^{(m+1)}) \cdots = \alpha(I^{(m+a)})$$

for some integers m and a. We conclude our paper presenting a Chudnovsky-type inequality.

2 Basic Notions and Auxiliary Facts

The original definition of the initial degree given in [3] was extended in [5] for an arbitrary smooth projective variety with an ample class.

Definition 2.1 *(Initial degree)* Let X be a smooth projective variety with an ample line bundle L on X and let Z be a reduced subscheme of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_Z$. For a positive integer m, the initial degree (with respect to L) of the subscheme mZ is the integer

$$\alpha(mZ) = \alpha(\mathcal{I}_Z^{(m)}) := \min\left\{d :\ H^0(X, dL \otimes \mathcal{I}_Z^{(m)}) \neq 0\right\}.$$

Analogously, the initial sequence (with respect to L) of a subscheme Z is the sequence

$$\alpha(Z), \alpha(2Z), \alpha(3Z), \ldots.$$

The initial sequence is a sequence of positive integers with the following properties:

Fact 2.2

1) The initial sequence is weakly growing, i.e., $\alpha(mZ) \leq \alpha(nZ)$ for $n \geq m$.
2) The initial sequence is subadditive, i.e., $\alpha((m+n)Z) \leq \alpha(mZ) + \alpha(nZ)$.
3) The initial sequence is monotonic with respect to the subscheme, i.e., if $Z \subset W$, then $\alpha(mZ) \leq \alpha(mW)$.

Properties in Fact 2.2 are generally known facts; thus, we take them for granted.

The choice of a line bundle L strictly depends on the variety X. In the projective plane, the initial degree was taken with respect to the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. This line bundle is $\frac{1}{3}$ of the anticanonical bundle $-K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3)$. Similarly, on $\mathbb{P}^1 \times \mathbb{P}^1$ it is natural to work with the α-invariant taken with respect to the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$. In this case, the line bundle is half of the anticanonical divisor as on $\mathbb{P}^1 \times \mathbb{P}^1$ we have $-K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)$.

The most natural choice of an ample line bundle on del Pezzo surfaces $\mathbb{S}_r$ seems to be the anticanonical bundle

$$-K_{\mathbb{S}_r} = 3H - E_1 - \cdots - E_r,$$

which for $r \geq 1$ is not divisible in the Picard group $Pic(\mathbb{S}_r)$.

The fattening effect can be also considered more generally for graded linear series. Let $V_\bullet = \bigoplus_{d \geq 0} V_d \subseteq \bigoplus_{d \geq 0} H^0(X, dL)$ be a graded linear system. We define

$$\alpha_{V_\bullet}(mZ) = \min\{d : \exists s \in V_d : \operatorname{mult}_Z(s) \geq m\}.$$

Then, we have the following property.

Lemma 2.3 *Let $V_\bullet \subseteq W_\bullet$ be graded linear systems. Then,*

$$\alpha_{V_\bullet}(mZ) \geq \alpha_{W_\bullet}(mZ).$$

Proof It follows immediately from the fact that $W_\bullet$ has more sections than $V_\bullet$. □

Corollary 2.4 *Let $t \geq s$ and let $V_\bullet = \bigoplus V_d$ and $W_\bullet = \bigoplus W_d$, where*

$$V_d = H^0\left(\mathbb{S}_t, 3dH - d \cdot \sum_{i=1}^{s} E_i\right) \subseteq H^0(\mathbb{S}_t, 3dH),$$

$$W_d = H^0\left(\mathbb{S}_t, 3dH - d \cdot \sum_{i=1}^{t} E_i\right) \subseteq H^0(\mathbb{S}_t, 3dH).$$

Then, $W_d \subseteq V_d$ for all $d \geq 0$. Lemma 2.3 implies that

$$\alpha_{W_\bullet}(mZ) \geq \alpha_{V_\bullet}(mZ).$$

Remark 2.5 Note that if $Z \subseteq \mathbb{S}_t \setminus \{E_{s+1}, \ldots, E_t\}$, then $\alpha_{W_\bullet}(mZ)$ is $\alpha(mZ)$ counted on the surface $\mathbb{S}_t$, whereas $\alpha_{V_\bullet}(mZ)$ can be regarded as either $\alpha(mZ)$ on $\mathbb{S}_t$ or $\alpha(mZ)$ on $\mathbb{S}_s$. This allows us to compare α's computed on various del Pezzo surfaces, provided that it makes sense to consider the underlying set Z on both surfaces.

The study of sequences of minimal growth can be paralleled by the investigation of ratios of the type

$$\frac{\alpha(mZ)}{m}.$$

There are some estimates for such quotients. The first result of this kind,

$$\frac{\alpha(mZ)}{m} \geq \frac{\alpha(Z)+1}{2}$$

appears in a work of Chudnovsky, and it concerns finite sets of points Z in the projective plane (see [9], Proposition 3.1 and [8]). It was generalized to other spaces, i.e., $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surfaces (see [1,2,5,6,10]). All these inequalities are known under the common name *Chudnovsky-type results*. In Section 4, we present our result of this kind for del Pezzo surfaces $\mathbb{S}_r$ with $r \leq 6$.

Our paper is concluded by a comparison of points fattening effect for surface $\mathbb{S}_1$ considered as a del Pezzo surface and on the other hand as a Hirzebruch surface.

3 The Points Fattening Effect on $\mathbb{S}_r$

In this section, we present some results concerning the fattening effect on $\mathbb{S}_r$. Let us recall that $\mathbb{S}_r$ arises as the blowing-up of the complex projective plane in fixed r general points $P_1, \ldots, P_r$. We denote by $f_r : \mathbb{S}_r \to \mathbb{P}^2$ the blow-up, where $E_1, \ldots, E_r$ are the exceptional divisors. If r is fixed, then we write simply f instead of f_r.

In further considerations, we will use the following observations about blow-ups.

Remark 3.1 If F is a plane curve of degree $3k$ in $\mathbb{P}^2$ passing through points $P_1, \ldots, P_r$, so that $\operatorname{mult}_{P_i}(F) = m_i \geq k$ for $i \in \{1, \ldots, r\}$, then E_i is a $(m_i - k)$-tuple component of the divisor $f_r^*(F) - kE_1 - \cdots - kE_r$ in the system

$$|3kH - kE_1 - \cdots - kE_r| = |-kK_{\mathbb{S}_r}|.$$

Definition 3.2 *(Adapted transform)* We keep the notation as in Remark 3.1. The adapted transform of F is the divisor

$$A(F) := f_r^*(F) - kE_1 - \cdots - kE_r = \widetilde{F} + \sum_{i=1}^{r}(m_i - k)E_i,$$

where $\widetilde{F}$ denotes the proper transform of F.

Lemma 3.3 *Let $D \in |-kK_{\mathbb{S}_r}|$ for fixed $1 \leq r \leq 8$ and let $Q \in \mathbb{S}_r$. Then,*

$$\operatorname{mult}_Q(D) \leq 2 \cdot \operatorname{mult}_{f_r(Q)}(f_r(D)) - k, \tag{1}$$

if $Q \in E_1 \cup \ldots \cup E_r$ and

$$\operatorname{mult}_Q(D) = \operatorname{mult}_{f_r(Q)}(f_r(D)) \leq 3k, \tag{2}$$

if $Q \notin E_1 \cup \ldots \cup E_r$. Furthermore, if equality holds in (2), then $f_r(D)$ is a union of lines through $f_r(Q)$.

Proof Let $D \in |-kK_{\mathbb{S}_r}|$ and $Q \in \mathbb{S}_r$. Then, $\deg(f_r(D)) = 3k$. Let us denote by $m = \operatorname{mult}_Q(D)$.

First, we consider the situation, when $Q \notin E_1 \cup \ldots \cup E_r$. Since f_r is an isomorphism away of points $\{P_1, \ldots, P_r\}$, $\operatorname{mult}_{f_r(Q)}(f_r(D)) = m$. The multiplicity of the singular point of the plane curve can be at most the degree of this curve; thus, $f_r(D)$ may have at most $3k$-tuple points, which finishes the proof of statement (2).

We assume now that $Q \in E_i$ for some $i \in \{1, \ldots, r\}$. Let us denote by $F = f_r(D)$. Then,

$$\operatorname{mult}_Q(D) = \operatorname{mult}_{P_i}(F) - k + \operatorname{mult}_Q(\widetilde{F}).$$

But $\operatorname{mult}_Q(\widetilde{F}) \leq \operatorname{mult}_{P_i}(F)$; thus, we finally obtain statement (1). □

A natural consequence of Lemma 3.3 is the following property for surfaces $\mathbb{S}_r$.

Corollary 3.4 *If $Z \subset \mathbb{S}_r$ for $1 \leq r \leq 8$ satisfies the condition*

$$\alpha(mZ) = \cdots = \alpha((m+t)Z)$$

for some positive integers m and $t \geq 3$, then $Z \subset E_1 \cup \cdots \cup E_r$.

Now we turn to the main subject of this paper, namely a characterization of subschemes Z with

$$\alpha(Z) = \alpha(2Z) \cdots = \alpha(mZ) = 1.$$

We begin with surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$.

3.1 Surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$

Theorem 3.5 *Let $Z \subset \mathbb{S}_1$ be a finite set of points. Then, the following conditions are equivalent*

i) $Z = \{Q\} \subset E_1$,
ii) $\alpha(Z) = \alpha(2Z) = \alpha(3Z) = \alpha(4Z) = \alpha(5Z) = 1$.

Proof The implication from $i)$ to $ii)$ is obvious. It is enough to consider the nonreduced curve $F = 3L \subset \mathbb{P}^2$ for some line passing through point P_1. Indeed, it gives rise to

$$A(F) = f^*F - E_1 = 3\widetilde{L} + 2E_1$$

in $\mathbb{S}_1$, which vanishes to order 5 at $Q \in \widetilde{L} \cap E_1$.

In order to prove the reverse implication, let $Z = \{Q_1, \ldots, Q_s\}$ and we assume that $D \in |-K_{\mathbb{S}_1}|$ is a divisor satisfying $\operatorname{mult}_{Q_i}(D) \geq 5$ for all points $Q_i \in Z$. By Corollary 3.4, we have that $Z \subset E_1$.

Let us consider possible types of cubic curves in the projective plane and their adapted transforms. The curve F has to pass through point P_1, and in order to get the highest possible multiplicities along the exceptional divisor E_1, it should have the highest possible multiplicity at P_1. We have the following types of cubic curves in $\mathbb{P}^2$:

a) an irreducible cubic (possibly singular);
b) a union of an irreducible conic and a line;
c) a union of three lines (possibly not distinct).

In case $a)$, the divisor $A(F)$ on $\mathbb{S}_1$ has points of multiplicity at most two. In case b), the highest possible multiplicity of a point on E_1 is three, and this happens in the case when the line is tangent to the conic at point P_1.

Let us pass to the case $c)$. We know that the adapted transform of a curve F consisting of some triple line L has a quintuple point. Except this arrangement of three lines, we never get the quintuple points, which completes the proof. □

Remark 3.6 In fact, one can weaken condition $ii)$ in Theorem 3.5. Assuming

$$\alpha(mZ) = \alpha((m+1)Z) = \alpha((m+2)Z) = \alpha((m+3)Z) = \alpha((m+4)Z)$$

for some $m \geq 1$, implies that $Z = \{Q\} \subset \mathbb{S}_1$.

Proof Let $Z = \{Q_1, \ldots, Q_t\}$ be such that $\alpha(mZ) = \cdots = \alpha((m+4)Z) = k$ for some integers k and t, and let $D \in |-kK_{\mathbb{S}_1}|$ be a divisor such that $\operatorname{mult}_{Q_i}(D) \geq m+4$ for any point $Q_i \in Z$. Firstly, by Corollary 3.4 we conclude that $Z \subset E_1$.

Let us denote by $F = f(D)$. Since F is of degree $3k$, its multiplicity at P_1 is at most $3k$. Hence, the multiplicity of E_1 in D is at most $2k$. This contributes to the multiplicity of D at every point $Q_1, \ldots, Q_t$. The remaining multiplicity at these points must come from components of F passing through P_1 at directions corresponding to $Q_1, \ldots, Q_t$. We have

$$t(m+4) \leq \sum_{i=1}^{t} \operatorname{mult}_{Q_i} D \leq 3k + 2kt. \tag{3}$$

On the other hand, since $\alpha(mZ) = k$, it must be

$$3(k-1) + 2(k-1)t < t \cdot m, \tag{4}$$

since otherwise one could find $3(k-1)$ lines through P_1. Their images in $\mathbb{P}^2$ would show $\alpha(mZ) \leq k-1$ contradicting the assumption. Combining (3) and (4), we get that

$$3k - 3 + 2kt - 2t + 4t < 3k + 2kt$$

and thus $t < \frac{3}{2}$, which finally means that Z is a single point. □

On $\mathbb{S}_1$, there also exist infinitely many sets satisfying a weaker condition, namely

$$\alpha(mZ) = \cdots = \alpha((m+3)Z),$$

and these sets are not necessarily the same as in Theorem 3.5.

Theorem 3.7 *Let $Z \subset \mathbb{S}_1$ be a finite set of points and let m be a positive integer. Then, the following conditions are equivalent:*

i) $\alpha(mZ) = \cdots = \alpha((m+3)Z)$;
ii) $Z = \{Q\} \subset E_1$ or $Z = \{Q_1, Q_2\} \subset E_1$, where $Q_1 \neq Q_2$.

Proof The sets in ii) satisfy the condition

$$\alpha(mZ) = \cdots = \alpha((m+3)Z),$$

for example, with $m = 1$ and $m = 4$, respectively. We will prove the opposite implication. Suppose now that $Z = \{Q_1, \ldots, Q_t\}$ is a set such that $\alpha(mZ) = \cdots = \alpha((m+3)Z) = k$ for some integers k and t and let $D \in |-kK_{\mathbb{S}_1}|$ be a divisor such that $\text{mult}_{Q_i}(D) \geq m+3$ for any point $Q_i \in Z$. Let us denote by $F = f(D)$, with $\deg(F) = 3k$.

In fact, we can repeat reasoning used in the proof of Remark 3.6, but this time with the following estimates

$$t(m+3) \leq \sum_{i=1}^{t} \text{mult}_{Q_i} D \leq 3k + 2kt, \tag{5}$$

$$3(k-1) + 2(k-1)t < t \cdot m. \tag{6}$$

By (5) combined with (6), we get

$$3k - 3 + 2kt - 2t + 3t < 3k + 2kt,$$

which gives $t < 3$. □

Corollary 3.8 *For a finite set of points $Z \subset \mathbb{S}_1$ and a positive integer m, we have*

$$\alpha(mZ) < \alpha((m+5)Z).$$

Proof Suppose to the contrary that $Z = \{Q_1, \dots, Q_t\}$ is such that $\alpha(mZ) = \dots = \alpha((m+5)Z) = k$ for some integers k and t, and let $D \in |-kK_{\mathbb{S}_1}|$ be a divisor such that $\text{mult}_{Q_i}(D) \geq m+3$ for $Q_i \in Z$. Let $F = f(D)$, with $\deg(F) = 3k$. In the spirit of the proof of Remark 3.6, we get estimates

$$t(m+5) \leq \sum_{i=1}^{t} \text{mult}_{Q_i} D \leq 3k + 2kt, \tag{7}$$

$$3(k-1) + 2(k-1)t < t \cdot m. \tag{8}$$

Combining (7) and (8), we obtain $t < 1$, but t is a positive integer, a contradiction. □

One may also consider sets with three initial values equal to 1. The list of possible types gets much longer, but the arguments used to obtain their classification are similar to those used above. We refer the interested reader to [11] (Subchapter 6.1).

In further considerations, lines joining points $P_1, \dots, P_r$ play an important role. From now on, we denote by L_{ij} the line passing through points P_i and P_j for fixed distinct $i, j \in \{1, \dots, r\}$. We are ready to formulate analogous results concerning the fattening effect on $\mathbb{S}_2$.

Theorem 3.9 *The following conditions are equivalent:*

i) $Z \subset \widetilde{L_{12}} \cap (E_1 \cup E_2)$;
ii) $\alpha(Z) = \dots = \alpha(5Z) = 1$.

Theorem 3.10 *The following conditions are equivalent:*

i) $Z = \{Q\} \subset (E_1 \cup E_2) \setminus \widetilde{L_{12}}$;
ii) $\alpha(Z) = \dots = \alpha(4Z) = 1$ *and* $\alpha(5Z) > 1$.

Theorem 3.11 *For any finite set of points $Z \subset \mathbb{S}_2$ and any positive integer m, we have $\alpha(mZ) < \alpha((m+5)Z)$.*

Theorem 3.12 *For any finite set of point $Z \subset \mathbb{S}_2$, the following conditions are equivalent:*

i) there exists a positive integer m such that $\alpha(mZ) = \dots = \alpha((m+4)Z)$;
ii) $Z \subset (E_1 \cup E_2) \cap \widetilde{L_{12}}$.

Theorem 3.13 *Let $Z \subset \mathbb{S}_2$ be a finite set of points such that $Z \subset E_i$ for $i \in \{1, 2\}$. Then, Z satisfies the condition*

$$\alpha(mZ) = \dots = \alpha((m+3)Z) < \alpha((m+4)Z), \tag{9}$$

for some positive integer m if and only if Z has the following form: either

a) $Z = \{Q\} \subset E_i \setminus \widetilde{L_{12}}$ *for* $i \in \{1, 2\}$, *or*
b) $Z = \{Q, Q'\} \subset E_i$ *for* $i \in \{1, 2\}$, *where* $Q' \in \widetilde{L_{12}}$.

All theorems from 3.9 to 3.13 can be proved analogously as in the case of surface $\mathbb{S}_1$, or the reader can find alternative proofs in [11]. In subchapter 2.4 of [11], reader can also find a description of sets satisfying the condition

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1.$$

Question about the maximal integer a satisfying condition

$$\alpha(mZ) = \alpha((m+1)Z) \cdots = \alpha((m+a)Z)$$

is not trivial, and it is still an open problem for some of studied before surfaces. For example, fattening effect for the Hirzebruch surfaces is described in [5] only with respect to the condition

$$\alpha(Z) = \cdots = \alpha(mZ) = 1.$$

The available tools are not useful in the case of greater r, in particular when considered sets have some points on the exceptional divisors. For that reason, for remaining surfaces $\mathbb{S}_r$ we only establish maximal m, for which $\alpha(Z) = \cdots = \alpha(mZ) = 1$ hold and we describe sets Z with that property.

3.2 Surfaces $\mathbb{S}_r$ for $r \geq 3$

The natural sequence of inclusions between linear systems

$$|-K_{\mathbb{S}_1}| \supset |-K_{\mathbb{S}_2}| \supset |-K_{\mathbb{S}_3}| \supset |-K_{\mathbb{S}_4}| \supset |-K_{\mathbb{S}_5}| \supset |-K_{\mathbb{S}_6}| \supset |-K_{\mathbb{S}_7}| \supset |-K_{\mathbb{S}_8}|$$

suggests that the sequence of equalities

$$\alpha(Z) = \alpha(2Z) = \cdots = 1$$

should become shorter with r growing. In the case of $r = 1$ and $r = 2$, we had $\alpha(5Z) = 1$ and we proved, moreover, that there it is not possible to obtain more than five consecutive initial values equal.

We present now such a characterization for remaining surfaces $\mathbb{S}_r$.

Theorem 3.14 *Let $Z \subset \mathbb{S}_3$ be a finite set of points. The following conditions are equivalent:*

i) $\alpha(Z) = \cdots = \alpha(4Z) = 1$;
ii) $Z = \{Q\} = E_i \cap \widetilde{L_{ij}}$ for distinct $i, j \in \{1, 2, 3\}$.

Theorem 3.15 *Let $Z \subset \mathbb{S}_4$ be a finite set of points. Then, Z satisfies equality $\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$, if and only if it is one of the following sets:*

a) $Z = \{Q\} \subset \widetilde{L_{ij}}$ for distinct $i, j \in \{1, 2, 3, 4\}$,
b) $Z \subset \{Q, Q_1, Q_2\} \subset \widetilde{L_{ij}}$, where $Q_1 \in E_i$, $Q_2 \in E_j$ and $Q = \widetilde{L_{ij}} \cap \widetilde{L_{kl}}$ for pairwise distinct $i, j, k, l \in \{1, 2, 3, 4\}$,

c) $Z \subset E_i \cap (\widetilde{L_{ij}} \cup \widetilde{L_{il}} \cup \widetilde{L_{ik}})$ *for pairwise distinct* $i, j, k, l \in \{1, 2, 3, 4\}$,
d) $Z = \{Q\} \subset E_i$ *and* $Q \in \widetilde{C} \cap \widetilde{L}$, *where* C *is an irreducible conic curve passing through points* P_1, P_2, P_3, P_4 *and* L *is the line tangent to* C *at point* P_i *for* $i \in \{1, 2, 3, 4\}$.

Theorem 3.16 *Let* $Z \subset \mathbb{S}_5$ *be a finite set of points. Then, the condition* $\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$ *is fulfilled if and only if* $Z = \{Q\}$ *and point* Q *satisfies one of the following two conditions:*

a) $Q \in E_i \cap \widetilde{L_i} \cap \widetilde{C}$ *for* $i \in \{1, 2, 3, 4, 5\}$, *where* C *is a conic passing through points* $P_1, \ldots, P_5$ *and* L_i *is a line tangent to* C *at point* P_i,
b) $Q \in \widetilde{L_{ij}} \cap \widetilde{L_{kl}}$ *for pairwise distinct* $i, j, k, l \in \{1, 2, 3, 4, 5\}$.

Theorem 3.17 *Let* $Z \subset \mathbb{S}_6$ *be a finite set of points. Then,* Z *satisfies equality* $\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$, *if and only if* $Z = \{Q\}$ *and point* Q *fulfills one of the following two conditions:*

a) $Q \in \widetilde{L_{ij}} \cap \widetilde{L_{kl}} \cap \widetilde{L_{mn}}$ *for pairwise distinct* $i, j, k, l, m, n \in \{1, 2, 3, 4, 5, 6\}$,
b) $Q \in E_i \cap \widetilde{L_{ij}} \cap \widetilde{C_j}$ *for distinct* $i, j \in \{1, 2, 3, 4, 5, 6\}$, *where* C_j *is a conic curve determined by five points of* $P_1, \ldots, P_6$ *excluding* P_j *and* L_{ij} *is of course the line passing through points* P_i *and* P_j, *but simultaneously* L_{ij} *is the tangent line to the curve* C_j *at point* P_i.

Let us note that $\mathbb{S}_6$ is the first example of surfaces $\mathbb{S}_r$, where the existence of a set Z satisfying the condition

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$$

depends on the geometry of points $P_1, \ldots, P_6$. For given six points in general position in the projective plane, there always exists a cubic curve consisting of three lines, passing through these points. But these lines do not have to intersect at one point (Fig. 1). It is a rather strong condition.

Similarly, when the cubic splits into a conic and a line, each five points determine a conic curve in a unique way. But the line joining the sixth point with one of the previous five does not necessarily need to be tangent to this conic (Fig. 2). It is also a situation, which may happen or not, and it depends on the arrangement of the starting six points (although they are always in general position). It is a quite interesting phenomenon. Especially that for remaining two surfaces $\mathbb{S}_r$, i.e., $\mathbb{S}_7$ and $\mathbb{S}_8$, the condition

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1$$

is never satisfied.

Theorem 3.18 *Let* $Z \subset \mathbb{S}_7$ *be a finite set of points. The equality* $\alpha(Z) = \alpha(2Z) = 1$ *holds if and only if* Z *is one of the following sets:*

a) $Z \subset \{Q_1, Q_2\} \subset E_i \cap \widetilde{F}$, *where* F *is an irreducible singular cubic curve with the singularity at point* P_i *for some* $i \in \{1, \ldots, 7\}$,

Fig. 1 The three lines intersecting at a single point

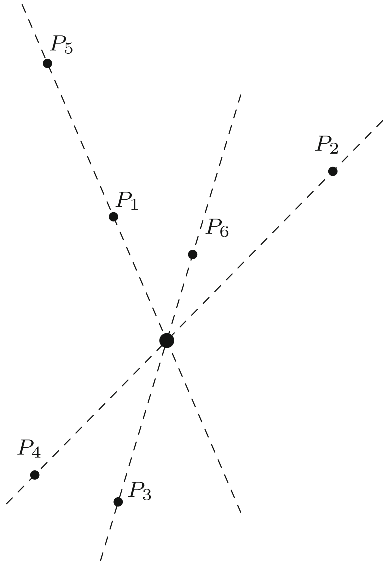
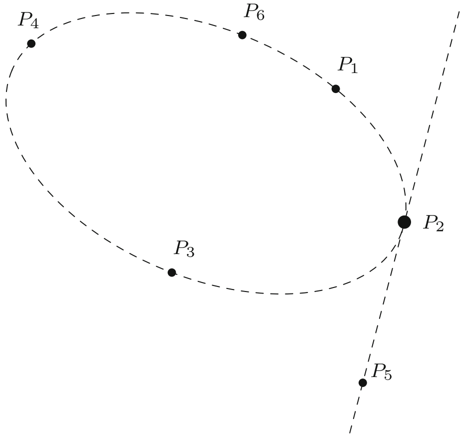


Fig. 2 The line tangent to a conic



b) $Z = \{Q\}$, where $f(Q)$ is the double point of a singular cubic passing through points $P_1, \ldots, P_7$ and $Q \notin E_1 \cup \cdots \cup E_7$,
c) $Z \subset \widetilde{L_{ij}} \cap \widetilde{C_{ij}}$ for distinct $i, j \in \{1, \ldots, 7\}$, where C_{ij} is the irreducible conic passing through the five points of $P_1, \ldots, P_7$, distinct from P_i and P_j.

Proofs of Theorems 3.14 to 3.18 are based on a review of plane cubics passing through points $P_1, \ldots, P_r$ (analogously as in the proof of Theorem 3.5). Thus, we skip details here. We refer the curious reader to [11].

The line bundle $-K_{\mathbb{S}_8}$ has the least number of sections of all line bundles $-K_{\mathbb{S}_r}$ considered so far, namely $h^0(-K_{\mathbb{S}_8}) = 1$. For that reason, we expected that

$$\alpha(2Z) \geq 2$$

here. Thus, the fact that there exists a set Z, where $\alpha(Z) = \alpha(2Z) = 1$, was surprising.

Theorem 3.19 *If $Z \subset \mathbb{S}_8$, then the equality $\alpha(Z) = \alpha(2Z) = 1$ holds iff $Z = \{Q\}$, where $f(Q)$ is the singular point of an irreducible cubic curve F passing through points $P_1, \ldots, P_8$ and $f(Q)$ is distinct of any point P_i for $i \in \{1, \ldots, 8\}$.*

Proof If $Z = \{Q\}$ and $f(Q)$ is double point on a cubic, distinct of any P_i, then it is obvious that Q has multiplicity 2 on $\mathbb{S}_8$ and of course $\alpha(Z) = \alpha(2Z) = 1$.

Let us focus on the opposite implication. Let $D \in |-K_{\mathbb{S}_8}|$ be such that $\text{mult}_Q(D) \geq 2$ for any point $Q \in Z$. The curve $F = f(D)$ has degree 3 and passes through eight distinct points $P_1, \ldots, P_8$ in general position. Then, F is irreducible. Irreducible cubic has at most one singular point, and it cannot be any of P_i's (general points). Thus, it must be $f(Q)$.

To finish the proof, we need to show that there always exists a singular cubic curve passing through eight given general points.

Recall that cubics passing through eight fixed points form a pencil, if no four points lie on a line and no seven lie on a conic. Since $P_1, \ldots, P_8$ are in general position, the family of cubics passing through these points is a pencil. We denote it by $\mathcal{S}$. Every two cubics in $\mathcal{S}$ meet in nine points; thus, the set $\{P_1, \ldots, P_8\}$ determines a new point. This point is determined uniquely (Cayley–Bacharach theorem, see [12], Theorem 1). Let us denote it by P_9.

Let $\widetilde{\mathbb{S}}$ be a blow-up of $\mathbb{P}^2$ in all nine points $P_1, \ldots, P_9$. Then, $\widetilde{\mathbb{S}}$ is the total space of the pencil $\mathcal{S}$, and we have the morphism

$$\varphi : \widetilde{\mathbb{S}} \to \mathbb{P}^1,$$

whose fibers are the elements of $\mathcal{S}$. Let $e(\cdot)$ be the topological Euler characteristic. Thus, we have

$$e(\mathcal{S}) = e(\mathbb{P}^2) + 9 = 12.$$

Suppose now, to the contrary, that φ has only smooth fibers. Then, from the topological point of view we have

$$\widetilde{\mathbb{S}} = \mathbb{P}^1 \times \mathcal{E},$$

where $\mathcal{E}$ is an elliptic curve. We have then

$$e(\mathcal{S}) = e(\mathbb{P}^1) \cdot e(\mathcal{E}) = 2 \cdot 0 = 0.$$

Thus, $\mathcal{S}$ must contain singular fibers. Since points $P_1, \ldots, P_8$ are in general position, these singular fibers are irreducible cubics, which ends the proof. □

4 The Chudnovsky-Type Result for Surfaces $\mathbb{S}_r$

We conclude our considerations by a lower bound on the growth rate of the initial sequence for surfaces $\mathbb{S}_r$. We present a general estimate for sets Z satisfying the condition $\alpha(Z) \geq 2$. The assumption of very ampleness of line bundle $-K_{\mathbb{S}_r}$ is significant; thus, our result concerns surfaces $\mathbb{S}_r$ with $r \leq 6$.

Theorem 4.1 *Let $1 \leq r \leq 6$ and $Z \subset \mathbb{S}_r$ be a finite set of points such that $\alpha := \alpha(Z) \geq 2$. Then, we have*

$$\frac{\alpha(mZ)}{m} \geq \frac{\alpha - 1}{2}.$$

Proof We have

$$h^0(-mK_{\mathbb{S}_r}) = \binom{3m+2}{2} - r \cdot \binom{m+1}{2} = \frac{(9-r)m^2 + (9-r)m + 2}{2}.$$

We choose a minimal subset $W \subseteq Z$, such that $\alpha(W) = \alpha$, i.e., there is no element in $(\alpha - 1) \cdot (-K_{\mathbb{S}_r})$. The minimality of W is taken with respect to the inclusion. (Thus, there can be several sets satisfying this condition.) It follows that the points in W impose independent conditions on the space of sections in $|(\alpha - 1) \cdot (-K_{\mathbb{S}_r})|$. Then,

$$\#W = t = \binom{3\alpha - 1}{2} - r \cdot \binom{\alpha}{2} = \frac{(9-r)\alpha^2 - (9-r)\alpha + 2}{2}.$$

We claim that $|\alpha \cdot (-K_{\mathbb{S}_r}) \otimes I_W|$ has no additional base points on $\mathbb{S}_r$, i.e., not contained in W. Let $W = \{Q_1, \ldots, Q_t\}$. For any Q_i, there exists a curve $C_i \in (\alpha - 1) \cdot (-K_{\mathbb{S}_r})$, such that C_i does not vanish at Q_i and it does vanish at all points in $W \setminus \{Q_i\}$. Let s_i denote the section in $H^0(\mathbb{S}_r, (\alpha - 1) \cdot (-K_{\mathbb{S}_r}))$ corresponding to C_i. Then, the sections $s_1, \ldots, s_t$ form a basis of $H^0(\mathbb{S}_r, (\alpha - 1) \cdot (-K_{\mathbb{S}_r}))$.

Suppose that $R \in \mathbb{S}_r \setminus W$ is a base point of $|\alpha \cdot (-K_{\mathbb{S}_r}) \otimes I_W|$. There exists a section $s_i \in \{s_1, \ldots, s_t\}$ not vanishing at R. Indeed, otherwise R would be a common zero of $|(\alpha - 1) \cdot (-K_{\mathbb{S}_r})|$, which is not possible by the choice of W. Since $-K_{\mathbb{S}_r}$ is very ample, the system $|-K_{\mathbb{S}_r} \otimes I_{Q_i}|$ is then base point free away from Q_i. Hence, there exists a section $s \in H^0(\mathbb{S}_r, -K_{\mathbb{S}_r} \otimes I_{Q_i})$ not vanishing at R. Then, in particular, $|\alpha \cdot (-K_{\mathbb{S}_r}) \otimes I_W|$ has no base component. Thus,

$$s_i \cdot s \in H^0(\mathbb{S}_r, (\alpha - 1) \cdot (-K_{\mathbb{S}_r}) \otimes I_{W \setminus \{Q\}} \otimes (-K_{\mathbb{S}_r}) \otimes I_{Q_i}) = H^0(\mathbb{S}_r, \alpha \cdot (-K_{\mathbb{S}_r}) \otimes I_W)$$

is a section not vanishing at R. Let $A \in |\alpha \cdot (-K_{\mathbb{S}_r})|$ and $B \in |\alpha(mZ) \cdot (-K_{\mathbb{S}_r})|$. Using Bezout theorem, we obtain

$$\alpha \cdot \alpha(mZ) \cdot (-K_{\mathbb{S}_r})^2 = A \cdot B \geq \frac{(9-r)\alpha^2 - (9-r)\alpha + 2}{2} \cdot m,$$

which finally implies

$$\frac{\alpha(mZ)}{m} \geq \frac{(9-r)\alpha^2 - (9-r)\alpha + 2}{2\alpha(9-r)} > \frac{(9-r)\alpha^2 - (9-r)\alpha}{2\alpha(9-r)} > \frac{\alpha - 1}{2}.$$

This ends the proof. □

Remark 4.2 Let us notice that if $\alpha(Z) = 1$, then $\frac{\alpha(mZ)}{m} \geq \frac{1}{5}$. Firstly, let us observe that if Z is the set from Theorem 3.5, then its initial sequence is of the form

$$1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, \dots .$$

The divisor $F = 3kL$ for the line L passing through P_1 and corresponding to point $Q \in E_1$ gives rise to $D = 3k\widetilde{L} + 2kE_1 \in |3kH - kE_1|$ on the blow-up $\mathbb{S}_1$ and $\operatorname{mult}_Q(D) = 5k$ for any $Q \in Z$. Hence, $\alpha(5kZ) \leq k$ for any positive integer k.

For $k = 1$, we then obtain $\alpha(5Z) \leq 1$, which means that $\alpha(Z) = \cdots = \alpha(5Z) = 1$. Moreover, by Remark 3.6 we conclude

$$\alpha(6Z) \geq 2. \tag{10}$$

On the other hand, for $k = 2$ we have

$$\alpha(10Z) \leq 2. \tag{11}$$

From (10) and (11), we then obtain $\alpha(6Z) = \cdots = \alpha(10Z) = 2$.

Using the same argumentation for the next k, we finally conclude that the initial sequence in this case is $\alpha(mZ) = \lceil \frac{m}{5} \rceil$ and indeed $\frac{\alpha(mZ)}{m} \geq \frac{1}{5}$.

Let $\{\alpha'(mZ)\}$ be another subadditive and weakly growing sequence of positive integers with $\alpha'(Z) = 1$. By Remark 3.6,

$$\alpha'(mZ) \geq \alpha(mZ)$$

for any m, thus

$$\frac{\alpha'(mZ)}{m} \geq \frac{\alpha(mZ)}{m} \geq \frac{1}{5}.$$

By Lemma 2.3, we conclude that estimate $\frac{\alpha(mZ)}{m} \geq \frac{1}{5}$ concerns any initial sequence $\{\alpha(mZ)\}$ with $\alpha(Z) = 1$ for all surfaces $\mathbb{S}_r$. In the case of surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$, we were able to show that this estimate is optimal (in the sense that $\frac{1}{5}$ is the borderline value). Probably, this estimate is not sharp for $r \geq 3$.

5 Surface $\mathbb{S}_1$ as a del Pezzo Surface and as a Hirzebruch Surface

The surface $\mathbb{S}_1$ was considered with respect to the fattening effect in [5] as a Hirzebruch surface. An interesting phenomenon is that from the point of view of Hirzebruch surfaces, the most natural choice of the reference line bundle for $\mathbb{S}_1$ is

$$2H - E_1,$$

while if we consider it as a del Pezzo surface, we work with the anticanonical line bundle, i.e.,

$$-K_{\mathbb{S}_1} = 3H - E_1.$$

Di Rocco, Lundman, and Szemberg proved in [5] that on the Hirzebruch surface $\mathbb{S}_1$ (denoted there by $\mathbb{F}_1$) with $2H - E_1$, there does not exist any finite set Z, such that

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = \alpha(4Z)$$

(see [5], Proposition 4.1). From point of view of del Pezzo surfaces with the bundle $-K_{\mathbb{S}_1}$, we can even get

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = \alpha(4Z) = \alpha(5Z),$$

and moreover, there exist infinitely many sets satisfying it (all singletons $Z = \{Q\}$ with $Q \in E_1$). Thus, the choice of line bundle is a fundamental factor affecting the shape of the initial sequence.

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